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The behavior of the principal distributions on a real-analytic surface

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Abstract. The purposes of this paper are: (a) to study the behavior of the principal distributions around an isolated umbilical point on a real-analytic surface of a certain type, which contains all the real-analytic, special Weingarten surfaces; (b) to present one condition such that if a real-analytic surface satisfies the condition, then the index of an isolated umbilical point on the surface is less than or equal to one.

1. Introduction.

Let S be a smooth surface in $\boldsymbol{R}^3$ and $\mathrm{Umb}(S)$ the set of the umbilical points of S and set $\mathrm{Reg}(S) := S \backslash \mathrm{Umb}(S)$. If $\mathrm{Reg}(S) \neq \emptyset$, i.e., if S is not totally umbilical, then there exists a *principal distribution* D_S on S, which is a continuous one-dimensional distribution on $\mathrm{Reg}(S)$ such that $\mathsf{D}_S(p)$ is one of the principal directions at each $p \in \mathrm{Reg}(S)$. Let p_0 be an isolated umbilical point of S. Then as a quantity in relation to the behavior of D_S around p_0, the *index* $\mathrm{ind}_{p_0}(S)$ of p_0 on S is defined ([**6**, pp. 137]).

For each positive integer $l \in \boldsymbol{N}$, let $\mathscr{A}_o^{(l)}$ be the set of real-analytic functions defined on a connected neighborhood of $(0,0)$ in $\boldsymbol{R}^2$ such that each $F \in \mathscr{A}_o^{(l)}$ satisfies $(\partial^{m+n} F / \partial x^m \partial y^n)(0,0) = 0$ for non-negative integers $m, n \geqq 0$ satisfying $0 \leqq m+n < l$. Let F be an element of $\mathscr{A}_o^{(2)}$ and G_F the graph of F. If the origin o of $\boldsymbol{R}^3$ is an element of $\mathrm{Umb}(\mathsf{G}_F)$ and if $\mathrm{Reg}(\mathsf{G}_F) \neq \emptyset$, then there exists a nonzero element f_F of $\mathscr{A}_o^{(3)}$ satisfying $\mathrm{Reg}(\mathsf{G}_{F-f_F}) = \emptyset$, and there exists a nonzero homogeneous polynomial g_F of degree $k_F \geqq 3$ satisfying $f_F - g_F \in \mathscr{A}_o^{(k_F+1)}$. Let $\mathscr{A}_o^l$ be the subset of $\mathscr{A}_o^{(l)}$ such that on the graph of each element of $\mathscr{A}_o^l$, o is an isolated umbilical point. For each positive integer $k \in \boldsymbol{N}$, let $\mathscr{P}^k$ be the set of the homogeneous polynomials of degree k in two variables and set $\mathscr{P}_o^k := \mathscr{P}^k \cap \mathscr{A}_o^2$. Let $\mathscr{A}_{oo}^l$ be the subset of $\mathscr{A}_o^l$ such that $g_F \in \mathscr{P}_o^{k_F}$ holds for each $F \in \mathscr{A}_{oo}^l$. The purposes of this paper are

(a) to study the behavior of the principal distributions around o on the graph G_F of $F \in \mathscr{A}_{oo}^2$;

(b) to present one condition such that if $F \in \mathscr{A}_o^2$ satisfies the condition, then $\mathrm{ind}_o(\mathsf{G}_F) \leqq 1$ holds.

For $F \in \mathscr{A}_o^2$, there exist two principal distributions $\mathsf{D}_F^{(1)}, \mathsf{D}_F^{(2)}$ on G_F which give the principal directions at each point of $\mathrm{Reg}(\mathsf{G}_F)$, and there exists a positive number $\rho_0 > 0$ satisfying $\{0 < x^2 + y^2 < \rho_0^2\} \subset \mathrm{Reg}(\mathsf{G}_F)$. Let $\phi_F^{(i)}$ be a continuous function defined on $(0, \rho_0) \times \boldsymbol{R}$ satisfying

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$$\cos\phi_F^{(i)}(\rho,\theta)\frac{\partial}{\partial x}+\sin\phi_F^{(i)}(\rho,\theta)\frac{\partial}{\partial y}\in \mathsf{D}_F^{(i)}(\rho\cos\theta,\rho\sin\theta)$$

for $(\rho,\theta)\in(0,\rho_0)\times\boldsymbol{R}$. In Section 3, we shall prove the following:

Proposition 1.1. *For $F\in\mathscr{A}_o^2$ and $\theta_0\in\boldsymbol{R}$,*
(a) *there exists a number $\phi_{F,o}^{(i)}(\theta_0)\in\boldsymbol{R}$ satisfying*

$$\lim_{\rho\to+0}\phi_F^{(i)}(\rho,\theta_0)=\phi_{F,o}^{(i)}(\theta_0);$$

(b) *there exist numbers $\phi_{F,o}^{(i)}(\theta_0+0)$, $\phi_{F,o}^{(i)}(\theta_0-0)\in\boldsymbol{R}$ satisfying*

$$\lim_{\theta\to\theta_0\pm0}\phi_{F,o}^{(i)}(\theta)=\phi_{F,o}^{(i)}(\theta_0\pm0);$$

(c) *there exists an element $\Gamma_{F,o}(\theta_0)$ of $\{n\pi/2\}_{n\in\boldsymbol{Z}}$ satisfying*

$$\Gamma_{F,o}(\theta_0)=\phi_{F,o}^{(i)}(\theta_0+0)-\phi_{F,o}^{(i)}(\theta_0-0)$$

for $i=1,2$.

For $k\geqq 3$, let g be an element of $\mathscr{P}^k$. For $\theta\in\boldsymbol{R}$, let $\mathrm{Hess}_g(\theta)$ be the Hessian of g at $(\cos\theta,\sin\theta)$. Let η_g be a continuous function on $\boldsymbol{R}$ such that ${}^t(\cos\eta_g(\theta),\sin\eta_g(\theta))$ is an eigenvector of $\mathrm{Hess}_g(\theta)$ for any $\theta\in\boldsymbol{R}$, and S_g the set of the numbers at each of which Hess_g is represented by the unit matrix up to a constant. In Section 3, we shall prove the following:

Proposition 1.2. *For $F\in\mathscr{A}_o^2$,*
(a) *if $\theta_0\in\boldsymbol{R}$ satisfies $\Gamma_{F,o}(\theta_0)\neq 0$, then $\theta_0\in S_{g_F}$ holds;*
(b) *$\mathrm{ind}_o(\mathsf{G}_F)$ is represented as*

$$\mathrm{ind}_o(\mathsf{G}_F)=\frac{\eta_{g_F}(\theta+2\pi)-\eta_{g_F}(\theta)}{2\pi}+\frac{1}{2\pi}\sum_{\theta_0\in S_{g_F}\cap[\theta,\theta+2\pi)}\Gamma_{F,o}(\theta_0),$$

where $\theta\in\boldsymbol{R}$.

In Section 4, we shall present one way of computing $\eta_{g_F}(\theta+2\pi)-\eta_{g_F}(\theta)$. In Section 5, we shall prove the following:

Theorem 1.3. *For $F\in\mathscr{A}_{oo}^2$,*
(a) *$-\pi/2\leqq\Gamma_{F,o}(\theta_0)\leqq\pi/2$ hold for any $\theta_0\in S_{g_F}$;*
(b) *$\mathrm{ind}_o(\mathsf{G}_{g_F})\leqq\mathrm{ind}_o(\mathsf{G}_F)\leqq 1$ hold.*

Remark 1.4. For $k\geqq 3$, let g be an element of $\mathscr{P}_o^k$. Then the following hold:
(a) $\Gamma_{g,o}(\theta_0)=-\pi/2$ for $\theta_0\in S_g$ ([**3**]);
(b) $\mathrm{ind}_o(\mathsf{G}_g)\in\{1-k/2+i\}_{i=0}^{[k/2]}$ ([**1**]).

There exists a conjecture which asserts that $\mathrm{ind}_o(\mathsf{G}_F)\leqq 1$ holds for any $F\in\mathscr{A}_o^2$. This is part of Loewner's conjecture ([**7**], [**8**]). In Section 5, we shall prove the following:

Theorem 1.5. *Let F be an element of $\mathscr{A}_o^2$ satisfying $\Gamma_{F,o}(\theta_0)\leqq\pi$ for any $\theta_0\in S_{g_F}$. Then $\mathrm{ind}_o(\mathsf{G}_F)\leqq 1$ holds.*

Let S be a real-analytic, Weingarten surface and w_S a function of two variables satisfying $w_S(K_S, H_S) \equiv 0$ on S, where K_S and H_S are the Gaussian curvature and the mean curvature of S, respectively. In this paper, we suppose that w_S is real-analytic, and according to [**5**], we call S *special* if w_S satisfies

$$H_S \frac{\partial w_S}{\partial X}(K_S, H_S) + \frac{1}{2}\frac{\partial w_S}{\partial Y}(K_S, H_S) \neq 0 \tag{1}$$

on $\mathrm{Umb}(S)$. In Section 6, we shall prove the following:

THEOREM 1.6. *Let F be an element of $\mathscr{A}_o^{(2)}$ such that the graph G_F of F is a special Weingarten surface satisfying $o \in \mathrm{Umb}(\mathsf{G}_F)$ and $\mathrm{Reg}(\mathsf{G}_F) \neq \varnothing$. Then the following hold*:

(a) $F \in \mathscr{A}_{oo}^2$,

(b) $\mathrm{ind}_o(\mathsf{G}_F) = \mathrm{ind}_o(\mathsf{G}_{g_F}) = 1 - k_F/2$.

REMARK 1.7. Since $k_F \geqq 3$, we see that if F is as in Theorem 1.6, then $\mathrm{ind}_o(\mathsf{G}_F) < 0$ holds, which was already obtained in [**5**].

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2. Preliminaries.

Let f be a smooth function of two variables and G_f the graph of f. We set $p_f := \partial f/\partial x$, $q_f := \partial f/\partial y$, and

$$E_f := 1 + p_f^2, \quad F_f := p_f q_f, \quad G_f := 1 + q_f^2. \tag{2}$$

The *first fundamental form* of G_f is a symmetric tensor field I_f on G_f of type $(0,2)$ represented in terms of the coordinates (x, y) as

$$\mathrm{I}_f := E_f\,dx^2 + 2F_f\,dxdy + G_f\,dy^2,$$

where

$$dx^2 := dx \otimes dx, \quad dxdy := \frac{1}{2}(dx \otimes dy + dy \otimes dx), \quad dy^2 := dy \otimes dy.$$

We set $r_f := \partial^2 f/\partial x^2$, $s_f := \partial^2 f/\partial x\partial y$, $t_f := \partial^2 f/\partial y^2$, and

$$L_f := \frac{r_f}{\sqrt{\det(\mathrm{I}_f)}}, \quad M_f := \frac{s_f}{\sqrt{\det(\mathrm{I}_f)}}, \quad N_f := \frac{t_f}{\sqrt{\det(\mathrm{I}_f)}}, \tag{3}$$

where $\det(\mathrm{I}_f) := E_f G_f - F_f^2$. The *second fundamental form* of G_f is a symmetric tensor field II_f on G_f of type $(0,2)$ represented in terms of the coordinates (x, y) as

$$\mathrm{II}_f := L_f\,dx^2 + 2M_f\,dxdy + N_f\,dy^2.$$

For a point $p \in \mathsf{G}_f$, let $T_p(\mathsf{G}_f)$ be the tangent plane to G_f at p and $U_p(\mathsf{G}_f)$ the subset of $T_p(\mathsf{G}_f)$ defined by

$$U_p(\mathsf{G}_f) := \{\boldsymbol{u} \in T_p(\mathsf{G}_f); \mathrm{I}_{f,p}(\boldsymbol{u}, \boldsymbol{u}) = 1\}.$$

Let $v_{f,p}$ be the function on $U_p(\mathsf{G}_f)$ defined by $v_{f,p}(\boldsymbol{u}) := \mathrm{II}_{f,p}(\boldsymbol{u}, \boldsymbol{u})$ for $\boldsymbol{u} \in U_p(\mathsf{G}_f)$. Suppose that $v_{f,p}$ attains an extremum at $\boldsymbol{u}_0 \in U_p(\mathsf{G}_f)$. Then the extremum $v_{f,p}(\boldsymbol{u}_0)$ is called a *principal curvature* of G_f at p and the one-dimensional subspace of $T_p(\mathsf{G}_f)$ determined by $\boldsymbol{u}_0$ is called a *principal direction* of G_f at p. The *Weingarten map* of G_f is a tensor field W_f on G_f of type $(1, 1)$ satisfying

$$\left[\mathsf{W}_f\left(\frac{\partial}{\partial x}\right), \mathsf{W}_f\left(\frac{\partial}{\partial y}\right)\right] = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] W_f,$$

where

$$W_f := \begin{pmatrix} E_f & F_f \\ F_f & G_f \end{pmatrix}^{-1} \begin{pmatrix} L_f & M_f \\ M_f & N_f \end{pmatrix}.$$

By Lagrange's method of indeterminate coefficients, we obtain

Proposition 2.1. *The principal curvatures and the principal directions of* G_f *are given by the eigenvalues and the one-dimensional eigenspaces of* W_f, *respectively*.

The *Gaussian curvature* K_f and the *mean curvature* H_f of G_f are given by $K_f := \det(\mathsf{W}_f)$ and $H_f := \mathrm{tr}(\mathsf{W}_f)/2$, respectively.

Let PD_f be a symmetric tensor field on G_f of type $(0, 2)$ represented in terms of the coordinates (x, y) as

$$\mathrm{PD}_f := \frac{1}{\sqrt{\det(\mathrm{I}_f)}}\{A_f\, dx^2 + 2B_f\, dxdy + C_f\, dy^2\},$$

where

$$A_f := E_f M_f - F_f L_f, \quad 2B_f := E_f N_f - G_f L_f, \quad C_f := F_f N_f - G_f M_f.$$

For two vector fields $\boldsymbol{V}_1, \boldsymbol{V}_2$ on G_f, the following holds:

$$\frac{1}{2}\sum_{\{i,j\}=\{1,2\}} \boldsymbol{V}_i \wedge \mathsf{W}_f(\boldsymbol{V}_j) = \frac{\mathrm{PD}_f(\boldsymbol{V}_1, \boldsymbol{V}_2)}{\sqrt{\det(\mathrm{I}_f)}}\left(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\right).$$

Therefore by Proposition 2.1, we obtain

Proposition 2.2. *A tangent vector* $\boldsymbol{v}_0$ *to* G_f *is in a principal direction if and only if* $\mathrm{PD}_f(\boldsymbol{v}_0, \boldsymbol{v}_0) = 0$ *holds*.

A point p_0 of G_f is called *umbilical* if v_{f,p_0} is constant.

Proposition 2.3. *For a point* $p_0 \in \mathsf{G}_f$, *the following hold*:

(a) *The condition* $p_0 \in \mathrm{Umb}(\mathsf{G}_f)$ *is equivalent to each of the following*:
  (i) *any one-dimensional subspace of* $T_{p_0}(\mathsf{G}_f)$ *is a principal direction*,
  (ii) $A_f(p_0) = B_f(p_0) = C_f(p_0) = 0$;

(b) *The condition* $p_0 \in \mathrm{Reg}(\mathsf{G}_f)(= \mathsf{G}_f \setminus \mathrm{Umb}(\mathsf{G}_f))$ *is equivalent to each of the following*:

(i) *The number of the principal directions at* p_0 *is equal to two and they are perpendicular to each other with respect to* I_f,

(ii) $A_f(p_0)C_f(p_0) - B_f(p_0)^2 < 0$.

PROOF. Noticing Proposition 2.2, we obtain (a). In addition, noticing that W_f is symmetric with respect to I_f, we obtain (b). □

Let $\mathrm{D}_f, \mathsf{N}_f$ be symmetric tensor fields on G_f of type $(0,2)$ represented in terms of the coordinates (x, y) as

$$\mathrm{D}_f := s_f\,dx^2 + (t_f - r_f)\,dxdy - s_f\,dy^2,$$

$$\mathsf{N}_f := (s_f p_f^2 - p_f q_f r_f)\,dx^2 + (t_f p_f^2 - r_f q_f^2)\,dxdy + (p_f q_f t_f - s_f q_f^2)\,dy^2.$$

By (2) together with (3), we obtain $\det(\mathrm{I}_f)\,\mathrm{PD}_f = \mathrm{D}_f + \mathsf{N}_f$. For a vector field $\boldsymbol{V}$ on G_f, we set

$$\tilde{\mathrm{D}}_f(\boldsymbol{V}) := \mathrm{D}_f(\boldsymbol{V}, \boldsymbol{V}), \quad \tilde{\mathsf{N}}_f(\boldsymbol{V}) := \mathsf{N}_f(\boldsymbol{V}, \boldsymbol{V}),$$

$$\widetilde{\mathrm{PD}}_f(\boldsymbol{V}) := \mathrm{PD}_f(\boldsymbol{V}, \boldsymbol{V}).$$

We set

$$\mathrm{grad}_f := \begin{pmatrix} p_f \\ q_f \end{pmatrix}, \quad \mathrm{grad}_f^{\perp} := \begin{pmatrix} -q_f \\ p_f \end{pmatrix}, \quad \mathrm{Hess}_f := \begin{pmatrix} r_f & s_f \\ s_f & t_f \end{pmatrix}.$$

For $\phi \in \boldsymbol{R}$, we set

$$u_\phi := \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \quad \boldsymbol{U}_\phi := \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y}.$$

Let $\langle\,,\rangle$ be the scalar product in $\boldsymbol{R}^2$. Then we obtain

LEMMA 2.4. *For* $\phi \in \boldsymbol{R}$, *the following hold*:

$$\tilde{\mathrm{D}}_f(\boldsymbol{U}_\phi) = \langle \mathrm{Hess}_f\,u_\phi, u_{\phi+\pi/2}\rangle,$$

$$\tilde{\mathsf{N}}_f(\boldsymbol{U}_\phi) = \langle \mathrm{grad}_f, u_\phi\rangle\langle \mathrm{grad}_f^{\perp}, \mathrm{Hess}_f\,u_\phi\rangle.$$

We set $\mathbf{Grad}_f := p_f\partial/\partial x + q_f\partial/\partial y$. By Lemma 2.4, we obtain

LEMMA 2.5. $\widetilde{\mathrm{PD}}_f(\mathbf{Grad}_f) = \tilde{\mathrm{D}}_f(\mathbf{Grad}_f)$.

3. The behavior of the principal distributions around an umbilical point.

For $F \in \mathscr{A}_o^{(2)}$, let Φ_F be a real-analytic function on $(-\rho_0, \rho_0) \times \boldsymbol{R} \times \boldsymbol{R}$ defined by

$$\Phi_F(\rho, \theta, \phi) := \det(\mathrm{I}_{F,(\rho\cos\theta,\rho\sin\theta)})\,\widetilde{\mathrm{PD}}_{F,(\rho\cos\theta,\rho\sin\theta)}(\boldsymbol{U}_\phi),$$

where $\rho_0 > 0$ satisfies $\{x^2 + y^2 < \rho_0^2\} \subset \mathsf{G}_F$. We see that $\Phi_F(\rho, \theta, \phi)$ is the value of $\tilde{\mathrm{D}}_F(\boldsymbol{U}_\phi) + \tilde{\mathsf{N}}_F(\boldsymbol{U}_\phi)$ at $(\rho\cos\theta, \rho\sin\theta)$. Let $\mathscr{A}_o^{\langle 2\rangle}$ be the subset of $\mathscr{A}_o^{(2)}$ such that for each $F \in \mathscr{A}_o^{\langle 2\rangle}$, $o \in \mathrm{Umb}(\mathsf{G}_f)$ and $\mathrm{Reg}(\mathsf{G}_f) \neq \varnothing$ hold. Then for $F \in \mathscr{A}_o^{\langle 2\rangle}$, the following hold:

$$\begin{aligned}
&\tilde{\mathsf{D}}_F(\boldsymbol{U}_\phi)+\tilde{\mathsf{N}}_F(\boldsymbol{U}_\phi)\\
&\quad=\langle(\mathrm{Hess}_{f_F}+\mathrm{Hess}_{F-f_F})u_\phi,u_{\phi+\pi/2}\rangle+\langle(\mathrm{grad}_{f_F}+\mathrm{grad}_{F-f_F}),u_\phi\rangle\\
&\qquad\times\langle(\mathrm{grad}_{f_F}^{\perp}+\mathrm{grad}_{F-f_F}^{\perp}),(\mathrm{Hess}_{f_F}+\mathrm{Hess}_{F-f_F})u_\phi\rangle\\
&\quad=\tilde{\mathsf{D}}_{f_F}(\boldsymbol{U}_\phi)+\tilde{\mathsf{N}}_{f_F}(\boldsymbol{U}_\phi)+\tilde{\mathsf{D}}_{F-f_F}(\boldsymbol{U}_\phi)+\tilde{\mathsf{N}}_{F-f_F}(\boldsymbol{U}_\phi)+\langle\mathrm{grad}_{f_F},u_\phi\rangle\langle\mathrm{grad}_{f_F}^{\perp},\mathrm{Hess}_{F-f_F}u_\phi\rangle\\
&\qquad+\langle\mathrm{grad}_{f_F},u_\phi\rangle\langle\mathrm{grad}_{F-f_F}^{\perp},\mathrm{Hess}_{f_F}u_\phi\rangle+\langle\mathrm{grad}_{f_F},u_\phi\rangle\langle\mathrm{grad}_{F-f_F}^{\perp},\mathrm{Hess}_{F-f_F}u_\phi\rangle\\
&\qquad+\langle\mathrm{grad}_{F-f_F},u_\phi\rangle\langle\mathrm{grad}_{f_F}^{\perp},\mathrm{Hess}_{f_F}u_\phi\rangle+\langle\mathrm{grad}_{F-f_F},u_\phi\rangle\langle\mathrm{grad}_{f_F}^{\perp},\mathrm{Hess}_{F-f_F}u_\phi\rangle\\
&\qquad+\langle\mathrm{grad}_{F-f_F},u_\phi\rangle\langle\mathrm{grad}_{F-f_F}^{\perp},\mathrm{Hess}_{f_F}u_\phi\rangle.
\end{aligned}$$

Since G_{F-f_F} is totally umbilical, we obtain $\Phi_{F-f_F}\equiv 0$. Therefore we obtain $\tilde{\mathsf{D}}_{F-f_F}+\tilde{\mathsf{N}}_{F-f_F}\equiv 0$. We represent f_F as $f_F:=\sum_{i\geqq k_F}f_F^{(i)}$, where $f_F^{(i)}\in\mathscr{P}^i$. Then the following hold:

$$\mathrm{grad}_{f_F}(\rho\cos\theta,\rho\sin\theta)=\sum_{i\geqq k_F}\rho^{i-1}\,\mathrm{grad}_{f_F^{(i)}}(\theta),$$
$$\mathrm{Hess}_{f_F}(\rho\cos\theta,\rho\sin\theta)=\sum_{i\geqq k_F}\rho^{i-2}\,\mathrm{Hess}_{f_F^{(i)}}(\theta),$$

where

$$\mathrm{grad}_{f_F^{(i)}}(\theta):=\mathrm{grad}_{f_F^{(i)}}(\cos\theta,\sin\theta),\quad \mathrm{Hess}_{f_F^{(i)}}(\theta):=\mathrm{Hess}_{f_F^{(i)}}(\cos\theta,\sin\theta).$$

Therefore we obtain

$$\begin{aligned}
\lim_{\rho\to 0}\frac{\Phi_F(\rho,\theta,\phi)}{\rho^{k_F-2}}&=\lim_{\rho\to 0}\frac{\tilde{\mathsf{D}}_{f_F,(\rho\cos\theta,\rho\sin\theta)}(\boldsymbol{U}_\phi)}{\rho^{k_F-2}}\\
&=\langle\mathrm{Hess}_{g_F}(\theta)u_\phi,u_{\phi+\pi/2}\rangle. \qquad (4)
\end{aligned}$$

Then we obtain

Proposition 3.1. *Let F be an element of $\mathscr{A}_o^{\langle 2\rangle}$ satisfying $S_{g_F}=\emptyset$. Then $F\in\mathscr{A}_{oo}^2$ holds.*

Suppose $F\in\mathscr{A}_o^2$, $\theta_0\in S_{g_F}$ and $F\equiv f_F$. We may represent $\tilde{\mathsf{D}}_F(\boldsymbol{U}_\phi)$ and $\tilde{\mathsf{N}}_F(\boldsymbol{U}_\phi)$ as

$$\tilde{\mathsf{D}}_{F,(\rho\cos\theta,\rho\sin\theta)}(\boldsymbol{U}_\phi)=\sum_{i\geqq k_F}\rho^{i-2}d_F^{(i)}(\theta,\phi),$$
$$\tilde{\mathsf{N}}_{F,(\rho\cos\theta,\rho\sin\theta)}(\boldsymbol{U}_\phi)=\sum_{i\geqq k_F}\rho^{i-2}n_F^{(i)}(\theta,\phi).$$

Then we see that $d_F^{(k_F)}(\theta_0,\phi)=0$ holds for any $\phi\in\boldsymbol{R}$ and that $n_F^{(i)}(\theta_0,\phi)=0$ holds for any $\phi\in\boldsymbol{R}$ and any integer $i\in[k_F,3k_F-2)$. Since $F\in\mathscr{A}_o^2$, there exists an integer $k>k_F$ satisfying $d_F^{(k)}(\theta_0,\phi)+n_F^{(k)}(\theta_0,\phi)\neq 0$ for some $\phi\in\boldsymbol{R}$. The minimum of such integers as k is denoted by k_{F,θ_0}. We shall prove

Lemma 3.2. There exists a symmetric matrix $M(\theta_0)$ which is not represented by the unit matrix up to any constant and satisfies

$$d_F^{(k_{F,\theta_0})}(\theta_0, \phi) + n_F^{(k_{F,\theta_0})}(\theta_0, \phi) = \langle M(\theta_0)u_\phi, u_{\phi+\pi/2}\rangle$$

for any $\phi \in \boldsymbol{R}$.

Proof. If $k_{F,\theta_0} \in (k_F, 3k_F - 2)$, then we see that $\mathrm{Hess}_{F^{(k_{F,\theta_0})}}(\theta_0)$ is suitable for $M(\theta_0)$. Suppose $k_{F,\theta_0} \geqq 3k_F - 2$. Noticing $\theta_0 \in S_{g_F}$, we see that there exists a symmetric matrix $M^{(3k_F-2)}(\theta_0)$ satisfying

$$n_F^{(3k_F-2)}(\theta_0, \phi) = \tilde{\mathsf{N}}_{g_F,(\cos\theta_0,\sin\theta_0)}(\boldsymbol{U}_\phi) = \langle M^{(3k_F-2)}(\theta_0)u_\phi, u_{\phi+\pi/2}\rangle$$

for any $\phi \in \boldsymbol{R}$. Therefore we see that if $k_{F,\theta_0} = 3k_F - 2$, then

$$\mathrm{Hess}_{F^{(3k_F-2)}}(\theta_0) + M^{(3k_F-2)}(\theta_0)$$

is suitable for $M(\theta_0)$. In the following, suppose $k_{F,\theta_0} > 3k_F - 2$. By Euler's identity, we obtain $n_F^{(3k_F-2)}(\theta_0, \theta_0) = 0$. Therefore we see that u_{θ_0} and $u_{\theta_0+\pi/2}$ are eigenvectors of $M^{(3k_F-2)}(\theta_0)$. In general, we see that if k is an integer in $[3k_F - 2, k_{F,\theta_0}]$ such that u_{θ_0} and $u_{\theta_0+\pi/2}$ are eigenvectors of $\mathrm{Hess}_{F^{(l)}}(\theta_0)$ for any integer $l \in [k_F, k-1]$, then there exists a symmetric matrix $M^{(k)}(\theta_0)$ satisfying the following:

(a) $n_F^{(k)}(\theta_0, \phi) = \langle M^{(k)}(\theta_0)u_\phi, u_{\phi+\pi/2}\rangle$ holds for any $\phi \in \boldsymbol{R}$,

(b) u_{θ_0} and $u_{\theta_0+\pi/2}$ are eigenvectors of $M^{(k)}(\theta_0)$.

Therefore noticing that $\mathrm{Hess}_{F^{(l)}}(\theta_0) + M^{(l)}(\theta_0)$ is represented by the unit matrix up to a constant for any integer $l \in [3k_F - 2, k_{F,\theta_0} - 1]$, we see that $\mathrm{Hess}_{F^{(k_{F,\theta_0})}}(\theta_0) + M^{(k_{F,\theta_0})}(\theta_0)$ is suitable for $M(\theta_0)$. Hence we obtain Lemma 3.2. □

Lemma 3.2 implies

$$\lim_{\rho\to 0} \frac{\Phi_F(\rho, \theta_0, \phi)}{\rho^{k_{F,\theta_0}-2}} = \langle M(\theta_0)u_\phi, u_{\phi+\pi/2}\rangle. \tag{5}$$

We may find such a symmetric matrix as $M(\theta_0)$ in Lemma 3.2, even if $F \not\equiv f_F$.

Proof of Proposition 1.1. By (4) together with (5), we obtain (a), and we see that for $\theta_1, \theta_2 \in \boldsymbol{R}$ satisfying $\theta_1 < \theta_2$ and $S_{g_F} \cap (\theta_1, \theta_2) = \emptyset$, there exists an element $z^{(i)}(\theta_1, \theta_2)$ of $\{n\pi/2\}_{n\in\boldsymbol{Z}}$ satisfying

$$\phi_{F,o}^{(i)}(\theta) = \eta_{g_F}(\theta) + z^{(i)}(\theta_1, \theta_2) \tag{6}$$

for any $\theta \in (\theta_1, \theta_2)$. Therefore noticing that the set S_{g_F} is empty or discrete, we obtain (b). By (4) together with (5) (or by (b) of Proposition 2.3), we obtain $\phi_{F,o}^{(2)} - \phi_{F,o}^{(1)} \equiv z_0$ for some $z_0 \in \{n\pi/2\}_{n\in\boldsymbol{Z}}$. Therefore by (6), we obtain (c). □

Proof of Proposition 1.2. By (6), we obtain (a). For $\theta \in \boldsymbol{R}$, the following holds:

$$\mathrm{ind}_o(\mathsf{G}_F) = \frac{\phi_{F,o}^{(i)}(\theta + 2\pi) - \phi_{F,o}^{(i)}(\theta)}{2\pi}. \tag{7}$$

By (6), we obtain

$$\phi^{(i)}_{F,o}(\theta+2\pi)-\phi^{(i)}_{F,o}(\theta)=\eta_{g_F}(\theta+2\pi)-\eta_{g_F}(\theta)+\sum_{\theta_0\in S_{g_F}\cap[\theta,\theta+2\pi)}\Gamma_{F,o}(\theta_0). \tag{8}$$

From (7) and (8), we obtain (b). □

4. Homogeneous polynomials.

Let k be a positive integer. For $g\in\mathscr{P}^k$, set $\tilde{g}(\theta):=g(\cos\theta,\sin\theta)$. A number $\theta_0\in\boldsymbol{R}$ is called a *root* of g if $(d\tilde{g}/d\theta)(\theta_0)=0$. The set of the roots of g is denoted by R_g. The straight line $L(\theta_0):=\{(\rho\cos\theta_0,\rho\sin\theta_0)\}_{\rho\in\boldsymbol{R}}$ in $\boldsymbol{R}^2$ determined by $\theta_0\in R_g$ is called a *root line* of g.

For $\theta,\phi\in\boldsymbol{R}$, we set

$$d_g(\theta,\phi):=\tilde{\mathrm{D}}_{g,(\cos\theta,\sin\theta)}(\boldsymbol{U}_\phi).$$

Then $d_g(\theta,\eta_g(\theta))=0$ holds for any $\theta\in\boldsymbol{R}$. Let $R(\mathrm{Hess}_g)$ be the set of numbers such that each $\theta_0\in R(\mathrm{Hess}_g)$ satisfies $\theta_0-\eta_g(\theta_0)\in\{n\pi/2\}_{n\in\boldsymbol{Z}}$. By Euler's identity, we see that for any $\theta\in\boldsymbol{R}$, the following holds:

$$d_g(\theta,\theta)=(k-1)\frac{d\tilde{g}}{d\theta}(\theta). \tag{9}$$

Therefore we obtain $R(\mathrm{Hess}_g)\subset R_g$. We also obtain $S_g\subset R_g$.

Suppose $R_g=\boldsymbol{R}$. Then k is even and g is represented by $(x^2+y^2)^{k/2}$ up to a constant ([**1**]). If g is nonzero, then by direct computations, we obtain $S_g=\emptyset$, and from (9), we see that $R(\mathrm{Hess}_g)=\boldsymbol{R}$ holds, i.e., there exists a number $z_0\in\{n\pi/2\}_{n\in\boldsymbol{Z}}$ satisfying $\eta_g\equiv\theta+z_0$. Therefore we obtain

$$\frac{\eta_g(\theta+2\pi)-\eta_g(\theta)}{2\pi}=1.$$

In the following, suppose $R_g\neq\boldsymbol{R}$. Then for each $\theta_0\in R_g$, there exists a positive integer $m\in\boldsymbol{N}$ satisfying $(d^{m+1}\tilde{g}/d\theta^{m+1})(\theta_0)\neq0$. The minimum of such integers as m is called the *multiplicity* of θ_0 and denoted by $\mu_g(\theta_0)$. A root $\theta_0\in R_g$ is said to be

(a) *related* if θ_0 satisfies $\tilde{g}(\theta_0)=0$ or if $\mu_g(\theta_0)$ is odd;

(b) *non-related* if θ_0 satisfies $\tilde{g}(\theta_0)\neq0$ and if $\mu_g(\theta_0)$ is even.

Suppose that $\theta_0\in R_g$ is related. Then it is said that the *critical sign* of θ_0 is positive (resp. negative) if the following holds:

$$\tilde{g}(\theta_0)\frac{d^{\mu_g(\theta_0)+1}\tilde{g}}{d\theta^{\mu_g(\theta_0)+1}}(\theta_0)\leqq0\quad(\text{resp.}>0).$$

The critical sign of θ_0 is denoted by $\text{c-sign}_g(\theta_0)$.

Lemma 4.1. *The following hold*:

(a) *The set* $R_g\backslash R(\mathrm{Hess}_g)$ *consists of the numbers at each of which* Hess_g *is represented by the unit matrix up to a nonzero constant*;

(b) *For* $\theta_0\in R_g\backslash R(\mathrm{Hess}_g)$, $\mu_g(\theta_0)=1$ *and* $\text{c-sign}_g(\theta_0)=-$ *hold.*

Proof. If $\mathrm{Hess}_g(\theta_0)$ is not represented by the unit matrix up to any constant for $\theta_0\in R_g$, then by (9), we obtain $\theta_0\in R(\mathrm{Hess}_g)$. Suppose $\tilde{g}(\theta_0)=0$ for $\theta_0\in R_g$. Then there exist an integer $l \geqq 2$ and an element $g_0 \in \mathscr{P}^{k-l}$ satisfying $\tilde{g}_0(\theta_0) \neq 0$ and $\tilde{g}(\theta) = \sin^l(\theta - \theta_0)\tilde{g}_0(\theta)$ for any $\theta \in \boldsymbol{R}$. The following holds:

$$\begin{aligned} \mathrm{Hess}_g(\theta) &= l(l-1)\tilde{g}_0(\theta)\sin^{l-2}(\theta-\theta_0)\begin{pmatrix} \sin^2\theta_0 & -\cos\theta_0\sin\theta_0 \\ -\cos\theta_0\sin\theta_0 & \cos^2\theta_0 \end{pmatrix} \\ &\quad + l\sin^{l-1}(\theta-\theta_0)\begin{pmatrix} -2(\sin\theta_0)\tilde{p}_{g_0}(\theta) & \tilde{c}_{g_0,\theta_0}(\theta) \\ \tilde{c}_{g_0,\theta_0}(\theta) & 2(\cos\theta_0)\tilde{q}_{g_0}(\theta) \end{pmatrix} \\ &\quad + \sin^l(\theta-\theta_0)\,\mathrm{Hess}_{g_0}(\theta), \end{aligned}$$

where

$$\tilde{c}_{g_0,\theta_0}(\theta) := -(\sin\theta_0)\tilde{q}_{g_0}(\theta) + (\cos\theta_0)\tilde{p}_{g_0}(\theta).$$

Therefore we obtain

$$\begin{aligned} \frac{d_g(\theta,\phi)}{\sin^{l-2}(\theta-\theta_0)} &= \binom{l}{2}\tilde{g}_0(\theta)\sin 2(\phi-\theta_0) + \tilde{a}_{g_0,\theta_0}(\theta,\phi)\sin(\theta-\theta_0) \\ &\quad + d_{g_0}(\theta,\phi)\sin^2(\theta-\theta_0), \end{aligned} \tag{10}$$

where $\tilde{a}_{g_0,\theta_0}$ satisfies

$$\tilde{g}_0(\theta_0)\tilde{a}_{g_0,\theta_0}(\theta_0,\theta_0) > 0. \tag{11}$$

Then we obtain $\sin 2(\eta_g(\theta_0) - \theta_0) = 0$, i.e., $\theta_0 \in R(\mathrm{Hess}_g)$. Suppose that $\mathrm{Hess}_g(\theta_0)$ is represented by the unit matrix up to a nonzero constant for $\theta_0 \in R_g$. Then we may suppose that $\mathrm{Hess}_g(\theta_0)$ is the unit matrix. In addition, we may suppose $\theta_0 = 0$. Then we may represent g as

$$g(x,y) = \frac{1}{k(k-1)}x^k + \frac{1}{2}x^{k-2}y^2 + \sum_{i=3}^{k} a_i x^{k-i}y^i.$$

Therefore the following holds:

$$\mathrm{Hess}_g(\theta) = (\cos^{k-2}\theta)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\cos^{k-3}\theta\sin\theta)\begin{pmatrix} 0 & k-2 \\ k-2 & 6a_3 \end{pmatrix} + (\sin^2\theta)M_2(\theta),$$

where M_2 is a continuous, matrix valued function. Therefore we obtain $\cot 2\eta_g(0) = -3a_3/(k-2)$. This implies $0 \notin R(\mathrm{Hess}_g)$. Hence we obtain (a). Then for $\theta_0 \in R_g \setminus R(\mathrm{Hess}_g)$, the following hold:

$$\frac{d^2\tilde{g}}{d\theta^2}(\theta_0) = k(k-2)\tilde{g}(\theta_0) \neq 0.$$

These imply $\mu_g(\theta_0) = 1$ and $\text{c-sign}_g(\theta_0) = -$. Hence we obtain (b). □

We set $U_\theta(\varepsilon) := (\theta - \varepsilon, \theta + \varepsilon)$ for $\theta \in \boldsymbol{R}$ and $\varepsilon > 0$. It is said that the *sign* of $\theta_0 \in R(\mathrm{Hess}_g)$ is positive (resp. negative) if there exists a positive number $\varepsilon_0 > 0$ satisfying

$$\{\theta - \eta_g(\theta) - (\theta_0 - \eta_g(\theta_0))\}(\theta - \theta_0) > 0 \quad (\text{resp. } < 0)$$

for any $\theta \in U_{\theta_0}(\varepsilon_0)\setminus\{\theta_0\}$. Let $n_{g,+}$ (resp. $n_{g,-}$) be the number of the root lines determined by the elements of $R(\mathrm{Hess}_g)$ with positive (resp. negative) sign. Then referring to [**1**], we obtain

Proposition 4.2. *For any $\theta \in \boldsymbol{R}$, the following holds*:

$$\frac{\eta_g(\theta+2\pi)-\eta_g(\theta)}{2\pi} = 1 - \frac{n_{g,+}-n_{g,-}}{2}.$$

We shall prove

Proposition 4.3. *For $\theta_0 \in R(\mathrm{Hess}_g)$, θ_0 is related if and only if the sign of θ_0 is positive or negative. In addition, for a related root $\theta_0 \in R(\mathrm{Hess}_g)$,*

(a) *if $\tilde{g}(\theta_0) \neq 0$, then the number*

$$\delta_g(\theta_0) := \frac{d^{\mu_g(\theta_0)+1}\tilde{g}}{d\theta^{\mu_g(\theta_0)+1}}(\theta_0)\frac{\partial d_g}{\partial\phi}(\theta_0,\theta_0)$$

is nonzero and the sign of θ_0 is given by the sign of $\delta_g(\theta_0)$;

(b) *if $\tilde{g}(\theta_0) = 0$, then the sign of θ_0 is positive.*

Proof. Suppose that for $\theta_0 \in R(\mathrm{Hess}_g)$, $\mathrm{Hess}_g(\theta_0)$ is not represented by the unit matrix up to any constant. Then $(\partial d_g/\partial\phi)(\theta_0,\theta_0) \neq 0$ holds. Therefore by the implicit function theorem, we see that η_g is infinitely differentiable at θ_0 and satisfies

$$\left.\frac{d^m}{d\theta^m}(\theta-\eta_g)\right|_{\theta=\theta_0} = (k-1)\frac{d^{m+1}\tilde{g}}{d\theta^{m+1}}(\theta_0)\bigg/\frac{\partial d_g}{\partial\phi}(\theta_0,\theta_0) \tag{12}$$

for $m = 1,\ldots,\mu_g(\theta_0)$. Therefore θ_0 is related if and only if the sign of θ_0 is positive or negative. By (12), we obtain (a).

Suppose $\tilde{g}(\theta_0) = 0$ for $\theta_0 \in R(\mathrm{Hess}_g)$. Then noticing (10) and (11), we obtain (b).

Hence we obtain Proposition 4.3. □

If $\theta_0 \in R(\mathrm{Hess}_g)$ is related, then the sign of θ_0 is denoted by $\mathrm{sign}_g(\theta_0)$.

Proposition 4.4. *Let θ_0 be a related root of g satisfying* $\text{c-sign}_g(\theta_0) = +$. *Then $\theta_0 \in R(\mathrm{Hess}_g)$ and* $\mathrm{sign}_g(\theta_0) = +$ *hold.*

Proof. Suppose that a related root θ_0 satisfies $\text{c-sign}_g(\theta_0) = +$. Then from (b) of Lemma 4.1, we obtain $\theta_0 \in R(\mathrm{Hess}_g)$. If $\tilde{g}(\theta_0) \neq 0$, then the number $\delta_g(\theta_0)$, which appears in (a) of Proposition 4.3, is positive ([**1**], [**2**]). Therefore we obtain $\mathrm{sign}_g(\theta_0) = +$. If $\tilde{g}(\theta_0) = 0$, then Proposition 4.3 says $\mathrm{sign}_g(\theta_0) = +$. Hence we obtain Proposition 4.4. □

Remark 4.5. Referring to [**2**], we see that if θ_0 is a related element of $R(\mathrm{Hess}_g)$ satisfying $\text{c-sign}_g(\theta_0) = -$, then the condition $\mathrm{sign}_g(\theta_0) = +$ (resp. $-$) is equivalent to each of the following:

(a) there does not exist (resp. exists) an umbilical point other than o on $L(\theta_0)$;

(b) $(d^2\tilde{g}/d\theta^2)(\theta_0)/\tilde{g}(\theta_0) \in (k(k-2),\infty)$ (resp. $[0,k(k-2))$).

An element $g \in \mathscr{P}^k$ is called *harmonic* if $\mathrm{tr}(\mathrm{Hess}_g) \equiv 0$ holds. If g is harmonic, then $\tilde{g}(\theta) = c\cos k(\theta-\alpha)$ holds, where $c,\alpha \in \boldsymbol{R}$. Then we immediately obtain

PROPOSITION 4.6. *For a nonzero harmonic element* $g \in \mathscr{P}^k$,
(a) *the number of the root lines of* g *is equal to* k;
(b) *any root* $\theta_0 \in R_g$ *is related and satisfies* $\text{c-sign}_g(\theta_0) = +$;
(c) $S_g = \emptyset$ *holds.*

5. Proof of Theorems 1.3 and 1.5.

Let F be an element of $\mathscr{A}_o^{\langle 2 \rangle}$. We set $\varpi_F := \tilde{\mathrm{D}}_F(\mathbf{Grad}_F)$ and

$$\tilde{\varpi}_F(\rho,\theta) := \varpi_F(\rho\cos\theta, \rho\sin\theta)$$

for $(\rho,\theta) \in (-\rho_0,\rho_0) \times \boldsymbol{R}$, where $\rho_0 > 0$ satisfies $\{x^2 + y^2 < \rho_0^2\} \subset \mathsf{G}_F$. We represent $\tilde{\varpi}_F$ as

$$\tilde{\varpi}_F(\rho,\theta) = \sum_{i \geqq k_0} \rho^i \tilde{\varpi}_F^{(i)}(\theta),$$

where

$$k_0 := \begin{cases} 3k_F - 4, & \text{if } F - f_F \equiv 0, \\ k_F, & \text{if } F - f_F \not\equiv 0. \end{cases}$$

For any $\theta \in \boldsymbol{R}$, the following holds:

$$\tilde{\varpi}_F^{(k_0)}(\theta) = \begin{cases} \dfrac{\det(\mathrm{Hess}_{g_F}(\theta))}{k_F - 1} \dfrac{d\tilde{g}_F}{d\theta}(\theta), & \text{if } F - f_F \equiv 0, \\ a_F^2(k_F - 1)\dfrac{d\tilde{g}_F}{d\theta}(\theta), & \text{if } F - f_F \not\equiv 0, \end{cases}$$

where $a_F := H_F(o)$. Let θ_0 be an element of $R_{g_F} \backslash R(\mathrm{Hess}_{g_F})$. Then noticing Lemma 4.1 and the implicit function theorem, we obtain

LEMMA 5.1. *There exist a neighborhood* V_{θ_0} *of* $(0,\theta_0)$ *in* $\boldsymbol{R}^2$ *and a real-analytic curve* C_{θ_0} *in* V_{θ_0} *through* $(0,\theta_0)$ *satisfying*
(a) $C_{\theta_0} = \{(\rho,\theta) \in V_{\theta_0}; \tilde{\varpi}_F(\rho,\theta)/\rho^{k_0} = 0\}$;
(b) C_{θ_0} *is not tangent to the* θ*-axis at* $(0,\theta_0)$.

PROOF OF THEOREM 1.3. Let F be an element of $\mathscr{A}_{oo}^2$. Then $S_{g_F} = R_{g_F} \backslash R(\mathrm{Hess}_{g_F})$ holds. Suppose $S_{g_F} = \emptyset$. Then by (b) of Proposition 1.2, we obtain $\mathrm{ind}_o(\mathsf{G}_F) = \mathrm{ind}_o(\mathsf{G}_{g_F})$. In addition, by (b) of Remark 1.4, we obtain $\mathrm{ind}_o(\mathsf{G}_F) \leqq 1$. In the following, suppose $S_{g_F} \neq \emptyset$ and $\theta_0 \in S_{g_F}$. Let ψ_F be a continuous function on $(0,\rho_0) \times \boldsymbol{R}$ such that for each $(\rho,\theta) \in (0,\rho_0) \times \boldsymbol{R}$, $\mathrm{grad}_F(\rho\cos\theta, \rho\sin\theta)$ is represented by $u_{\psi_F(\rho,\theta)}$ up to a constant. Noticing Lemma 2.5, we suppose $\phi_F^{(1)} = \psi_F$ on $\{(0,\rho_0) \times \boldsymbol{R}\} \cap C_{\theta_0}$. Noticing $\theta_0 \in R_{g_F} \backslash R(\mathrm{Hess}_{g_F})$ and (a) of Lemma 4.1, we see that ψ_F may be continuously extended to $\overline{\{(0,\rho_0) \times \boldsymbol{R}\} \cap V_{\theta_0}}$. Let ε be a positive number satisfying $\{0\} \times U_{\theta_0}(\varepsilon) \subset V_{\theta_0}$. We set

$$\psi_{F,o}(\theta) := \psi_F(0,\theta), \quad \chi_{F,o}(\theta) := \phi_{F,o}^{(1)}(\theta) - \psi_{F,o}(\theta)$$

for $\theta \in U_{\theta_0}(\varepsilon)$. Then by Euler's identity, we obtain $\theta_0 - \psi_{F,o}(\theta_0) \in \{n\pi\}_{n \in \boldsymbol{Z}}$ and we see that there exists a number $\varepsilon_0 \in (0, \varepsilon)$ satisfying $\chi_{F,o}(\theta) \neq 0$ for any $\theta \in \overline{U_{\theta_0}(\varepsilon_0)} \backslash \{\theta_0\}$. In addition, noticing $\theta_0 \in R_{g_F} \backslash R(\mathrm{Hess}_{g_F})$, we obtain

$$\Gamma_{F,o}(\theta_0) = \begin{cases} \pi/2, & \text{if } \chi_{F,o}(\theta)(\theta - \theta_0) > 0 \text{ for any } \theta \in U_{\theta_0}(\varepsilon_0)\backslash\{\theta_0\}, \\ -\pi/2, & \text{if } \chi_{F,o}(\theta)(\theta - \theta_0) < 0 \text{ for any } \theta \in U_{\theta_0}(\varepsilon_0)\backslash\{\theta_0\}, \\ 0, & \text{if } \chi_{F,o}(\theta_0 + \varepsilon_0)\chi_{F,o}(\theta_0 - \varepsilon_0) > 0. \end{cases}$$

Therefore we obtain (a).

We may suppose $\phi^{(1)}_{g_F}(\rho, \theta_0) = \psi_{g_F,o}(\theta_0) = \theta_0$. By (b) of Lemma 4.1, we obtain

$$\{\psi_{g_F,o}(\theta) - \theta\}(\theta - \theta_0) > 0 \tag{13}$$

for any $\theta \in U_{\theta_0}(\varepsilon_0)\backslash\{\theta_0\}$. We set $\phi^{(1)}_{g_F,\rho}(\theta) := \phi^{(1)}_{g_F}(\rho, \theta)$. By Lemma 2.4 together with Euler's identity, we obtain

$$\frac{\partial \Phi_{g_F}}{\partial \phi}(\rho, \theta_0, \theta_0) = k_F^3(k_F - 1)\rho^{3k_F-4}\tilde{g}_F(\theta_0)^3 \neq 0.$$

Therefore we see that $\phi^{(1)}_{g_F,\rho}$ is differentiable at θ_0 and by (9), we obtain

$$\left.\frac{d}{d\theta}(\theta - \phi^{(1)}_{g_F,\rho})\right|_{\theta=\theta_0} = \frac{d^2\tilde{g}_F}{d\theta^2}(\theta_0) \Big/ k_F^3\rho^{2k_F-2}\tilde{g}_F(\theta_0)^3.$$

Then from (b) of Lemma 4.1, we obtain

$$\{\theta - \phi^{(1)}_{g_F,\rho}(\theta)\}(\theta - \theta_0) > 0 \tag{14}$$

for any $\theta \in U_{\theta_0}(\varepsilon_0)\backslash\{\theta_0\}$. By (13) together with (14), we obtain $\chi_{g_F,o}(\theta)(\theta - \theta_0) < 0$ for any $\theta \in U_{\theta_0}(\varepsilon_0)\backslash\{\theta_0\}$, and $\Gamma_{g_F,o}(\theta_0) = -\pi/2$. Therefore by (b) of Proposition 1.2 together with (a) of Theorem 1.3, we obtain $\mathrm{ind}_o(\mathsf{G}_{g_F}) \leqq \mathrm{ind}_o(\mathsf{G}_F)$.

For $\theta \in \boldsymbol{R}$, set $n_{g_F,s} := \#(S_{g_F} \cap [\theta, \theta + \pi))$, and let $\{\theta_n\}_{n=0}^{n_{g_F,s}}$ be a subset of S_{g_F} satisfying $\theta_{n-1} < \theta_n$ and $(\theta_{n-1}, \theta_n) \cap S_{g_F} = \varnothing$ for $n = 1, \dots, n_{g_F,s}$. Then by (b) of Lemma 4.1, we see that for any $n \in \{1, \dots, n_{g_F,s}\}$, the number of the related roots in $R_{g_F} \cap (\theta_{n-1}, \theta_n)$ with positive critical sign is more than the number of the related roots in $R_{g_F} \cap (\theta_{n-1}, \theta_n)$ with negative critical sign. Therefore by Proposition 4.4, we obtain $n_{g,+} - n_{g,-} \geqq n_{g_F,s}$. Then by Proposition 4.2, we obtain

$$\frac{\eta_{g_F}(\theta + 2\pi) - \eta_{g_F}(\theta)}{2\pi} \leqq 1 - n_{g_F,s}/2. \tag{15}$$

Therefore by (b) of Proposition 1.2, (a) of Theorem 1.3 and (15), we obtain $\mathrm{ind}_o(\mathsf{G}_F) \leqq 1$.

Hence we obtain (b). □

Remark 5.2. In [**4**], we studied the behavior of the principal distributions around o on the graph G_F of $F \in \mathscr{A}_o^{(2)}$ satisfying $\varpi_F \equiv 0$. In particular, we showed that for an element $F \in \mathscr{A}_o^2$ satisfying $\varpi_F \equiv 0$, G_F is part of a surface of revolution such that o lies on the axis of rotation. This implies $\mathrm{ind}_o(\mathsf{G}_F) = 1$ for $F \in \mathscr{A}_o^2$ satisfying $\varpi_F \equiv 0$.

Remark 5.3. In [**3**], we proved $\Gamma_{g,o}(\theta_0) = -\pi/2$ for $\theta_0 \in S_g$ and $g \in \mathscr{P}_o^k$ $(k \geqq 3)$ in another way different from that in the above proof.

Proof of Theorem 1.5. Let $S_{g_F}^{(0)}$ be the set of the elements of S_{g_F} at each of which Hess_{g_F} is the zero matrix, and set $S_{g_F}^{(1)} := S_{g_F} \setminus S_{g_F}^{(0)}$. For $\theta \in \boldsymbol{R}$ and $i \in \{0, 1\}$, set $n_{g_F,s}^{(i)} := \#(S_{g_F}^{(i)} \cap [\theta, \theta + \pi))$. Then $n_{g_F,s} = n_{g_F,s}^{(0)} + n_{g_F,s}^{(1)}$ holds. By Lemma 4.1, Proposition 4.2 and Proposition 4.4, we obtain

$$\frac{\eta_{g_F}(\theta + 2\pi) - \eta_{g_F}(\theta)}{2\pi} \leqq 1 - n_{g_F,s}^{(0)} - n_{g_F,s}^{(1)}/2. \tag{16}$$

If $\Gamma_{F,o}(\theta_0) \leqq \pi$ holds for any $\theta_0 \in S_{g_F}^{(0)}$, then by (b) of Proposition 1.2, $-\pi/2 \leqq \Gamma_{F,o}(\theta_0) \leqq \pi/2$ for any $\theta_0 \in S_{g_F}^{(1)}$ and (16), we obtain $\mathrm{ind}_o(\mathsf{G}_F) \leqq 1$. Hence we obtain Theorem 1.5. □

6. Special Weingarten surfaces.

We shall prove

Proposition 6.1. *Let F be an element of $\mathscr{A}_o^{\langle 2 \rangle}$ whose graph is a special Weingarten surface. Then g_F is harmonic.*

To prove Proposition 6.1, we need lemmas.

For $F \in \mathscr{A}_o^{\langle 2 \rangle}$, we have set $a_F := H_F(o)$ (in Section 5). This implies $K_F(o) = a_F^2$. We represent $K_F - a_F^2$ and $H_F - a_F$ as

$$K_F - a_F^2 := \sum_{i \geqq 1} K_F^{(i)}, \quad H_F - a_F := \sum_{i \geqq 1} H_F^{(i)},$$

where $K_F^{(i)}$ and $H_F^{(i)}$ are elements of $\mathscr{P}^i$. Since G_{F-f_F} is totally umbilical, we obtain $K_{F-f_F} \equiv a_F^2$ and $H_{F-f_F} \equiv a_F$. Therefore we obtain

Lemma 6.2. *For $F \in \mathscr{A}_o^{\langle 2 \rangle}$,*

(a) (i) *if $a_F = 0$, then the following holds:*

$$K_F^{(i)} = \begin{cases} 0, & \text{if } i \in \{1, \dots, 2k_F - 5\}, \\ \det(\mathrm{Hess}_{g_F}), & \text{if } i = 2k_F - 4, \end{cases}$$

(ii) *if $a_F \neq 0$, then the following holds:*

$$K_F^{(i)} = \begin{cases} 0, & \text{if } i \in \{1, \dots, k_F - 3\}, \\ a_F \,\mathrm{tr}(\mathrm{Hess}_{g_F}), & \text{if } i = k_F - 2, \end{cases}$$

(b) *the following holds:*

$$H_F^{(i)} = \begin{cases} 0, & \text{if } i \in \{1, \dots, k_F - 3\}, \\ \mathrm{tr}(\mathrm{Hess}_{g_F})/2, & \text{if } i = k_F - 2. \end{cases}$$

Let w be an element of $\mathscr{A}_o^{(1)}$ satisfying

$$C_{F,w} := a_F \frac{\partial w}{\partial X}(0, 0) + \frac{1}{2} \frac{\partial w}{\partial Y}(0, 0) \neq 0,$$

and set

$$\Delta_{F,w}(x,y) := w(K_F(x,y) - a_F^2, H_F(x,y) - a_F).$$

We represent $\Delta_{F,w}$ as $\Delta_{F,w} := \sum_{i \geqq 1} \Delta_{F,w}^{(i)}$, where $\Delta_{F,w}^{(i)}$ is an element of $\mathscr{P}^i$. By Lemma 6.2, we obtain

LEMMA 6.3. *The following holds*:

$$\Delta_{F,w}^{(i)} = \begin{cases} 0, & \text{if } i \in \{1, \dots, k_F - 3\}, \\ C_{F,w} \operatorname{tr}(\mathrm{Hess}_{g_F}), & \text{if } i = k_F - 2. \end{cases}$$

PROOF OF PROPOSITION 6.1. If the graph of $F \in \mathscr{A}_o^{\langle 2 \rangle}$ is a special Weingarten surface, then by (1) together with Lemma 6.3, we see that g_F is harmonic. □

PROOF OF THEOREM 1.6. Since g_F is harmonic, from Proposition 3.1 and (c) of Proposition 4.6, we obtain (a) of Theorem 1.6. In addition, by (b) of Proposition 1.2 together with (c) of Proposition 4.6, we obtain

$$\mathrm{ind}_o(\mathsf{G}_F) = \mathrm{ind}_o(\mathsf{G}_{g_F}) = \frac{\eta_{g_F}(\theta + 2\pi) - \eta_{g_F}(\theta)}{2\pi}. \tag{17}$$

By Proposition 4.4 together with (a) and (b) of Proposition 4.6, we obtain $(n_{g_F,+}, n_{g_F,-}) = (k_F, 0)$. Therefore by Proposition 4.2 together with (17), we obtain (b) of Theorem 1.6. □

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