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Universal functions on Stein manifolds

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Abstract. We study universal holomorphic functions on a Stein manifold M with projective compactification. Let $\{\varphi_n\}$ be a sequence of holomorphic automorphisms of M. We prove that if $\{\varphi_n^{-1}\}$ is A run-away, then the set of all universal functions with respect to $\{\varphi_n\}$ in $\mathscr{A}(K)$ for all compact subsets K with a certain property is the intersection of countable number of open dense subsets in the space of all holomorphic functions on M. We also note that there is a close connection between the direction of run-awayness and a family of compact sets for which there exists a universal function.

1. Introduction.

The study of universal functions is initiated by the following theorem due to Birkhoff [**6**].

Theorem 1.1. *There exists an entire function $f(z)$ with the property that for any entire function $g(z)$ there exists a sequence $\{a_n\}$ such that*

$$\lim_{n\to\infty} f(z+a_n) = g(z)$$

uniformly on compact sets.

This function $f(z)$ is a universal function for the Euclidean translations $T_a(z) = z + a$. In 1941, Seidel and Walsh [**26**] obtained an analogous theorem for the unit disk in which the Euclidean translations are replaced by the non-Euclidean translations $T_\alpha(z) = (z+\alpha)/(1+\bar{\alpha}z)$. In 1976, Luh [**21**] proved the following theorem.

Theorem 1.2. *Let $\{a_n\}$ be a sequence in $\boldsymbol{C}$ with limit ∞. Then there is an entire function $f(z)$ such that for every compact set K with connected complement in $\boldsymbol{C}$ and for every function $g(z)$ holomorphic in the interior of K and continuous on K, there exists a subsequence $\{a_{n_k}\}$ such that $\{f(z+a_{n_k})\}$ converges to $g(z)$ uniformly on K.*

In 1988 the second author [**28**] obtained an analogous theorem on $\boldsymbol{C}^*$, and pointed out its generalization for noncompact Riemann surfaces.

The above two types of theorems are generalized for noncompact Riemann surfaces by Montes-Rodríguez [**24**]. There are many other contributions for this subject and related problems ([**9**], [**12**], [**14**], [**16**], [**19**], [**22**], [**27**] etc.). Große-Erdmann [**15**] is a good survey article not only for the analytic setting but also for the general aspect.

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On the contrary, there are only few papers concerning universal functions in several variables. The former type (Birkhoff-Seidel-Walsh type) theorem holds in $\boldsymbol{C}^n$, the unit ball and the polydisk (Abe [**1**], Abe and Zappa [**3**], Chee [**8**] and Godefroy and Shapiro [**13**]). These are all results for this type of theorem which are known till now.

Let D be a domain in $\boldsymbol{C}^n$. We denote by $\mathcal{O}(D)$ the set of all holomorphic functions on D. For a compact set $K \subset D$ we define

$$\hat{K}_D := \left\{ z \in D; |f(z)| \leqq \max_K |f| \ \text{for all} \ f \in \mathcal{O}(D) \right\}.$$

We say that a compact set $K \subset D$ is $\mathcal{O}(D)$-convex if $K = \hat{K}_D$. A domain $D \subset \boldsymbol{C}^n$ is said to be Stein if for any compact set K, $\hat{K}_D$ is compact. Let $\mathrm{Aut}(D)$ be the group of holomorphic automorphisms of D.

Definition 1.3. Let $\{\varphi_n\}$ be a sequence in $\mathrm{Aut}(D)$. A function $f \in \mathcal{O}(D)$ is called universal with respect to $\{\varphi_n\}$ in $\mathcal{O}(D)$ if $\{f \circ \varphi_n\}$ is dense in $\mathcal{O}(D)$, where the topology of $\mathcal{O}(D)$ is given by the compact-open topology.

A sequence $\{\varphi_n\}$ in $\mathrm{Aut}(D)$ is said to be run-away if for any compact set $K \subset D$ there exists $n_0 \in \boldsymbol{N}$ such that $K \cap \varphi_{n_0}(K) = \emptyset$. León-Saavedra [**20**] showed that the run-awayness is necessary for the existence of universal functions. This is an analogy to the one-dimensional case ([**5**], [**24**]).

The authors extended the result on $\boldsymbol{C}^*$ to complex general linear groups ([**3**]). The first author obtained an analogous result also in complex special linear groups ([**2**]). The purpose of this paper is to consider the latter type (Luh type) theorem for Stein manifolds with projective compactification (see Section 3 for the definition), and to improve the previous results.

We shall begin with giving some remarks for the case of noncompact Riemann surfaces in order to clarify our point of view.

2. Noncompact Riemann surfaces.

We first summarize the results of Montes-Rodríguez [**24**]. Although he proved the results for run-away sequences of self-mappings, we state them for run-away sequences of automorphisms for the simplicity of statements. Let R be a noncompact Riemann surface. We denote by $\mathcal{O}(R)$ the space of holomorphic functions on R. If K is a compact subset of R, $\mathcal{A}(K)$ denotes the set of all functions which are holomorphic in the interior K° of K and continuous on K. $\mathcal{K}(R)$ is the set of all compact subsets whose complements in R have no connected, relatively compact component. We denote by $\mathcal{K}_1(R)$ the set of all compact subsets whose complements are connected. Let $\mathrm{Aut}(R)$ be the group of holomorphic automorphisms of R. A sequence $\{\varphi_n\}$ in $\mathrm{Aut}(R)$ is called a run-away sequence if for every compact subset K there exists $n_0 \in \boldsymbol{N}$ such that $K \cap \varphi_{n_0}(K) = \emptyset$. A function $f \in \mathcal{O}(R)$ is said to be universal with respect to $\{\varphi_n\}$ in $\mathcal{O}(R)$ if the orbit $\{f \circ \varphi_n\}$ is dense in $\mathcal{O}(R)$. We similarly say that f is universal with respect to $\{\varphi_n\}$ in $\mathcal{A}(K)$ for a compact subset K if $\{f \circ \varphi_n\}$ is dense in $\mathcal{A}(K)$. We note that $\{\varphi_n\}$ is run-away if and only if $\{\varphi_n^{-1}\}$ is run-away. Every noncompact Riemann surface R has the Freudenthal compactificatin $\hat{R} = R \cup \mathcal{F}(R)$, where $\mathcal{F}(R)$ is the space of (Freudenthal) ends. This compactification is also known as Stoilow's compactification (see [**4**] and [**25**]). We refer to [**24**], [7] or to [**10**], [**11**] for the precise definition and its properties. For an open set $\hat{U} \subset \hat{R}$, we put $U := \hat{U} \cap R$. We denote by $\mathscr{K}'(R)$ the set of all compact subsets K satisfying the following conditions (i), (ii) and (iii):

(i) K is a compact bordered surface.

(ii) Each connected component of $R \backslash K$ is not relatively compact and is either planar or of infinte genus.

(iii) The closure of each connected component of $R \backslash K$ intersected with K is a topological circle.

Then $\mathscr{K}'(R) \subset \mathscr{K}(R)$. We define $\mathscr{K}_1'(R) := \mathscr{K}_1(R) \cap \mathscr{K}'(R)$.

The main result of Montes-Rodríguez is the following theorem.

Theorem 2.1 (Theorem 3.1 in [**24**]). *Let R be a noncompact Riemann surface, and let $\{\varphi_n\} \subset \mathrm{Aut}(R)$ be a sequence. Then the following statements hold.*

(a) *If $\mathscr{F}(R)$ is not a two-point set and $\{\varphi_n\}$ is run-away, then the set of all functions in $\mathscr{O}(\mathscr{R})$ which are universal with respect to $\{\varphi_n\}$ in $\mathscr{O}(R)$ is the intersection of countable number of open dense subsets in $\mathscr{O}(R)$.*

(b) *If $\{\varphi_n\}$ is run-away, then the set of all functions in $\mathscr{O}(R)$ which are universal with respect to $\{\varphi_n\}$ in $\mathscr{A}(K)$ for all $K \in \mathscr{K}_1(R)$ is the intersection of countable number of open dense subsets in $\mathscr{O}(R)$.*

Since $\mathscr{O}(R)$ is a Baire space, the sets of universal functions in the above theorem are dense in $\mathscr{O}(R)$. The following lemma is important in its proof.

Lemma 2.2 (Lemma 2.15 in [**24**]). *Let R be a noncompact Riemann surface with an infinite space $\mathscr{F}(R)$ of ends. If $\{\varphi_n\} \subset \mathrm{Aut}(R)$ is run-away, then there exist a nonisolated end e and a run-away subsequence $\{\varphi_{n_k}\}$ such that for any compact subset $K \subset R$ and for any neighbourhood $\hat{U} \subset \hat{R}$ of e there exists $k_0 \in \mathbf{N}$ such that $\varphi_{n_k}(K) \subset U$ for $k \geqq k_0$.*

An end e and a run-away subsequence $\{\varphi_{n_k}\}$ in the above lemma play an essential role in the proof of the part (a) in Theorem 2.1. This shows that the direction of run-awayness is important. Then we give the following definition.

Definition 2.3. Let $e \in \mathscr{F}(R)$. A sequence $\{\varphi_n\} \subset \mathrm{Aut}(R)$ is e run-away if for any compact subset $K \subset R$ and for any neighbourhood $\hat{U} \subset \hat{R}$ of e there exists $n_0 \in \mathbf{N}$ such that $\varphi_{n_0}(K) \subset U$.

Proposition 2.4. *Let R be a noncompact Riemann surface. Every run-away sequence $\{\varphi_n\} \subset \mathrm{Aut}(R)$ has an e run-away subsequence $\{\varphi_{n_k}\}$, where e is an end.*

Proof. If $\mathscr{F}(R)$ is an infinite space, this is Lemma 2.2. R has no run-away sequence when $\mathscr{F}(R)$ is a finite space with at least 3 elements (Theorem 2.16 in [**24**]). Then it suffices to consider the case that $\mathscr{F}(R)$ is a one- or two-point set.

The proposition is trivial for the case $\mathscr{F}(R) = \{e\}$. Let $\mathscr{F}(R) = \{e_1, e_2\}$. There exists a sequence $\{K_n\}$ of connected compact subsets of R such that

(i) $K_n \subset (K_{n+1})^\circ$ and $R = \bigcup K_n$,

(ii) $R \backslash K_n = U_n \cup V_n$ (disjoint union),

(iii) $\{U_n\}$ and $\{V_n\}$ determine e_1 and e_2 respectively (see p. 670 in [**24**]). Since $\{\varphi_n\}$ is run-away, there exists $k(n) \in \boldsymbol{N}$ for any n such that $\varphi_{k(n)}(K_n) \cap K_n = \emptyset$, then $\varphi_{k(n)}(K_n) \subset U_n$ or $\varphi_{k(n)}(K_n) \subset V_n$. If there are infinitely many n with $\varphi_{k(n)}(K_n) \subset U_n$, then we can take an e_1 run-away subsequence. □

For the proof of the part (b) in Theorem 2.1, another lemma (Lemma 2.11 in [**24**]) is needed. If we use the term "e run-away", then its proof becomes easier.

Lemma 2.5 (cf. Lemma 2.11 in [**24**]). *Let R be a noncompact Riemann surface. Suppose that $\{\varphi_n\} \subset \mathrm{Aut}(R)$ is e run-away, where $e \in \mathscr{F}(R)$. Then, for any $K_1 \in \mathscr{K}_1(R)$ and for any $K \in \mathscr{K}(R)$ there exists $n_0 \in \boldsymbol{N}$ such that $K \cap \varphi_{n_0}(K_1) = \emptyset$ and $K \cup \varphi_{n_0}(K_1) \in \mathscr{K}(R)$.*

Proof. Take a neighbourhood $\hat{U}$ of e such that $U = \hat{U} \cap R$ is connected and $K \cap \overline{U} = \emptyset$. There exists $K_1' \in \mathscr{K}_1'(R)$ with $K_1 \subset (K_1')^\circ$ (Lemma 2.10 in [**24**]). By the assumption there exists $n_0 \in \boldsymbol{N}$ such that $\varphi_{n_0}(K_1') \subset U$. Since $(K_1')^\circ \backslash K_1$ is connected, $(\varphi_{n_0}(K_1') \backslash \varphi_{n_0}(K_1))^\circ = \varphi_{n_0}((K_1')^\circ \backslash K_1)$ is also connected. Hence $U \backslash \varphi_{n_0}(K_1)$ is connected. This means $K \cup \varphi_{n_0}(K_1) \in \mathscr{K}(R)$. □

Along the argument in [**24**], we can show that if $\{\varphi_n\}$ is an e run-away sequence, the part (b) in Theorem 2.1 holds for the set $\mathscr{K}_2(R, e)$ of compact subsets $K \in \mathscr{K}(R)$ with the following property: there exists a neighbourhood $\hat{U}$ of e such that $(R \backslash \overline{U}) \backslash K$ is connected and $\overline{U} \cap K = \emptyset$.

It is obvious that $\mathscr{K}_1(R) \subset \mathscr{K}_2(R, e) \subset \mathscr{K}(R)$ for any $e \in \mathscr{F}(R)$. But, $\mathscr{K}_1(R) \neq \mathscr{K}_2(R, e)$ in general. We see also in the multi-dimensional case that there is a relation between the direction of run-awayness and a family of compact subsets for which an analogous statement to the part (b) in Theorem 2.1 holds (see Section 7).

3. Stein manifolds with projective compactification.

Let X be a connected projective algebraic variety, and let Y be an analytic subvariety of X. We call the paire (X, Y) a projective compactification of a complex manifold M if $X \backslash Y$ is biholomorphic to M. We may assume by the resolution of singularities that X is smooth.

Let M be a Stein manifold which has a projective compactification (X, Y). For example, affine algebraic Stein manifolds are such ones. It is known that Stein groups are affine algebraic (Matsushima [**23**]). Let $\mathrm{Aut}(M)$ be the group of holomorphic automorphisms of M. We denote by $\mathcal{O}(M)$ the set of all holomorphic functions on M. For every analytic subset $A \subset Y$, we define

$$\mathcal{O}_A(M) := \{f \in \mathcal{O}(M); f \text{ has a holomorphic extension to } A\}.$$

Definition 3.1. Let $A \subset Y$ be an analytic subset. A sequence $\{\varphi_n\} \subset \mathrm{Aut}(M)$ is called A run-away if for any compact set $K \subset M$ and for any neighbourhood U of A in X there exists $n_0 \in \boldsymbol{N}$ such that $\varphi_{n_0}(K) \subset U$.

Lemma 3.2. *Suppose that an analytic subset $A \subset Y$ has the decomposition $A = \bigcup_{i=1}^m A_i$ into the union of connected components. If $\{\varphi_n\} \subset \mathrm{Aut}(M)$ is A run-away, then there exist i $(1 \leqq i \leqq m)$ and a subsequence $\{\varphi_{n_k}\}$ such that $\{\varphi_{n_k}\}$ is A_i run-away.*

Proof. We can take a sequence $\{K_i\}$ of compact subsets such that
(i) K_i is connected,
(ii) $K_i \subset K_{i+1}$ and $M = \bigcup_{i=1}^{\infty} K_i$.
Each component A_j has a countable basis $\{U_i^j\}_{i=1}^{\infty}$ of neighbourhoods such that U_i^j is connected and $U_i^1, \dots, U_i^m$ are mutually disjoint for any i. Since $\{\varphi_n\}$ is A run-away, there exists $n_i \in \boldsymbol{N}$ for any i such that $\varphi_{n_i}(K_i) \subset \bigcup_{j=1}^{m} U_i^j$. By the connectedness of K_i, there exists j depending on i such that $\varphi_{n_i}(K_i) \subset U_i^j$. For some j_0 $(1 \leqq j_0 \leqq m)$, there exist infinitely many i with $\varphi_{n_i}(K_i) \subset U_i^{j_0}$. Then we can take a subsequence $\{\varphi_{n_k}\}$ which is A_{j_0} run-away. □

By the above lemma, it is sufficient to consider the case that an analytic subset $A \subset Y$ is connected.

Definition 3.3. For a compact subset $K \subset M$ we define

$$\hat{K}_M := \left\{ x \in M; |f(x)| \leqq \max_K |f| \text{ for all } f \in \mathcal{O}(M) \right\},$$

$$\hat{K}_A := \left\{ x \in M \cup A; |f(x)| \leqq \max_K |f| \text{ for all } f \in \mathcal{O}_A(M) \right\}.$$

$\hat{K}_M$ (resp. $\hat{K}_A$) is called the $\mathcal{O}(M)$ (resp. $\mathcal{O}_A(M)$)-hull of K. When $\hat{K}_M = K$, K is called $\mathcal{O}(M)$-convex.

In general, $\hat{K}_M \subset \hat{K}_A$ for $\mathcal{O}_A(M) \subset \mathcal{O}(M)$. In the case $M = \boldsymbol{C}$, we write $\hat{K} = \hat{K}_{\boldsymbol{C}}$.

Definition 3.4. Let $A \subset Y$ be a connected analytic subset. $\mathcal{K}(M, A)$ is the set of all compact sets $K \subset M$ such that $\hat{K}_M = K$ and $\hat{K}_A \cap A = \emptyset$.

4. Basic properties of $\mathcal{O}_A(M)$.

Let M be a Stein manifold with projective compactification (X, Y), and let A be a connected analytic subset of Y.

The set $\mathcal{O}(M)$ of all holomorphic functions on M is a topological space with the compact-open topology. This topology coincides with the topology of $\mathcal{O}(M)$ as a Fréchet space. Then $\mathcal{O}(M)$ is a complete metric space. On the other hand, we may assume that M is a closed complex submanifold of $\boldsymbol{C}^N$ for some N by the embedding theorem of Bishop-Narasimhan. Since the set of all polynomials on $\boldsymbol{C}^N$ is dense in $\mathcal{O}(\boldsymbol{C}^N)$, $\mathcal{O}(\boldsymbol{C}^N)$ is separable. By the extension theorem (for example, see Theorem 7.4.8 in [**17**]), every holomorphic function on M is the restriction to M of a function in $\mathcal{O}(\boldsymbol{C}^N)$. Then $\mathcal{O}(M)$ is also separable. A metric space is separable if and only if it satisfies the second axiom of countability. Therefore $\mathcal{O}(M)$ has a countable open basis $\{U_i; i \in \boldsymbol{N}\}$. Let $U_{A,i} := U_i \cap \mathcal{O}_A(M)$. Then $\{U_{A,i}; i \in \boldsymbol{N}\}$ is a countable open basis of $\mathcal{O}_A(M)$ for $\mathcal{O}_A(M)$ is a metric subspace of $\mathcal{O}(M)$. Therefore $\mathcal{O}_A(M)$ satisfies the second axiom of countability. Hence it is separable.

Any $f \in \mathcal{O}_A(M)$ is constant on A. Therefore we mean without confusion that $f(A)$ is the constant value of f on A.

Lemma 4.1. *Let $A \subset Y$ be a connected analytic subset. If a sequence $\{f_k\} \subset \mathcal{O}_A(M)$ converges to $f \in \mathcal{O}_A(M)$, then*

$$\lim_{k\to\infty} f_k(A) = f(A).$$

Proof. Take a point $x_0 \in A$. Then $f_k(x_0) = f_k(A)$ and $f(x_0) = f(A)$. There exists a one-dimensional complex submanifold L in a neighbourhood of x_0 such that $L \subset X$ and $L \cap Y = \{x_0\}$. Let ζ be a coordinate on L around x_0 with $\zeta = 0$ at x_0. We can take $R > 0$ such that $\{0 < |\zeta| < R\} \subset M$. We obtain by Cauchy's integral formula

$$f_k(x_0) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f_k(\zeta)}{\zeta}\,d\zeta,$$

$$f(x_0) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta)}{\zeta}\,d\zeta,$$

where $0 < r < R$. Since $\{f_k\}$ converges to f uniformly on $|\zeta| = r$, we obtain

$$\lim_{k\to\infty} f_k(x_0) = f(x_0). \qquad \square$$

Lemma 4.2. *For any compact subset K of M the following two conditions are equivalent*:

(i) *There exists $f \in \mathcal{O}_A(M)$ such that* $\max_K |f| < |f(A)|$.

(ii) *There exists $f \in \mathcal{O}_A(M)$ such that* $f(A) \notin \widehat{f(K)}$.

Proof. Assume (i) holds. Let $R := \max_K |f|$. Then $\widehat{f(K)} \subset \{|z| < R\}$. Hence we have $f(A) \notin \widehat{f(K)}$.

Conversely we assume that there exists $f \in \mathcal{O}_A(M)$ with $f(A) \notin \widehat{f(K)}$. Then there exists $g \in \mathcal{O}(\boldsymbol{C})$ such that

$$|g(f(A))| > \max_{\widehat{f(K)}} |g|.$$

Letting $h := g \circ f \in \mathcal{O}_A(M)$, we obtain

$$\max_K |h| < |h(A)|. \qquad \square$$

Corollary 4.3. *Let K be an $\mathcal{O}(M)$-convex compact subset of M, and let A be a connected analytic subset of Y. Then $K \in \mathcal{K}(M, A)$ if and only if there exists $f \in \mathcal{O}_A(M)$ such that $f(A) \notin \widehat{f(K)}$.*

Proof. We note that for an $\mathcal{O}(M)$-convex compact subset K of M, $K \in \mathcal{K}(M, A)$ if and only if there exists $f \in \mathcal{O}_A(M)$ with

$$\max_K |f| < |f(A)|.$$

Then the assertion follows from Lemma 4.2. $\square$

5. Lemmas.

Let M be a Stein manifold with projective compactification (X, Y), and let $A \subset Y$ be a connected analytic subset. Take a C^∞ strictly plurisubharmonic exhaustion function φ on M. For any $r > 0$ we define

$$\overline{M}_r := \{x \in M; \varphi(x) \leqq r\}.$$

Then $\overline{M}_r$ is an $\mathcal{O}(M)$-convex compact set (Theorem 5.2.10 in [**17**]). We denote by $\mathcal{K}(\boldsymbol{C})$ the set of all compact subsets $L \subset \boldsymbol{C}$ with connected complement in $\boldsymbol{C}$. It is Runge's result (cf. Theorem 1.3.1 in [**17**]) that any compact set in $\mathcal{K}(\boldsymbol{C})$ is $\mathcal{O}(\boldsymbol{C})$-convex.

Lemma 5.1 (Lemma 1 in [**3**]). *There exists a sequence $\mathcal{L} = \{L_\ell\}$ in $\mathcal{K}(\boldsymbol{C})$ such that for any $L \in \mathcal{K}(\boldsymbol{C})$ and any neighbourhood U of L there exists L_ℓ in $\mathcal{L}$ with $L \subset L_\ell \subset U$.*

Lemma 5.2. *There exists a sequence $\mathcal{K} = \{K_k\}$ in $\mathcal{K}(M, A)$ such that for any $K \in \mathcal{K}(M, A)$ there exists K_k with $K \subset K_k$.*

Proof. Let $\{c_\alpha\}$ be a strictly increasing sequence of positive numbers such that $c_\alpha \to \infty$ $(\alpha \to \infty)$. Since $\mathcal{O}_A(M)$ is separable, there exists a dense countable subset $\{f_i\}_{i=1}^\infty \subset \mathcal{O}_A(M)$. Let $\mathcal{L} = \{L_\ell\}$ be a sequence in $\mathcal{K}(\boldsymbol{C})$ in Lemma 5.1.

For any $f \in \mathcal{O}_A(M)$, we denote by $\{L_{\ell_j}(f)\}$ the subsequence of $\{L_\ell\}$ consisting of L_ℓ with $f(A) \notin L_\ell$. We define

$$K_{\ell_j}^\alpha(f) := f^{-1}(L_{\ell_j}(f)) \cap \overline{M}_{c_\alpha}.$$

Then $K_{\ell_j}^\alpha(f)$ is an $\mathcal{O}(M)$-convex compact set, and satisfies

$$f(A) \notin (f(K_{\ell_j}^\alpha(f)))\hat{} \subset L_{\ell_j}(f).$$

By Corollary 4.3 we obtain $K_{\ell_j}^\alpha(f) \in \mathcal{K}(M, A)$.

We set a countable subset

$$\mathcal{K} := \{K_{\ell_j}^\alpha(f_i); i, j, \alpha \in \boldsymbol{N}\}$$

in $\mathcal{K}(M, A)$. We show that $\mathcal{K}$ has the desired property.

For any $K \in \mathcal{K}(M, A)$, there exists $f \in \mathcal{O}_A(M)$ such that $f(A) \notin (\widehat{f(K)})$ (Corollary 4.3). $(\widehat{f(K)})$ has two relatively compact simply connected neighbourhoods U_1 and U_2 such that $\overline{U_1} \subset U_2$ and $f(A) \notin \overline{U_2}$. By Lemma 4.1 we can take f_i such that

$$f_i(K) \subset U_1 \quad \text{and} \quad f_i(A) \notin \overline{U_2}.$$

Since $(\widehat{\overline{U_1}}) = \overline{U_1} \subset U_2$, there exists L_{ℓ_j} such that $\overline{U_1} \subset L_{\ell_j} \subset U_2$ by Lemma 5.1. It is obvious that $(\widehat{f_i(K)}) \subset \overline{U_1}$. Then we have $K_{\ell_j}^\alpha(f_i) \supset K$ for a sufficiently large α. □

6. Main theorem.

Let M be a Stein manifold with projective compactification (X, Y), and let $A \subset Y$ be a connected analytic subset. For a compact subset $K \subset M$, we denote by $\mathcal{A}(K)$ the set of all functions which are holomorphic in a neighbourhood of K. We define $\|f\|_K := \max_{x \in K}|f(x)|$ for any $f \in \mathcal{O}(M)$ and any compact subset K.

Let $\{\varphi_n\}$ be a sequence in $\mathrm{Aut}(M)$. A function $f \in \mathcal{O}(M)$ is universal with respect to $\{\varphi_n\}$ in $\mathscr{A}(K)$ for all $K \in \mathscr{K}(M, A)$ if every function in $\mathscr{A}(K)$ is approximated uniformly on K by $\{f \circ \varphi_n\}$ for all $K \in \mathscr{K}(M, A)$.

Theorem 6.1. *Assume $\mathscr{K}(M, A) \neq \emptyset$. Let $\{\varphi_n\} \subset \mathrm{Aut}(M)$. If $\{\varphi_n^{-1}\}$ is A run-away, then the set of all functions in $\mathscr{O}(M)$ which are universal with respect to $\{\varphi_n\}$ in $\mathscr{A}(K)$ for all $K \in \mathscr{K}(M, A)$ is the intersection of countable number of open dense subsets in $\mathscr{O}(M)$.*

Proof. Let $T_n : \mathscr{O}(M) \to \mathscr{O}(M)$ be a composition operator defined by $T_n(f) := f \circ \varphi_n$. Obviously, T_n is continuous. For any compact set $K \subset M$, for any $f \in \mathscr{O}(M)$ and for any $\varepsilon > 0$, we set

$$G(f, \varepsilon, K) := \{g \in \mathscr{O}(M);\ \text{there exists } n \in \boldsymbol{N} \text{ with } \|T_n(g) - f\|_K < \varepsilon\},$$

$$O(f, \varepsilon, K) := \{h \in \mathscr{O}(M); \|h - f\|_K < \varepsilon\}.$$

The family $\{O(f, \varepsilon, K)\}$ is an open basis in the topology of $\mathscr{O}(M)$. Since

$$G(f, \varepsilon, K) = \bigcup_{n=1}^{\infty} T_n^{-1}(O(f, \varepsilon, K))$$

and T_n is continuous, $G(f, \varepsilon, K)$ is an open set in $\mathscr{O}(M)$.

Let $\{f_i\}$ be a dense countable subset of $\mathscr{O}(M)$. We take a strictly decreasing sequence $\{\varepsilon_j\}$ of positive numbers with $\lim \varepsilon_j = 0$. Let $\mathscr{K} = \{K_k\} \subset \mathscr{K}(M, A)$ be a sequence in Lemma 5.2. We denote by G the set of all universal functions with respect to $\{\varphi_n\}$ in $\mathscr{A}(K)$ for all $K \in \mathscr{K}(M, A)$. Then we can write

$$G = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{k=1}^{\infty} G(f_i, \varepsilon_j, K_k).$$

Therefore it suffices to prove that $G(f, \varepsilon, K)$ is dense in $\mathscr{O}(M)$ for any $f \in \mathscr{O}(M)$, any $\varepsilon > 0$ and any $K \in \mathscr{K}(M, A)$. To do this, we show that for any $h \in \mathscr{O}(M)$, any $\varepsilon' > 0$ and any compact subset $K' \subset M$,

$$G(f, \varepsilon, K) \cap O(h, \varepsilon', K') \neq \emptyset.$$

Since $K \in \mathscr{K}(M, A)$, there exists $g \in \mathscr{O}_A(M)$ such that $g(A) \notin \widehat{g(K)}$ by Corollary 4.3. Let $\bar{B}$ be a closed ball with center at $g(A)$ such that

$$\bar{B} \cap \widehat{g(K)} = \emptyset.$$

Then $g^{-1}(B)$ is a neighbourhood of A in X. We can take $r > 0$ such that $K' \subset \overline{M}_r$. Since $\{\varphi_n^{-1}\}$ is A run-away, there exists $n_0 \in \boldsymbol{N}$ such that

$$\varphi_{n_0}^{-1}(\overline{M}_r) \subset g^{-1}(B).$$

We define a holomorphic function $\tilde{g}$ on M by $\tilde{g}(x) := (g \circ \varphi_{n_0}^{-1})(x)$. Then we have

$$\tilde{g}(\varphi_{n_0}(K)) = g(K),$$

$$\tilde{g}(\overline{M}_r) = g(\varphi_{n_0}^{-1}(\overline{M}_r)) \subset g(g^{-1}(B)) \subset B.$$

Hence

$$(\tilde{g}(\varphi_{n_0}(K)))\hat{} \cap (\tilde{g}(\overline{M}_r))\hat{} = \varnothing.$$

It follows from the Separation Lemma (Kallin [**18**], see [**3**] for Stein manifolds) that

$$\begin{aligned}(\varphi_{n_0}(K) \cup \overline{M}_r)\hat{}_M &= (\varphi_{n_0}(K))\hat{}_M \cup (\widehat{\overline{M}_r})_M \\ &= \varphi_{n_0}(K) \cup \overline{M}_r.\end{aligned}$$

Let

$$h_1(x) := \begin{cases} h(x) & \text{if } x \in \overline{M}_r, \\ f(\varphi_{n_0}^{-1}(x)) & \text{if } x \in \varphi_{n_0}(K). \end{cases}$$

Then $h_1 \in \mathscr{A}(\varphi_{n_0}(K) \cup \overline{M}_r)$. By a Mergelyan type theorem in several variables (see Corollary 5.2.9 in [**17**]), there exists $\psi \in \mathcal{O}(M)$ which approximates h_1 on $\varphi_{n_0}(K) \cup \overline{M}_r$, i.e.

$$\|h - \psi\|_{\overline{M}_r} < \varepsilon',$$

$$\|f \circ \varphi_{n_0}^{-1} - \psi\|_{\varphi_{n_0}(K)} < \varepsilon.$$

As $\|f \circ \varphi_{n_0}^{-1} - \psi\|_{\varphi_{n_0}(K)} = \|f - T_{n_0}(\psi)\|_K$, we obtain the conclusion. □

Remark. In the situation of Theorem 6.1, the set of all universal functions with respect to $\{\varphi_n\}$ in $\mathscr{A}(K)$ for all $K \in \mathscr{K}(M, A)$ is dense in $\mathcal{O}(M)$, for $\mathcal{O}(M)$ is a Baire space.

Next we consider a general analytic subset $A \subset Y$. Let $A = \bigcup_{i=1}^m A_i$ be the decomposition into the union of connected components. For each component A_i we take $\mathscr{K}(M, A_i)$. Let

$$\mathscr{K}(M, A) := \bigcup_{i=1}^m \mathscr{K}(M, A_i).$$

Then we obtain the following corollary.

Corollary 6.2. *Assume $\mathscr{K}(M, A) \neq \varnothing$. Let $\{\varphi_n\} \subset \mathrm{Aut}(M)$. If $\{\varphi_n^{-1}\}$ is A_i run-away for any component A_i, then the set of all functions in $\mathcal{O}(M)$ which are universal with respect to $\{\varphi_n\}$ in $\mathscr{A}(K)$ for all $K \in \mathscr{K}(M, A)$ is the intersection of countable number of open dense subsets in $\mathcal{O}(M)$.*

7. Examples.

Example 1. Let $G = GL(n, \boldsymbol{C})$ be the complex general linear group of degree n. It is considered as an open submanifold of the space $M = M(n, \boldsymbol{C})$ of all square matrices of degree n with complex coefficients. Since $M \cong \boldsymbol{C}^N$ $(N = n^2)$, we have the natural inclusion $\iota : M \to \boldsymbol{P}^N$,

$$Z = (z_{ij}) \mapsto [1 : z_{11} : \cdots : z_{1n} : \cdots : z_{nn}].$$

We identify G and M with $\iota(G)$ and $\iota(M)$ respectively. Let $[x_0 : x_1 : \cdots : x_N]$ be the homogeneous coordinates of $\boldsymbol{P}^N$. We set

$$P(x) = x_0 \det \begin{pmatrix} x_1 & \cdots & x_n \\ x_{n+1} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots \\ x_{(n-1)n+1} & \cdots & x_{n^2} \end{pmatrix}.$$

Letting $Y := \{[x] \in \boldsymbol{P}^N; P(x) = 0\}$, we obtain a projective compactification $(\boldsymbol{P}^N, Y)$ of G.

Let $A_0 := \{[1 : 0 : \cdots : 0]\} \subset Y$. In [**3**] we defined the set $B(G)$ of compact sets $K \subset G$ with $\hat{K}_G = K$ such that there exists $f \in \mathcal{O}(M)$ with $f(A_0) \notin \widehat{f(K)}$. We essentially proved the following theorem in [**3**].

Theorem 7.1 (cf. Theorem 4 in [**3**]). *Let $\{C_i\} \subset G$ be a sequence with $C_i^{-1} \to A_0$. Then there exists $f \in \mathcal{O}(G)$ such that f is universal with respect to $\{C_i\}$ in $\mathcal{A}(K)$ for all $K \in B(G)$.*

Since $B(G) \subset \mathcal{K}(G, A_0)$ and $\{C_i^{-1}\}$ is A_0 run-away if $C_i^{-1} \to A_0$, the above theorem is contained in Theorem 6.1.

We see more interesting facts. Let $K := \{\lambda I; |\lambda| = 1\}$, where I is the unit matrix of degree n. By a mapping $i : \boldsymbol{C}^* \to G$, $\lambda \mapsto \lambda I$, $\boldsymbol{C}^*$ is embedded as a closed submanifold of G. Let $N := i(\boldsymbol{C}^*)$. K is $\mathcal{O}(N)$-convex, then $\mathcal{O}(G)$-convex. We take a sequence $\{c_i\}$ in $\boldsymbol{C}^*$ with $c_i \to \infty$, and set $C_i := i(c_i) = c_i I$. Then $\{C_i^{-1}\}$ is A_0 run-away. We see $K \notin \mathcal{K}(G, A_0)$ because of the following assertion.

Assertion. *There does not exist a universal function with respect to $\{C_i\}$ in $\mathcal{A}(K)$.*

Proof. Assume that there exists a universal function $F \in \mathcal{O}(G)$ with respect to $\{C_i\}$ in $\mathcal{A}(K)$. Then, for any $c \in \boldsymbol{C}$ there exists a subsequence $\{C_{i_k}\}$ such that $F(C_{i_k} Z) \to c$ uniformly on K. We define $f \in \mathcal{O}(\boldsymbol{C}^*)$ by $f := F \circ i$. We note that $f(c_j z) = F(C_j(zI))$ for any j and $z \in \boldsymbol{C}^*$, and that $zI \in K$ if and only if $z \in S^1 := \{|\zeta| = 1\}$. Then $f(c_{i_k} z) \to c$ uniformly on S^1. Especially, for any $c \in \boldsymbol{C}$ there exists a subsequence $\{c_{i_k}\}$ with $c_{i_k} \to \infty$ such that $f(c_{i_k}) \to c$. Then the point at infinity ∞ is an essential singularity of f.

Consider a subsequence $\{c_{i_k}\}$ with $f(c_{i_k} z) \to 0$ uniformly on S^1. Let $R = \max_{z \in S^1} |f(c_{i_1} z)|$. Since f is not a constant function, $R > 0$. Then there exists $\tilde{k} \in \boldsymbol{N}$ such that $|f(c_{i_k} z)| < R$ for all $z \in S^1$ and for all $k > \tilde{k}$. By the maximum principle $|f(z)| < R$ on $\{|c_{i_1}| \leqq |z| \leqq |c_{i_k}|\}$ for all $k > \tilde{k}$, hence also $|f(z)| < R$ on $\{|c_{i_1}| \leqq |z|\}$. Then ∞ is a removable singularity of f. This is a contradiction. □

On the other hand, we can find an A run-away sequence $\{D_i\} \subset G$ such that $K \in \mathcal{K}(G, A)$. This shows there is a relation between the direction of run-awayness and a family of compact subsets for which there exist universal functions (see Section 2).

For the sake of simplicity, we consider the case $n = 2$. Let $G = GL(2, \boldsymbol{C})$, $M = M(2, \boldsymbol{C})$ and $\boldsymbol{P}^4$ be as above. A rational function

$$f : \boldsymbol{P}^4 \to \boldsymbol{P}^1, \quad [x_0 : \cdots : x_4] \mapsto [x_1 x_4 - x_2 x_3 : x_1^2]$$

is holomorphic on G, but not defined on the set

$$V := \{[x_0 : \cdots : x_4]; x_1 = x_2 = 0 \text{ or } x_1 = x_3 = 0\}.$$

Consider the graph

$$\Gamma := \{([x_0 : \cdots : x_4], [x_1x_4 - x_2x_3 : x_1^2]) \in \boldsymbol{P}^4 \times \boldsymbol{P}^1; [x_0 : \cdots : x_4] \in \boldsymbol{P}^4 \backslash V\}.$$

Let X be the closure of Γ in $\boldsymbol{P}^4 \times \boldsymbol{P}^1$, and let

$$\pi_1 : X \to \boldsymbol{P}^4, \quad \pi_2 : X \to \boldsymbol{P}^1$$

be the first and second projections. Let

$$W := \{[x_0 : \cdots : x_4]; P(x) = 0\},$$

where $P(x) = x_0(x_1x_4 - x_2x_3)$. Then $V \subset W$. If we set $Y := \pi_1^{-1}(W)$, (X, Y) is a projective compactification of G for π_1 is biholomorphic on $\pi_1^{-1}(\boldsymbol{P}^4 \backslash V)$.

Now we consider a sequence $\{D_i\}$ with

$$D_i = \begin{pmatrix} 1 & 0 \\ 0 & 1/i \end{pmatrix} \quad \text{for } i \in \boldsymbol{N}.$$

For any $Z = (z_{ij}) \in G$ we have

$$\begin{aligned} \lim_{i \to \infty} \pi_1^{-1}(D_i^{-1}Z) &= \lim_{i \to \infty}([1 : z_{11} : z_{12} : iz_{21} : iz_{22}], [i \det Z : z_{11}^2]) \\ &= ([0 : 0 : 0 : z_{21} : z_{22}], [1 : 0]). \end{aligned}$$

If we set

$$A := \{([0 : 0 : 0 : x_3 : x_4], [1 : 0]); [x_3 : x_4] \in \boldsymbol{P}^1\},$$

then $\{D_i^{-1}\}$ is A run-away.

We show that $K = \{\lambda I; |\lambda| = 1\}$ is contained in $\mathscr{K}(G, A)$. Let $\pi_2 = [f_0 : f_1]$. We define a function $g := (f_1/f_0) - 1$ which is holomorphic on G and extends holomorphically to A. Since $|g(A)| = 1$ and $|g(K)| = 0$, $K \in \mathscr{K}(G, A)$.

Example 2. Let $S = SL(n, \boldsymbol{C})$ be the complex special linear group of degree n. It is an algebraic submanifold of $M = M(n, \boldsymbol{C})$. We identify $\boldsymbol{C}^N$ with $\{[x] \in \boldsymbol{P}^N; x_0 \neq 0\}$, where $N = n^2$. S has the algebraic extension X in $\boldsymbol{P}^N$. Let

$$\begin{aligned} Y &:= X \cap \{x_0 = 0\} \\ &= \left\{ [0 : x_1 : \cdots : x_N] \in \boldsymbol{P}^N; \det \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_{n+1} & x_{n+2} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{(n-1)n+1} & x_{(n-1)n+2} & \cdots & x_{n^2} \end{pmatrix} = 0 \right\}. \end{aligned}$$

Then the pair (X, Y) is a projective compactification of S. In [**2**], the first author considered the following subspaces

$$X_j := \{Z = (z_{\alpha\beta}) \in M; z_{\alpha\beta} = 0 \text{ for all } \alpha, \beta \text{ except } \alpha = j\},$$

$$Y_j := \{Z = (z_{\alpha\beta}) \in M; z_{j1} = \cdots = z_{jn} = 0\}.$$

Let $\overline{X_j}$ be the closure of X_j in $\boldsymbol{P}^N$. We define

$$A_j := \overline{X_j} \cap Y.$$

There is the natural inclusion $\mathcal{O}(Y_j) \subset \mathcal{O}(M)$. In [**2**], we defined the set $B(S)$ of all compact sets $K \subset S$ with $\hat{K}_S = K$ for which there exist $f \in \mathcal{O}(Y_j)$ for some $j = 1, \dots, n$ with $f(0) \notin \widehat{f(K)}$. Let $A := \bigcup_{j=1}^{m} A_j$. By Corollary 6.2 we obtain the following extension of the result in [**2**] for $B(S) \subset \mathcal{K}(S, A)$: *If $\{C_i^{-1}\} \subset S$ is A_j run-away for any j, then the set of all functions in $\mathcal{O}(S)$ which are universal with respect to $\{C_i\}$ in $\mathcal{A}(K)$ for all $K \in \mathcal{K}(S, A)$ is the intersection of countable number of open dense subsets in $\mathcal{O}(S)$.*

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