J. Math. Soc. Japan
Vol. 56, No. 1, 2004

Invariant fiber measures of angular flows and the Ruelle invariant

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(Received Feb. 13, 2001)
(Revised Jun. 12, 2002)

Abstract. The Ruelle invariants for non-singular flows of a 3-dimensional manifold and diffeomorphisms of the disc are described by invariant fiber measures, which are families of probability measures on the fibers of the projectivized bundle invariant under the holonomies among almost all fibers. The dynamical properties of invariant fiber measures are also given, which show the benefit of this description.

1. Introduction.

Let M be a 3-dimensional Riemannian manifold, and X a non-singular C^1 vector field of M. Denote by ψ_t the flow generated by X. Let NX be the quotient bundle of the tangent bundle TM by the line bundle determined by X. For any t, the differential $D\psi_t$ of ψ_t induces a flow on NX, denoted by $N\psi_t$, which represents the infinitesimal behavior of ψ_t and was studied in various points of view ([**5**], [**6**]). Taking account of the variation of the angles along the orbits of $N\psi_t$, we define the projectivized bundle PX by $\bigcup_{z \in M}(NX_z - 0/v \sim kv)$ $(v \in NX_z - 0, k \in \boldsymbol{R} - 0)$, where NX_z is the fiber of NX at z. Then $N\psi_t$ also induces a flow, say φ_t, on PX. The bundle PX has a natural $PSO(2)$-structure induced from the Riemannian metric of M and the time t map φ_t restricted to each fiber of PX is a projective transformation.

In order to estimate the twist along the orbits, we use the Ruelle invariant defined as follows: We assume that PX is a trivial bundle, and is parametrized as $M \times \boldsymbol{P}^1$. We define the projection from $M \times \boldsymbol{R}$ to $PX = M \times \boldsymbol{P}^1$ by $(z, x) \mapsto (z, [x])$, where $\boldsymbol{P}^1$ is parametrized as $\boldsymbol{R}/\boldsymbol{Z}$ and $x \mapsto [x]$ is the natural projection from $\boldsymbol{R}$ to $\boldsymbol{P}^1$. Then there is a flow $\tilde{\varphi}_t$ of $M \times \boldsymbol{R}$ which is a lift of φ_t ([**6**]). Let p_i denote the projection to the i-th factor of $M \times \boldsymbol{P}^1$ and $M \times \boldsymbol{R}$ $(i = 1, 2)$. We define $\tau_{(z,t)} : \boldsymbol{R} \to \boldsymbol{R}$ by $\tau_{(z,t)}(x) = p_2\tilde{\varphi}_t(z, x)$. For an invariant measure μ of ψ_t, the Ruelle invariant $R_\mu(\psi_t)$ is defined by $(1/2)\int_M (\lim_{t \to \infty} \tau_{(z,t)}(x)/t)\, d\mu$.

In order to examine the Ruelle invariant, we use a family of probability measures on the fibers invariant under the holonomies among almost all fibers, which is called an 'invariant fiber measure' (The precise definition is given in §2). This was constructed in [**8**], [**9**] and [**10**] by using Markov-Kakutani theorem for measure valued functions. In this paper, we will construct an invariant fiber measure from an invariant measure of φ_t by using the disintegration.

In §3 and §4, we will characterize the invariant fiber measure by using Poincaré

2000 *Mathematics Subject Classification*. Primary 37C40; Secondary 37E45, 37B20, 37C10.
Key Words and Phrases. Ruelle invariant, angular flow, invariant fiber measure.
Partially supported by Grant-in-Aid for Scientific Research (No. 11640062[†], No. 11554001[‡]), Japan Society for the Promotion of Science.

recurrence theorem and Furstenberg's theorem, which show that the supports on almost all fibers consist of one point or two points or the whole fiber (Theorem 4). This shows the geometric aspects of invariant fiber measures, and it motivates the description of the Ruelle invariant by it.

Let ν be a probability measure on $PX = M \times \boldsymbol{P}^1$ invariant under φ_t such that $(p_1)_* \nu = \mu$. Denote by $\tilde{\nu}$ the lift of ν on $M \times \boldsymbol{R}$. Then the Ruelle invariant is described as follows:

Theorem 1. *Let* $\Omega_+ = \{(z, x) \in M \times \boldsymbol{R}; 0 \leqq x < p_2 \tilde{\varphi}_1(\psi_{-1}(z), 0), p_2 \tilde{\varphi}_1(\psi_{-1}(z), 0) > 0\}$ *and* $\Omega_- = \{(z, x) \in M \times \boldsymbol{R}; 0 > x \geqq p_2 \tilde{\varphi}_1(\psi_{-1}(z), 0), p_2 \tilde{\varphi}_1(\psi_{-1}(z), 0) < 0\}$. *Then the Ruelle invariant* $R_\mu(\psi_t)$ *is equal to* $(1/2)\tilde{\nu}(\Omega_+) - (1/2)\tilde{\nu}(\Omega_-)$.

Here the Ruelle invariant is given by ν instead of the invariant fiber measure as a result, which can be understood as a benefit of the construction of the invariant fiber measure from ν.

The same argument as above is available for diffeomorphisms of the 2-dimensional disc D^2. Let G denote the set of diffeomorphisms of D^2 which are the identity near the boundary and preserve the canonical measure μ. Denote by PD^2 the projectivized bundle of the tangent bundle of D^2, which is parametrized as $D^2 \times \boldsymbol{P}^1$. Let $p_1 : PD^2 \to D^2$ denote the projection to the first factor. For the induced diffeomorphism Pf of the differential Df on PD^2, the equation similar to Theorem 1 holds (Theorem 5), and furthermore we obtain the following results.

Theorem 2. *Let* f *and* g *be elements of* G. *If there is a probability measure* ν *of* PD^2 *invariant under both* Pf *and* Pg *satisfying* $(p_1)_* \nu = \mu$, *then* $R_\mu(g \circ f) = R_\mu(f) + R_\mu(g)$.

Corollary 1. *Let* Γ *be an amenable subgroup of* G. *Then* $R_\mu : \Gamma \to \boldsymbol{R}$ *is a homomorphism.*

The authors wish to thank Masahiko Kanai for his various helpful suggestions for their attempts to apply Zimmer's theory to dynamical systems and also Shigenori Matsumoto for his remark on the disintegration and the application of the invariant fiber measure for diffeomorphisms of the disc.

2. Definition of invariant fiber measures.

Let M be a compact metric space and $p : N \to M$ a $\boldsymbol{P}^1$-bundle over M. Let ψ_t and φ_t be topological flows of M and N respectively such that $p \circ \varphi_t = \psi_t \circ p$. In this section and also in §3, the holonomies among the fibers along φ_t is not assumed to be projective.

Let ν be a φ_t-invariant measure on N, and let $\mu = p_* \nu$, i.e. $\mu(E) = \nu(p^{-1}(E))$ if $p^{-1}(E)$ is a ν-measurable set. Then μ is invariant under ψ_t (μ is not always ergodic in this section).

Let $C(N)$ be the set of continuous functions of N endowed with the uniform convergence topology. Denote by $\mathscr{M}(N)$ the set of Radon probability measures of N, endowed with the weak* topology (i.e. $\sigma_n \to \sigma$ $(\sigma_n, \sigma \in \mathscr{M}(N))$ if $\langle \sigma_n, f \rangle \to \langle \sigma, f \rangle$ for

$f \in C(N)$). Then $\mathscr{M}(N)$ is a compact metrizable space. A map $\lambda : M \to \mathscr{M}(N)$ is called scalarwise measurable if the function $z \mapsto \langle \lambda(z), f \rangle = \int_N f \, d\lambda(z)$ is μ-measurable for all $f \in C(N)$. A map $\lambda : M \to \mathscr{M}(N)$ is called measurable if, for any $\varepsilon > 0$, there is a compact subset K of M such that $\mu(M - K) < \varepsilon$ and $\lambda | K$ is continuous. Note that the measurability of λ implies the scalarwise measurability of λ. Conversely, λ is measurable if it is scalarwise measurable by Lemma 3 of Chapter 6, §3, $\mathrm{N}^0 1$ in [**1**]. Let N_z denote the fiber $p^{-1}(z)$ for $z \in M$. By disintegration (Chapter 6, §3, Theorem 1 of [**1**]), there is a scalarwise measurable map $\lambda : M \to \mathscr{M}(N)$ satisfying that $\operatorname{supp} \lambda(z) \subset N_z$ for all $z \in M$ and that $\nu = \int d\mu \int d\lambda(z)$ and such a scalarwise measurable map is unique μ-a.e.z.

For any homeomorphism h of N and $\sigma \in \mathscr{M}(N)$, we define $h_* \sigma$ by $\langle h_* \sigma, f \rangle = \langle \sigma, f \circ h \rangle$ for $f \in C(N)$. In other words, $h_* \sigma(E) = \sigma(h^{-1}(E))$ for any measurable set E of N. Then the map $(h, \sigma) \mapsto h_* \sigma$ is continuous with respect to the uniform convergence topology and the weak* topology.

Lemma 1. *For a scalarwise measurable map $\lambda : M \to \mathscr{M}(N)$ satisfying that $\operatorname{supp} \lambda(z) \subset N_z$ for all $z \in M$ and that $\nu = \int d\mu \int d\lambda(z)$, the equation $(\varphi_t)_* \lambda(z) = \lambda(\psi_t(z))$ μ-a.e.z holds for any $t \in \boldsymbol{R}$.*

Proof. Let $\eta_t : M \to \mathscr{M}(N)$ $(t \in \boldsymbol{R})$ denote the map defined by $\eta_t(z) = (\varphi_t^{-1})_* \lambda(\psi_t(z))$. For any $\varepsilon > 0$, there is a compact set K such that $\mu(M - K) < \varepsilon$ and $\lambda \,|\, \psi_t(K)$ is continuous. Then η_t is continuous on K. Therefore η_t is measurable, and hence scalarwise measurable.

For any measurable set W of N,

$$\begin{aligned}
\int_M d\mu \int_W d\eta_t(z) &= \int_M d\mu \int_{\varphi_t(W)} d\lambda(\psi_t(z)) \\
&= \int_M d((\psi_t)_* \mu) \int_{\varphi_t(W)} d\lambda(z) \\
&= \int_M d\mu \int_{\varphi_t(W)} d\lambda(z) \\
&= \nu(\varphi_t(W)) = \nu(W).
\end{aligned}$$

Therefore, $\eta_t(z)$ coincides with $\lambda(z)$ μ-a.e.z for any $t \in \boldsymbol{R}$ by the uniqueness of such a scalarwise measurable map. That is, $(\varphi_t^{-1})_* \lambda(\psi_t(z)) = \lambda(z)$ μ-a.e.z for any $t \in \boldsymbol{R}$. □

A set E of M is called conull if E is measurable and $\mu(E) = 1$. By the above consideration, $(\varphi_t^{-1})_* \lambda(\psi_t(z)) = \lambda(z)$ on a conull set for any $t \in \boldsymbol{R}$. However this conull set may vary with respect to t. The following lemma shows the existence of conull sets on which $(\varphi_t^{-1})_* \lambda(\psi_t(z)) = \lambda(z)$ holds for any $t \in \boldsymbol{R}$ after a slight modification of λ, which is proved in the same way as Appendix B.5 of [**10**].

Lemma 2. *For any scalarwise measurable map $\lambda_1 : M \to \mathscr{M}(N)$ satisfying that $\operatorname{supp} \lambda_1(z) \subset N_z$ for all $z \in M$ and that $\nu = \int d\mu \int d\lambda_1(z)$, there are an invariant conull set B of M and a scalarwise measurable map $\lambda_2 : M \to \mathscr{M}(N)$ such that $\lambda_2(z) = \lambda_1(z)$ μ-a.e.z and $\lambda_2(\psi_t(z)) = (\varphi_t)_* \lambda_2(z)$ for any $t \in \boldsymbol{R}$ and any $z \in B$. In particular, $\nu = \int d\mu \int d\lambda_2(z)$.*

Proof. Let σ_1 denote the ordinary measure $\int dt$ of $\boldsymbol{R}$. Denote by B the set $\{z \in M; t \mapsto (\varphi_t^{-1})_* \lambda_1(\psi_t(z))$ is essentially constant with respect to $\sigma_1\}$.

Here we claim that $(z,t) \mapsto (\varphi_t^{-1})_* \lambda_1(\psi_t(z))$ is measurable with respect to $\mu \times \sigma_1$. Let n be a positive integer. For any $\varepsilon > 0$, there is a compact set K such that $\mu(M - K) < \varepsilon/(2n)$ and $\lambda_1 | K$ is continuous. Let K_n denote the compact set $\{(z,t) \in M \times \boldsymbol{R}; -n \leqq t \leqq n, \psi_t(z) \in K\}$. Then $(z,t) \mapsto (\varphi_t^{-1})_* \lambda_1(\psi_t(z))$ is continuous on K_n. On the other hand, we have $(\mu \times \sigma_1)(M \times [-n,n] - K_n) = 2n - \int_{-n}^{n} d\sigma_1 \int \psi_{-t}(K)\, d\mu < \varepsilon$. Thus $(z,t) \mapsto (\varphi_t^{-1})_* \lambda_1(\psi_t(z))$ is measurable with respect to $\mu \times \sigma_1$, and hence is scalarwise measurable.

Let $\mathscr{E}$ denote the set $\{(z,t) \in M \times \boldsymbol{R}; (\varphi_t^{-1})_* \lambda_1(\psi_t(z)) \neq \lambda_1(z)\}$. For a countable dense set $\{f_n\}_{n=1,2,\dots}$ of $C(N)$, the set $\mathscr{E}$ coincides with the set $\bigcup_{n=1,2,\dots} \{(z,t) \in M \times \boldsymbol{R}; \langle (\varphi_t^{-1})_* \lambda_1(\psi_t(z)), f_n \rangle \neq \langle \lambda_1(z), f_n \rangle\}$. Since $(z,t) \mapsto \langle (\varphi_t^{-1})_* \lambda_1(\psi_t(z)), f_n \rangle$ and $(z,t) \mapsto \langle \lambda_1(z), f_n \rangle$ are measurable as above, we obtain that $\mathscr{E}$ is a measurable set with respect to $\mu \times \sigma_1$. Let $\mathscr{E}_z = \{t \in \boldsymbol{R}; (z,t) \in \mathscr{E}\}$ $(z \in M)$ and $\mathscr{E}_t = \{z \in M; (z,t) \in \mathscr{E}\}$ $(t \in \boldsymbol{R})$. Then $\mu(\mathscr{E}_t) = 0$ by Lemma 1. Using Fubini's theorem, we have $\int_M \sigma_1(\mathscr{E}_z)\, d\mu = (\mu \times \sigma_1)(\mathscr{E}) = \int_{\boldsymbol{R}} \mu(\mathscr{E}_t)\, d\sigma_1 = 0$. Hence $\sigma_1(\mathscr{E}_z) = 0$ μ-a.e.z. Let E be a conull set of M such that $\sigma_1(\mathscr{E}_z) = 0$ for $z \in E$. Then E is contained in B. This implies that B is a conull set.

We define a map $\lambda_2 : M \to \mathscr{M}(N)$ such that $\lambda_2(z)$ is the essential constant of $(\varphi_t^{-1})_* \lambda_1(\psi_t(z))$ with respect to σ_1 if $z \in B$. Since E is contained in B, we have $\lambda_2(z) = \lambda_1(z)$ μ-a.e.z. For any $f \in C(N)$ and $z \in B$, we have $\langle \lambda_2(z), f \rangle = \int_{\boldsymbol{R}} \langle (\varphi_t^{-1})_* \lambda_1(\psi_t(z)), f \rangle\, d\sigma_2$, where σ_2 is a probability measure on $\boldsymbol{R}$ with the same null sets with those of σ_1. Hence λ_2 is scalarwise measurable by Fubini's theorem.

For any $s \in \boldsymbol{R}$ and $z \in B$,

$$
\begin{aligned}
\sigma_1(\{t; (\varphi_t^{-1})_* \lambda_1(\psi_t \psi_s(z)) \neq (\varphi_s)_* \lambda_2(z)\}) \\
&= \sigma_1(\{t; (\varphi_s)_* (\varphi_{s+t}^{-1})_* \lambda_1(\psi_{s+t}(z)) \neq (\varphi_s)_* \lambda_2(z)\}) \\
&= \sigma_1(\{u - s; (\varphi_u^{-1})_* \lambda_1(\psi_u(z)) \neq \lambda_2(z)\}) \\
&= \sigma_1(\{u; (\varphi_u^{-1})_* \lambda_1(\psi_u(z)) \neq \lambda_2(z)\}) = 0.
\end{aligned}
$$

Thus we have $\psi_s(z) \in B$ and $\lambda_2(\psi_s(z)) = (\varphi_s)_* \lambda_2(z)$. □

Definition. A scalarwise measurable map $\lambda : M \to \mathscr{M}(N)$ such that $\operatorname{supp} \lambda(z) \subset N_z$ for all $z \in M$ is called an *invariant fiber measure* for a probability measure μ of M if there is an invariant conull set B of M with respect to μ such that $\lambda(\psi_t(z)) = (\varphi_t)_* \lambda(z)$ for any $z \in B$ and $t \in \boldsymbol{R}$, where the set B is called a *basic set*. By the above consideration, we obtain the following key lemma needed later.

Fundamental lemma. Let ψ_t and φ_t be topological flows of M and N respectively satisfying $p \circ \varphi_t = \psi_t \circ p$. For a probability measure ν on N invariant under φ_t, there are a scalarwise measurable map $\lambda : M \to \mathscr{M}(N)$ and an invariant conull set B of M satisfying

(1) $\operatorname{supp} \lambda(z) \subset N_z$,

(2) $\nu = \int d\mu \int d\lambda(z)$ $(\mu = p_* \nu)$ and

(3) $\lambda(\psi_t(z)) = (\varphi_t)_* \lambda(z)$ for any $z \in B$ and $t \in \boldsymbol{R}$,

which is called an invariant fiber measure derived from ν.

Remark. Let μ be a probability measure on M invariant under ψ_t. Denote by $\mathscr{M}_\mu(N)$ the set of probability measures ν on N satisfying $p_* \nu = \mu$. Then $\mathscr{M}_\mu(N)$ is a closed subset of the set of probability measures, and hence is compact. By Markov-Kakutani theorem, there is a probability measure ν on N invariant under φ_t satisfying $p_* \nu = \mu$.

Remark. The existence of a scalarwise measurable map $\lambda : M \to \mathscr{M}(N)$ satisfying $\operatorname{supp} \lambda(z) \subset N_z$ and $\lambda(\psi_t(z)) = (\varphi_t)_* \lambda(z)$ μ-a.e.z for any $t \in \boldsymbol{R}$ was already given for an ergodic measure μ in the case of product bundles in Theorem 2.1 of [**8**], where the existence was more generally shown for amenable actions instead of $\boldsymbol{R}$-actions.

3. Recurrency.

In §3 and §4, we give properties of the invariant fiber measure, which show that it is useful to study the dynamical properties of φ_t and ψ_t. Though many parts of these sections are more or less known in the fields of ergodic theory of amenable group actions, the authors have not found them in the literature. In order to explain the benefit of the description of the Ruelle invariant by the invariant fiber measure, we give a self-contained explanation in these sections.

Let M be a compact metric space and $p : N \to M$ a $\boldsymbol{P}^1$-bundle over M. Let ψ_t be a topological flow of M, and let φ_t be a topological flow of N satisfying $p \circ \varphi_t = \psi_t \circ p$. Let μ be an ergodic probability measure on M invariant under ψ_t, i.e. a ψ_t-invariant measurable set has a full or null measure. Under these conditions, we will give a relation among almost all $\lambda(z)$ (Lemma 4). The essential part of its proof is given in Lemma 3, which is proved by using the method of Poincaré recurrence theorem.

Definition. Let S be a subset of M. For $z \in M$, we define the subset $\omega_S(z)$ by $\bigcap_{s \geq 0} \overline{\{\psi_t(z); t \geq s\} \cap S}$. Then, for any $y \in \omega_S(z)$, there exists a sequence $\{t_n\}_{n=1,2,\ldots}$ such that $\lim_{n\to\infty} t_n = \infty$, $\psi_{t_n}(z) \in S$ and $y = \lim_{n\to\infty} \psi_{t_n}(z)$.

Lemma 3. *Let λ be an invariant fiber measure with a basic set B, then there is a closed set F with $\mu(F) > 0$ such that*

(1) *λ is continuous on $F \cap B$, and*

(2) *$\omega_F(z) = F$ μ-a.e.z in F.*

Proof. Since λ is measurable, there is a compact set F_0 such that $\mu(F_0) > 0$ and $\lambda|_{F_0}$ is continuous. Let $F = \operatorname{supp}(\mu|F_0)$. In other words, $z \in F_0$ is an element of F if and only if $\mu(F_0 \cap U) > 0$ for any open set U of M satisfying $z \in U$. Then F is a closed subset of M and $\mu(F_0 - F) = 0$. In particular, F has a positive measure.

Denote by $\{V_m\}_{m=1,2,\ldots}$ the subfamily of the countable base $\{U_n\}$ of M satisfying $F \cap V_m \neq \emptyset$. Then $\mu(F \cap V_m) > 0$. Let E_m $(m = 1,2,\ldots)$ denote $\bigcap_{t \leqq 0} \bigcup_{s \leqq t} \psi_s (F \cap \overline{V_m})$. Then we have $\mu(\bigcup_{s \leqq t} \psi_s(F \cap \overline{V_m})) = \mu(\psi_{-t} \bigcup_{s \leqq t} \psi_s(F \cap \overline{V_m})) = \mu(\bigcup_{s \leqq 0} \psi_s (F \cap \overline{V_m})) \geqq \mu(F \cap \overline{V_m})$. Hence $\mu(E_m) \geqq \mu(F \cap \overline{V_m}) > 0$ because $\bigcup_{s \leqq t} \psi_s(F \cap \overline{V_m})$ decreases as $t \to -\infty$. On the other hand, E_m is invariant under ψ_t. Since we assume that ψ_t is ergodic, we obtain $\mu(E_m) = 1$, and hence $\mu(F - E_m) = 0$. Therefore, we have $\mu(F - \bigcap_{m=1}^{\infty} E_m) = \mu(\bigcup_{m=1}^{\infty}(F - E_m)) = 0$, and $\mu(F \cap \bigcap_{m=1}^{\infty} E_m) = \mu(F)$.

Finally we will show that $\omega_F(z) = F$ on $F \cap \bigcap_{m=1}^{\infty} E_m$. Let z be an element of $F \cap \bigcap_{m=1}^{\infty} E_m$. Suppose that $F - \omega_F(z)$ is not empty. Let w be a point of $F - \omega_F(z)$. Since $\omega_F(z)$ is a closed set, we can choose an open set V_k from $\{V_m\}$ such that $w \in V_k$ and $\overline{V_k} \cap \omega_F(z) = \emptyset$. Then $\overline{F \cap V_k} \subset F \cap \overline{V_k} \subset F - \omega_F(z)$. By the choice of E_k, there is a sequence $\{t_n\}_{n=1,2,\dots}$ such that $\lim_{n\to\infty} t_n = -\infty$ and $z \in \psi_{t_n}(F \cap V_k)$. By taking a subsequence of $\{t_n\}$, we can assume that $\{\psi_{-t_n}(z)\}$ converges to some point of $\overline{F \cap V_k}$, which is also contained in $\omega_F(z)$ by the definition of $\omega_F(z)$. However, this contradicts the choice of V_k. Thus $\omega_F(z) = F$ for $z \in F \cap \bigcap_{m=1}^{\infty} E_m$. □

Lemma 4. *Let λ be an invariant fiber measure with a basic set B. Then there exists an invariant conull set E of M contained in B such that, for any z and w of E, there is a sequence $\{t_n\}_{n=1,2,\dots}$ of $\boldsymbol{R}$ satisfying $\lim_{n\to\infty} t_n = +\infty$, $\lim_{n\to\infty} \psi_{t_n}(z) = w$ and $\lim_{n\to\infty} \lambda(\psi_{t_n}(z)) = \lambda(w)$.*

Proof. Let F be the closed set obtained in Lemma 3, and E_0 a closed subset of F such that $\mu(E_0) > 0$ and $\omega_F(z) = F$ for any $z \in E_0$. Let $E = (\bigcup_{s\in\boldsymbol{R}} \psi_s(E_0)) \cap B$. Since $\bigcup_{s\in\boldsymbol{R}} \psi_s(E_0)$ is invariant under ψ_t, the set E is also a conull set.

Let z and w be points of E. Then there are s_0 and s_1 of $\boldsymbol{R}$ such that $\psi_{s_0}(z) \in E_0$ and $\psi_{s_1}(w) \in F$. Hence there is a sequence $\{u_n\}_{n=1,2,\dots}$ such that $\lim_{n\to\infty} u_n = +\infty$, $\lim_{n\to\infty} \psi_{u_n}(\psi_{s_0}(z)) = \psi_{s_1}(w)$ and $\psi_{u_n}(\psi_{s_0}(z)) \in F$. In particular, $\lim_{n\to\infty} \psi_{u_n+s_0-s_1}(z) = w$.

Let $t_n = u_n + s_0 - s_1$. Then $\lim_{n\to\infty} t_n = +\infty$ and $\lim_{n\to\infty} \psi_{t_n}(z) = w$. Since λ is continuous on F, we have $\lim_{n\to\infty} \lambda(\psi_{u_n+s_0}(z)) = \lambda(\psi_{s_1}(w))$. Hence $\lim_{n\to\infty} (\varphi_{s_1})_* \lambda(\psi_{t_n}(z)) = (\varphi_{s_1})_* \lambda(w)$, and thus, $\lim_{n\to\infty} \lambda(\psi_{t_n}(z)) = \lambda(w)$. □

By the above lemmas, we obtain the following theorem.

Theorem 3. *Let M be a compact metric space and $p : N \to M$ a $\boldsymbol{P}^1$-bundle over M. Let ψ_t and φ_t be flows of M and N respectively such that $p \circ \varphi_t = \psi_t \circ p$. Let ν be a probability measure on N invariant under φ_t. Denote by μ the probability measure $p_*\nu$ on M. If μ is ergodic, then there are a scalarwise measurable map $\lambda : M \to \mathscr{M}(N)$ and an invariant measurable set E of M with $\mu(E) = 1$ such that*

(1) *For any $z \in M$, $\operatorname{supp} \lambda(z)$ is contained in N_z,*

(2) $\nu = \int d\mu \int d\lambda(z)$,

(3) $(\varphi_t)_* \lambda(z) = \lambda(\psi_t(z))$ *for any $z \in E$ and $t \in \boldsymbol{R}$, and*

(4) *For any points z and w of E, there is a sequence $\{t_n\}_{n=1,2,\dots}$ of $\boldsymbol{R}$ satisfying that $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} \psi_{t_n}(z) = w$, and $\lim_{n\to\infty} \lambda(\psi_{t_n}(z)) = \lambda(w)$.*

4. Extension of supports.

In this section, we will assume that our $\boldsymbol{P}^1$-bundle $p : N \to M$ has a $PSO(2)$-structure and that each φ_t induces projective transformations among the fibers as follows.

Let $\{(U_i, \xi_i)\}_{i=1,2,\dots,r}$ be a $PSO(2)$-structure of a $\boldsymbol{P}^1$-bundle $p : N \to M$. Namely, $\{U_i\}$ is an open cover of M, and $\xi_i : p^{-1}(U_i) \to U_i \times \boldsymbol{P}^1$ is a homeomorphism such that $p = pr \circ \xi_i$, where $pr : U_i \times \boldsymbol{P}^1 \to U_i$ is the projection to the first factor, and that

$\xi_j \circ \xi_i^{-1} : (U_i \cap U_j) \times \boldsymbol{P}^1 \to (U_i \cap U_j) \times \boldsymbol{P}^1$ is of the form $\xi_j \circ \xi_i^{-1}(z, x) = (z, \gamma_{ji}(z)x)$ where $\gamma_{ji} : U_i \cap U_j \to PSO(2)$.

Let $z \in U_i$ and $w \in U_j$. We say that a transformation $g : N_z \to N_w$ is *projective* if $(\xi_j|N_w) \circ g \circ (\xi_i|N_z)^{-1} : \boldsymbol{P}^1 \to \boldsymbol{P}^1$ is an element of $PSL(2, \boldsymbol{R})$. We readily see that this definition is well-defined.

For $A \in PSL(2, \boldsymbol{R})$, let $\|A\| = \sup_{\|v\|=1, v \in \boldsymbol{R}^2} \|Av\|$. Then, for a projective transformation $g : N_z \to N_w$ we can define the norm of g by $\|g\| = \|(\xi_j|N_w) \circ g \circ (\xi_i|N_z)^{-1}\|$, which is also independent of the choice of (U_i, ξ_i) and (U_j, ξ_j).

We assume throughout that $\varphi_t|N_z$ is projective for all $z \in M$ and $t \in \boldsymbol{R}$. For an invariant fiber measure λ, we abbreviate $\lambda(z) \,|\, N_z$ as $\lambda(z)$.

By using Furstenberg's theorem, we obtain the following.

Lemma 5. *Let λ be an invariant fiber measure and let E be an invariant conull set contained in a basic set satisfying the condition* (4) *of Theorem* 3. *Then one of the following properties holds.*

(1) $\operatorname{supp} \lambda(z)$ *consists of one point for any* $z \in E$.

(2) $\operatorname{supp} \lambda(z)$ *consists of two points for any* $z \in E$.

(3) *For any points z and w of E, there is a projective transformation $g : N_z \to N_w$ satisfying $g_* \lambda(z) = \lambda(w)$.*

Proof. By assumption, for any points z and w of E, there is a sequence $\{t_n\}_{n=1,2,\dots}$ of $\boldsymbol{R}$ satisfying $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} \psi_{t_n}(z) = w$ and $\lim_{n\to\infty} \lambda(\psi_{t_n}(z)) = \lambda(w)$.

If there are two points z and w of E such that $\|\varphi_{t_n}|N_z\|$ is not bounded, then $\operatorname{supp} \lambda(w)$ consists of one or two points by Furstenberg's theorem ([**2**], see also Lemma 3.2.1 of [**10**], Theorem 4.1 of [**9**]). (A precise argument goes as follows: Let $z \in U_i$ and $w \in U_j$. We assume without loss of generality that $\psi_{t_n}(z) \in U_j$ for all n. Then apply Furstenberg's theorem to the sequence $(\xi_j|N_{\psi_{t_n}(z)}) \circ \varphi_{t_n} \,|\, N_z \circ (\xi_i|N_z)^{-1}$ in $PSL(2, \boldsymbol{R})$.) For any p in E, there is a sequence $\{s_n\}_{n=1,2,\dots}$ such that $\lim_{n\to\infty} \psi_{s_n}(w) = p$ and $\lim_{n\to\infty} \lambda(\psi_{s_n}(w)) = \lambda(p)$ by assumption. Then $\operatorname{supp} \lambda(p)$ also consists of one or two points. Furthermore, if $\operatorname{supp} \lambda(w)$ consists of one point, then $\operatorname{supp} \lambda(p)$ also consists of one point. Therefore, $\operatorname{supp} \lambda(z)$ consists of one point for any $z \in E$ or $\operatorname{supp} \lambda(z)$ consists of two points for any $z \in E$.

On the other hand, if $\|\varphi_{t_n}|N_z\|$ is bounded for any z and w in E, then we can take a subsequence $\{s_n\}_{n=1,2,\dots}$ of $\{t_n\}$ such that $\varphi_{s_n}|N_z$ converges to some projective transformation $g : N_z \to N_w$. Then, we have $g_* \lambda(z) = \lim_{n\to\infty} (\varphi_{s_n})_* \lambda(z) = \lim_{n\to\infty} \lambda(\psi_{s_n}(z)) = \lambda(w)$. Thus λ is of type (3). □

Lemma 6. *Let σ be a probability measure of $\boldsymbol{P}^1$. If* $\operatorname{supp} \sigma$ *contains at least three points and* $\operatorname{supp} \sigma \neq \boldsymbol{P}^1$, *then the stabilizer $St(\sigma) = \{f \in PSL(2, \boldsymbol{R}); f_* \sigma = \sigma\}$ is generated by a unique periodic elliptic element of $PSL(2, \boldsymbol{R})$.*

Proof. If $\{\|f\|; f \in St(\sigma)\}$ is not bounded, then $\operatorname{supp} \sigma$ consists of one or two points by Furstenberg's theorem ([**2**], see also Lemma 3.2.1 of [**10**]). Since this contradicts the assumption, the set $\{\|f\|; f \in St(\sigma)\}$ is bounded. In particular, all the elements of $St(\sigma)$ are elliptic. Thus they are individually conjugate to rotations.

We claim that $St(\sigma)$ is abelian. Let x_0 be a point of $\boldsymbol{P}^1$. We define $\Phi : \boldsymbol{P}^1 \to$

$\boldsymbol{R}/\boldsymbol{Z}$ by $\Phi(x) = \int_{x_0}^{x} d\sigma = \sigma([x_0, x))$, where $[x_0, x)$ denotes the half-open subarc of $\boldsymbol{P}^1$ from x_0 to x in the counterclockwise order. Let f and g be arbitrary elements of $St(\sigma)$. Then $\Phi(f(x)) = \Phi(x) + \Phi(f(x_0))$ and $\Phi(g(x)) = \Phi(x) + \Phi(g(x_0))$ ([**4**]). Hence there are two rotations R_f and R_g satisfying $\Phi f = R_f \Phi$ and $\Phi g = R_g \Phi$. Since $\Phi(f^{-1}(x)) = \Phi(x) - \Phi(f(x_0))$, we obtain $\Phi f g f^{-1} g^{-1} = R_f R_g R_f^{-1} R_g^{-1} \Phi = \Phi$. Here we assume that there is x_1 of $\operatorname{supp} \sigma$ such that $fgf^{-1}g^{-1}(x_1)$ is different from x_1. Then $fgf^{-1}g^{-1}(x_1)$ and x_1 are the endpoints of a component of $\boldsymbol{P}^1 - \operatorname{supp} \sigma$ and $\mu(\{fgf^{-1}g^{-1}(x_1)\}) = \mu(\{x_1\}) = 0$ because $\Phi(fgf^{-1}g^{-1}(x_1)) = \Phi(x_1)$. Now $fgf^{-1}g^{-1}$ is an orientation preserving homeomorphism preserving $\operatorname{supp} \sigma$. Hence $fgf^{-1}g^{-1}$ maps this component to the adjacent component of $\boldsymbol{P}^1 - \operatorname{supp} \sigma$ with the common boundary $fgf^{-1}g^{-1}(x_1)$. But then $fgf^{-1}g^{-1}(x_1)$ is an isolated point in $\operatorname{supp} \sigma$ and hence $\mu(\{fgf^{-1}g^{-1}(x_1)\})$ must be positive. This is a contradiction. Therefore $fgf^{-1}g^{-1} = \mathrm{id}$ on $\operatorname{supp} \sigma$. Since an element of $PSL(2, \boldsymbol{R})$ is determined by the image of given three points, we conclude that $St(\sigma)$ is abelian.

Let f be an element of $St(\sigma)$ which is not the identity. Let h be an element of $PSL(2, \boldsymbol{R})$ such that $h^{-1}fh$ is a rotation R_f. Since $\operatorname{supp} \sigma$ is not the whole $\boldsymbol{P}^1$, R_f is a rational rotation. Suppose that the period of R_f as an action on $\boldsymbol{P}^1$ is greater than two. For any element g of $St(\sigma)$, we have $R_f^n(h^{-1}gh) = (h^{-1}gh)R_f^n$ for any $n \in \boldsymbol{Z}$ because $fg = gf$. Let x_2 be a point of $\boldsymbol{P}^1$, and let R denote the rotation satisfying $R(x_2) = h^{-1}gh(x_2)$. Then $h^{-1}gh(R_f^n x_2) = R_f^n(R x_2) = R(R_f^n x_2)$. Since the orbit of R_f passing through x_2 contains at least three points, $h^{-1}gh$ also coincides with the rotation R. Since the abelian group $\{h^{-1}gh; g \in St(\sigma)\}$ is generated by a unique rational rotation, $St(\sigma)$ is generated by a unique periodic element.

In case where all elements of $St(\sigma)$ except id have period two, we claim that $St(\sigma)$ consists of id and the other unique element. Let f and g be any elements of $St(\sigma)$ with period two. Suppose that there is a point x_3 of $\boldsymbol{P}^1$ such that $f(x_3) \neq g(x_3)$. Let I be a component of $\boldsymbol{P}^1 - \{x_3, g(x_3)\}$ containing $f(x_3)$, and J a component of $\boldsymbol{P}^1 - \{x_3, f(x_3)\}$ containing $g(x_3)$. Since $f(x_3)$ is contained in I and is also a boundary of $f(J), f(J)$ intersects I. On the other hand, $f(J)$ is a component of $\boldsymbol{P}^1 - \{x_3, f(x_3)\}$ disjoint from J because $f^2 = \mathrm{id}$. Hence $g(x_3) \notin f(J)$ and $x_3 \notin f(J)$. Therefore, $f(J)$ is contained in I because I is a component bounded by $g(x_3)$ and x_3. Thus $f(x_3) = g^2 f(x_3) = gfg(x_3) \in gf(J) \subset g(I)$. However this contradicts the assumption that $f(x_3)$ is contained in I. Thus $St(\sigma) = \{\mathrm{id}, f\}$. □

COROLLARY 2. *Let σ_1 be a probability measure on $\boldsymbol{P}^1$ whose support contains at least three points. Then there is a probability measure σ_2 with full support such that $f_*\sigma_2 = \sigma_2$ for any f of $St(\sigma_1)$.*

PROOF. Suppose that $\operatorname{supp} \sigma_1 \neq \boldsymbol{P}^1$. By Lemma 6, there is a periodic elliptic element f of $PSL(2, \boldsymbol{R})$ generating $St(\sigma_1)$. We choose an element h from $PSL(2, \boldsymbol{R})$ so that $h^{-1}fh$ is a rotation R_f. Let σ_2 denote $h_*\sigma_3$, where σ_3 is the ordinary measure on $\boldsymbol{P}^1$ invariant under rotations. Then σ_2 is a measure with full support satisfying $g_*\sigma_2 = \sigma_2$ for any g of $St(\sigma_1)$. □

THEOREM 4. *Let M be a compact metric space and $p : N \to M$ a $\boldsymbol{P}^1$-bundle over M with structural group $PSO(2)$. Let $\psi_t, \varphi_t, v, \mu, \lambda$ and E be as in Theorem* 3. *Suppose that*

$\varphi_t|N_z : N_z \to N_{\psi_t(z)}$ *is a projective transformation for all* $(z,t) \in M \times \boldsymbol{R}$. *Then we have the following trichotomy*:

(a) $\operatorname{supp} \lambda(z)$ *consists of one point for any* $z \in E$.

(b) $\operatorname{supp} \lambda(z)$ *consists of two points for any* $z \in E$.

(c) *There exist a* φ_t*-invariant probability measure* ν' *on* N *and a scalarwise measurable map* $\lambda' : M \to \mathcal{M}(N)$ *such that the properties* (1), (2), (3) *and* (4) *in Theorem* 3 *hold for* ν' *and* λ' *instead of for* ν *and* λ, *and* $\operatorname{supp} \lambda'(z) = N_z$ *for any* $z \in E$.

Proof. The following proof is essentially the same as that of Lemma 5.3 of [**9**]. Let λ be an invariant fiber measure satisfying the conclusion of Theorem 3. If $\operatorname{supp} \lambda(z)$ does not satisfy the conditions (a) and (b) of Theorem 4, then, for any z and w in E, there is a projective transformation $g : N_z \to N_w$ satisfying $g_* \lambda(z) = \lambda(w)$ and furthermore $\operatorname{supp} \lambda(z)$ contains at least three points for any $z \in E$ (Lemma 5).

We construct an invariant measure λ' such that $\operatorname{supp} \lambda'(z) = N_z$ for any $z \in E$. Let z_0 be a point of E. If $\operatorname{supp} \lambda(z_0) = N_{z_0}$, then, by Lemma 5, $\operatorname{supp} \lambda(z) = N_z$ at any $z \in E$, and we are done. Thus we assume that $\operatorname{supp} \lambda(z_0)$ is not N_{z_0}. For each $z \in E$, choose an element $U_{i(z)}$ of the open cover $\{U_i\}$ such that $z \in U_{i(z)}$, and denote by ξ_z the map $\xi_{i(z)}|N_z : N_z \to \boldsymbol{P}^1$. Let $\sigma_0 = (\xi_{z_0})_* \lambda(z_0)$. Since $H = St(\sigma_0)$ is a finite subgroup of $PSL(2,\boldsymbol{R})$, the quotient map $\theta : PSL(2,\boldsymbol{R}) \to PSL(2,\boldsymbol{R})/H$ is a covering projection. Let O denote the subset $\{g_* \sigma_0; g \in PSL(2,\boldsymbol{R})\}$ of the set of probability measures of $\boldsymbol{P}^1$. Denote by η the map from $PSL(2,\boldsymbol{R})$ to O defined by $\eta(g) = g_* \sigma_0$. Then η is continuous with respect to the weak* topology. Denote by $\hat{\eta}$ the map from $PSL(2,\boldsymbol{R})/H$ to O induced from η. Then $\hat{\eta}$ is bijective. Furthermore $\hat{\eta}$ is a homeomorphism, because, by using Furstenberg's theorem again, we see that if a sequence $\{g_n\}$ in $PSL(2,\boldsymbol{R})$ is not bounded, $(g_n)_* \sigma_0$ does not converge in O. By Corollary 2, there is a probability measure σ with full support such that $g_* \sigma = \sigma$ for any g in H. By assumption, $(\xi_z)_* \lambda(z)$ is contained in O for any $z \in E$. We define a map $\lambda' : M \to \mathcal{M}(N)$ by $\lambda'(z) = (\xi_z)_*^{-1} g_* \sigma$ $(z \in E)$ for an element g of $PSL(2,\boldsymbol{R})$ satisfying $\theta(g) = \hat{\eta}^{-1} (\xi_z)_* \lambda(z)$. Note that $\lambda'(z)$ neither depends on the choice of g nor on the choice of ξ_z.

Now we claim that λ' is scalarwise measurable. Since μ is regular, for any $\varepsilon > 0$ there is a compact set contained in E such that $\mu(M - K) < \varepsilon$ and λ is continuous on K. Let $\{z_n\}$ be a sequence in K such that z_n converges to some point z_∞ on K as $n \to \infty$. We assume without loss of generality that $z_n \in U_{i(z_\infty)}$ for all n. Since $\lambda'(z)$ does not depend on the choice of ξ_z, by changing the choice of $i(z_n)$ if necessary, we also assume that $i(z_n) = i(z_\infty)$ for all n. For brevity we write ξ for ξ_{z_∞}. Then $\hat{\eta}^{-1} \xi_* \lambda(z_n)$ converges to $\hat{\eta}^{-1} \xi_* \lambda(z_\infty)$. Since θ is a covering map, there is a sequence $\{g_n\}$ in $PSL(2,\boldsymbol{R})$ converging to some g such that $\theta(g_n) = \hat{\eta}^{-1} \xi_* \lambda(z_n)$ and that $\theta(g) = \hat{\eta}^{-1} \xi_* \lambda(z_\infty)$. Thus $\lambda'(z_n) = \xi_*^{-1} (g_n)_* \sigma$ converges to $\lambda'(z_\infty) = \xi_*^{-1} g_* \sigma$. Therefore, λ' is measurable, and is also scalarwise measurable.

Next we show that λ' is invariant. Let z be any point of E and let $t \in \boldsymbol{R}$. Then there are elements g_1 and g_2 of $PSL(2,\boldsymbol{R})$ satisfying $\theta(g_1) = \hat{\eta}^{-1} (\xi_z)_* \lambda(z)$ and $\theta(g_2) = \hat{\eta}^{-1} (\xi_{\psi_t(z)})_* \lambda(\psi_t(z))$. These mean, by definition, that $\lambda(z) = (\xi_z)_*^{-1} (g_1)_* \sigma_0$ and that $\lambda(\psi_t(z)) = (\xi_{\psi_t(z)})_*^{-1} (g_2)_* \sigma_0$. Since $(\varphi_t)_* \lambda(z) = \lambda(\psi_t(z))$, it follows that $(g_2)_*^{-1} (\xi_{\psi_t(z)})_* (\varphi_t)_* (\xi_z)_*^{-1} (g_1)_* \sigma_0 = \sigma_0$. Hence we see that $(g_2)^{-1} \circ \xi_{\psi_t(z)} \circ \varphi_t \circ (\xi_z)^{-1} \circ g_1$ belongs to

H. Thus we have $(g_2)_*^{-1}(\xi_{\psi_t(z)})_*(\varphi_t)_*(\xi_z)_*^{-1}(g_1)_*\sigma = \sigma$, which implies that $(\varphi_t)_*\lambda'(z) = \lambda'(\psi_t(z))$. Thus λ' is an invariant fiber measure.

By Lemma 4, λ' can be assumed to satisfy the property (4) of Theorem 3. Now define a probability measure ν' on N by $\int d\mu \int d\lambda'(z)$. Then ν' is invariant under φ_t, which can be shown in the same way as Lemma 1. □

Remark. The authors were communicated that this theorem can also be shown by using the barycenter of the measure instead of Furstenberg's theorem.

5. Ruelle invariant.

In this section, we will calculate the Ruelle invariant by means of invariant fiber measures.

Let M be a closed orientable 3-manifold, and X a non-singular vector field of M. Denote by ψ_t the flow generated by X. Let φ_t denote the flow of the projectivized bundle PX defined in the introduction. We assume that PX is a trivial bundle, and is parametrized as $M \times \boldsymbol{P}^1$. We define the projection from $M \times \boldsymbol{R}$ to $PX = M \times \boldsymbol{P}^1$ by $(z, x) \mapsto (z, [x])$, where $\boldsymbol{P}^1$ is parametrized as $\boldsymbol{R}/\boldsymbol{Z}$ and $x \mapsto [x]$ is the natural projection from $\boldsymbol{R}$ to $\boldsymbol{P}^1$. Then there is a flow $\tilde{\varphi}_t$ of $M \times \boldsymbol{R}$ which is a lift of φ_t ([**6**]). Let p_i denote the projection to the i-th factor of $M \times \boldsymbol{P}^1$ and $M \times \boldsymbol{R}$ $(i = 1, 2)$. We define $\tau_{(z,t)} : \boldsymbol{R} \to \boldsymbol{R}$ by $\tau_{(z,t)}(x) = p_2\tilde{\varphi}_t(z, x)$.

Let ν be a probability measure of $M \times \boldsymbol{P}^1$ invariant under φ_t and let $\mu = (p_1)_*\nu$. By Ruelle's theorem, there is a limit of $(1/(2t))(\tau_{(z,t)}(x) - x)$ as $t \to +\infty$ μ-a.e.z for any $x \in \boldsymbol{R}$, denoted by $\rho(z)$ ([**7**], and also [**3**]), where $\rho(z)$ is independent of the choice of x and $\rho(z)$ is measurable with respect to μ. Then the Ruelle invariant $R_\mu(\psi_t)$ is defined by $\int_M \rho(z)\, d\mu$.

Let $\tilde{\nu}$ denote the measure on $M \times \boldsymbol{R}$ which is the lift of ν (i.e. $\tilde{\nu}(E) = \sum_{n \in \boldsymbol{Z}} \nu(\{(z, [x]); (z, x) \in E, n \leqq x < n + 1\})$ for any measurable set E of $M \times \boldsymbol{R}$). Let λ be an invariant fiber measure derived from ν with a basic set B (Fundamental lemma). Let $\mathscr{M}(\boldsymbol{R})$ denote the set of measures on $\boldsymbol{R}$, where these are not always probability measures. Denote by $\tilde{\lambda} : M \to \mathscr{M}(\boldsymbol{R})$ the lift of λ defined by $\tilde{\lambda}(z)(E) = \sum_{n \in \boldsymbol{Z}} \lambda(z)$ $(\{(z, [x]); x \in E, n \leqq x < n + 1\})$ for any measurable set E of $\boldsymbol{R}$. For $\sigma \in \mathscr{M}(\boldsymbol{R})$, we define $\int_a^b d\sigma$ by $\sigma([a, b))$ if $a < b$ and by $-\sigma([b, a))$ if $b < a$ and furthermore by 0 if $a = b$. Then we obtain the following properties:

Proposition 1.
(1) $\int_{\tau_{(z,t)}(a)}^{\tau_{(z,t)}(b)} d\tilde{\lambda}(\psi_t(z)) = \int_a^b d\tilde{\lambda}(z)$ *for* $a, b, t \in \boldsymbol{R}$ *and* $z \in B$.
(2) $\tilde{\nu} = \int d\mu \int d\tilde{\lambda}(z)$.

Let $\Delta : M \to \boldsymbol{R}$ denote the measurable function defined by $\Delta(z) = \int_0^{\tau_{(\psi_{-1}(z),1)}(0)} d\tilde{\lambda}(z)$.

Lemma 7. *For any positive integer n,*

$$\left| \tau_{(z,n)}(0) - \sum_{i=1}^{n} \Delta(\psi_i(z)) \right| \leqq 1$$

holds for any $z \in B$.

Proof. Since $\tilde{\lambda}(z)([m, m+1))$ is equal to 1 for any $m \in \boldsymbol{Z}$ and $z \in B$, we obtain

$$\lfloor \tau_{(z,n)}(0) \rfloor \leqq \int_0^{\tau_{(z,n)}(0)} d\tilde{\lambda}(\psi_n(z)) \leqq \lfloor \tau_{(z,n)}(0) \rfloor + 1$$

for any $n \in \boldsymbol{Z}$, where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x.
On the other hand, we have

$$\begin{aligned}\int_0^{\tau_{(z,n)}(0)} d\tilde{\lambda}(\psi_n(z)) &= \sum_{i=1}^{n} \int_{\tau_{(\psi_i(z),n-i)}(0)}^{\tau_{(\psi_{i-1}(z),n-i+1)}(0)} d\tilde{\lambda}(\psi_n(z)) \\ &= \sum_{i=1}^{n} \int_0^{\tau^{-1}_{(\psi_i(z),n-i)}\tau_{(\psi_{i-1}(z),n-i+1)}(0)} d\tilde{\lambda}(\psi_i(z)) \\ &= \sum_{i=1}^{n} \int_0^{\tau_{(\psi_{i-1}(z),1)}(0)} d\tilde{\lambda}(\psi_i(z)) \\ &= \sum_{i=1}^{n} \Delta(\psi_i(z))\end{aligned}$$

for $z \in B$. □

Remark. $|\Delta(z)|$ is bounded because $|\Delta(z)| \leqq |\tau_{(\psi_{-1}(z),1)}(0)| + 1$.

Let $\Omega_+ = \{(z,x) \in M \times \boldsymbol{R}; 0 \leqq x < \tau_{(\psi_{-1}(z),1)}(0), \tau_{(\psi_{-1}(z),1)}(0) > 0\}$ and $\Omega_- = \{(z,x) \in M \times \boldsymbol{R}; \tau_{(\psi_{-1}(z),1)}(0) \leqq x < 0, \tau_{(\psi_{-1}(z),1)}(0) < 0\}$.

Theorem 1.

$$R_\mu(\psi_t) = \frac{1}{2}\tilde{\nu}(\Omega_+) - \frac{1}{2}\tilde{\nu}(\Omega_-)$$

Proof. Let E be a conull set of M on which $(1/(2t))(\tau_{(z,t)}(x) - x)$ converges as $t \to +\infty$ for any $x \in \boldsymbol{R}$. For any $z \in B \cap E$, we have $\rho(z) = \lim_{n\to\infty}(1/(2n))\tau_{(z,n)}(0) = \lim_{n\to\infty}(1/(2n))\sum_{i=1}^{n}\Delta(\psi_i(z))$ by Lemma 7.
Let $M_+ = \{z; 0 < \tau_{(\psi_{-1}(z),1)}(0)\}$ and $M_- = \{z; \tau_{(\psi_{-1}(z),1)}(0) < 0\}$. Since $(1/(2n)) \sum_{i=1}^{n}\Delta(\psi_i(z))$ is bounded and μ is invariant under ψ_t, we obtain

$$\begin{aligned}R_\mu(\psi_t) &= \int_M \rho(z)\,d\mu = \lim_{n\to\infty}\frac{1}{2n}\sum_{i=1}^{n}\int_M \Delta(\psi_i(z))\,d\mu \\ &= \frac{1}{2}\int_M \Delta(z)\,d\mu = \frac{1}{2}\int_M d\mu \int_0^{\tau_{(\psi_{-1}(z),1)}(0)} d\tilde{\lambda}(z) \\ &= \frac{1}{2}\int_{M_+} d\mu \int_0^{\tau_{(\psi_{-1}(z),1)}(0)} d\tilde{\lambda}(z) + \frac{1}{2}\int_{M_-} d\mu \int_0^{\tau_{(\psi_{-1}(z),1)}(0)} d\tilde{\lambda}(z) \\ &= \frac{1}{2}\int_M d\mu \int_{\{y;(z,y)\in\Omega_+\}} d\tilde{\lambda}(z) - \frac{1}{2}\int_M d\mu \int_{\{y;(z,y)\in\Omega_-\}} d\tilde{\lambda}(z) \\ &= \frac{1}{2}\tilde{\nu}(\Omega_+) - \frac{1}{2}\tilde{\nu}(\Omega_-).\end{aligned}$$

□

Remark. In the proof of Theorem 1, we used the fact that $(1/(2n))\sum_{i=1}^{n}\int_M \Delta(\psi_i(z))\,d\mu$ is constant to $(1/2)\int_M \Delta(z)\,d\mu$. Thus we have

$$\begin{aligned}\left|\frac{1}{2n}\int_M \tau_{(z,n)}(0)\,d\mu - R_\mu(\psi_t)\right| &\leqq \left|\frac{1}{2n}\int_M \tau_{(z,n)}(0)\,d\mu - \frac{1}{2}\int_M \Delta(z)\,d\mu\right| \\ &\leqq \left|\frac{1}{2n}\int_M \tau_{(z,n)}(0)\,d\mu - \frac{1}{2n}\sum_{i=1}^{n}\int_M \Delta(\psi_i(z))\,d\mu\right| \\ &\leqq \frac{1}{2n}\end{aligned}$$

by Lemma 7. This implies that the Ruelle invariant depends continuously on μ with respect to the weak* topology and on ψ_t with respect to C^1 topology, which was already shown in [**7**] without using the invariant fiber measure. On the other hand, Gambaudo-Ghys showed that the Ruelle invariant does not depend continuously on ψ_t with respect to C^0 topology ([**3**]).

Remark. If ψ_t is ergodic, $\rho(z) = \lim_{n\to+\infty}(1/n)\sum_{i=1}^{n}\Delta(\psi_i(z))$ is constant μ-a.e.z. Therefore $\lim_{t\to+\infty}\tau_{(z,t)}(x) = +\infty$ μ-a.e.z if $R_\mu(\psi_t)$ is positive. On the other hand, $\lim_{t\to-\infty}(1/(2t))(\tau_{(z,t)}(x) - x)$ exists μ-a.e.z and $\int_M(\lim_{t\to-\infty}(1/(2t))(\tau_{(z,t)}(x) - x))\,d\mu$ is equal to $R_\mu(\psi_t)$ (See [**7**]. This can also be shown by using the invariant fiber measure). Thus we have $\lim_{t\to-\infty}\tau_{(z,t)}(x) = -\infty$ μ-a.e.z if $R_\mu(\psi_t)$ is positive. Therefore, there is an invariant conull set E of M such that $\lim_{t\to+\infty}\tau_{(z,t)}(x) = +\infty$ $(z \in E)$ and $\lim_{t\to-\infty}\tau_{(z,t)}(x) = -\infty$ $(z \in E)$ if $R_\mu(\psi_t) > 0$. Such an orbit of $\tilde{\varphi}_t$ is called proper and is considered in [**6**].

Let D^2 denote the closed disc $\{(z_1, z_2) \in \boldsymbol{R}^2; z_1^2 + z_2^2 \leqq 1\}$. Denote by μ the canonical measure $dz_1 \wedge dz_2$. Let G denote the set of diffeomorphisms of D^2 preserving μ which are the identity near the boundary. Denote by PD^2 the projectivized bundle of the tangent bundle of D^2, which is parametrized as $D^2 \times \boldsymbol{P}^1$. For an element f of G, the differential Df induces a diffeomorphism of PD^2, denoted by Pf, and it has a lift $\widetilde{Pf}$ on the infinite cyclic cover $D^2 \times \boldsymbol{R}$ corresponding to the isotopy from the identity to f. Then the Ruelle invariant $R_\mu(f)$ is defined by $(1/2)\int_{D^2}(\lim_{n\to\infty} p_2\widetilde{Pf}^n(z,x)/n)\,d\mu$ $(n \in \boldsymbol{Z}, z \in D^2, x \in \boldsymbol{R})$, where p_i is the projection to the i-th factor $(i = 1, 2)$.

Let ν be a probability measure of $D^2 \times \boldsymbol{P}^1$ invariant under Pf satisfying $(p_1)_*\nu = \mu$. By the same way as Theorem 1, we can prove the following.

Theorem 5. *Let* $\Omega_+ = \{(z,x) \in D^2 \times \boldsymbol{R}; 0 \leqq x < p_2\widetilde{Pf}(f^{-1}(z),0), p_2\widetilde{Pf}(f^{-1}(z),0) > 0\}$ *and* $\Omega_- = \{(z,x) \in D^2 \times \boldsymbol{R}; p_2\widetilde{Pf}(f^{-1}(z),0) \leqq x < 0, p_2\widetilde{Pf}(f^{-1}(z),0) < 0\}$. *Then* $R_\mu(f) = (1/2)\tilde{\nu}(\Omega_+) - (1/2)\tilde{\nu}(\Omega_-)$ *for the lift* $\tilde{\nu}$ *of* ν.

Theorem 2. *Let f and g be elements of G. If there is a probability measure ν of PD^2 invariant under both Pf and Pg satisfying $(p_1)_*\nu = \mu$, then $R_\mu(g \circ f) = R_\mu(f) + R_\mu(g)$.*

Proof. This theorem is proved by the following equations:

$$
\begin{aligned}
R_\mu(g \circ f) &= \frac{1}{2}\int_M d\mu \int_0^{p_2 \widetilde{Pg \cdot f}((g \cdot f)^{-1}(z),0)} d\tilde{\lambda}(z) \\
&= \frac{1}{2}\int_M d\mu \int_{p_2 \widetilde{Pg}(g^{-1}(z),0)}^{p_2 \widetilde{Pg}\widetilde{Pf}((g \cdot f)^{-1}(z),0)} d\tilde{\lambda}(z) + \frac{1}{2}\int_M d\mu \int_0^{p_2 \widetilde{Pg}(g^{-1}(z),0)} d\tilde{\lambda}(z) \\
&= \frac{1}{2}\int_M d\mu \int_0^{p_2 \widetilde{Pf}(f^{-1}g^{-1}(z),0)} d\tilde{\lambda}(g^{-1}(z)) + R_\mu(g) \\
&= R_\mu(f) + R_\mu(g)
\end{aligned}
$$

□

Let Γ be an amenable subgroup of G. By using the definition of the amenable subgroup, we can construct a probability measure ν of $D^2 \times \boldsymbol{P}^1$ invariant under Γ satisfying $(p_1)_*\nu = \mu$. Thus we conclude Corollary 1 in the introduction.

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