L^2-torsion invariants of a surface bundle over S^1

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Abstract. In the present paper, we introduce L^2-torsion invariants $\tau_k(k \geq 1)$ for surface bundles over the circle and investigate them from the view point of the mapping class group of a surface. It is conjectured that they converge to the L^2-torsion for the regular representation of the fundamental group. Further we give an explicit and computable formula of the first two invariants by using the Mahler measure.

1. Introduction.

Let M be a complete hyperbolic 3-manifold. By the famous Mostow rigidity theorem, its fundamental group $\pi_1 M$ dominates all geometric information of M. Hence its hyperbolic volume can be computed theoretically from a presentation of $\pi_1 M$. The L^2-torsion, which is an invariant of 3-manifolds, gives a method to do actually it.

Historically, L^2-analogues of Reidemeister-Ray-Singer torsion were initiated by Mathai [23], Carey-Mathai [5] and Lott [16]. They are defined for a manifold with trivial L^2-(co)homology and positive Novikov-Shubin invariants by using the Fuglede-Kadison determinant of von Neumann algebras. The first one called combinatorial L^2-torsion is a piecewise linear invariant and the second one called analytic L^2-torsion is a smooth invariant. Burghelea, Friedlander, Kappeler and McDonald [3] proved the equality between the combinatorial L^2-torsion and the analytic L^2-torsion for compact Riemannian manifolds. Afterward, Lück and Schick [21] generalized the equality for hyperbolic manifolds with finite volume. In the following, based on these facts, we simply call these two types invariants the L^2-torsion.

Moreover it is shown in [16], [11] and [21] that the L^2-torsion for the regular representation is equal to Gromov's simplicial volume up to a constant. Thus, for a hyperbolic manifold, the L^2-torsion is essentially equal to its hyperbolic volume. On the one hand, Lück [18] gave a formula of the L^2-torsion $\tau(M)$ which is computable from certain presentation of the fundamental group. Therefore we could compute the hyperbolic volume of M from a presentation of $\pi_1 M$. However, in general, it seems to be difficult to evaluate directly the exact value of hyperbolic volumes from Lück's formula.

Now from recent works of Kashaev-Murakami-Murakami [25], it is conjectured that certain asymptotic behavior of the colored Jones polynomial of a knot gives the simplicial volume of the complement of a given knot in S^3. Then it seems to be natural

to raise the following problem: *Does there exist another sequence of invariants which approximates the simplicial volume?*

The purpose of this paper is to construct a sequence of L^2-invariants, which approximates the original L^2-torsion (namely the simplicial volume), for once-holed surface bundles over S^1. The point to do is the lower central series of the surface group. Roughly speaking, the regular representation of $\pi_1 M$ could be approximated by representations on the l^2-spaces defined by the nilpotent quotients of $\pi_1 M$.

Here the contents of this paper is the following. In the next section, we recall the definition of the L^2-torsion of 3-manifolds and Lück's formula. In Section 3, we define L^2-torsion invariants for surface bundles over S^1 and state our volume conjecture for them. In particular, we show that the conjecture is true for a certain special case. In fact, if a given surface bundle has a finite covering so that it is topologically the product, then our L^2-torsion invariants approximate the simplicial volume. In Section 4, we describe these invariants from the view point of the Magnus representations of the mapping class group of a surface. In Section 5, we state certain integral formula on the first term of these invariants. More precisely, we show that it is essentially equal to the Mahler measure of the characteristic polynomial of the homology representation. By using the formula, we give some numerical calculations for low genus. Further, we explain a geometric meaning of the formula for the first term. In the final section, we give a similar formula of the second term of our L^2-torsion invariants for surface bundles with monodromies in the Torelli subgroup.

2. L^2-torsion and Lück's formula.

In this section, we review the combinatorial definition of L^2-torsion $\tau(M)$ and its basic properties. See [**18**] for details.

Let π be a discrete group and $C\pi$ denote its group ring over C. For an element $\sum_{g \in \pi} \lambda_g g \in C\pi$, we define the $C\pi$-trace by $\mathrm{tr}_{C\pi}(\sum_{g \in \pi} \lambda_g g) = \lambda_e \in C$, where e is the unit element in π. Next, for a matrix $B = (b_{ij}) \in M(n, C\pi)$, we extend the definition of $C\pi$-trace by means of

$$\mathrm{tr}_{C\pi}(B) = \sum_{i=1}^{n} \mathrm{tr}_{C\pi}(b_{ii}).$$

Now let us recall the definition of the L^2-Betti number of $B \in M(n, C\pi)$. We consider the bounded π-equivariant operator

$$R_B : \bigoplus_{i=1}^{n} l^2(\pi) \to \bigoplus_{i=1}^{n} l^2(\pi)$$

defined by natural right action of B. Here $l^2(\pi)$ denotes the regular representation of π. Namely it is the complex Hilbert space of the formal sums $\sum_{g \in \pi} \lambda_g g$ which are square

summable. This is the Hilbert completion of a pre-Hilbert space $C\pi$ with respect to the natural inner product

$$\left\langle \sum_{g\in\pi} \lambda_g g, \sum_{g\in\pi} \mu_g g \right\rangle = \sum_{g\in\pi} \lambda_g \bar{\mu}_g.$$

We fix a positive real number K so that $K \geq \|R_B\|_\infty$ holds where $\|R_B\|_\infty$ is the operator norm of the bounded π-equivariant operator R_B.

DEFINITION 2.1. The L^2-Betti number of a matrix B is defined by

$$b(B) = \lim_{p\to\infty} \text{tr}_{C\pi}((I - K^{-2}BB^*)^p) \in \mathbf{R}_{\geq 0},$$

where I is the identity matrix and B^* denotes the adjoint of B. That is, for any element $\sum \lambda_g g \in C\pi$, we define $\overline{\sum \lambda_g g} = \sum \bar{\lambda}_g g^{-1}$. By using this, we define $B^* = (\bar{b}_{ji})$.

Roughly speaking, the L^2-Betti number measures the size of the kernel of a matrix B. Hereafter we assume $b(B) = 0$. Then, for a matrix with a coefficient in a non-commutative ring, we introduce the determinant as follows.

DEFINITION 2.2. The Fuglede-Kadison determinant of a matrix B is defined by

$$\det{}_{C\pi}(B) = K^n \exp\left(-\frac{1}{2}\sum_{p=1}^{\infty}\frac{1}{p}\,\text{tr}_{C\pi}((I - K^{-2}BB^*)^p)\right) \in \mathbf{R}_{>0},$$

if the infinite sum of non-negative real numbers $\sum(1/p)\,\text{tr}_{C\pi}((I - K^{-2}BB^*)^p)$ converges to a real number.

REMARK 2.3. It is shown that the L^2-Betti number $b(B)$ and the Fuglede-Kadison determinant $\det_{C\pi}(B)$ are independent of a choice of a constant K. A possible choice is given by

$$K = \sqrt{n} \cdot \max\{\|b_{ij}\|_1 \mid 1 \leq i, j \leq n\},$$

where $\|u\|_1 = \sum_{g\in\pi} |\lambda_g|$ $(u = \sum_{g\in\pi} \lambda_g g \in C\pi)$.

Now we define the L^2-torsion of 3-manifolds. Let M be a compact connected orientable 3-manifold. We fix a CW-complex structure on M. Then we may assume that the action of $\pi_1 M$ on the universal covering $\tilde{M}$ is cellular (if necessary, we have only to take a subdivision of the original structure). Let us consider the $C\pi_1 M$-chain complex of $\tilde{M}$:

$$0 \to C_3(\tilde{M}, C) \xrightarrow{\partial_3} C_2(\tilde{M}, C) \xrightarrow{\partial_2} C_1(\tilde{M}, C) \xrightarrow{\partial_1} C_0(\tilde{M}, C) \to 0.$$

Since the boundary operator ∂_i is a matrix with coefficients in $C\pi_1 M$, if we take the adjoint operator $\partial_i^* : C_{i-1}(\tilde{M}, C) \to C_i(\tilde{M}, C)$ as above, we can define the i-th (combinatorial) Laplace operator $\Delta_i : C_i(\tilde{M}, C) \to C_i(\tilde{M}, C)$ by

$$\Delta_i = \partial_{i+1} \circ \partial_{i+1}^* + \partial_i^* \circ \partial_i.$$

Let us suppose that all the L^2-Betti numbers $b(\Delta_i)$ vanish. Thereby as a generalization of classical Reidemeister torsion, an (combinatorial) L^2-torsion $\tau(M)$ is defined by

Definition 2.4.

$$\tau(M) = \prod_{i=0}^{3} \det{}_{C\pi_1 M}(\Delta_i)^{(-1)^{i+1}i} \in \boldsymbol{R}.$$

Here it should be noted that the L^2-torsion $\tau(M)$ is a positive real number.

To ensure the well-definedness of the above definition, we need the positivity of all the Novikov-Shubin invariant $\alpha(\Delta_i)$ for the Laplace operator Δ_i. Namely, this condition guarantees the convergence of the infinite sum in the Fuglede-Kadison determinant. For the precise definition, we refer to [17], [18], [27].

For instance, it is known that if a compact connected orientable 3-manifold M satisfies the following all conditions, the L^2-Betti numbers $b(\Delta_i)$ vanish and the Novikov-Shubin invariants $\alpha(\Delta_i)$ are positive:

(i) $\pi_1 M$ is infinite.

(ii) M is homotopy equivalent to an irreducible 3-manifold or $S^1 \times S^2$ or $\boldsymbol{R}P^3 \# \boldsymbol{R}P^3$.

(iii) If $\partial M \neq \varnothing$, it consists of tori.

(iv) If $\partial M = \varnothing$, M is finitely covered by a 3-manifold which is homotopy equivalent to a hyperbolic, Seifert or Haken 3-manifold.

As a notable property of the L^2-torsion τ, it is known that $\log \tau(M)$ can be interpreted as Gromov's simplicial volume [10] of M (see [18], [16], [11], [21]).

Theorem 2.5. *Let M be a compact connected orientable irreducible 3-manifold with an infinite fundamental group such that ∂M is empty or a disjoint union of incompressible tori. Then it holds*

$$\log \tau(M) = C \cdot \|M\|,$$

where C is a universal constant not depending on M and $\|M\|$ denotes the simplicial volume of M. In particular, if M is a hyperbolic 3-manifold,

$$\log \tau(M) = -\frac{1}{3\pi} \operatorname{Vol}(M)$$

holds.

Finally we describe Lück's formula for the L^2-torsion (see [18] Theorem 2.4).

Theorem 2.6. *Let M be as in the above theorem. Further we suppose that ∂M is non-empty and $\pi_1 M$ has a presentation $\langle s_1, \ldots, s_{n+1} \mid r_1, \ldots, r_n \rangle$. Define A to be the $n \times n$-matrix with entries in $\boldsymbol{Z}\pi_1 M$ obtained from the Fox matrix $(\partial r_i / \partial s_j)$ by deleting one of the columns. Then the logarithm of the L^2-torsion of M is given by*

$$\log \tau(M) = -2 \log \det{}_{C\pi_1 M}(R_A)$$

$$= -2n \log K + \sum_{p=1}^{\infty} \frac{1}{p} \operatorname{tr}_{C\pi_1 M}((I - K^{-2}AA^*)^p),$$

where K is a constant satisfying $K \geq \|R_A\|_{\infty}$.

These two theorems give a method to compute the simplicial volume of 3-manifolds in terms of the presentation of the fundamental group.

3. Definition of τ_k.

From now on, we restrict ourselves to the L^2-torsion of a surface bundle over the circle S^1.

Let $\Sigma_{g,1}$ be a compact oriented smooth surface of genus $g \geq 1$ with one boundary component. For an orientation preserving diffeomorphism φ of $\Sigma_{g,1}$, we form the mapping torus W_φ by taking $\Sigma_{g,1} \times I$ and gluing $\Sigma_{g,1} \times \{0\}$ and $\Sigma_{g,1} \times \{1\}$ via φ. This gives a surface bundle over S^1. Its diffeomorphism type is determined by its mon-odromy diffeomorphism, up to conjugacy and isotopy. Here an isotopy fixes setwisely the points of the boundary of $\Sigma_{g,1}$. However from here, we assume that a diffeo-morphism on $\Sigma_{g,1}$ fixes pointwisely them and an isotopy also does. It is the technical reason from the view point of the mapping class group.

For simplicity, we put $\pi = \pi_1(W_\varphi, *)$ and $\Gamma = \pi_1(\Sigma_{g,1}, *)$, where the base point $*$ of π and Γ is the same one on the fiber $\Sigma_{g,1} \times \{0\} \subset W_\varphi$. Then π is isomorphic to the semi-direct product of Γ and $\pi_1 S^1 \cong \boldsymbol{Z} = \langle t \rangle$.

In order to construct a sequence of L^2-torsion invariants which approximates the original one, we consider the lower central series of Γ. Since Γ is the free group of rank $2g$, it is residually nilpotent. Namely, we have an infinite sequence

$$\Gamma_1 = \Gamma \supset \Gamma_2 \supset \cdots \supset \Gamma_k \supset \cdots,$$

where $\Gamma_k = [\Gamma_{k-1}, \Gamma_1]$ for $k \geq 2$. Let N_k be the k-th nilpotent quotient $N_k = \Gamma/\Gamma_k$ and $p_k : \Gamma \to N_k$ be the natural projection.

In the previous section, we considered a chain complex $C_*(\tilde{W}_\varphi, \boldsymbol{C})$ as a chain complex of $\boldsymbol{C}\pi$-modules. Instead of this complex, we can use the following chain complex

$$C_*(W_\varphi, l^2(\pi)) = l^2(\pi) \otimes_{\boldsymbol{C}\pi} C_*(\tilde{W}_\varphi, \boldsymbol{C})$$

to define the same L^2-torsion $\tau(W_\varphi)$. The group Γ_k is a normal subgroup of π, so that we can take the quotient group $\pi(k) = \pi/\Gamma_k$. It should be noted that $\pi(k)$ is isomorphic to the semi-direct product $N_k \rtimes \boldsymbol{Z}$. We denote the induced homomorphism $\pi \to \pi(k)$ by the same letter p_k. Thereby we can consider the chain complex

$$C_*(W_\varphi, l^2(\pi(k))) = l^2(\pi(k)) \otimes_{\boldsymbol{C}\pi} C_*(\tilde{W}_\varphi, \boldsymbol{C})$$

through the projection p_k. By using the Laplace operator on this complex, we can formally define the k-th L^2-torsion $\tau_k(W_\varphi)$ as follows.

DEFINITION 3.1.

$$\tau_k(W_\varphi) = \prod_{i=0}^{3} \det{}_{\boldsymbol{C}\pi(k)}(\Delta_i^{(k)})^{(-1)^{i+1}i},$$

where $\Delta_i^{(k)} : C_i(W_\varphi, l^2(\pi(k))) \to C_i(W_\varphi, l^2(\pi(k)))$ is the Laplace operator over $l^2(\pi(k))$.

Of course, the above definition is well-defined if every L^2-Betti number $b(\Delta_i^{(k)})$ vanishes and every Novikov-Shubin invariant $\alpha(\Delta_i^{(k)})$ is positive for any $i = 0, 1, 2, 3$. As for the L^2-Betti numbers, we can show the following.

LEMMA 3.2.   *The L^2-Betti numbers of W_φ with coefficients in $p_k^* l^2(\pi(k))$ are all zero.*

PROOF.   We can directly apply Lück's result [19] Theorem 2.1 for a factorization $\pi \xrightarrow{p_k} \pi(k) \to \mathbf{Z}$ of the canonical map $\pi \to \mathbf{Z}$, so that the assertion immediately follows.
$\square$

On the other hand, at the time of writing, the positivity of the Novikov-Shubin invariant for general operators seems to be an open problem. However, fortunately, it is known that a weaker condition (namely, the operator $\Delta_i^{(k)}$ is $\pi(k)$-determinant class) guarantees the well-definedness of τ_k (see [3] and [29] for details). In fact, the groups $\pi(k) \cong N_k \rtimes \mathbf{Z}$ are all contained in a large class $\mathscr{G}$ of groups, which is determinant class, considered in [29]. Therefore our L^2-torsion invariants τ_k can be defined for any $k \geq 1$ and they are all homotopy invariants.

Let $x_1, \ldots, x_{2g}$ be a generating system of the free group $F_{2g} = \Gamma$. Then the fundamental group π is presented by

$$\pi = \langle x_1, \ldots, x_{2g}, t \,|\, r_i = t x_i t^{-1} (\varphi_*(x_i))^{-1}, 1 \leq i \leq 2g \rangle,$$

where $\varphi_* : \Gamma \to \Gamma$ is a homomorphism induced by $\varphi : \Sigma_{g,1} \to \Sigma_{g,1}$. Applying the free differential calculus to relators $r_1, \ldots, r_{2g}$, we obtain an Alexander matrix

$$A = \left(\frac{\partial r_i}{\partial x_j} \right) \in M(2g, \mathbf{Z}\pi).$$

Then Lück's formula for a surface bundle over S^1 is described as follows:

$$\log \tau(W_\varphi) = -2 \log \det_{\mathbf{C}\pi}(R_A)$$

$$= -4g \log K + \sum_{p=1}^{\infty} \frac{1}{p} \operatorname{tr}_{\mathbf{C}\pi}((I - K^{-2} A A^*)^p),$$

where K is a constant satisfying $K \geq \|R_A\|_\infty$.

Now let us derive a formula for the k-th L^2-torsion $\tau_k(W_\varphi)$. Let $p_{k*} : \mathbf{C}\pi \to \mathbf{C}\pi(k)$ be an induced homomorphism over the group rings. Thereby we put

$$A_k = \left(p_{k*} \left(\frac{\partial r_i}{\partial x_j} \right) \right) \in M(2g, \mathbf{C}\pi(k)).$$

Moreover we fix a constant K_k satisfying $K_k \geq \|R_{A_k}\|_\infty$. Then we have

$$\log \tau_k(W_\varphi) = -2 \log \det_{\mathbf{C}\pi(k)}(R_{A_k})$$

$$= -4g \log K_k + \sum_{p=1}^{\infty} \frac{1}{p} \operatorname{tr}_{\mathbf{C}\pi(k)}((I - K_k^{-2} A_k A_k^*)^p),$$

by virtue of the same argument as Theorem 2.6.

REMARK 3.3. It is easy to see that we can take constants K and K_k so that $K = K_k$ for all $k \geq 1$ (see [6] Part 1, Chapter 1, Proposition 8).

REMARK 3.4. By using the above formulas, we have numerically computed approximate values of τ_1 and τ_2 for some examples in [14].

Here let us state our volume conjecture for a surface bundle over S^1.

CONJECTURE 3.5. *For any orientation preserving diffeomorphism φ of $\Sigma_{g,1}$, the sequence $\{\tau_k(W_\varphi)\}$ converges to $\tau(W_\varphi)$ when we take the limit on k.*

It would be an interesting problem to compare Conjecture 3.5 with the Kashaev-Murakami-Murakami volume conjecture when W_φ is the complement of a fibered knot in S^3.

REMARK 3.6. As was discussed in [29], $\log \det_{C\pi}(R_A)$ is not smaller than the superior limit of $\log \det_{C\pi(k)}(R_{A_k})$, which are all bounded below by zero. However, the reverse inequality seems to be an open problem in general.

Now we consider the following case. That is, we assume that there exists an integer n such that W_{φ^n} is topologically the product of $\Sigma_{g,1}$ and S^1. Here, in general, its bundle structure is not trivial. Namely, the n-th power φ^n of a given monodromy φ is not trivial. A typical example is a diffeomorphism φ so that some power of φ becomes the Dehn twist along a simple closed curve on $\Sigma_{g,1}$ which is parallel to the boundary. The difference between an isotopy fixes pointwisely and such one fixes setwisely, it gives birth to the difference between a bundle structure and a topological type.

Let us consider a more general setting. By abuse of notation, we use the same letter φ for the mapping class of an orientation preserving diffeomorphism φ of $\Sigma_{g,1}$. Naturally there exists the projection $W_{\varphi^n} \to W_\varphi$ of an n-fold cyclic covering. We then take a representation of $\pi_1(W_{\varphi^n})$ on $l^2(N_k \rtimes_{\varphi^n} \mathbf{Z})$. Further we take the induced representation of this, which is a representation of $\pi_1(W_\varphi)$ on $\mathbf{C}\pi_1(W_\varphi) \otimes_{\mathbf{C}\pi_1(W_{\varphi^n})} l^2(N_k \rtimes_{\varphi^n} \mathbf{Z})$. The following lemma is straightforward by using the fixed presentation of groups.

LEMMA 3.7. *This induced representation is equivalent to the representation of $\pi_1(W_\varphi)$ on $l^2(N_k \rtimes_\varphi \mathbf{Z})$.*

Thereby we can show the next proposition by using this lemma. This is a well-known property of the torsion invariants. (for example, see [5]).

PROPOSITION 3.8. $\tau_k(W_{\varphi^n}) = \tau_k(W_\varphi)^n$.

PROPOSITION 3.9. *For the product bundle $\Sigma_{g,1} \times S^1$, it holds $\tau_k(\Sigma_{g,1} \times S^1) = 1$.*

PROOF. In this case, we see that $A_k = tI - I$ and then by an easy computation, $\det_{C\pi(k)}(tI - I) = \det_{C\pi(1)}(tI - I) = 1$. Therefore, we have $\tau_k(\Sigma_{g,1} \times S^1) = 1$. □

Now we assume that W_φ has a finite covering $W_{\varphi^n} \to W_\varphi$ such that it is topologically the product $\Sigma_{g,1} \times S^1$. Combining these two propositions, we obtain the following one.

PROPOSITION 3.10. *It holds $\tau_k(W_\varphi) = 1$ for all $k \geq 1$.*

It is easy to see that such a 3-manifold does not admit a hyperbolic structure, then it has the trivial simplicial volume. This gives an affirmative example in our conjecture. To sum up, we obtain the following theorem.

THEOREM 3.11. *For a surface bundle W_φ over S^1 as above, the sequence $\{\tau_k(W_\varphi)\}$ converges to the original L^2-torsion $\tau(W_\varphi)$.*

REMARK 3.12. In some sense, this result corresponds to an affirmative answer, due to Kashaev and Tirkkonen [**13**], of Kashaev-Murakami-Murakami volume conjecture for torus knots.

4. Relation to the Magnus representation.

In this section, we reconsider our L^2-torsion invariants from the view point of the Magnus representation of the mapping class group. See [**24**], [**33**] as references for the Magnus representation.

Let $\mathcal{M}_{g,1}$ be the mapping class group of $\Sigma_{g,1}$, that is, the group of all isotopy classes of orientation preserving diffeomorphisms of $\Sigma_{g,1}$ relative to the boundary. From the well-known result of Nielsen, one can consider $\mathcal{M}_{g,1}$ to be a subgroup of the automorphism group of $\Gamma = \langle x_1, \ldots, x_{2g} \rangle$.

DEFINITION 4.1. The Magnus representation $\rho : \mathcal{M}_{g,1} \to GL(2g, \mathbf{Z}\Gamma)$ of $\mathcal{M}_{g,1}$ is defined by

$$\rho : \mathcal{M}_{g,1} \ni \varphi \mapsto \left(\overline{\frac{\partial \varphi_*(x_j)}{\partial x_i}} \right)_{ij} \in GL(2g, \mathbf{Z}\Gamma),$$

where the map $^- : \mathbf{Z}\Gamma \to \mathbf{Z}\Gamma$ implies $\overline{\sum \lambda_g g} = \sum \lambda_g g^{-1}$.

REMARK 4.2. To say fact, this is a crossed homomorphism, not a homomorphism. However one calls this simply the Magnus representation of $\mathcal{M}_{g,1}$.

By using this Magnus representation ρ, we can explain the previous Lück's formula as a characteristic polynomial with respect to the Fuglede-Kadison determinant as follows.

In Lück's formula, we applied the free differential calculus to relators $r_i = tx_it^{-1}(\varphi_*(x_i))^{-1}$ of $\pi_1(W_\varphi)$. As a result, we obtain $A = tI - {}^t\overline{\rho(\varphi)}$. Namely, in some sense, this Alexander matrix A is nothing but a characteristic matrix of the Magnus representation. Then if we take the Fuglede-Kadison determinant in $M(2g, \mathbf{C}\pi)$, its value is equal to the one for a matrix defined by $t^{-1}I - \rho(\varphi)$ (see [**18**] Lemma 4.2). Therefore the L^2-torsion is interpreted as the characteristic polynomial of $\rho(\varphi)$.

For the k-th term τ_k, we have taken the lower central series $\{\Gamma_k\}$ of Γ and the nilpotent quotients $\{N_k = \Gamma/\Gamma_k\}$. These quotients induce a sequence of representations

$$\rho_k : \mathcal{M}_{g,1} \to GL(2g, \mathbf{Z}N_k)$$

for $k \geq 1$ (see [**24**]). Consequently, by the similar observation as above, the k-th term $\tau_k(W_\varphi)$ can be regarded as the characteristic polynomial of $\rho_k(\varphi)$ with respect to the Fuglede-Kadison determinant in $M(2g, \mathbf{C}\pi(k))$.

5. A formula of τ_1.

First we try to formulate the first term τ_1 of our L^2-torsion invariants. It means that we consider a representation

$$\rho_1 : \mathcal{M}_{g,1} \to GL(2g, \mathbf{Z}N_1).$$

Here N_1 is the trivial group and then the above representation is the same one as the usual homological action of $\mathcal{M}_{g,1}$ on $H_1(\Sigma_{g,1}, \mathbf{Z})$. Namely we have the representation

$$\rho_1 : \mathcal{M}_{g,1} \to \mathrm{Aut}(H_1(\Sigma_{g,1}, \mathbf{Z}), \langle\,,\,\rangle) \cong Sp(2g, \mathbf{Z}),$$

where $\langle\,,\,\rangle$ denotes the intersection form on the homology. Further $\pi(1) = \pi/\Gamma_1 \cong \mathbf{Z} = \langle t \rangle$ and its group ring $\mathbf{C}\langle t \rangle$ is a commutative Laurent polynomial ring $\mathbf{C}[t, t^{-1}]$. Then the matrix A_1 is nothing but the usual characteristic matrix of ${}^t\rho_1(\varphi)$. In Section 4, we saw that $\tau_1(W_\varphi)$ is the characteristic polynomial in the sense of the Fuglede-Kadison determinant in $GL(2g, \mathbf{C}\langle t \rangle)$. In this case, it is described by the usual determinant for a matrix with commutative entries.

In order to state the theorem, we recall a definition from the number theory (see [**8**] and its references). For a Laurent polynomial $F(\mathbf{t}) \in \mathbf{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, the Mahler measure of F is defined by

$$m(F) = \int_0^1 \cdots \int_0^1 \log|F(e^{2\pi\sqrt{-1}\theta_1}, \ldots, e^{2\pi\sqrt{-1}\theta_n})|\, d\theta_1 \cdots d\theta_n,$$

where we assume that undefined terms are omitted. Namely we define the integrand to be zero whenever we hit a zero of F.

THEOREM 5.1. *The logarithm of the first invariant τ_1 is given by*

$$\log \tau_1(W_\varphi) = -2m(\Delta_{\rho_1(\varphi)}),$$

where $\Delta_{\rho_1(\varphi)}(t) = \det(tI - \rho_1(\varphi))$. Further if $\Delta_{\rho_1(\varphi)}$ factorizes as $\Delta_{\rho_1(\varphi)}(t) = \prod_{i=1}^{2g}(t - \alpha_i)$ ($\alpha_i \in \mathbf{C}$), then we have

$$\log \tau_1(W_\varphi) = -2\sum_{i=1}^{2g} \log\max\{1, |\alpha_i|\}.$$

REMARK 5.2. In [**18**], Lück points out

$$\log \tau_1(S^3 \backslash K) = \int_{S^1} \log(\Delta_K(z)\Delta_K(\bar{z}))\, dvol$$

for a knot K in S^3 without proof. Here Δ_K is the Alexander polynomial of K. In the fibered knot case, it is the same with $\Delta_{\rho_1(\varphi)}$. Probably this integral formula is well-known to experts.

PROOF. First of all, we remark that the Novikov-Shubin invariants $\alpha(\Delta_i^{(1)})$ are positive in this case. Because $\pi(1) = \mathbf{Z}$ is an abelian group (see [**16**]). Thus the well-definedness of τ_1 also follows from this fact.

Next we recall that the Hilbert space $l^2(\mathbf{Z})$ can be identified with $L^2(\mathbf{R}/\mathbf{Z})$ in terms of the Fourier transforms. For example, an element t^n is identified with the L^2-function $e^{2\pi\sqrt{-1}n\theta}$. Then the trace

$$\mathrm{tr}_{\mathbf{C}\langle t\rangle} : l^2(\mathbf{Z}) \ni \sum a_n t^n \mapsto a_0 \in \mathbf{C}$$

can be realized as the integration

$$L^2(\mathbf{R}/\mathbf{Z}) \ni f(\theta) \mapsto \int_0^1 f(\theta)\,d\theta \in \mathbf{C}.$$

Now from Lück's formula for the first term, we have

$$\log \tau_1(W_\varphi) = -2\log\det{}_{\mathbf{C}\langle t\rangle}(R_{A_1})$$

$$= -4g\log K + \sum_{p=1}^\infty \frac{1}{p}\,\mathrm{tr}_{\mathbf{C}\langle t\rangle}((I - K^{-2}A_1 A_1^*)^p)$$

$$= -4g\log K + \mathrm{tr}_{\mathbf{C}\langle t\rangle}\left(\sum_{p=1}^\infty \frac{1}{p}\,\mathrm{tr}((I - K^{-2}A_1 A_1^*)^p)\right).$$

Here $\mathrm{tr} : M(2g, \mathbf{Z}\langle t\rangle) \to \mathbf{Z}\langle t\rangle$ is the usual trace. By using the above identification $l^2(\mathbf{Z}) \cong L^2(\mathbf{R}/\mathbf{Z})$, an infinite series $\sum(1/p)\,\mathrm{tr}((I - K^{-2}A_1 A_1^*)^p)$ gives an L^2-function on $\mathbf{R}/\mathbf{Z}$. Therefore we obtain

$$\log\tau_1(W_\varphi) = -4g\log K + \int_0^1\left(\sum_{p=1}^\infty \frac{1}{p}\,\mathrm{tr}((I - K^{-2}A_1 A_1^*)^p)\right)d\theta$$

$$= \int_0^1\left(-4g\log K + \sum_{p=1}^\infty \frac{1}{p}\,\mathrm{tr}((I - K^{-2}A_1 A_1^*)^p)\right)d\theta$$

$$= -\int_0^1 \log\det(A_1 A_1^*)\,d\theta$$

$$= -\int_0^1 \log\det(A_1)\det(A_1^*)\,d\theta$$

$$= -\int_0^1 \log|\Delta_{\rho_1(\varphi)}(e^{2\pi\sqrt{-1}\theta})|^2\,d\theta$$

$$= -2m(\Delta_{\rho_1(\varphi)}),$$

completing the proof of the desired formula.

As for the second assertion, it is nothing but Mahler's theorem (see [8] Theorem 1). This completes the proof of Theorem 5.1. ∎

From this description, we obtain the following notable corollary.

COROLLARY 5.3. *The logarithm of the first term $\tau_1(W_\varphi)$ vanishes if and only if every eigenvalue of $\rho_1(\varphi) \in Sp(2g, \mathbf{Z})$ is a root of unity.*

PROOF. We easily see the coefficient of the leading term of $\Delta_{\rho_1(\varphi)}(t)$ is one, so that it is a primitive polynomial. Hence the claim follows from Kronecker's theorem (see [**8**] Theorem 2). $\qquad\square$

This corollary seems to be interesting. Because in some case, we can say that the first term τ_1 already approximates the simplicial volume. In particular, Corollary 5.3 implies that a torus bundle W_φ with the hyperbolic structure (namely, $|\mathrm{tr}(\rho_1(\varphi))| \geq 3$) has always non-trivial L^2-torsion invariant $\tau_1(W_\varphi)$. We explain this fact in the next example.

EXAMPLE 5.4. We apply Theorem 5.1 to torus bundles. In order to compute τ_1 numerically, we have used the Maple 6. It is well-known that the mapping class group of the two dimensional torus T^2 is isomorphic to $SL(2, \mathbf{Z})$. Then we take a matrix $\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$. Naturally, this matrix gives a diffeomorphism φ on T^2 and we may assume it is the identity on some embedded 2-disk by an isotopic deformation. For this map φ, we make a calculation of a hyperbolic volume by using the SnapPea [**34**].

trace($\rho_1(\varphi)$)	$-3\pi \log \tau_1$	hyperbolic volume
0	0	0
1	0	0
2	0	0
3	18.1412584724	2.0298832128
4	24.8240715245	2.6667447835
5	29.5334698358	2.9891202829
6	33.2270014461	3.1772932786
7	36.2825168855	3.2969024143
8	38.8948730158	3.3775974082
9	41.1795720326	3.4345408859
10	43.2113662660	3.4761739892

EXAMPLE 5.5. We consider some surface bundles of genus two. Let $t_1, \ldots, t_5$ be the Lickorish generators of $\mathscr{M}_{2,1}$ and $T_1, \ldots, T_5$ their images in $Sp(4, \mathbf{Z})$ with respect to the corresponding symplectic basis of $H_1(\Sigma_{2,1}, \mathbf{Z})$. They are explicitly given by

$$T_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Penner's results [**28**], mapping classes $t_1^l t_3^m t_5^n t_2^{-1} t_4^{-1}$ for any $l, m, n \geq 1$ are pseudo-

Anosov maps.   By applying the above formula to these cases, we compute τ_1 for some triples of (l,m,n) by using the Maple 6 again.   Their hyperbolic volumes can be computed by two methods.   The first one is to use the SnapPea with Ichihara's algorithm [12] to describe a surface bundles by means of link surgeries.   The second one is to use also the SnapPea with the XTrain [1], [2].

(l,m,n)	$-3\pi \log \tau_1$	hyperbolic volume
$(1,1,1)$	47.6747282482	10.6497813754
$(1,1,2)$	52.9544769222	11.4666578757
$(1,1,3)$	56.9524589673	11.8937138137
$(1,1,4)$	60.2003564513	12.1434702788
$(1,1,5)$	62.9462610289	12.3010254753
$(1,2,1)$	54.4237752394	11.9187558233
$(1,2,2)$	59.2291561987	12.7824557985
$(1,2,3)$	62.9462610289	13.2306812552
$(1,2,4)$	66.0036368428	13.4904289941
$(1,2,5)$	68.6103164760	13.6529808192
$(1,3,1)$	59.3208237316	12.4291049018

By the way, if we consider only the first term τ_1, we can define it to knot exteriors, as well, other than surface bundles.   Namely, $\log \tau_1$ is described as the integral of the Alexander polynomial of knots.   In this case, the above integral formula of τ_1 is related with the next classical result on knots.   The following argument was informed us by Porti.

Let K be a knot in S^3 and $M(n) \to S^3$ the n-fold cyclic branched covering along K. Here we define $\mathrm{ord}(n)$ to be the order of $H_1(M(n), \mathbf{Z})$.   If the order of $H_1(M(n), \mathbf{Z})$ is infinity, then we put $\mathrm{ord}(n) = 0$.   The following result is classically well-known (see [30]):

$$\mathrm{ord}(n) = \prod_{i=1}^{n} |\Delta_K(\zeta_i)|,$$

where Δ_K is the Alexander polynomial of K and $\zeta_1, \ldots, \zeta_n$ are the n-th roots of unity. Then we can modify this equality as follows:

$$\frac{2}{n} \log \mathrm{ord}(n) = \sum_{i=1}^{n} \frac{1}{n} \log |\Delta_K(\zeta_i)|^2.$$

If we take the limit on n, we obtain

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \log |\Delta_K(\zeta_i)|^2 = \int_0^1 \log |\Delta_K(e^{2\pi\sqrt{-1}\theta})|^2 \, d\theta.$$

It means that the logarithm of τ_1 can be described by the asymptotic behavior of the order of the first homology group of branched coverings up to sign.

REMARK 5.6. As for a related work to the above interpretation, see [**31**]. X. S. Lin informed us the literature at the workshop "Invariants of Knots and 3-Manifolds" held in Kyoto 2001.

6. A formula of τ_2.

If φ is an element of the Torelli group $\mathscr{I}_{g,1}$, that is, φ acts trivially on the first homology group $H_1(\Sigma_{g,1}, \mathbf{Z})$, we can give an explicit formula of the second invariant $\tau_2(W_\varphi)$. Here it should be noted that $\log \tau_1(W_\varphi) = 0$ for $\varphi \in \mathscr{I}_{g,1}$ (see Corollary 5.3). The second term is described by the representation

$$\rho_2 : \mathscr{M}_{g,1} \to GL(2g, \mathbf{Z}N_2),$$

where $N_2 = \Gamma/[\Gamma, \Gamma] \cong H_1(\Sigma_{g,1}, \mathbf{Z})$. If we restrict ρ_2 to the Torelli group $\mathscr{I}_{g,1}$, this is really a homomorphism (see [**24**] Corollary 5.4). Then our second formula is the following.

THEOREM 6.1. *For any mapping class $\varphi \in \mathscr{I}_{g,1}$, the logarithm of the second invariant $\tau_2(W_\varphi)$ is given by*

$$\log \tau_2(W_\varphi) = -2m(\Delta_{\rho_2(\varphi)}),$$

where $\Delta_{\rho_2(\varphi)}(y_1, \ldots, y_{2g}, t) = \det(tI - \overline{\rho_2(\varphi)})$ and y_i denotes the homology class corresponding to x_i.

PROOF. If φ belongs to the Torelli group, we easily notice that the group

$$\pi(2) = N_2 \rtimes \mathbf{Z} = H_1(\Sigma_{g,1}, \mathbf{Z}) \times \mathbf{Z}$$

is isomorphic to the abelian group $\mathbf{Z}^{2g+1}$. Accordingly we see that the Novikov-Shubin invariants $\alpha(\Delta_i^{(2)})$ are positive in this case (see [**16**]).

On the one hand, we can identify $l^2(\mathbf{Z}^{2g+1})$ with $L^2(T^{2g+1})$, the space of L^2-functions on the $(2g+1)$-dimensional torus, by Fourier transforms. Under this identification, the trace is just equal to the multiple integral

$$\int_0^1 \cdots \int_0^1 d\theta_1 \cdots d\theta_{2g+1}.$$

Hence we can prove the desired formula by the similar argument as in the case of the first term τ_1. This completes the proof. □

Now we suppose $F(\mathbf{t}) \in \mathbf{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is primitive. Then define F to be a generalized cyclotomic polynomial if it is a monomial times a product of one-variable cyclotomic polynomials evaluated at monomials.

COROLLARY 6.2. *For any mapping class $\varphi \in \mathscr{I}_{g,1}$, the logarithm of the second invariant $\tau_2(W_\varphi)$ vanishes if and only if $\Delta_{\rho_2(\varphi)}$ is a generalized cyclotomic polynomial.*

PROOF. Since the characteristic polynomial $\Delta_{\rho_2(\varphi)}$ is primitive as before, the assertion immediately follows from the theorem of Boyd, Lawton and Smyth (see [**8**] Theorem 4). □

Example 6.3.   Let φ_h $(1 \le h \le g)$ denote a BSCC-map of genus h, that is, a Dehn twist along a bounding simple closed curve on $\Sigma_{g,1}$ which separates $\Sigma_{g,1}$ into $\Sigma_{h,1}$ and genus $g - h$ surface with two boundaries.   This is a typical element of the Torelli group $\mathscr{I}_{g,1}$.   Thereby we see from [32] that

$$\Delta_{\rho_2(\varphi_h)} = (t-1)^{2g}.$$

Thus by virtue of Corollary 6.2, we can conclude $\log \tau_2(W_{\varphi_h}) = 0$.

Next we consider the BP-map $\psi_h = D_c D_{c'}^{-1}$ of genus h $(1 \le h \le g-1)$, where c and c' are disjoint homologous simple closed curves on $\Sigma_{g,1}$ as in Figure 1 and D_c denotes the Dehn twist along c.   It is known that the Torelli group $\mathscr{I}_{g,1}$ is normally generated
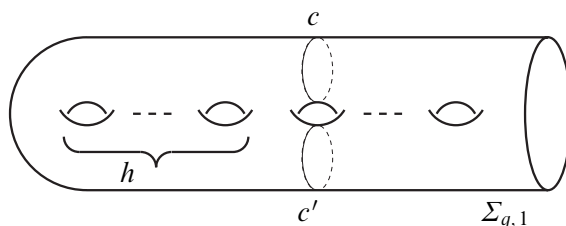

FIGURE 1.

in $\mathscr{M}_{g,1}$ by ψ_1.   Then from [32] Proposition 3.5, we obtain

$$\Delta_{\rho_2(\psi_h)} = (t-1)^{2g-2h}(t - y_{g+h+1})^{2h},$$

where y_{g+h+1} denotes the homology class corresponding to the $(h+1)$-th meridian of $\Sigma_{g,1}$.   This is also a generalized cyclotomic polynomial, so that its L^2-torsion vanishes.

Example 6.4.   Let $\varphi = D_{c_2} D_{c_1} D_{c_2}^{-1} D_{c_1} \in \mathscr{I}_{g,1}$.   Here c_1 and c_2 are simple closed curves on $\Sigma_{g,1}$ as in Figure 2 below.   Then we see from a computation of [32] that

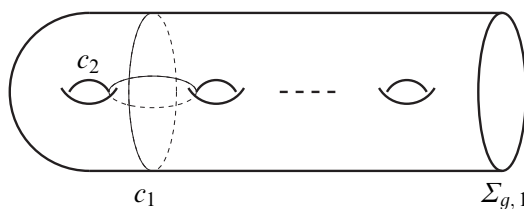$$\Delta_{\rho_2(\varphi)} = (t-1)^4 + t(t-1)^2(y_{g+1} - 2 + y_{g+1}^{-1})(y_{g+2} - 2 + y_{g+2}^{-1}).$$

FIGURE 2.

This is not a generalized cyclotomic polynomial, so that the mapping torus W_φ has a non-trivial L^2-torsion invariant $\tau_2(W_\varphi)$.   In fact we can compute the second term as follows.   By means of Lawton's result (see [15]), $m(\Delta_{\rho_2(\varphi)})$ can be expressed as the limit of the Mahler measure in a single variable.   More precisely we have

$$m(\Delta_{\rho_2(\varphi)}) = \lim_{r \to \infty} m(\Delta_{\rho_2(\varphi)}(u, u, u^r)).$$

In Figure 3, we plot the Mahler measure $m(\Delta_{\rho_2(\varphi)}(u, u, u^r))$ versus r. The convergence is evident and we obtain the approximate value

$$-3\pi \log \tau_2(W_\varphi) = 6\pi m(\Delta_{\rho_2(\varphi)}) = 19.28\ldots.$$
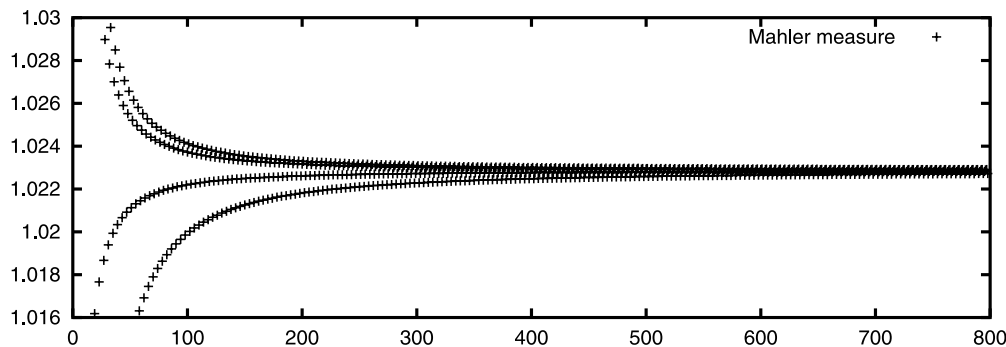

FIGURE 3. Mahler measure $m(\Delta_{\rho_2(\varphi)}(u, u, u^r))$ vs. r.

References

[1] P. Brinkmann and S. Schleimer, Computing triangulations of mapping tori of surface homeomorphisms, Experiment. Math., **10** (2001), 571–581.

[2] P. Brinkmann, XTrain, http://www.math.uiuc.edu/~brinkman/.

[3] D. Burghelea, L. Friedlander, T. Kappeler and P. McDonald, Analytic and Reidemeister torsion for representations in finite type Hilbert modules, Geom. Funct. Anal., **6** (1996), 751–859.

[4] A. Carey, M. Farber and V. Mathai, Determinant lines, von Neumann algebras and L^2 torsion, J. Reine. Angew. Math., **484** (1997), 153–181.

[5] A. Carey and V. Mathai, L^2-torsion invariants, J. Funct. Anal., **110** (1992), 377–409.

[6] J. Diximier, Von Neumann Algebras, North-Holland, 1981.

[7] J. Dodziuk, L^2-harmonic forms on rotationally symmetric Riemannian manifolds, Proc. Amer. Math. Soc., **77** (1979), 395–400.

[8] G. Everest, Measuring the height of a polynomial, Math. Intelligencer, **20** (1998), 9–16.

[9] R. Fenn, Techniques of Geometric Topology, London Math. Soc. Lect. Notes, **57** (1983).

[10] M. Gromov, Volume and bounded cohomology, Publ. Math. Inst. Hautes Études Sci., **56** (1982), 5–100.

[11] E. Hess and T. Schick, L^2-torsion of hyperbolic manifolds, Manuscripta Math., **97** (1998), 329–334.

[12] K. Ichihara, On framed link presentations of surface bundles, J. Knot Theory Ramifications, **7** (1998), 1087–1092.

[13] R. M. Kashaev and O. Tirkkonen, A proof of the volume conjecture on torus knots, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), **269** (2000), Vopr. Kvant. Teor. Polya i Stat. Fiz. 16, 262–268, 370.

[14] T. Kitano, T. Morifuji and M. Takasawa, Numerical calculation of L^2-torsion invariants, Interdiscip. Inform. Sci., **9** (2003), 35–42.

[15] W. M. Lawton, A problem of Boyd concerning geometric means of polynomials, J. Number Theory, **16** (1983), 356–362.

[16] J. Lott, Heat kernels on covering spaces and topological invariants, J. Differential Geom., **35** (1992), 471–510.

[17] J. Lott and W. Lück, L^2-topological invariants of 3-manifolds, Invent. Math., **120** (1995), 15–60.

[18] W. Lück, L^2-torsion and 3-manifolds, Proc. Low-Dimensional Topology, Knoxville, TN, 1992, (ed. K. Johanson), Internat. Press, 1994, 75–107.

[19] W. Lück, L^2-Betti numbers of mapping tori and groups, Topology, **33** (1994), 203–214.

[20] W. Lück and M. Rothenberg, Reidemeister torsion and the K-theory of von Neumann algebras, K-Theory, **5** (1991), 213–264.

[21] W. Lück and T. Schick, L^2-torsion of hyperbolic manifolds of finite volume, Geom. Funct. Anal., **9** (1999), 518–567.

[22] W. Magnus, A. Karrass and D. Solitar, Combinatorial Group Theory, Dover, 1976.

[23] V. Mathai, L^2-analytic torsion, J. Funct. Anal., **107** (1992), 369–386.

[24] S. Morita, Abelian quotients of subgroups of the mapping class group of surfaces, Duke Math. J., **70** (1993), 699–726.

[25] H. Murakami and J. Murakami, The colored Jones polynomials and the simplicial volume of a knot, Acta Math., **186** (2001), 85–104.

[26] S. Novikov and M. Shubin, Morse inequalities and von Neumann invariants of non-simply connected manifolds, Uspekhi. Mat. Nauk., **46** (1986), 222–223.

[27] S. Novikov and M. Shubin, Morse inequalities and von Neumann II$_1$-factors, Soviet Mathematics Doklady, **34** (1987), 79–82.

[28] R. C. Penner, A construction of pseudo-Anosov homeomorphisms, Trans. Amer. Math. Soc., **310** (1988), 179–197.

[29] T. Schick, L^2-determinant class and approximation of L^2-Betti numbers, Trans. Amer. Math. Soc., **353** (2001), 3247–3265 (electronic).

[30] H. Seifert, Über das Geshlecht von Knoten, Math. Ann., **110** (1934), 571–592.

[31] D. S. Silver and S. G. Williams, Mahler measure, links and homology growth, Topology, **41** (2002), 979–991.

[32] M. Suzuki, Some properties of the Magnus representation of the Torelli group; reducibility and a class function, Master thesis at Univ. Tokyo, 2000.

[33] M. Suzuki, Irreducible decomposition of the Magnus representation of the Torelli group, Bull. Austral. Math. Soc., **67** (2003), 1–14.

[34] J. Weeks, SnapPea, http://geometrygames.org/SnapPea.

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