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Decay estimates of solutions for dissipative wave equations in $\boldsymbol{R}^N$ with lower power nonlinearities

By Ryo Ikehata, Yasuaki Miyaoka and Takashi Nakatake

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Abstract. Optimal energy decay estimates will be derived for weak solutions to the Cauchy problem in $\boldsymbol{R}^N$ $(N = 1, 2, 3)$ of dissipative wave equations, which have lower power nonlinearities $|u|^{p-1}u$ satisfying $1 + 2/N < p \leq N/[N-2]^+$.

1. Introduction.

In this paper we are concerned with the following Cauchy problem in $\boldsymbol{R}^N$:

$$u_{tt}(t,x) - \Delta u(t,x) + u_t(t,x) = |u(t,x)|^{p-1}u(t,x), \quad (t,x) \in (0,\infty) \times \boldsymbol{R}^N, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \boldsymbol{R}^N. \tag{1.2}$$

First let us introduce some notations used throughout this paper. The total energy $E_u(t)$ to the equation (1.1) is defined as follows:

$$E_u(t) = \frac{1}{2}\|u_t(t,\cdot)\|^2 + \frac{1}{2}\|\nabla u(t,\cdot)\|^2,$$

where in these cases, $\|\cdot\|_q$ and $\|\cdot\|_{H^1}$ denote the usual $L^q(\boldsymbol{R}^N)$-norm and $H^1(\boldsymbol{R}^N)$-norm, respectively. Furthermore we use $\|\cdot\|$ for $\|\cdot\|_2$.

For the Cauchy problem (1.1)–(1.2) in $\boldsymbol{R}^N$ $(N \geq 1)$ with the usual nonlinearity:

$$1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}, \tag{1.3}$$

for the small initial data without $L^1 \times L^1$ assumption Nakao-Ono [**13**] has already derived the global existence of small weak solutions $u(t,x)$ and the decay estimates:

$$\|u(t,\cdot)\|^2 \leq C, \quad E_u(t) \leq C(1+t)^{-1}. \tag{1.4}$$

Their argument is based on the so called (modified) potential well method combined with the energy method whose idea originally comes from Payne-Sattinger [**15**] and Sattinger [**17**].

On the other hand, in Kawashima-Nakao-Ono [**5**] (see also Matsumura [**7**]) they dealt with the Cauchy problem (1.1)–(1.2) in $\boldsymbol{R}^N$ $(N \geq 1)$ with $|u(t,x)|^{p-1}u(t,x)$ replaced

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by $-|u(t,x)|^{p-1}u(t,x)$, and obtained the $L^p - L^q$ estimates of global weak solutions in the framework of "higher" power (1.3):

$$\|u(t,\cdot)\|^2 \leq C(1+t)^{-N(1/r-1/2)}, \quad E_u(t) \leq C(1+t)^{-1-N(1/r-1/2)} \tag{1.5}$$

for a rather stronger assumption of the initial data:

$$(u_0, u_1) \in (H^1(\boldsymbol{R}^N) \cap L^r(\boldsymbol{R}^N)) \times (L^2(\boldsymbol{R}^N) \cap L^r(\boldsymbol{R}^N))$$

for $r \in [1,2]$. Their argument depends on the monotonicity of the nonlinearity. But, it is easy to see that the same result can be also derived for the non-monotone case of nonlinearity by combining the (modified) potential well method with the $L^p - L^q$ estimates for the linear equation of (1.1)–(1.2) with $|u(t,x)|^{p-1}u(t,x)$ replaced by $f(t,x)$. In any case the condition (1.3) plays an essential role in deriving the various decay or bounded estimates of solutions. For the other type of decay property for the linear and nonlinear problems, we refer the reader to Ikehata [**3**], Ikehata-Matsuyama [**4**], Mochizuki-Nakazawa [**9**], Saeki-Ikehata [**16**], Shibata [**19**], Zuazua [**22**] and the references therein.

On the other hand, quite recently in [**21**] Todorova-Yordanov have already found the critical exponent $1 + 2/N$ $(N \geq 1)$ to the Cauchy problem (1.1)–(1.2) under rather stronger assumptions on the initial data such as compactness of the support.

The purpose of this paper is to relax the condition (1.3) to the lower power of nonlinearity:

$$1 + \frac{2}{N} < p \leq 1 + \frac{4}{N},$$

and to derive the optimal decay estimates like (1.5) for small global solutions to the Cauchy problem (1.1)–(1.2) without compactness of the support of the initial data. Since we consider the small solutions, it seems difficult to relax below the condition (1.3) of the power because the nonlinearity is hard to vanish as $t \to +\infty$. In particular, in Li-Zhou [**6**] the small data blowup results to the Cauchy problem (1.1)–(1.2), which has the power $1 < p \leq 1 + 2/N$ $(N = 1,2)$ have been already discussed. So our study will become a kind of interpolation of the power p between $(1, 1 + 2/N]$ and $(1 + 4/N, N/(N-2)]$ in the case when (at least) $N = 1,2$.

Our idea is to combine the $L^p - L^q$ estimates in Kawashima-Nakao-Ono [**5**] and Matsumura [**7**] with the method in Ikehata [**2**] which has recently relaxed below the condition (1.3) to

$$1 + \frac{6}{N+2} < p \leq \frac{N}{[N-2]^+},$$

for the case when $N = 2,3$ concerning the exterior mixed problem of (1.1), where $[a]^+ = \max\{a, 0\}$. We do not rely on the potential well method as in [**13**] so that even in the "monotone" nonlinearity case we must restrict the size of the initial data sufficiently small.

Before introducing our results we shall impose the following assumptions:

$$1+\frac{2}{N}<p<+\infty, \quad (N=1,2), \tag{1.6}$$

$$2<p\leq\frac{N}{N-2}, \quad (N=3). \tag{1.7}$$

Note that in the case when $N=3$ we see

$$1+\frac{2}{N}<\frac{3}{2}+\frac{1}{N}<2<1+\frac{6}{N+2}<1+\frac{4}{N}<\frac{N}{N-2}. \tag{1.8}$$

Our main result reads as follows.

THEOREM 1.1. *Suppose that* (1.6) *in the case when* $N=1,2$ *and* (1.7) *in the case when* $N=3$ *are satisfied. Then there exists a real number* $\varepsilon_0>0$ *such that if the initial data* $(u_0,u_1)\in(H^1(\boldsymbol{R}^N)\cap L^1(\boldsymbol{R}^N))\times(L^2(\boldsymbol{R}^N)\cap L^1(\boldsymbol{R}^N))$ *further satisfies*

$$I_{0,u}\equiv\|u_0\|_1+\|u_0\|_{H^1}+\|u_1\|_1+\|u_1\|\leq\varepsilon_0,$$

the problem (1.1)–(1.2) *admits a global solution* $u\in C([0,\infty);H^1(\boldsymbol{R}^N))\cap C^1([0,\infty);L^2(\boldsymbol{R}^N))$ *satisfying the decay property*:

$$\|u(t,\cdot)\|^2\leq CI_{0,u}^2(1+t)^{-N/2},$$

$$E_u(t)\leq CI_{0,u}^2(1+t)^{-1-N/2}$$

with some generous constant $C>0$.

REMARK 1.1. For the solutions to the problem (1.1)–(1.2) with (at least) small initial data, our result is much sharper than that in [**5**, Theorem 1] and thus, these results imply that the exponent $1+4/N$ is just the technical value and we can conjecture that for all $N\geq1$, $1+2/N<p<(N+2)/[N-2]^+$ is the region for which the small data global existence property occurs. The exponent $1+2/N$ is the so called Fujita exponent in the semilinear heat equation case (see Li-Zhou [**6**]). In a sense we can say that the result in Theorem 1.1 implies the so called diffusion structure of the equation (1.1) (see Han-Milani [**1**], Nishihara [**14**]). Furthermore, in the case when $N=1,2$ Theorem 1.1 completely includes the result in [**21**]. Finally we can also deal with the other type of nonlinearities $\pm|u|^p$, $-|u|^{p-1}u$ and so on.

By the way, in the occasion of the proof of Theorem 1.1 we shall proceed our argument based on the following well-known result. (cf. Strauss [**20**] and Nakao-Ono [**13**]):

PROPOSITION 1.1. *Suppose* $1<p\leq N/[N-2]^+$ *with* $N\geq1$. *For each* $(u_0,u_1)\in H^1(\boldsymbol{R}^N)\times L^2(\boldsymbol{R}^N)$, *there exists a maximal existence time* $T_m>0$ *such that the problem* (1.1)–(1.2) *has a unique solution* $u\in C([0,T_m);H^1(\boldsymbol{R}^N))\cap C^1([0,T_m);L^2(\boldsymbol{R}^N))$. *If* $T_m<+\infty$, *then*

$$\lim_{t\uparrow T_m}[\|u(t,\cdot)\|+\|\nabla u(t,\cdot)\|+\|u_t(t,\cdot)\|]=+\infty.$$

2. Proof of Theorem 1.1.

In this section let us prove Theorem 1.1. For this aim we first prepare the following lemma, that is the Gagliardo-Nirenberg inequality.

Lemma 2.1. *Let* $1 \le r < q \le 2N/[N-2]^+$, $2 \le q$ *and* $N \ge 1$. *Then the inequality*

$$\|v\|_q \le K_0 \|\nabla v\|^{\theta} \|v\|_r^{1-\theta}, \quad v \in H^1(\boldsymbol{R}^N)$$

holds with some constant $K_0 > 0$ *and*

$$\theta = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{r} - \frac{1}{2} + \frac{1}{N}\right)^{-1}$$

provided that $0 < \theta \le 1$.

Now we shall consider the linear wave equation:

$$v_{tt}(t,x) - \Delta v(t,x) + v_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \boldsymbol{R}^N, \tag{2.1}$$

$$v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \quad x \in \boldsymbol{R}^N. \tag{2.2}$$

For this linear problem, in [**5**] and [**7**] the so called $L^p - L^q$ estimates of solutions have been already derived. Thus by using these $L^p - L^q$ estimates and the so called Duhamel principle we shall handle the semilinear problem (1.1)–(1.2).

Set

$$I_{1,v} = \|v_0\| + \|v_0\|_1 + \|v_1\| + \|v_1\|_1.$$

Proposition 2.1 ([**5**, Proposition 3.2]). *Let* $N \ge 1$. *Then for each* $(v_0, v_1) \in (H^1(\boldsymbol{R}^N) \cap L^1(\boldsymbol{R}^N)) \times (L^2(\boldsymbol{R}^N) \cap L^1(\boldsymbol{R}^N))$, *the solution* v *of* (2.1)–(2.2) *satisfies*

$$\|v(t,\cdot)\|^2 \le C I_{1,v}^2 (1+t)^{-N/2}.$$

Proposition 2.2 ([**7**, Lemma 2]). *Let* $N \ge 1$. *Then for each* $(v_0, v_1) \in C_0^\infty(\boldsymbol{R}^N) \times C_0^\infty(\boldsymbol{R}^N)$, *the solution* v *of* (2.1)–(2.2) *satisfies*

$$E_v(t) \le C I_{0,v}^2 (1+t)^{-1-(N/2)}.$$

So, based on these Propositions 2.1 and 2.2 we can derive the following total energy decay estimates to the weak solution of the linear problem (2.1)–(2.2).

Proposition 2.3. *Let* $N \ge 1$. *Then for each* $(v_0, v_1) \in (H^1(\boldsymbol{R}^N) \cap L^1(\boldsymbol{R}^N)) \times (L^2(\boldsymbol{R}^N) \cap L^1(\boldsymbol{R}^N))$, *the weak solution* $v \in C([0,+\infty); H^1(\boldsymbol{R}^N)) \cap C^1([0,+\infty); L^2(\boldsymbol{R}^N))$ *to the problem* (2.1)–(2.2) *has the decay estimates*:

$$\|v(t,\cdot)\|^2 \le C I_{1,v}^2 (1+t)^{-N/2},$$

$$E_v(t) \le C I_{0,v}^2 (1+t)^{-1-(N/2)}.$$

Proof. Since $v_0 \in H^1(\boldsymbol{R}^N) \cap L^1(\boldsymbol{R}^N)$ and $v_1 \in L^2(\boldsymbol{R}^N) \cap L^1(\boldsymbol{R}^N)$ can be approximated by smooth functions $\{\phi_n\} \subset C_0^\infty(\boldsymbol{R}^N)$ and $\{\psi_n\} \subset C_0^\infty(\boldsymbol{R}^N)$ satisfying

$$\|\phi_n - v_0\|_{H^1} + \|\phi_n - v_0\|_1 \to 0 \quad (n \to +\infty),$$

$$\|\psi_n - v_1\| + \|\psi_n - v_1\|_1 \to 0 \quad (n \to +\infty),$$

the desired statement is rather standard. □

Under these preparations we can prove Theorem 1.1. The proof will be done along the same way as in [**2**].

Proof of Theorem 1.1. First define a semigroup $S(t) : H^1(\boldsymbol{R}^N) \times L^2(\boldsymbol{R}^N) \to H^1(\boldsymbol{R}^N) \times L^2(\boldsymbol{R}^N)$ by

$$S(t) : \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \mapsto \begin{bmatrix} v(t,\cdot) \\ v_t(t,\cdot) \end{bmatrix},$$

where $v(t,\cdot) \in C([0,+\infty); H^1(\boldsymbol{R}^N)) \cap C^1([0,+\infty); L^2(\boldsymbol{R}^N))$ is a unique solution to the "linear" problem (2.1)–(2.2).

The following well-known inequalities are useful in order to derive some decay rate (see Segal [**18**]).

Lemma 2.2. *If $\beta > 1$ and $\eta \le \beta$, then there exists a constant $C_{\beta,\eta} > 0$ depending only on β and η such that*

$$\int_0^t (1+t-s)^{-\eta}(1+s)^{-\beta}\,ds \le C_{\beta,\eta}(1+t)^{-\eta}$$

for all $t \ge 0$.

In the following paragraph we set $I_{0,u} = I_0$ for simplicity. Now we shall derive the decay property of a nonlinear problem (1.1)–(1.2). By a standard semigroup theory, the nonlinear problem (1.1)–(1.2) is rewritten as the integral equation:

$$U(t) = S(t)U_0 + \int_0^t S(t-s)F(s)\,ds, \tag{2.3}$$

where $U(t) = \begin{bmatrix} u(t,\cdot) \\ u_t(t,\cdot) \end{bmatrix}$, $U_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$, $F(s) = \begin{bmatrix} 0 \\ f(u(s,\cdot)) \end{bmatrix}$ with $f(u)(x) = |u(x)|^{p-1}u(x)$.

We proceed our argument based on the way of Nakao [**12**] (the small data perturbation method). In order to show the global existence, it suffices to obtain the a priori estimates for $E_u(t)$ and $\|u(t,\cdot)\|$ in the interval of existence $[0, T_m)$ (see Proposition 1.1). As a result of Proposition 2.3, first one has

Lemma 2.3. *Under the assumptions as in Theorem* 1.1, *we have*

$$\|S(t)U_0\|_E \le CI_0(1+t)^{-1/2-N/4}$$

on $[0, T_m)$, where we set

$$\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_E = \|v\| + \|\nabla u\|.$$

Furthermore, if

$$I(s) = \|f(u(s,\cdot))\| + \|f(u(s,\cdot))\|_1 < +\infty \tag{2.4}$$

for each $s \in [0,t]$ with $t \in [0,T_m)$, then from Proposition 2.3 we have

$$\|S(t-s)F(s)\|_E \le CI(s)(1+t-s)^{-1/2-N/4}. \tag{2.5}$$

Thus from (2.3) one can estimate $U(t)$ as follows:

$$\|U(t)\|_E \le CI_0(1+t)^{-1/2-N/4} + C\int_0^t (1+t-s)^{-1/2-N/4} I(s)\,ds. \tag{2.6}$$

Take $K > 0$ so large (if necessary) such as

$$\|U_0\|_E < KI_0, \quad \|u_0\| < KI_0.$$

Because of the continuity of functions $t \mapsto \|U(t)\|_E$ and $t \mapsto \|u(t)\|$ we assume that there exists a real number $T \in (0,T_m)$ such that

$$\|U(t)\|_E < KI_0(1+t)^{-(1/2+N/4)}, \quad \text{and} \quad \|u(t)\| < KI_0(1+t)^{-N/4} \quad \text{on } [0,T), \tag{2.7}$$

and

$$\|U(T)\|_E = KI_0(1+T)^{-(1/2+N/4)}, \quad \text{or} \quad \|u(T)\| = KI_0(1+T)^{-N/4}. \tag{2.8}$$

By Lemma 2.1 and the assumption (1.6) or (1.7) we have

$$\|f(u(s,\cdot))\|_1 = \|u(t,\cdot)\|_p^p \le K_0\|u(s,\cdot)\|^{p(1-\theta_1)}\|\nabla u(s,\cdot)\|^{p\theta_1}$$

with $\theta_1 = N(p-2)/2p \in (0,1]$. Similarly one has

$$\|f(u(s,\cdot))\| \le K_0\|u(s,\cdot)\|^{p(1-\theta_2)}\|\nabla u(s,\cdot)\|^{p\theta_2}$$

with $\theta_2 = N(p-1)/2p \in (0,1]$. Therefore, as long as (2.7)–(2.8) holds one gets

$$\begin{aligned}\|f(u(s,\cdot))\|_1 &\le K_0\{KI_0(1+s)^{-N/4}\}^{p(1-\theta_1)}\{KI_0(1+s)^{-1/2-N/4}\}^{p\theta_1}\\ &= K_0K^pI_0^p(1+s)^{-p(\theta_1/2+N/4)},\\ \|f(u(s,\cdot))\| &\le K_0\{KI_0(1+s)^{-N/4}\}^{p(1-\theta_2)}\{KI_0(1+s)^{-1/2-N/4}\}^{p\theta_2}\\ &= K_0K^pI_0^p(1+s)^{-p(\theta_2/2+N/4)},\end{aligned}$$

for $s \in [0,T]$. Here we set

$$\gamma_1 = pN/4 + N(p-2)/4, \quad \gamma_2 = pN/4 + N(p-1)/4.$$

Summing up these calculations we have the following lemma, which shows the validity of the condition (2.4).

Lemma 2.4. *As long as* (2.7)–(2.8) *hold on* $[0,T]$ *we have*

$$\|f(u(t,\cdot))\|_1 \le K_0K^pI_0^p(1+t)^{-\gamma_1},$$
$$\|f(u(t,\cdot))\| \le K_0K^pI_0^p(1+t)^{-\gamma_2}.$$

By applying Lemmas 2.3 and 2.4 to (2.3) we see that

$$\begin{aligned}\|U(t)\|_E &\le CI_0(1+t)^{-1/2-N/4} + C\int_0^t (1+t-s)^{-1/2-N/4} K_0 K^p I_0^p \{(1+s)^{-\gamma_1} + (1+s)^{-\gamma_2}\}\, ds \\ &\le CI_0(1+t)^{-1/2-N/4} + CK_0K^pI_0^p \int_0^t (1+t-s)^{-1/2-N/4}(1+s)^{-\gamma_1}\, ds.\end{aligned}$$

Note that $\gamma_1 > 1$ because of the assumptions (1.6) or (1.7) and $\gamma_1 < \gamma_2$. Thus from Lemma 2.2 and again (1.6) or (1.7) it follows that

$$\|U(t)\|_E \le CI_0(1+t)^{-1/2-N/4} + CK_0K^pI_0^p(1+t)^{-1/2-N/4}$$

with some constant $C > 0$ independent of K and I_0 (see also (1.8)). Setting

$$Q_0(I_0, K) = C + CK_0K^pI_0^{p-1},$$

we get the following lemma.

LEMMA 2.5. *As long as* (2.7)–(2.8) *hold on* $[0, T]$ *we get*

$$\|U(t)\|_E \le I_0 Q_0(I_0, K)(1+t)^{-1/2-N/4}.$$

Next let us derive the L^2-estimates for the local solution $u(t,x)$ to the problem (1.1)–(1.2). Indeed, we have from (2.3) and Proposition 2.3 that

$$\|u(t,\cdot)\| \le CI_0(1+t)^{-N/4} + C\int_0^t (1+t-s)^{-N/4} I(s)\, ds.$$

Therefore, it follows from Lemma 2.4 and the similar argument to Lemma 2.5 that

$$\begin{aligned}\|u(t,\cdot)\| &\le CI_0(1+t)^{-N/4} + C\int_0^t (1+t-s)^{-N/4} K_0K^pI_0^p[(1+s)^{-\gamma_1} + (1+s)^{-\gamma_2}]\, ds \\ &\le CI_0(1+t)^{-N/4} + CK_0K^pI_0^p \int_0^t (1+t-s)^{-N/4}(1+s)^{-\gamma_1}\, ds\end{aligned}$$

with some constant $C > 0$ independent of K and I_0, where $\gamma_1 > 1$ because of (1.6) or (1.7). This together with Lemma 2.2 implies

$$\|u(t,\cdot)\| \le CI_0(1+t)^{-N/4} + CK_0K^pI_0^p(1+t)^{-N/4}.$$

Thus we have

LEMMA 2.6. *As long as* (2.7)–(2.8) *hold on* $[0, T]$ *it follows that*

$$\|u(t,\cdot)\| \le I_0Q_0(I_0, K)(1+t)^{-N/4}.$$

Take $K > C$ further so large and take I_0 so small such as

$$CK_0K^pI_0^{p-1} < K - C. \tag{2.9}$$

For such $K > 0$ and I_0 we have

$$Q_0(I_0, K) < K.$$

Therefore, by combining this with Lemmas 2.5 and 2.6 we see that

$$\|U(t)\|_E < KI_0(1+t)^{-1/2-N/4}, \tag{2.10}$$

$$\|u(t,\cdot)\| < KI_0(1+t)^{-N/4} \tag{2.11}$$

on $[0,T]$, which yields a contradiction to (2.8). Thus the a priori estimates (2.10) and (2.11) hold on $[0,T_m)$ under the assumption (2.9). Because of Proposition 1.1 the local solution $u(t,\cdot)$ exists globally in time and these estimates hold in fact for all $t \geq 0$. Taking $\varepsilon_0 = ((K-C)/CK_0K^p)^{1/(p-1)}$, the proof of Theorem 1.1 is now finished. □

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Ryo Ikehata
Department of Mathematics
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739-8524
Japan
E-mail: ikehatar@hiroshima-u.ac.jp

Yasuaki Miyaoka
Jyôsei Junior Highschool
Matsuyama 790-0053
Japan

Takashi Nakatake
Department of Mathematics
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739-8524
Japan