J. Math. Soc. Japan
Vol. 56, No. 3, 2004

Classification of normal quartic surfaces with irrational singularities

By Yuji Ishii and Noboru Nakayama

(Received Jul. 29. 2002)
(Revised Apr. 14, 2003)

Abstract. If a normal quartic surface admits a singular point that is not a rational double point, then the surface is determined by the triplet (M, D, E) consisting of the minimal desingularization M, the pullback D of a general hyperplane section, and a non-zero effective anti-canonical divisor E of M. Geometric constructions of all the possible triplets (M, D, E) are given.

Introduction.

The purpose of this paper is to classify the complex normal quartic surfaces in the 3-dimensional projective space $\boldsymbol{P}^3$ with irrational singularities by determining their minimal desingularizations. Let S be a normal quartic surface and let $\sigma : M \to S$ be the minimal desingularization. Then M is known to be one of the following surfaces (cf. [**8**]):

(1) a K3 surface;
(2) a $\boldsymbol{P}^1$-bundle over a smooth quartic curve of $\boldsymbol{P}^2$;
(3) a ruled surface over an elliptic curve;
(4) a rational surface.

In the case (1), S has only rational double points as singularities. In the case (2), S is nothing but the cone over the quartic curve. Umezu [**8**] have determined the structure of M and the minimal desingularization $\sigma : M \to S$ in the case (3).

The classification problem has been studied by a number of algebraic geometers for more than half a century. Umezu [**8**] and Urabe [**9**], [**10**] considered the problem from a viewpoint of singularities. Umezu studied the singularities of a normal Gorenstein surface with trivial dualizing sheaf in [**7**]. In our case, the singularities which are not rational double points are studied by the configuration of the effective anti-canonical divisor E of M determined by $K_M \sim \sigma^* K_S - E$. In the next paper [**8**], Umezu described the pair (M, E) by determining the blowing-up process from a relative minimal model of M. Urabe [**9**], [**10**] applied Looijenga's argument in [**5**] to the pair (M, E) in which E is irreducible. By using Dynkin diagrams, Urabe determined possible singularities on $S \backslash \sigma(E)$. On the other hand, Degtyarev [**2**] considered the problem by types of equations of hypersurface singularities listed in [**1**].

Our approach is different from theirs. We consider a triplet (M, D, E) called a *basic triplet* which consists of a non-singular projective surface M, a smooth non-hyperelliptic curve D of genus 3 on M, and a non-zero anti-canonical divisor E of

---

2000 *Mathematics Subject Classification*. 14J25, 14J26, 14J70, 14E30.
Key Words and Phrases. quartic surface, ruled surface, extremal ray.

M. If $\sigma : M \to S$ is the minimal desingularization of a normal quartic surface S with irrational singularities, then (M, D, E) is a basic triplet for the pullback D of a general hyperplane section of S and for the anti-canonical divisor E with $K_M \sim \sigma^* K_S - E$. The basic triplet satisfies the condition $\mathscr{C}$ in §1. Conversely, if a basic triplet (M, D, E) satisfies $\mathscr{C}$, then it is induced from a normal quartic surface with irrational singularities (cf. Proposition 1.4). Therefore, it is enough to determine all the basic triplets satisfying $\mathscr{C}$. We apply the theory of extremal rays [**6**] to $K_M + D$ and $2K_M + D$. If $K_M + D$ is not nef, then we infer that M is a $\boldsymbol{P}^1$-bundle over a smooth non-hyperelliptic curve of genus 3 and S is nothing but the cone over a smooth quartic curve. If $K_M + D$ is nef, then we consider an extremal curve Γ with $(2K_M + D) \cdot \Gamma < 0$. If Γ is a (-1)-curve and if $\phi : M \to M'$ is the contraction of Γ, then $D' = \phi(D)$ is isomorphic to D and $E' = \phi_* E$ is an anti-canonical divisor with $K_M + D \sim \phi^*(K_{M'} + D')$. The morphism ϕ is the blowing-up at the unique point $D' \cap E'$. The new triplet (M', D', E') satisfies the condition $\mathscr{C}_1$ in §1. Next, we consider another (-1)-curve Γ' with $(2K_{M'} + D') \cdot \Gamma' < 0$ and its contraction. In this way, we finally have a basic triplet (X, B, G) and a birational morphism $\rho : M \to X$ such that $K_M + D \sim \rho^*(K_X + B)$, $D \simeq B$, and $(2K_X + B) \cdot \Xi \geq 0$ for any (-1)-curve Ξ on X. The basic triplet (X, B, G) is called a *minimal* basic triplet and M is obtained canonically from (X, B, G) by the method called *separation* (cf. §1.2). By the structure of (X, B, G), the triplets (M, D, E) are classified into Types A to D in Theorem 1.7.

We shall give examples of the triplets (M, D, E) and (X, B, G) in §2 and we shall show in §3 that any basic triplet (M, D, E) satisfying $\mathscr{C}$ is one of the triplets given in §2. For the proof, we need some well-known facts on generalized del Pezzo surfaces, rational elliptic surfaces, elliptic ruled surfaces, double-coverings, and extremal rays.

Our classification is very rough compared to Umezu's work [**8**]. Because, firstly, it is not the classification modulo isomorphisms. We need a hyperplane section as an additional datum. Secondly, by the use of separation, we avoid studying the configuration of centers (including infinitesimally near points) of related blowings-up. It is related to the description of singular points on S. However, we can give a geometric construction of any normal quartic surface with irrational singularities. It might be useful for the fine classification.

Acknowledgement. This paper is based on the first author's Master's thesis [**4**] under the guidance of the second author at Dept. of Math. Sci., Univ. Tokyo in 1993. The original version of this paper which completes the classification in [**4**] was written in 1993. The authors thank Professors Yumiko Umezu and Tohsuke Urabe for informing kindly the background and the resent progress of the classification problem of quartic surfaces. The first author left the university and became a bank clerk just after finishing the Master course. Seminar talks on the paper were given at several universities in Japan in 1993–1994. The second author moved to RIMS. Kyoto Univ. in 1995 and has kept the manuscript almost unchanged until 2002. The present modified version was written up in July 2002. The second author apologizes for the delay in submitting to a journal and thanks Professor Hiromichi Takagi for his careful reading of an old version. The second author also thanks the referee for the valuable comment.

Notation. Every varieties are defined over the complex number field $\boldsymbol{C}$. *Curves* and *surfaces* are always assumed to be irreducible, reduced, and projective. These are *smooth* (over $\boldsymbol{C}$) if and only if they are *non-singular*.

Divisors: Let X be a normal surface or a smooth curve.

— $\mathcal{O}_X(D)$ denotes the invertible sheaf associated with a Cartier divisor D. We write $H^p(X,D) = H^p(X,\mathcal{O}_X(D))$ for short. We write also $h^p(X,D) = \dim H^p(X,D)$.

— For a non-zero global rational section φ of an invertible sheaf $\mathscr{L}$ of X, we define

$$\operatorname{div}(\varphi) := \sum \operatorname{ord}_\Gamma(\varphi)\Gamma$$

in which $\operatorname{ord}_\Gamma(\varphi)$ is the order of zeros or the minus of the order of poles of φ along a prime divisor $\Gamma \subset X$.

— K_X denotes the *canonical divisor* of X. If K_X is Cartier, then X is called *Gorenstein*. The *dualizing sheaf* ω_X of X is isomorphic to $\mathcal{O}_X(K_X)$. A divisor E is called *anti-canonical* if $E \sim -K_X$.

— $|D|$ denotes the *complete linear system* associated with a divisor D. The associated rational map $X \cdots\to |D|^\vee$ into the dual space $|D|^\vee$ is denoted by $\Phi_{|D|}$. If we fix a basis $(\varphi_0, \varphi_1, \dots, \varphi_n)$ of $H^0(X,D)$, then $\Phi_{|D|}$ is equivalent to the map given by

$$x \mapsto (\varphi_0(x) : \varphi_1(x) : \cdots : \varphi_n(x)).$$

The *base locus* of $|D|$ is denoted by $\operatorname{Bs}|D|$.

— A Cartier divisor D of a surface is called *nef* if the intersection number $D \cdot C$ is non-negative for any curve $C \subset X$. If in addition the self-intersection number D^2 is positive, then D is called *nef and big*. *Kawamata-Viehweg's vanishing theorem* states that $H^p(X, K_X + D) = 0$ for a nef and big Cartier divisor D and for $p > 0$.

— Let D be a Cartier divisor of a surface X. If there is a morphism $f : X \to Y$ into another variety and if $D \cdot C \geq 0$ for any curves $C \subset X$ contained in fibers of f, then D is called *f-nef*.

— Let D and A be Cartier divisors on a surface X. If $D \cdot A \leq 0$ and $A^2 > 0$, then $D^2 \leq 0$, in which $D^2 = 0$ if and only if D is *numerically trivial*: $D \cdot C = 0$ for any curves C. This result is referred as the *Hodge index theorem*.

Curves: Let C be a curve.

— The *arithmetic genus* $p_a(C)$ is defined as $h^1(C,\mathcal{O}_C)$. The *genus* $g(C)$ is defined as $p_a(\tilde{C})$ for the normalization $\tilde{C} \to C$.

— A *rational curve* is a curve C with $g(C) = 0$. An *elliptic curve* is a smooth curve with $g(C) = 1$.

— A smooth curve C of genus $g(C) \geq 2$ is called *hyperelliptic* if the image of the *canonical map*

$$\Phi_{|K_C|} : C \to \boldsymbol{P}^{g(C)-1}$$

is $\boldsymbol{P}^1$. In this case, $C \to \boldsymbol{P}^1$ is a double-covering. If C is a non-hyperelliptic curve, then the canonical map is an embedding of C.

— A *quartic curve* is a curve $C \subset \boldsymbol{P}^2$ with degree 4. A smooth quartic curve is nothing but a non-hyperelliptic curve of genus 3.

— Let $\mathscr{E}$ be a locally free sheaf on C. We denote by $\boldsymbol{P}_C(\mathscr{E})$ the projective bundle associated with $\mathscr{E}$. The *tautological* invertible sheaf $\mathcal{O}_{\mathscr{E}}(1)$ associated with $\mathscr{E}$ is defined as the invertible sheaf on $\boldsymbol{P}_C(\mathscr{E})$ satisfying $p_*\mathcal{O}_{\mathscr{E}}(1) \simeq \mathscr{E}$ for the structure morphism $p : \boldsymbol{P}_C(\mathscr{E}) \to C$. A *tautological divisor* $H_{\mathscr{E}}$ is a Cartier divisor with $\mathcal{O}(H_{\mathscr{E}}) \simeq \mathcal{O}_{\mathscr{E}}(1)$.

Surfaces: Let X be a smooth surface and C a smooth curve.

— $q(X)$ denotes the *irregularity* of X: $q(X) = h^1(M, \mathcal{O}_X)$.

— A (-1)-*curve* of X is a smooth rational curve $C \subset X$ with $C^2 = -1$. It is usually called the exceptional curve of the first kind. A smooth rational curve $C \subset X$ with $C^2 = -2$ is called a (-2)-*curve*.

— Let $f : X \to Y$ be a morphism into another variety. If K_X is f-nef, then X is called *minimal* over Y or f is called *minimal*. If K_X is not f-nef, then one of the following cases occurs (cf. [**6**]):

(1) There is a (-1)-curve contained in a fiber of f;

(2) f is isomorphic to a $\boldsymbol{P}^1$-bundle over a smooth curve C defined over Y;

(3) $f(X)$ is a point and $X \simeq \boldsymbol{P}^2$.

— The *Hirzebruch surface* Σ_r of degree $r \geq 0$ is defined as the $\boldsymbol{P}^1$-bundle associated with $\mathcal{O} \oplus \mathcal{O}(r)$ on $\boldsymbol{P}^1$.

— X is called a *generalized del Pezzo surface* of degree d if $-K_X$ is nef and big with $K_X^2 = d$. The following properties are known:

(1) A generalized del Pezzo surface is a rational surface.

(2) If $d = 2$, then $\mathrm{Bs}|-K_X| = \varnothing$, $h^0(X, -K_X) = 3$, and $\Phi_{|-K_X|}$ is a generically finite surjective morphism onto $\boldsymbol{P}^2$ of mapping degree 2.

(3) If $d = 1$, then $\mathrm{Bs}|-K_X|$ consists of one point, $h^0(-K_X) = 2$, and $\Phi_{|-K_X|}$ induces an elliptic fibration $Z \to \boldsymbol{P}^1$ from the blown-up Z of X at $\mathrm{Bs}|-K_X|$.

For the readers' convenience, we shall give a proof.

Proof. (1) We have $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$ by Kawamata-Viehweg's vanishing theorem. In particular, $q(X) = 0$. Thus X is rational since the Kodaira dimension $\kappa(X)$ is negative.

(2), (3). We first show that the fixed divisor $F = |-K_X|_{\mathrm{fix}}$ is zero. If $|M|$ is the movable part, i.e., $|M| + F = |-K_X|$, then M is nef and hence $M - K_X$ is nef and big. Thus $\chi(M) = h^0(M) = h^0(-K_X) = \chi(-K_X) = K_X^2 + 1 = d + 1$ by Kawamata-Viehweg's vanishing theorem. This implies $M \cdot (M - K_X) = 2K_X^2$. We have inequalities

$$M^2 \leq M \cdot (M + F) = M \cdot (-K_X) \leq (-K_X)^2.$$

Hence, $M \cdot F = (-K_X) \cdot F = F^2 = 0$. The Hodge index theorem implies $F = 0$.

Secondly, we show that a general member of $|-K_X|$ is an elliptic curve. Let $f : Y \to X$ be a succession of blowings-up at points such that $\Phi_{|f^*(-K_X)|}$ is holomorphic. Let $|C|$ and G be the movable part and the fixed divisor of $f^*|-K_X|$, respectively. We may assume that $E \leq G$ for the exceptional divisor $E = K_Y - f^*K_X$. Since $C + E - K_Y$ is nef and big, we have $\chi(Y, C + E) = \chi(-K_X)$, equivalently, $(C + E) \cdot (C + f^*(-K_X)) = 2K_X^2$. We infer that $C \cdot (G - E) = 0$ from inequalities

$$(C + E) \cdot C \leq (C + G) \cdot C = f^*(-K_X) \cdot (C + E) \leq f^*(-K_X)^2.$$

Thus a general member C of $|C|$ is a disjoint union of elliptic curves and $C \to f(C)$ is an isomorphism by

$$(K_Y + C)|_C = f^*(K_X + (-K_X))|_C - (G - E)|_C \sim 0.$$

For a member D of $|-K_X|$, we have $h^0(D, \mathcal{O}_D) = 1$ by $q(X) = 0$. Thus a general member D is an elliptic curve.

Finally, we show (2) and (3). For a general member $D \in |-K_X|$, $H^0(X, -K_X) \to H^0(D, (-K_X)|_D)$ is surjective by $q(X) = 0$. Thus if $d \geq 2$, then $|-K_X|$ is base point free. Thus (2) follows. If $d = 1$, then $\mathrm{Bs}|-K_X|$ consists of one point which is the intersection of general two members. Thus (3) follows. □

The referee pointed out that more systematic argument on generalized del Pezzo surfaces are written in [**3**].

Desingularization: Let S be a normal surface and let $\sigma : M \to S$ be a birational morphism from a non-singular surface.

— If $R^1\sigma_*\mathcal{O}_M = 0$, then the singularities of S are called *rational*. If S is Gorenstein in addition, then the singularities are called *rational double points*. The dual graph defined by the exceptional locus of the minimal desingularization of a rational double point is one of Dynkin diagrams A_n, D_n, E_6, E_7, E_8. Rational double points are also called ADE-singularities, simple singularities, Du Val singularities, and so on.

— If $S \subset \boldsymbol{P}^3$ with $\deg S = 4$, then S is called a *quartic surface*. Here, $\omega_S \simeq \mathcal{O}_S$ and $h^1(S, \mathcal{O}_S) = 0$. If σ is the minimal desingularization and if M is not a ruled surface, then S has only rational double points as singularities and M is a K3 surface.

§1. Condition $\mathscr{C}$ and separation.

§1.1. Condition $\mathscr{C}$ and quartic surfaces.

Proposition 1.1. *Let M be a non-singular projective surface admitting a non-zero effective anti-canonical divisor E. Then $h^0(E, \mathcal{O}_E) = q(M) + 1$.*

Proof. In view of the exact sequence

$$0 \to \omega_M \to \mathcal{O}_M \to \mathcal{O}_E \to 0,$$

we infer that $h^0(E, \mathcal{O}_E) = 1$ when $q(M) = 0$. Assume that M contains a (-1)-curve Γ. Let $h : M \to M'$ be the contraction of Γ and let $E' = h_*E$ be the image of E as a divisor. Then $\mathcal{O}_{E'} \simeq h_*\mathcal{O}_E$ by the vanishing $R^1h_*\mathcal{O}_M(-E) = 0$. In particular, $h^0(E, \mathcal{O}_E) = h^0(E', \mathcal{O}_{E'})$. Therefore, we may assume that M is an irrational relatively minimal surface. Hence M has a $\boldsymbol{P}^1$-bundle structure $p : M \to C$ over a smooth curve C with $g(C) = q(M) > 0$. Here, we have the following exact sequence:

$$0 \to \mathcal{O}_C \simeq p_*\mathcal{O}_M \to p_*\mathcal{O}_E \to R^1p_*\omega_M \simeq \omega_C \to 0.$$

Since some component of E dominates C, there is a splitting of $\mathcal{O}_C \to p_*\mathcal{O}_E$. Thus $p_*\mathcal{O}_E \simeq \mathcal{O}_C \oplus \omega_C$. Therefore $h^0(E, \mathcal{O}_E) = 1 + g(C) = 1 + q(M)$. □

Corollary 1.2 (cf. [**7**]). *A non-singular projective surface admitting an irreducible and reduced anti-canonical divisor is rational.*

Lemma 1.3. *Let M be a non-singular projective surface admitting a non-zero effective anti-canonical divisor E. If any prime component of E is a rational curve, then M is rational.*

Proof. Assume the contrary. Then the Albanese map induces a surjective morphism $\pi : M \to C$ into a smooth curve C of genus $q(M) > 0$ whose general fibers F are rational curves. Since $E \cdot F = -\deg K_F = 2$, some component of E dominates C. Thus $C \simeq \boldsymbol{P}^1$. This is a contradiction. □

Definition. A *basic triplet* (M, D, E) is a triplet consisting of a non-singular projective surface M, a smooth non-hyperelliptic curve D of genus 3 on M, and a non-zero effective anti-canonical divisor E of M. The condition $\mathscr{C}$ for (M, D, E) is the collection of the following two conditions:

$\mathscr{C}$-1: $D \cdot \Gamma > 0$ for any (-1)-curve Γ on M;

$\mathscr{C}$-2: $D \cap E = \varnothing$.

If (M, D, E) satisfies $\mathscr{C}$, then

$$D^2 = (K_M + D) \cdot D = 2g(D) - 2 = 4.$$

Let $\sigma : M \to S$ be the minimal desingularization of a normal quartic surface S with irrational singularities. Let E be the σ-exceptional anti-canonical divisor such that $K_M \sim \sigma^* K_S - E \sim -E$ and let D be the pullback of a general hyperplane section of S. Then the basic triplet (M, D, E) satisfies the condition $\mathscr{C}$. Conversely, we have:

Proposition 1.4. *If a basic triplet (M, D, E) satisfies the condition $\mathscr{C}$, then there exist a normal quartic surface S and a birational morphism $\sigma : M \to S$ such that*

(1) *S has irrational singular points,*

(2) *σ is the minimal desingularization of S,*

(3) *D is the pullback of a general hyperplane section of S,*

(4) *E is the σ-exceptional divisor satisfying $K_M \sim \sigma^*(K_S) - E$.*

Proof. Since D is a nef and big divisor, $H^i(M, K_M + D) = 0$ for $i > 0$ by Kawamata-Viehweg's vanishing theorem. Hence

$$\begin{aligned} h^0(M, D) &= h^0(M, K_M + D) + h^0(E, \mathcal{O}_E) \\ &= \chi(M, K_M + D) + 1 + q(M) \\ &= g(D) + 1 = 4 \end{aligned}$$

by Proposition 1.1. In view of the exact sequence

$$0 \to \mathcal{O}_M \to \mathcal{O}_M(D) \to \mathcal{O}_D(D) \simeq \omega_D \to 0,$$

we infer that $H^0(M, D) \to H^0(D, K_D)$ is surjective. Hence $\mathrm{Bs}|D| = \varnothing$. Thus we have a generically finite morphism

$$\sigma := \Phi_{|D|} : M \to \boldsymbol{P}^3.$$

Let S be the image. Then $\deg \sigma = 1$ or $\deg \sigma = 2$ since $D^2 = 4$. We note that the restriction $\sigma|_D : D \to \boldsymbol{P}^2$ is the canonical map of D. This is an embedding since D is non-hyperelliptic. Thus $\deg \sigma = 1$. Hence $\sigma : M \to S$ is a birational morphism and S is a quartic surface. Now $\omega_S \simeq \mathcal{O}_S$, E is σ-exceptional, and $K_M \sim -E$. Thus S is a normal surface and $\sigma : M \to S$ is the minimal desingularization by $\mathscr{C}$-1. □

Therefore, the classification of normal quartic surfaces with irrational singularities is reduced to that of basic triplets (M, D, E) satisfying the condition $\mathscr{C}$.

§1.2. Separation.

Let (X, B, G) be a basic triplet with $B \not\leq G$. Let $\rho : Y \to X$ be a birational morphism from a non-singular projective variety and let B_Y and G_Y be effective divisors on Y.

Definition. The triplet (Y, B_Y, G_Y) or the birational morphism $\rho : Y \to X$ is called the *separation* of (X, B, G) if the following conditions are satisfied:

(1) $K_Y + B_Y \sim \rho^*(K_X + B)$;
(2) $K_Y + G_Y \sim 0$;
(3) $B_Y \leq \rho^*(B)$, $G_Y \leq \rho^*(G)$;
(4) $B_Y \cap G_Y = \varnothing$.

Lemma 1.5. *The separation exists and is unique.*

Proof. First, we shall show the existence. If $B \cap G = \varnothing$, then the identity mapping $X \to X$ is the separation. Assume that $B \cap G \neq \varnothing$. Let $\rho_1 : Y_1 \to X$ be the blowing-up at a point $x_1 \in B \cap G$ and let Γ be the exceptional divisor $\rho_1^{-1}(x_1)$. We consider divisors $B_{Y_1} := \rho_1^*(B) - \Gamma$ and $G_{Y_1} := \rho_1^*(G) - \Gamma$. Here, B_{Y_1} is the proper transform of B and $B \cdot G = B_{Y_1} \cdot G_{Y_1} + 1$. If $B_{Y_1} \cap G_{Y_1} = \varnothing$, then ρ_1 is the separation of (X, B, G). If $B_{Y_1} \cap G_{Y_1} \neq \varnothing$, then we blow up at a point $x_2 \in B_{Y_1} \cap G_{Y_1}$, and we define B_{Y_2} and G_{Y_2} similarly. By continuing this procedure, we finally get the separation.

Next, we shall show the uniqueness. Let (Y, B_Y, G_Y) be a separation of (X, B, G). If Γ is a ρ-exceptional curve, then $K_Y \cdot \Gamma = -B_Y \cdot \Gamma \leq 0$. If $K_Y \cdot \Gamma < 0$, then Γ is a (-1)-curve. Otherwise, Γ is a (-2)-curve. Let $\pi : Y \to V$ be the contraction of all the ρ-exceptional (-2)-curves. Then V has only rational double points as singularities, and

$$B_Y = \pi^*(B_V) \quad \text{and} \quad G_Y = \pi^*(G_V)$$

for effective Cartier divisors B_V and G_V on V, respectively. There is an effective Cartier divisor E on V such that

$$B_V = \tau^*(B) - E \quad \text{and} \quad G_V = \tau^*(G) - E,$$

for the induced morphism $\tau : V \to X$. Here, $-E$ is τ-ample and $B_V \cap G_V = \varnothing$. Hence, τ is the normalization of the blowing-up of X along the ideal $\mathcal{O}_X(-B) + \mathcal{O}_X(-G)$. Moreover, $\pi : Y \to V$ is the minimal desingularization. Therefore, $Y \to X$ is uniquely determined. □

Definition. Let (X, B, G) be a basic triplet and let r be a non-negative integer. The condition $\mathscr{C}_r$ for (X, B, G) is the collection of the following two conditions:

$\mathscr{C}_r$-1: $K_X + B$ is nef;
$\mathscr{C}_r$-2: $B \cdot G = r$.

If (X, B, G) satisfies the condition $\mathscr{C}_r$, then

$$B^2 = (K_X + B) \cdot B + G \cdot B = 2g(B) - 2 + r = 4 + r.$$

In particular $B \not\leq G$, since $B^2 > B \cdot G$. Note that the condition $\mathscr{C}_0$ implies the condition $\mathscr{C}$.

Lemma 1.6. *Let (X, B, G) be a basic triplet satisfying the condition $\mathscr{C}_r$.*

(1) *Suppose that $r > 0$. Let $\varphi : Y \to X$ be the blowing-up at a point $x \in B \cap G$, Γ the exceptional divisor $\varphi^{-1}(x)$, B_Y the proper transform of B, and $G_Y := \varphi^* G - \Gamma$. Then (Y, B_Y, G_Y) satisfies the condition $\mathscr{C}_{r-1}$.*

(2) *Suppose that there is a (-1)-curve Ξ with $B \cdot \Xi = 1$. Let $\phi : X \to Z$ be the blowing-down of Ξ, $B_Z := \phi(B)$, and $G_Z := \phi_* G$. Then (Z, B_Z, G_Z) satisfies the condition $\mathscr{C}_{r+1}$.*

Proof. (1) We infer that $B_Y \simeq B$ and that G_Y is a non-zero effective anti-canonical divisor of Y. Here, $K_Y + B_Y \sim \varphi^*(K_X + B)$ is nef and $B_Y \cdot G_Y = B \cdot G - 1$.

(2) We infer that $B \simeq B_Z$, $G \cdot \Xi = 1$, and that G_Z is a non-zero effective anti-canonical divisor. Here, $\phi^*(K_Z + B_Z) \sim K_X + B$ is nef and $B_Z \cdot G_Z = B \cdot G + 1$. □

Definition. A basic triplet (X, B, G) is called *minimal* if it satisfies the condition $\mathscr{C}_r$ for some r and $B \cdot \Gamma > 1$ for any (-1)-curve Γ on X.

Theorem 1.7. *Let (M, D, E) be a basic triplet satisfying the condition $\mathscr{C}$. Then one of the following four possibilities occurs*:

Type A: *$K_M + D$ is not nef*;

Type B: *(M, D, E) is the separation of a minimal basic triplet (X, B, G) in which $2K_X + B$ is nef*;

Type C: *(M, D, E) is the separation of a minimal basic triplet (X, B, G) in which X has a $\boldsymbol{P}^1$-bundle structure over a smooth curve and $(2K_X + B) \cdot \ell < 0$ for a fiber ℓ*;

Type D: *(M, D, E) is the separation of a minimal basic triplet (X, B, G) in which $X \simeq \boldsymbol{P}^2$ and* $\deg(2K_X + B) < 0$.

Proof. If (M, D, E) does not satisfy $\mathscr{C}_0$, then it is of Type A. Thus we assume that the triplet satisfies $\mathscr{C}_0$. By Lemma 1.6, we have a minimal basic triplet (X, B, G) whose separation is (M, D, E). Note that this (X, B, G) is not necessarily uniquely determined by (M, D, E). If $2K_X + B$ is nef, then (M, D, E) is of Type B. If $2K_X + B$ is not nef, then there is an extremal ray R such that $(2K_X + B) \cdot R < 0$ (cf. [**6**]). Now the contraction of R can not be birational, since $(2K_X + B) \cdot \Gamma \geq 0$ for any (-1)-curve Γ on X. Thus X has a $\boldsymbol{P}^1$-bundle structure over a smooth curve or $X \simeq \boldsymbol{P}^2$. □

§2. Examples.

We shall give examples of basic triplets (M, D, E) satisfying the condition $\mathscr{C}$ and examples of minimal basic triplets (X, B, G).

§2.1. Examples of Type A.

We take a hyperplane H in $\boldsymbol{P}^3$ and a smooth quartic curve C in $H \simeq \boldsymbol{P}^2$. For a point $v \notin H$, let $S := S_v$ be the union of all lines through v and a point of C. Then S

is a normal quartic surface and v is the unique singular point. Let $\sigma : M \to S$ be the blowing-up at v. Then σ is the minimal desingularization of S and M is isomorphic to the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}_C(\mathcal{O}_C \oplus \omega_C)$ over the curve C. In this case, σ^*H is a tautological divisor with respect to $\mathcal{O}_C \oplus \omega_C$, thus $K_M + \sigma^*H$ is not nef. Let C_0 be the minimal section of the $\boldsymbol{P}^1$-bundle. If we take a general member D of $|\sigma^*H|$, then the basic triplet $(M, D, 2C_0)$ does not satisfy the condition $\mathscr{C}_0$ but $\mathscr{C}$. Thus it is of Type A.

Next, we consider the defining equation of S. Let $\Phi_4(x, y, z) \in \boldsymbol{C}[x, y, z]$ be a homogeneous polynomial of degree 4 defining C in $\boldsymbol{P}^2 = \operatorname{Proj} \boldsymbol{C}[x, y, z]$. Then $S = S_v$ is defined as

$$\Phi_4(X_1, X_2, X_3) = 0$$

in $\boldsymbol{P}^3 = \operatorname{Proj} \boldsymbol{C}[X_0, X_1, X_2, X_3]$ in which v corresponds to the point $(1:0:0:0)$. The projection $\boldsymbol{P}^3 \cdots\to \boldsymbol{P}^2$ from v induces the rational map

$$S \overset{\sigma^{-1}}{\cdots\to} M = \boldsymbol{P}_C(\mathcal{O}_C \oplus \omega_C) \to C \subset \boldsymbol{P}^2.$$

§2.2. Examples of Type B.

§2.2.1. A generalized del Pezzo surface of degree two.

Let X be a generalized del Pezzo surface of degree 2. Then $|-K_X|$ has no base points and defines a generically finite morphism $\tau : X \to \boldsymbol{P}^2$ of degree 2.

LEMMA 2.1. *A general member of* $|-2K_X|$ *is a non-hyperelliptic curve of genus* 3.

PROOF. A general member B of $|-2K_X|$ is a smooth curve of genus 3. In view of the exact sequence

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(-K_X) \to \mathcal{O}_B(-K_X) \simeq \omega_B \to 0,$$

we infer that the restriction of τ to B is the canonical mapping of B. If $B \in |-2K_X|$ is a smooth hyperelliptic curve, then $\tau(B)$ is a smooth conic of $\boldsymbol{P}^2$ and $B = \tau^*(\tau(B))$. Now $h^0(\boldsymbol{P}^2, \mathcal{O}(2)) = 6$ and $h^0(X, -2K_X) = \chi(X, -2K_X) = 7$. Thus the pullback $H^0(\boldsymbol{P}^2, \mathcal{O}(2)) \to H^0(X, -2K_X)$ is not surjective. Therefore, a general member B is non-hyperelliptic. □

Let $B \in |-2K_X|$ be a non-hyperelliptic curve of genus 3 and let G be a member of $|-K_X|$. Then (X, B, G) satisfies the condition $\mathscr{C}_4$ and $2K_X + B \sim 0$. For the separation (M, D, E), M is a rational surface with the Picard number 12. In particular, $E^2 = -2$. The triplet (M, D, E) is called of Type B1.

We shall give a defining equation of S as follows: Let $\tau : X \to V \to \boldsymbol{P}^2$ be the Stein factorization of τ. Then V has only rational double points as singularities and $\tau_*\mathcal{O}_X \simeq \mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(-2)$. The $\mathcal{O}_{\boldsymbol{P}^2}$-algebra structure of $\tau_*\mathcal{O}_X$ is given by an element $\delta \in H^0(\boldsymbol{P}^2, \mathcal{O}(4))$ in such a way that

$$(u_1, v_1) \cdot (u_2, v_2) = (u_1u_2 + v_1v_2\delta,\ u_2v_1 + u_1v_2)$$

for local holomorphic sections u_1, u_2 of $\mathcal{O}_{\boldsymbol{P}^2}$ and v_1, v_2 of $\mathcal{O}_{\boldsymbol{P}^2}(-2)$. Let $\eta \in H^0(X, -2K_X)$ be an element corresponding to $(0, 1)$ under the isomorphism

$$H^0(X, -2K_X) \simeq H^0(\boldsymbol{P}^2, \mathcal{O}(2)) \oplus H^0(\boldsymbol{P}^2, \mathcal{O}).$$

Then $\eta^2 = \tau^*\delta$ in $H^0(X, -4K_X)$. The smooth curve B is defined as $\mathrm{div}(\eta + \tau^* q)$ for some $q \in H^0(\boldsymbol{P}^2, \mathcal{O}(2))$. The effective divisor G is defined as $\mathrm{div}(\tau^* l)$ for some $l \in H^0(\boldsymbol{P}^2, \mathcal{O}(1))$. For a suitable choice of homogeneous coordinate system $(x : y : z)$ of $\boldsymbol{P}^2$, we may assume that $l = x$. Then

$$\xi_0 = \eta + \tau^* q, \quad \xi_1 = \tau^*(x^2), \quad \xi_2 = \tau^*(xy), \quad \xi_3 = \tau^*(xz)$$

form a basis of the vector subspace $H^0(M, D) \subset H^0(X, -2K_X)$. We have the following relation:

$$\xi_0\xi_1 - q(\xi_1, \xi_2, \xi_3) = \eta\xi_1.$$

By taking square, we have

$$(\xi_0\xi_1 - q(\xi_1, \xi_2, \xi_3))^2 = \delta(\xi_1, \xi_2, \xi_3).$$

Therefore, S is defined in $\boldsymbol{P}^3 = \mathrm{Proj}\, \boldsymbol{C}[X_0, X_1, X_2, X_3]$ by

$$(X_0X_1 - q(X_1, X_2, X_3))^2 - \delta(X_1, X_2, X_3) = 0,$$

in which $\deg q = 2$ and $\deg \delta = 4$. The conditions required for q and δ are as follows:

(1) $\mathrm{div}(\delta)$ is a reduced divisor;

(2) the double-covering branched along $\mathrm{div}(\delta)$ has only rational double points as singularities;

(3) $\mathrm{div}(\delta - q^2)$ is a smooth curve.

The image $\sigma(E)$ of E under $\sigma : M \to S$ is just the point $(1 : 0 : 0 : 0)$. The projection $\boldsymbol{P}^3 \cdots\to \boldsymbol{P}^2$ from the singular point gives the composite

$$S \overset{\sigma^{-1}}{\cdots\to} M \to X \to \boldsymbol{P}^2.$$

§2.2.2. Blowing-up of a generalized del Pezzo surface of degree one at one point.

Let Y be a generalized del Pezzo surface of degree one and let G_Y be a member of $|-K_Y|$. It is well-known that $|-K_Y|$ has a unique base point b and the linear system induces a minimal elliptic fibration $\pi : Z \to \boldsymbol{P}^1$ from the blown-up Z of Y at b. Thus b is a smooth point of G_Y. Let $G_{Y,0}$ be the irreducible component of G_Y containing b. Here, we take a point $q \in G_{Y,0}$ such that

(1) q is a smooth point of G_Y,

(2) $\mathcal{O}_{G_Y}(b - q) \not\simeq \mathcal{O}_{G_Y}$ and $\mathcal{O}_{G_Y}(2b - 2q) \not\simeq \mathcal{O}_{G_Y}$.

There exists uniquely a point $q_1 \in G_{Y,0}$ satisfying $\mathcal{O}_{G_Y}(q_1) \simeq \mathcal{O}_{G_Y}(3b - 2q)$. Then $q_1 \neq b$ by the condition (2) above. Let $f : X \to Y$ be the blowing-up at q, $\Gamma = f^{-1}(q)$, and G the proper transform of G_Y in X. Then we have the following:

Lemma 2.2. *A general member of the linear system $|3G + \Gamma|$ is a smooth non-hyperelliptic curve of genus* 3. *In particular*, (X, B, G) *satisfies the condition $\mathscr{C}_1$ for a general member $B \in |3G + \Gamma|$.*

Proof. We consider the exact sequence:

$$0 \to \mathcal{O}_X(2G + \Gamma) \to \mathcal{O}_X(3G + \Gamma) \to \mathcal{O}_G(3G + \Gamma) \simeq \mathcal{O}_G(3b' - 2q') \to 0,$$

where $b' = f^{-1}(b)$ and $\{q'\} = G \cap \Gamma$. Note that b' and q' are contained in the proper transform G_0 of $G_{Y,0}$. There is a smooth point $q_1' \in G$ with $\mathcal{O}_G(3b' - 2q') \simeq \mathcal{O}_G(q_1')$, since $\deg \mathcal{O}_G(3b' - 2q') = 1$. Then $q_1' \in G_0$, $f(q_1') = q_1$, and $q_1' \in \mathrm{Bs}|3G + \Gamma|$. The divisor $3G + \Gamma$ is nef and big, since the restrictions of $3G + \Gamma$ to any component of G and Γ are nef and since $(3G + \Gamma)^2 = 5 > 0$. Hence, $H^i(X, 2G + \Gamma) = H^i(X, K_X + 3G + \Gamma) = 0$ for $i > 0$. Therefore $h^0(X, 2G + \Gamma) = 3$ and $h^0(X, 3G + \Gamma) = 4$. We can calculate $h^0(X, G) = h^0(X, 2G) = 1$ and $h^1(X, G) = h^1(X, 2G) = 0$ from the following three exact sequences:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(G) \to \mathcal{O}_G(G) \simeq \mathcal{O}_G(b' - q') \to 0,$$

$$0 \to \mathcal{O}_X(G) \to \mathcal{O}_X(2G) \to \mathcal{O}_G(2G) \simeq \mathcal{O}_G(2b' - 2q') \to 0,$$

$$0 \to \mathcal{O}_X(2G) \to \mathcal{O}_X(3G) \to \mathcal{O}_G(3G) \simeq \mathcal{O}_G(3b' - 3q') \to 0.$$

We consider the following two cases:

B2-1: $h^0(X, 3G) = 1$;

B2-2: $h^0(X, 3G) > 1$.

In Case B2-2, $3b' \sim 3q'$ and equivalently, $q_1' = q'$. Thus $h^0(X, 3G) = 2$. In Case B2-1, $3b' \not\sim 3q'$ and equivalently, $q_1' \neq q'$.

Claim 2.3. $\{q_1'\} = \mathrm{Bs}|3G + \Gamma|$. In particular, a general member of $|3G + \Gamma|$ is a smooth curve of genus 3.

Proof. *Case* B2-1: In view of the exact sequence

$$0 \to \mathcal{O}_X(3G) \to \mathcal{O}_X(3G + \Gamma) \to \mathcal{O}_\Gamma(3G + \Gamma) \simeq \mathcal{O}_{\boldsymbol{P}^1}(2) \to 0,$$

we infer that $H^0(X, 3G + \Gamma) \to H^0(\Gamma, \mathcal{O}_\Gamma(3G + \Gamma))$ is surjective. Therefore $\Gamma \cap \mathrm{Bs}|3G + \Gamma| = \emptyset$. Hence $\mathrm{Bs}|3G + \Gamma| = \{q_1'\}$.

Case B2-2: The image of

$$H^0(X, 3G + \Gamma) \to H^0(\Gamma, \mathcal{O}_\Gamma(3G + \Gamma)) \simeq \boldsymbol{C}^3$$

is contained in the two-dimensional subspace

$$H^0(\Gamma, \mathcal{O}_\Gamma(3G + \Gamma) \otimes \mathcal{O}_\Gamma(-q'))$$

since $q' = q_1'$. Hence $\mathrm{Bs}|3G + \Gamma| = \{q_1'\}$ by $h^0(X, 3G) = 2$. □

We shall show a general member B of $|3G + \Gamma|$ is non-hyperelliptic. Let $\rho : M \to X$ be the blowing-up at q_1'. Then ρ is the separation of (X, B, G). Let D and E be the proper transforms of B and G, respectively. Since $h^0(M, D) = h^0(X, B) = 4$, the homomorphism $H^0(X, D) \to H^0(D, D|_D) \simeq H^0(D, K_D)$ is surjective. Thus $\mathrm{Bs}|D| = \emptyset$. If the morphism $\Phi_{|D|} : M \to \boldsymbol{P}^3$ is a birational morphism onto its image, then $D \simeq B$ is a non-hyperelliptic curve. Let Ξ be the ρ-exceptional divisor $\rho^{-1}(q_1')$ and let Γ' be the proper transform of Γ.

Case B2-1: We have $D \sim 3E + \Gamma' + 2\Xi$. Thus $3E + \Gamma' + 2\Xi$ is the pullback of a hyperplane section. Now $\Phi_{|D|}$ maps E to a point and Ξ to a line of $\boldsymbol{P}^3$ isomorphically.

Since $H^0(M, D) \to H^0(\Gamma', D|_{\Gamma'}) \simeq H^0(\Gamma, B|_{\Gamma})$ is surjective, the restriction of $\Phi_{|D|}$ to Γ' is an isomorphism to a conic in $\boldsymbol{P}^3$. Therefore $\Phi_{|D|}$ is birational.

Case B2-2: We have $D \sim 3E + \Gamma' + 3\Xi$. Thus Γ' and Ξ are mapped to lines in $\boldsymbol{P}^3$ by $\Phi_{|D|}$. Therefore $\Phi_{|D|}$ is birational. $\square$

Therefore, (X, B, G) satisfies the condition $\mathscr{C}_1$ and M is a rational surface with the Picard number 11. In particular, $E^2 = -1$. The basic triplet (M, D, E) is called of Type B2; more precisely, of Type B2-1 or Type B2-2.

§2.2.3. Blowing-down from a double-covering over Σ_1.

Let $\Sigma := \Sigma_1$ be the Hirzebruch surface $\boldsymbol{P}_{\boldsymbol{P}^1}(\mathcal{O} \oplus \mathcal{O}(1))$. We denote the ruling by $p : \Sigma \to \boldsymbol{P}^1$ and a fiber of p by ℓ. Let Ξ be the unique (-1)-curve of Σ and let $\nu : \Sigma \to \boldsymbol{P}^2$ be the blowing-down of Ξ. We now fix a point $x_0 \in \Xi$ and we denote by ℓ_0 the fiber of p passing through x_0.

Lemma 2.4. *There is an effective divisor $\Delta \in |2\Xi + 6\ell|$ satisfying the following conditions*:

(b-1) *Δ is reduced and if $V \to \Sigma$ is the double covering branched just along Δ, then V has only rational double points as singularities.*

(b-2) $\Xi \not\subset \Delta$.

(b-3) $x_0 \in \Delta \cap \Xi$ *and* $\mathrm{mult}_{x_0}(\Delta|_{\Xi}) = 1$.

(b-4) $\ell_0 \not\subset \Delta$ *and* $\ell_0 \cap \Delta = \{x_0\}$.

Proof. Let $\mu_0 : S_0 \to \Sigma$ be the blowing-up at x_0 and let Γ_0 be the exceptional curve $\mu_0^{-1}(x_0)$. Let ℓ_0' be the proper transforms of ℓ_0. We next blow-up S_0 at $x_1 := \ell_0' \cap \Gamma_0$. Let $\mu_1 : S_1 \to S_0$ be the blowing-up and let ℓ_0'' be the proper transform of ℓ_0'. We look at the linear system $|\mu_1^*\mu_0^*(2\Xi + 5\ell) + \ell_0''|$. Let us consider the exact sequence:

$$0 \to \mathcal{O}(\mu_1^*\mu_0^*(2\Xi + 5\ell)) \to \mathcal{O}(\mu_1^*\mu_0^*(2\Xi + 5\ell) + \ell_0'') \to \mathcal{O}_{\ell_0''} \to 0.$$

Since $H^1(\Sigma, 2\Xi + 5\ell) = 0$ and $\mathrm{Bs}|2\Xi + 5\ell| = \emptyset$, we see that $\mathrm{Bs}|\mu_1^*\mu_0^*(2\Xi + 5\ell) + \ell_0''| = \emptyset$. Let Δ'' be a general member of the linear system $|\mu_1^*\mu_0^*(2\Xi + 5\ell) + \ell_0''|$ and let $\Delta := \mu_{0*}\mu_{1*}\Delta''$. Then Δ satisfies the conditions above. $\square$

We fix a divisor $\Delta \in |2\Xi + 6\ell|$ satisfying (b-1) to (b-4). Let $\lambda : Y \to \Sigma$ be the minimal desingularization of the double covering of Σ branched just along Δ. Then we have $K_Y \sim \lambda^*(K_\Sigma + \Xi + 3\ell) \sim \lambda^*(-\Xi)$. We set $G_Y := \lambda^*\Xi$. We infer that Y is a rational surface by $h^1(Y, \mathcal{O}) = h^1(\Sigma, \mathcal{O} \oplus \mathcal{O}(-\Xi - 3\ell)) = 0$. By the conditions (b-2) and (b-3), there is an irreducible component $G_{Y,0} \subset G_Y$ such that the induced morphism $G_{Y,0} \to \Xi$ is a double covering. Then other components of G_Y are contracted to points of Ξ by λ. Furthermore, λ is a finite morphism over an open neighborhood of x_0 and $\lambda^{-1}(x_0)$ consists of only one point, which we denote by $b' \in G_{Y,0}$. Moreover, we can write $\lambda^*\ell_0 = F_1 + F_2$ for (-1)-curves F_1 and F_2 such that $\{b'\} = F_1 \cap F_2$. Since $(\Xi + \ell)|_\Xi \sim 0$, we have $(\lambda^*\ell + G_Y)|_{G_Y} \sim 0$ and $\mathcal{O}_{G_Y}(G_Y) \simeq \lambda^*\mathcal{O}_\Xi(-x_0) \simeq \mathcal{O}_{G_Y}(-2b')$.

Let $\mu : Y \to M$ be the blowing-down of F_1. We set $E := \mu_* G_Y \sim -K_M$, $F := \mu_* F_2$, and $b := \mu(b')$. Then $\mu^*E = G_Y + F_1$ and $\lambda^*\ell \sim \mu^*F$. Hence, $(F + 2E)|_E \sim 0$.

LEMMA 2.5. *$|F+2E|$ is base point free and its general members are non-hyperelliptic curves of genus* 3.

PROOF. We consider the following two exact sequences:

$$0 \to \mathcal{O}(F) \to \mathcal{O}(F+E) \to \mathcal{O}_E(F+E) \simeq \mathcal{O}_E(b) \to 0,$$

$$0 \to \mathcal{O}(F+E) \to \mathcal{O}(F+2E) \to \mathcal{O}_E(F+2E) \simeq \mathcal{O}_E \to 0.$$

Since F is a fiber of the ruling $p \circ \lambda \circ \mu^{-1} : M \to \boldsymbol{P}^1$ and since b is a smooth point of the anti-canonical divisor E, we infer that $h^0(M, F+2E) = 4$ and $H^0(M, F+2E) \to H^0(E, \mathcal{O}_E)$ is surjective. Thus $\mathrm{Bs}|F+2E| = \emptyset$. Let D be a general member of the linear system. Then D is a smooth curve of genus 3. We have only to show that D is non-hyperelliptic.

We set $B_Y := \mu^* D$. Then $B_Y \simeq D$, since $b \notin D$. Here we have an isomorphism $H^0(Y, K_Y + B_Y) \simeq H^0(B_Y, K_{B_Y}) \simeq \boldsymbol{C}^{\oplus 3}$. Now $K_Y + B_Y \sim \lambda^*\ell + G_Y + 2F_1$. Since

$$H^0(Y, \lambda^*\ell + G_Y) \simeq H^0(Y, \lambda^*(\Xi + \ell)) \simeq H^0(\Sigma, \mathcal{O}(\Xi + \ell) \oplus \mathcal{O}(-2\ell)) \simeq \boldsymbol{C}^{\oplus 3},$$

we infer that $2F_1$ is the fixed part of $|K_Y + B_Y|$ and that $|K_Y + B_Y - 2F_1|$ is a base point free linear system inducing the morphism $\nu \circ \lambda : Y \to \boldsymbol{P}^2$. Thus its restriction $B_Y \to \boldsymbol{P}^2$ is the canonical map of B_Y. Let $\tau : Y \to Y$ be a generator of the Galois group of λ. If B_Y is hyperelliptic, then B_Y must be τ-invariant. However, $\tau_* B_Y \sim \tau_*(\lambda^*(\ell + 2\Xi) + 2F_1) \sim \lambda^*(\ell + 2\Xi) + 2F_2$. Since $2F_1 \not\sim 2F_2$, we infer that $B_Y \simeq D$ is non-hyperelliptic. □

Therefore, the triplet (M, D, E) is satisfying the condition $\mathscr{C}_0$ and M has the Picard number 11 in which $E^2 = -1$. The triplet (M, D, E) is called of Type B3.

§2.3. Examples of Type C.

§2.3.1. Minimal triplet (X, B, G) satisfying $\mathscr{C}_4$.

Let C be an elliptic curve with an ample divisor A of degree 2. Let X be the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(A))$ and let $p : X \to C$ be the structure morphism. Then $|-K_X|$ is non-empty. We take an effective divisor $G \sim -K_X$. Note that $h^0(G, \mathcal{O}_G) = 2$ by Proposition 1.1.

LEMMA 2.6. *A general member B of $|p^*A - K_X|$ is a smooth non-hyperelliptic curve of genus* 3.

PROOF. Let C_0 be the negative section of $p : X \to C$. Then $p^*A - K_X \sim 2(p^*A + C_0)$, $(p^*A + C_0)|_{C_0} \sim 0$, and $\mathrm{Bs}|p^*A + C_0| = \emptyset$. Thus $|p^*A - K_X|$ is also base point free. Hence, B is smooth with $B \cap C_0 = \emptyset$. Since $K_B \sim (K_X + B)|_B \sim p^*A|_B \sim (p^*A + C_0)|_B$, we have $g(B) = 3$ and an exact sequence:

$$0 \to \mathcal{O}_X(K_X + C_0) \to \mathcal{O}_X(p^*A + C_0) \to \mathcal{O}_B(K_B) \to 0.$$

This induces an isomorphism $H^0(X, p^*A + C_0) \simeq H^0(B, K_B)$, since $H^1(X, K_X + C_0) = 0$. Hence, for the morphism $\Phi := \Phi_{|p^*A + C_0|} : X \to \boldsymbol{P}^2$, the restriction $\Phi|_B : B \to \boldsymbol{P}^2$ is the canonical map. Here $B \sim p^*A - K_X \sim 2(p^*A + C_0) \sim \Phi^*\mathcal{O}(2)$. Hence, B is hyperelliptic if and only if B is the pullback of a conic by Φ. However, we have $h^0(\boldsymbol{P}^2, \mathcal{O}(2)) = 6$ and $h^0(X, B) = h^0(C, \mathcal{O} \oplus \mathcal{O}(A) \oplus \mathcal{O}(2A)) = 7$. Therefore, B is non-hyperelliptic. □

Thus we obtain a minimal triplet (X, B, G) with $B \sim p^*A - K_X$ and $G \sim -K_X$. Since $B \cdot G = 4$, the condition $\mathscr{C}_4$ is satisfied. Let (M, D, E) be the separation of (X, B, G). Then M is an elliptic ruled surface with Picard number 6. The basic triplet (M, D, E) is called of Type C1.

§2.3.2. Minimal triplet (X, B, G) satisfying $\mathscr{C}_2$.

Let C be an elliptic curve and let $\mathscr{E}$ be one of the following locally free sheaves:

C2-1: $\mathscr{E} = \mathscr{O}_C(q_1) \oplus \mathscr{O}_C(q_2)$ for two points $q_1 \neq q_2$ of C with $2q_1 \not\sim 2q_2$.

C2-2: There is a non-splitting exact sequence:

$$0 \to \mathscr{O}_C(q) \to \mathscr{E} \to \mathscr{O}_C(q) \to 0$$

for a point q of C.

Case C2-1: Let $p : X \to C$ be the $\boldsymbol{P}^1$-bundle associated with $\mathscr{E}$. For a point $q \in C$, we denote the fiber $p^{-1}(q)$ by ℓ_q. Then we have two sections C_1 and C_2 with $C_1 \sim H_{\mathscr{E}} - \ell_{q_1}$ and $C_2 \sim H_{\mathscr{E}} - \ell_{q_2}$, respectively. Here $H_{\mathscr{E}}$ denotes a tautological divisor with respect to $\mathscr{E}$. Then $C_1 \cap C_2 = \varnothing$. Furthermore, $|-K_X| = \{C_1 + C_2\}$, since $C_1|_{C_1} \sim q_2 - q_1$ and $C_2|_{C_2} \sim q_1 - q_2$. We set $G := C_1 + C_2$.

Lemma 2.7. *A general member B of $|H_{\mathscr{E}} - K_X|$ is a smooth non-hyperelliptic curve of genus* 3.

Proof. First, we shall prove that B is a smooth curve of genus 3. From the exact sequence

$$0 \to \mathscr{O}_{\mathscr{E}}(1) \to \mathscr{O}_X(H_{\mathscr{E}} - K_X) \to \mathscr{O}_{C_1}(2q_2 - q_1) \oplus \mathscr{O}_{C_2}(2q_1 - q_2) \to 0,$$

we infer that $h^0(X, H_{\mathscr{E}} - K_X) = 4$ and

$$\mathrm{Bs}|H_{\mathscr{E}} - K_X| = (\ell_{q_3} \cap C_1) \sqcup (\ell_{q_4} \cap C_2),$$

where q_3 and q_4 are points of C determined by $q_3 \sim 2q_2 - q_1$ and $q_4 \sim 2q_1 - q_2$, respectively. Therefore, $B \in |H_{\mathscr{E}} - K_X|$ must intersect C_1 and C_2, transversally. Thus B is smooth. The genus $g(B)$ is 3, since $(K_X + B) \cdot B = H_{\mathscr{E}} \cdot (H_{\mathscr{E}} - K_X) = 4$.

Next, we shall prove that B is non-hyperelliptic. Here, we note that $\{q_3, q_4\} \cap \{q_1, q_2\} = \varnothing$ by assumption. We consider the separation $\rho : (M, D, E) \to (X, B, G)$. Since B intersects C_1 and C_2 transversally at points $x_1 := \ell_{q_3} \cap C_1$ and $x_2 := \ell_{q_4} \cap C_2$, ρ is just the blowing-up at $\{x_1, x_2\}$. Let $\Gamma_i = \rho^{-1}(x_i)$ and let C_i' be the proper transform of C_i in M for $i = 1, 2$, respectively. Then $D = \rho^*B - \Gamma_1 - \Gamma_2$ and $E = \rho^*G - \Gamma_1 - \Gamma_2 = C_1' + C_2'$. The linear system $|D|$ is base point free by the proof of Proposition 1.4. Let Φ be the morphism $\Phi_{|D|} : M \to \boldsymbol{P}^3$. Suppose that $M \to \Phi(M)$ is not birational. Then this is a generically finite morphism of degree 2 and $\Phi(M)$ is a quadric surface in $\boldsymbol{P}^3$. Now $D \cap E = \varnothing$, $D \cdot \Gamma_1 = D \cdot \Gamma_2 = 1$, and $D \cdot \rho^*\ell_{q_1} = D \cdot \rho^*\ell_{q_2} = 3$. Thus $\Phi(C_1')$ is a point, $\Phi|_{\Gamma_1} : \Gamma_1 \to \Phi(\Gamma_1)$ is an isomorphism onto the line $\Phi(\Gamma_1)$, and $\rho^*\ell_{q_1} \to \Phi(\rho^*\ell_{q_1})$ is birational. This is a contradiction, since $2C_1' + \Gamma_1 + \rho^*\ell_{q_1} + C_2'$ is supposed to be the pullback of a conic of $\boldsymbol{P}^2$. Therefore, $\Phi : M \to \Phi(M)$ is birational and $D \simeq B$ is non-hyperelliptic. □

For B and G above, (X, B, G) is a minimal triplet satisfying the condition $\mathscr{C}_2$. The separation (M, D, E) is called of Type C2-1. Here, M is an elliptic ruled surface with Picard number 4.

Case C2-2: Let $p : X \to C$ be the $\boldsymbol{P}^1$-bundle associated with $\mathscr{E}$. Then we have a minimal section C_0 with $C_0 \sim H_{\mathscr{E}} - \ell_q$. Then $-K_X \sim 2C_0$. Moreover, $|-K_X| = \{2C_0\}$, since the exact sequence

$$0 \to \mathcal{O}(q) \to \mathscr{E} \to \mathcal{O}(q) \to 0$$

admits no splitting. We set $G := 2C_0$.

Lemma 2.8. *A general member B of $|H_{\mathscr{E}} - K_X|$ is a smooth non-hyperelliptic curve of genus* 3.

Proof. First, we show that B is a smooth curve of genus 3. Note that $H_{\mathscr{E}} - K_X \sim 3C_0 + \ell_q$. Let us consider the following exact sequences:

$$0 \to \mathcal{O}_X(2C_0 + \ell_q) \to \mathcal{O}_X(3C_0 + \ell_q) \to \mathcal{O}_{C_0}(q) \to 0,$$

$$0 \to \mathcal{O}_{\mathscr{E}}(1) \to \mathcal{O}_X(2C_0 + \ell_q) \to \mathcal{O}_{C_0}(q) \to 0,$$

$$0 \to \mathcal{O}_X(3C_0) \to \mathcal{O}_X(3C_0 + \ell_q) \to \mathcal{O}_{\ell_q}(3) \to 0.$$

Then $H^1(X, 2C_0 + \ell_q) = 0$, $h^0(X, 3C_0 + \ell_q) = 4$, and the image of $H^0(X, 3C_0 + \ell_q) \to H^0(\ell_q, \mathcal{O}(3))$ is a 3-dimensional subspace. Hence $\mathrm{Bs}|3C_0 + \ell_q|$ consists of only one point $b := \ell_q \cap C_0$. Furthermore, a general member $B \in |3C_0 + \ell_q|$ intersects C_0 and ℓ_q at b. This is because $B \cdot C_0 = 1$ and the image of $H^0(X, 3C_0 + \ell_q) \to H^0(\ell_q, \mathcal{O}(3))$ is just the subspace $H^0(\ell_q, \mathcal{O}(3C_0 + \ell_q|_{\ell_q}) \otimes \mathcal{O}(-b))$. Thus B is smooth. The genus $g(B)$ is 3, since $(K_X + B) \cdot B = (C_0 + \ell_q) \cdot (3C_0 + \ell_q) = 4$.

Next, we shall prove that B is non-hyperelliptic. Let $\rho_1 : X_1 \to X$ be the blowing-up at the point b and Γ_1 the exceptional divisor $\rho_1^{-1}(b)$. We set $G_1 := \rho_1^* G - \Gamma_1$. Let B_1, C_0', and ℓ_q' be the proper transforms of B, C_0, and ℓ_q, respectively. Then $B_1 \cdot G_1 = 1$. Thus there is a smooth point b_1 of G_1 with $\mathcal{O}_{G_1}(B_1) \simeq \mathcal{O}_{G_1}(b_1)$. Now $G_1 = 2C_0' + \Gamma_1$ and $B_1 \cdot C_0' = 0$. Therefore $b_1 \in \Gamma_1$. From the exact sequences

$$0 \to \mathcal{O}_{C_0'}(-\rho_1^* C_0) \simeq \mathcal{O}_{C_0'} \to \mathcal{O}_{2C_0'}(-\Gamma_1) \to \mathcal{O}_{C_0'}(-\Gamma_1) \to 0,$$

$$0 \to \rho_1^* \mathcal{O}_C(C_0 + \ell_q) \to \mathcal{O}_{X_1}(B_1 - \Gamma_1) \to \mathcal{O}_{2C_0'}(-\Gamma_1) \to 0,$$

we infer that $h^0(X_1, B_1 - \Gamma_1) = 3$ and hence the image of

$$H^0(X_1, B_1) \to H^0(\Gamma_1, B_1|_{\Gamma_1}) \simeq H^0(\Gamma_1, \mathcal{O}_{\Gamma_1}(b_1))$$

is one-dimensional. Thus $b_1 \in \mathrm{Bs}|B_1|$. Moreover, we have $b_1 \notin \ell_q'$, since $H^0(X_1, B_1) \to H^0(\ell_q', \mathcal{O}(2))$ is surjective. Let $\rho_2 : M \to X_1$ be the blowing-up at b_1 and Γ_2 the exceptional curve $\rho_2^{-1}(b_1)$. Let C_0'', ℓ_q'', and Γ_1' be the proper transforms of C_0', ℓ_q' and Γ_1, respectively. Then we get the separation $(M, D, E) \to (X, B, G)$, where D and E are the proper transforms of B_1 and G_1, respectively. Here, $E = 2C_0'' + \Gamma_1'$, $D \sim 3C_0'' + \ell_q'' + 3\Gamma_1' + 2\Gamma_2$, and $D \cdot \ell_q'' = 2$. We have $\mathrm{Bs}|D| = \varnothing$ since $h^0(M, D) = h^0(X, B) = 4$ and $H^0(M, D) \to H^0(D, D|_D) \simeq H^0(D, K_D)$ is surjective. Therefore, it is enough to show that the morphism $\Phi := \Phi_{|D|} : M \to \boldsymbol{P}^3$ is birational onto its image. The divisor $3C_0'' + \ell_q'' + 3\Gamma_1' + 2\Gamma_2$ is the pullback of an effective Cartier divisor of $\Phi(M)$, since it is linearly equivalent to D. All the components C_0'' and Γ_1' are contracted to points by Φ, since these are also components of E. The restriction $\Gamma_2 \to \Phi(\Gamma_2)$ is an isomorphism in which $\Phi(\Gamma_2)$ is a line of $\boldsymbol{P}^3$, since $D \cdot \Gamma_2 = 1$. The restriction $\Phi_{\ell_q''} : \ell_q'' \to \boldsymbol{P}^3$ is a closed embedding, since $H^0(M, D) \to H^0(\ell'', \mathcal{O}(2))$ is surjective. Therefore, Φ is birational onto its image. $\square$

For B and G above, (X, B, G) is a minimal triplet satisfying the condition $\mathscr{C}_2$. The separation (M, D, E) is called of Type C2-2. Here, M is an elliptic ruled surface with Picard number 4.

§2.4. Examples of Type D.

Let B and G be a smooth quartic curve and an effective divisor of degree 3 in $\boldsymbol{P}^2$, respectively. Then $(\boldsymbol{P}^2, B, G)$ is a minimal triplet satisfying the condition $\mathscr{C}_{12}$. The separation (M, D, E) is called of Type D. Here, M is a rational surface with Picard number 13. In particular, $E^2 = -3$.

Next, we consider the defining equation of $S \subset \boldsymbol{P}^3$. Let $\Phi_3(x, y, z) = 0$ and $\Phi_4(x, y, z) = 0$ be the defining equations of G and B, respectively, in $\boldsymbol{P}^2 = \operatorname{Proj} \boldsymbol{C}[x, y, z]$. Let $\rho : M \to \boldsymbol{P}^2$ be the separation. Then

$$\rho^* \Phi_3 = \varphi_3 e \quad \text{and} \quad \rho^* \Phi_4 = \varphi_4 e,$$

where $\varphi_3 \in H^0(M, E)$ is a defining equation of E, $\varphi_4 \in H^0(M, D)$ is a defining equation of D, and $e \in H^0(M, K_M - \rho^* K_{\boldsymbol{P}^2})$. The vector space $H^0(M, D)$ is spanned by

$$\xi_0 := \varphi_4, \quad \xi_1 := \varphi_3 \rho^* x, \quad \xi_2 := \varphi_3 \rho^* y, \quad \xi_3 := \varphi_3 \rho^* z.$$

We have a relation

$$\begin{aligned} \xi_0 \Phi_3(\xi_1, \xi_2, \xi_3) - \Phi_4(\xi_1, \xi_2, \xi_3) &= \varphi_4 \varphi_3^3 \rho^* \Phi_3 - \varphi_3^4 \rho^* \Phi_4 \\ &= \varphi_4 \varphi_3^4 e - \varphi_3^4 \varphi_4 e = 0. \end{aligned}$$

Therefore, $S \subset \boldsymbol{P}^3 = \operatorname{Proj} \boldsymbol{C}[X_0, X_1, X_2, X_3]$ is defined by

$$X_0 \Phi_3(X_1, X_2, X_3) = \Phi_4(X_1, X_2, X_3).$$

The image $\sigma(E)$ of E under $\sigma : M \to S$ consists of the point $(1 : 0 : 0 : 0)$. The rational map $S \cdots\to \boldsymbol{P}^2$ induced by the projection $\boldsymbol{P}^3 \cdots\to \boldsymbol{P}^2$ from the point is the birational map

$$S \overset{\sigma^{-1}}{\cdots\to} M \to X = \boldsymbol{P}^2.$$

§3. Theorem.

In what follows, we shall prove the following:

Main Theorem. *A normal quartic surface with irrational singularities is obtained from one of the examples of basic triplets in* §2.

§3.1. Proof in the case of Type A.

Proposition 3.1. *Let (M, D, E) be a basic triplet satisfying $\mathscr{C}$ such that $K_M + D$ is not nef. Then M is isomorphic to the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}_D(\mathcal{O}_D \oplus \omega_D)$ and $E = 2C_0$ for the negative section C_0 of $M \to D$. Moreover, the corresponding quartic surface S is a cone over D.*

Proof. By the cone theorem [6], there is an extremal curve Γ such that $(K_M + D) \cdot \Gamma < 0$. If the contraction morphism of Γ is birational, then Γ is a (-1)-curve with $D \cdot \Gamma = 0$; it contradicts the condition $\mathscr{C}$-1. On the other hand, M is not isomorphic to $\boldsymbol{P}^2$, since M is a desingularization of a normal quartic surface S. Therefore, the contraction morphism $p : M \to C$ of Γ is a $\boldsymbol{P}^1$-bundle structure over a curve C. Then $D \cdot \Gamma = D \cdot \ell = 1$ for a fiber ℓ of p. Therefore $D \simeq C$ and $M \simeq \boldsymbol{P}_C(\mathscr{E})$ for the locally free sheaf $\mathscr{E} := p_*\mathcal{O}_M(D)$. In view of the exact sequence

$$0 \to \mathcal{O}_M(K_M + D) \to \mathcal{O}_M(D) \to \mathcal{O}_E \to 0,$$

we have an isomorphism $\mathscr{E} \simeq p_*\mathcal{O}_E$. Thus $\mathscr{E} \simeq \mathcal{O}_C \oplus \omega_C$ by Proposition 1.1. Let C_0 be the negative section of p. Then $C_0 \sim D - p^*K_C$ and $E = 2C_0$, since $D \cap E = \varnothing$. Therefore the morphism $\Phi_{|D|} : M \to \boldsymbol{P}^3$ maps E to a point v and a fiber ℓ of p to a line of $\boldsymbol{P}^3$. Hence the image S is the join of v and the quartic curve $\Phi_{|D|}(D)$. Thus S is a cone over D. □

§3.2. Proof in the case of Type B.

Suppose that (X, B, G) satisfies $\mathscr{C}_r$ and $2K_X + B$ is nef. Then $(2K_X + B) \cdot G = -2K_X^2 + r \geq 0$ and $(2K_X + B)^2 = 4K_X^2 - 3r + 4 \geq 0$. Therefore

$$3r - 4 \leq 4K_X^2 \leq 2r.$$

Hence $r \leq 4$ and $-1 \leq K_X^2 \leq 2$.

Lemma 3.2. *(X, B, G) satisfies one of the following conditions:*
B1: *$r = 4$ and X is a generalized del Pezzo surface of degree 2 with $B \sim -2K_X$;*
B2: *$r = 1$ and X is a rational surface with $K_X^2 = 0$;*
B3: *$r = 0$ and X is a rational surface with $K_X^2 = -1$.*

Proof. Assume that $r = 3$. Then $5 \leq 4K_X^2 \leq 6$. This is a contradiction. Next assume that $r = 2$. Then $K_X^2 = 1$, $(2K_X + B) \cdot G = 0$, and $(2K_X + B)^2 = 2$. This contradicts the Hodge index theorem. Hence $r = 0, 1$, or 4.

Case $r = 4$: Now $K_X^2 = 2$. Since $G^2 = 2 > 0$ and $(2K_X + B) \cdot G = (2K_X + B)^2 = 0$, we infer that $2K_X + B$ is numerically trivial by the Hodge index theorem. In particular, $-K_X$ is nef and big. Hence X is a generalized del Pezzo surface of degree 2. Since X is rational, we have $B \sim -2K_X$.

Case $r = 1$: Now $K_X^2 = 0$. Furthermore, $(2K_X + B) \cdot G = (2K_X + B)^2 = 1$. Thus we have an irreducible component G_0 of G such that $B \cdot G_0 = 1$ and $B \cap G_1 = \varnothing$ for the effective divisor $G_1 := G - G_0$. Note that G_1 is not necessarily a non-zero divisor. The inequality $(2K_X + B) \cdot G_0 \geq 0$ implies $K_X \cdot G_0 \geq 0$ and hence $(2K_X + B) \cdot G_0 \geq 1$. Another inequality $(2K_X + B) \cdot G_1 \geq 0$ implies $(2K_X + B) \cdot G_0 = 1$, $(2K_X + B) \cdot G_1 = 0$, and $K_X \cdot G_0 = K_X \cdot G_1 = 0$. By the Hodge index theorem, every component of G_1 is a (-2)-curve. Now $G_0^2 = -G_1 \cdot G_0 \le 0$.

If $G_0^2 < 0$, then G_0 is also a (-2)-curve and hence every component of G is a rational curve. Thus X is rational by Lemma 1.3.

If $G_0^2 = 0$, then $G_1^2 = 0$ and hence $G_1 = 0$ by the Hodge index theorem. Thus $G = G_0$ is an irreducible and reduced anti-canonical divisor. Therefore, X is rational by Corollary 1.2.

Case $r = 0$: Now $K_X^2 = 0$ or -1. If $K_X^2 = 0$, then $(2K_X + B)^2 = 4 > 0$ and $(2K_X + B) \cdot G = 0$. Thus $G = 0$ by the Hodge index theorem. This is a contradiction. Therefore $K_X^2 = -1$. The equality $B \cdot G = r = 0$ implies that $K_X \cdot \Gamma \ge 0$ for any component Γ of G. Thus, there is an irreducible component G_0 of G such that $K_X \cdot G_0 = 1$ and $K_X \cdot G_1 = 0$ for the effective divisor $G_1 := G - G_0$. We infer that the intersection matrix of the prime components of G is negative definite by applying the Hodge index theorem to $B \cdot G = 0$ and $B^2 = 4$. If $G_1 \ne 0$, then any component of G_1 is a (-2)-curve. On the other hand, $p_a(G_0) \le 1$ by $(K_X + G_0) \cdot G_0 < 1$.

If $p_a(G_0) = 0$, then X is a rational surface by Lemma 1.3.

If $p_a(G_0) = 1$, then $G_0^2 = -1$, $G_0 \cdot G_1 = 0$, and $G_1^2 = 0$. Therefore $G_1 = 0$ and $G = G_0$ is an irreducible and reduced anti-canonical divisor. Thus X is rational by Corollary 1.2. □

§3.2.1. Proof in Case B1.

In this case, X is a generalized del Pezzo surface of degree 2, B is a member of $|-2K_X|$. Hence (X, B, G) is obtained as Type B1 in §2.

§3.2.2. Proof in Case B2.

In this case, X is a rational surface with $K_X^2 = 0$, $r = B \cdot G = 1$ and $B^2 = 5$. Since $2K_X + B$ is nef and big, $h^0(X, 3K_X + B) = \chi(X, 3K_X + B) = 1$. Let Γ be the unique member of $|3K_X + B|$.

Lemma 3.3. *Γ is a (-1)-curve.*

Proof. We have $K_X \cdot \Gamma = -1$, $\Gamma^2 = -1$, and $(K_X + B) \cdot \Gamma = 1$. We can take a prime component Γ_0 of Γ with $(K_X + B) \cdot \Gamma_0 = 1$. Then $(K_X + B) \cdot (\Gamma - \Gamma_0) = 0$ and $(2K_X + B) \cdot (\Gamma - \Gamma_0) \ge 0$. Hence $K_X \cdot (\Gamma - \Gamma_0) \ge 0$, thus $K_X \cdot \Gamma_0 \le -1$. By $(2K_X + B) \cdot \Gamma_0 \ge 0$, we have $K_X \cdot \Gamma_0 = -1$. Therefore, $(2K_X + B) \cdot \Gamma_0 = 0$ and Γ_0 is a (-1)-curve by the Hodge index theorem. Moreover, $(3K_X + B) \cdot \Gamma_0 = \Gamma \cdot \Gamma_0 = -1$. This implies that $(\Gamma - \Gamma_0)^2 = 0$. Thus $\Gamma = \Gamma_0$ by the Hodge index theorem. □

Let $f : X \to Y$ be the contraction of Γ. Then $B_Y := f_* B$ has a singularity at $q := f(\Gamma)$ with $\mathrm{mult}_q B_Y = 2$. The push-forward $G_Y := f_* G$ is an anti-canonical divisor and $3K_Y + B_Y \sim f_* \Gamma = 0$. Hence Y is a generalized del Pezzo surface of degree 1. Therefore $\dim|-K_Y| = 1$ and $\mathrm{Bs}|-K_Y|$ consists of a unique point b. Let $g : Z \to Y$ be the blowing-up at b and let Ξ be the exceptional curve $g^{-1}(b)$. Then $|-K_Z|$ is base point free and we have an elliptic fibration $\pi := \Phi_{|-K_Z|} : Z \to \boldsymbol{P}^1$ in which Ξ is a section of π.

Lemma 3.4. *There is a component $G_{Y,0}$ of G_Y such that*
(1) $\mathrm{mult}_{G_{Y,0}} G_Y = 1$,
(2) *b and q are not contained in the divisor $G_Y - G_{Y,0}$.*
Furthermore, b is not contained in B_Y.

Proof. Let G_0 be a component of G with $B \cdot G_0 = 1$ and let $G_1 := G - G_0$. Then $2K_X \cdot G_0 \geq -B \cdot G_0 = -1$. Hence $K_X \cdot G_0 \geq 0$. Thus $(2K_X + B) \cdot G_0 = 1$ and $(2K_X + B) \cdot G_1 = 0$. Therefore $K_X \cdot G_0 = K_X \cdot G_1 = 0$. Since $(3K_X + B) \cdot G_0 = \Gamma \cdot G_0 = 1$ and $\Gamma \cdot G_1 = 0$, the push-forward $G_{Y,0} := f_* G_0$ is the unique component of G_Y containing q, and q is a smooth point of G_Y. On the other hand, $K_Y \cdot f_* G_1 = 0$, since $K_X \cdot G_1 = 0$ and G_1 is away from the (-1)-curve Γ. In particular, a general member of $|-K_Y|$ does not intersect $f_* G_1$. Thus the base point b is not contained in $f_* G_1$ but $G_{Y,0}$. We shall show that $b \notin B_Y$. If $b \in B_Y$, then the proper transform B_Z of B_Y in Z is linearly equivalent to $g^* B_Y - m\Xi$ for some $m \geq 1$. Thus

$$(-K_Z) \cdot B_Z = (-K_Y) \cdot B_Y - m = 3 - m \leq 2.$$

This implies that π induces a double-covering $B_Z \to \boldsymbol{P}^1$. This contradicts the assumption: B is non-hyperelliptic. □

Lemma 3.5. *Let q', q'_1, and b' be the points of X defined by*

$$\{q'\} = G \cap \Gamma, \quad \{q'_1\} = B \cap G, \quad \textit{and} \quad \{b'\} = f^{-1}(b),$$

respectively. Then the following properties hold:
(1) $\mathcal{O}_G(G) \simeq \mathcal{O}_G(b' - q')$;
(2) $\mathcal{O}_G(G) \not\simeq \mathcal{O}_G$ *and* $\mathcal{O}_G(2G) \not\simeq \mathcal{O}_G$;
(3) $\mathcal{O}_G(3b') \simeq \mathcal{O}_G(2q' + q'_1)$.

Proof. Let G_Z be the proper transform of G_Y in Z: $G_Z = g^* G_Y - \Xi$. Then G_Z is a fiber of π. Thus $\mathcal{O}_{G_Z}(G_Z) \simeq \mathcal{O}_{G_Z}$. Hence $\mathcal{O}_{G_Y}(G_Y) \simeq \mathcal{O}_{G_Y}(b)$. Since $G \sim f^* G_Y - \Gamma$, the isomorphism $\mathcal{O}_G(G) \simeq \mathcal{O}_G(b' - q')$ in (1) follows. The linear equivalences $B_Y \sim 3G_Y$ and $B \sim f^* B_Y - 2\Gamma$ imply the isomorphism $\mathcal{O}_G(B) \simeq \mathcal{O}_G(q'_1) \simeq \mathcal{O}_G(3b' - 2q')$. Thus (3) follows. Since the dualizing sheaf of G is trivial, $h^0(G, \mathcal{O}_G(b)) = \chi(G, \mathcal{O}_G(b)) = 1$. Hence $b \neq q$ implies $\mathcal{O}_G(G) \not\simeq \mathcal{O}_G$. If $\mathcal{O}_G(2G) \simeq \mathcal{O}_G$, then $b' = q'_1$ by (3). However, $b' \neq q'_1$ since $b = f(b') \notin B_Y$ by Lemma 3.4. Thus (2) follows. □

Therefore, the minimal triplet (X, B, G) is constructed as Type B2 in §2.

§3.2.3. Proof in Case B3.

Lemma 3.6. *The linear system $|2K_X + B|$ is base point free and it defines a morphism $\Phi : X \to \boldsymbol{P}^1$ whose general fibers are rational curves.*

Proof. We have $h^0(X, 2K_X + B) = \chi(X, 2K_X + B) = 2$, since $K_X + B$ is nef and big. Thus $\Phi = \Phi_{|2K_X + B|}$ is a rational map into $\boldsymbol{P}^1$. Let $\nu : X' \to X$ be a proper birational morphism such that the composite $\Phi \circ \nu : X' \to X \cdots \to \boldsymbol{P}^1$ is a morphism. Then $\nu^*(2K_X + B) \sim F + N$ for a fiber F of $\Phi \circ \nu$ and an effective divisor N. The inequality $0 = (2K_X + B)^2 \geq (F + N) \cdot F$ implies $N \cdot F = 0$. Hence $\Phi : X \to \boldsymbol{P}^1$ is a morphism. Thus we may write $2K_X + B \sim F + N$. Since X is rational, a general fiber

F of Φ is irreducible. Moreover, $F \simeq \boldsymbol{P}^1$ by $(2K_X + B) \cdot F = 0$ and $B \cdot F > 0$. Thus $B \cdot F = 4$, $B \cdot N = 0$, and $N^2 = 0$. Therefore, $N = 0$ by the Hodge index theorem and hence $2K_X + B \sim F$. □

Therefore $B \sim F - 2K_X \sim F + 2G$ for a fiber F of $\Phi : X \to \boldsymbol{P}^1$. Since $B \cap G = \emptyset$, we have $\mathcal{O}_G(F + 2G) \simeq \mathcal{O}_G$.

Claim 3.7. There is a non-singular point b of G such that $\mathcal{O}_G(-G) \simeq \mathcal{O}_G(b)$.

Proof. If G is irreducible and reduced, then $H^0(G, -G|_G) \simeq \boldsymbol{C}$ and its non-zero section defines such a point b. Thus we may assume G is reducible or non-reduced. Note that $-2G|_G \sim F|_G$ is nef on G. Thus there exist an irreducible component $G_0 \subset G$ and an non-zero effective divisor Θ such that

(1) $G = G_0 + \Theta$ with $G_0 \not\subset \Theta$,
(2) $-G \cdot G_0 = 1$,
(3) $G \cdot \Theta_i = 0$ for any component $\Theta_i \subset \Theta$.

We have $h^0(G, \mathcal{O}_G) = 1$ by $h^1(X, -G) = q(X) = 0$. Thus G is connected. Note that G is contractible by $G \cdot B = 0$, where $B^2 = 4 > 0$. It follows that all Θ_i are (-2)-curves and $(K_X + G_0) \cdot G_0 = -\Theta \cdot G_0 < 0$ implies that G_0 is a smooth rational curve with $G_0^2 = -3$ and $\Theta \cdot G_0 = 2$. We have $\mathcal{O}_\Theta(K_X) \simeq \mathcal{O}_\Theta$ since the contraction of Θ gives a minimal resolution of rational double points. The exact sequence

$$0 \to \mathcal{O}_\Theta(-G_0) \to \mathcal{O}_G \to \mathcal{O}_{G_0} \to 0,$$

induces $h^1(\mathcal{O}_\Theta(-G_0)) = h^1(\mathcal{O}_G) = 1$ since $H^0(G, \mathcal{O}_G) \to H^0(G_0, \mathcal{O}_{G_0})$ is surjective. Thus $h^0(\Theta, \mathcal{O}_\Theta) = h^0(\Theta, \omega_\Theta(G_0)) = h^1(\mathcal{O}_\Theta(-G_0)) = 1$ by duality. From another exact sequence

$$0 \to \mathcal{O}_{G_0}(-G - \Theta) \simeq \mathcal{O}_{\boldsymbol{P}^1}(-1) \to \mathcal{O}_G(-G) \to \mathcal{O}_\Theta(-G) \simeq \mathcal{O}_\Theta \to 0,$$

we infer that $H^0(G, \mathcal{O}_G(-G)) \simeq \boldsymbol{C}$ and its non-zero section does not vanish on Θ. Thus $\mathcal{O}_G(-G) \simeq \mathcal{O}_G(b)$ for some $b \in G_0 \backslash \Theta$. □

We consider the following exact sequence:

$$0 \to \mathcal{O}_X(F) \to \mathcal{O}_X(F + G) \to \mathcal{O}_G(F + G) \simeq \mathcal{O}_G(b) \to 0.$$

Then $\mathrm{Bs}|F + G| = \mathrm{Bs}|(F + G)|_G| = \{b\}$. Thus a general member of $|F + G|$ is smooth. Let $\mu : Y \to X$ be the blowing-up at b and let Λ be the exceptional divisor $\mu^{-1}(b)$. Then $\mu^* G = G_Y + \Lambda$ for the proper transform G_Y of G. We set $F_Y := \mu^* F$. Then $F_Y + G_Y \sim \mu^*(F + G) - \Lambda$ and $\mathcal{O}_{G_Y}(F_Y + G_Y) \simeq \mathcal{O}_{G_Y}$. Thus $\mathrm{Bs}|F_Y + G_Y| = \emptyset$ by the exact sequence:

$$0 \to \mathcal{O}_Y(F_Y) \to \mathcal{O}_Y(F_Y + G_Y) \to \mathcal{O}_{G_Y} \to 0.$$

Let f be the morphism $\Phi_{|F_Y + G_Y|} : Y \to \boldsymbol{P}^2$. Then f is a generically finite surjective morphism of degree 2, since $(F_Y + G_Y)^2 = 2$.

Lemma 3.8. *Let $\tau : \Sigma \to \boldsymbol{P}^2$ be the blowing-up at the point $f(G_Y)$ and let Ξ be the exceptional curve. Let ℓ be a fiber of the ruling $p : \Sigma \to \boldsymbol{P}^1$ of the Hirzebruch surface $\Sigma \simeq \Sigma_1$.*

(1) *There is a generically finite morphism* $\lambda : Y \to \Sigma$ *such that* $f = \tau \circ \lambda$ *and* $G_Y = \lambda^* \Xi$.

Let $Y \to V \to \Sigma$ *be the Stein factorization of* λ.

(2) V *has only rational double points as singularities.*

(3) $Y \to V$ *is the minimal desingularization of* V.

(4) V *is the double-covering of* Σ *branched along a reduced divisor* $\Delta \sim 2\Xi + 6\ell$.

Proof. (1) Let π be the composite $\Phi \circ \mu : Y \to X \to \boldsymbol{P}^1$. Then $\mathscr{E} := \pi_* \mathcal{O}_Y(G_Y)$ is a locally free sheaf of $\boldsymbol{P}^1$ of rank 3 and there is the following exact sequence on $\boldsymbol{P}^1$:

$$0 \to \mathcal{O}_{\boldsymbol{P}^1}(1) \to \mathscr{E} \otimes \mathcal{O}_{\boldsymbol{P}^1}(1) \to \pi_* \mathcal{O}_{G_Y} \to 0.$$

Since $h^0(\mathcal{O}_{G_Y}) = 1$, by pulling back the injection $\mathcal{O}_{\boldsymbol{P}^1} \hookrightarrow \pi_* \mathcal{O}_{G_Y}$, we have a subsheaf $\mathscr{F}$ of $\mathscr{E}$ and an exact sequence

$$0 \to \mathcal{O}_{\boldsymbol{P}^1}(1) \to \mathscr{F} \otimes \mathcal{O}_{\boldsymbol{P}^1}(1) \to \mathcal{O}_{\boldsymbol{P}^1} \to 0.$$

Furthermore, we have a surjection $\pi^* \mathscr{F} \to \mathcal{O}_Y(G_Y)$, since $\mathrm{Bs}|F_Y + G_Y| = \varnothing$. Therefore, we have a morphism $\lambda : Y \to \Sigma \simeq \boldsymbol{P}_{\boldsymbol{P}^1}(\mathscr{F})$ over $\boldsymbol{P}^1$ such that $\lambda^* \ell \sim F_Y$ and $\lambda^* \Xi = G_Y$. In particular, $f = \tau \circ \lambda$.

(2)–(4) We have $K_Y - \lambda^* K_\Sigma \sim \lambda^*(\Xi + 3\ell)$ by (1). Therefore, $V \to \Sigma$ is the double-covering branched along a reduced divisor $\Delta \sim 2\Xi + 6\ell$. Moreover, V has only rational double points and $Y \to \Sigma$ is the minimal desingularization, since K_Y is relatively trivial over V. □

The point $b' = G_Y \cap \Lambda$ is a smooth point of G_Y. Thus b' is contained in a unique component $G_{Y,0}$ of G_Y. We have an isomorphism $\mathcal{O}_{G_Y}(F_Y) \simeq \mathcal{O}_{G_Y}(-G_Y) \simeq \mathcal{O}_{G_Y}(2b')$ by $G_Y \sim \mu^* G - \Lambda$ and $\mathcal{O}_G(G) \simeq \mathcal{O}_G(-b)$. Thus λ induces a double-covering $G_{Y,0} \to \Xi$ and contracts the other components of G_Y to points of Ξ. We set $x_0 := \lambda(b') \in \Xi$. Then λ is a finite morphism over a neighborhood of x_0, and x_0 is contained in the branch locus. Hence x_0 is a smooth point of Δ. Let ℓ_0 be the fiber of $p : \Sigma \to \boldsymbol{P}^1$ passing through x_0. Note that $\Lambda \leq \lambda^* \ell_0$. In particular, $\lambda^* \ell_0$ is reducible. Since $\Lambda \cdot \lambda^* \Xi = -\Lambda \cdot K_Y = 1$, we infer that $\Lambda \to \ell_0$ is an isomorphism.

Lemma 3.9. ℓ_0 *is not a component of* Δ.

Proof. Assume the contrary. Then $\lambda^* \ell_0 = 2\Lambda + J$ for a non-zero effective divisor J. Here any component Γ of J is a (-2)-curve contracted to a point by λ, since $\Gamma \cdot K_Y = -\Gamma \cdot \lambda^* \Xi = 0$. Now $J \cdot \Lambda = \lambda^* \ell \cdot \Lambda - 2\Lambda^2 = 2$. If $\Gamma \cdot \Lambda = 2$, then $\mu^*(\mu_* \Gamma) = \Gamma + 2\Lambda$ and thus $(\mu_* \Gamma)^2 = 2 > 0$. This is a contradiction, since $\mu_* \Gamma$ is contained in a fiber of $\Phi : X \to \boldsymbol{P}^1$. Hence we have $\Gamma \cdot \Lambda \leq 1$. Let Γ_1 be a component of J with $\Gamma_1 \cdot \Lambda = 1$ and let $C_1 := \mu_* \Gamma_1$. Then C_1 is a (-1)-curve on X. By Proposition 1.4, $\mathrm{Bs}|B| = \mathrm{Bs}|F + 2G| = \varnothing$ and $\Phi_{|B|}$ is a birational morphism into a normal quartic surface in $\boldsymbol{P}^3$. Now $B \cdot C_1 = (F + 2G) \cdot C_1 = 2$. Hence $\Phi_{|B|}(C_1)$ is a conic of $\boldsymbol{P}^3$. On the other hand, we have $H^0(X, F + G) \simeq H^0(X, B - G) \simeq \boldsymbol{C}^{\oplus 3}$. Thus the rational map $\Phi_{|F+G|} : X \cdots\to \boldsymbol{P}^2$ is the composite of $\Phi_{|B|} : X \to \boldsymbol{P}^3$ and the projection $\boldsymbol{P}^3 \cdots\to \boldsymbol{P}^2$ from the point $\Phi_{|B|}(G)$. The image of C_1 under $\Phi_{|F+G|} : X \to \boldsymbol{P}^2$ is the point $f(\Gamma_1)$ of $\boldsymbol{P}^2$. Therefore $\Phi_{|B|}(C_1)$ is a line of $\boldsymbol{P}^3$. This is a contradiction. □

The divisor $\Delta|_{\ell_0}$ on ℓ_0 is $2x_0$, since $\ell_0 \cdot \Delta = 2$. In other words, Δ intersects ℓ_0 only at x_0 and the intersection is tangential. Hence, Δ and ℓ_0 satisfy the conditions (b-1) to (b-4) of Lemma 2.4. Thus the triplet (X, B, G) is obtained as Type B3 in §2.

§3.3. Proof in the case of Type C.

Let (X, B, G) be a minimal basic triplet satisfying $\mathscr{C}_r$ such that X has a $\boldsymbol{P}^1$-bundle structure $p : X \to C$ in which $-(2K_X + B)$ is relatively ample. Let us denote a fiber of p by ℓ. Since $K_X + B$ is nef, we have $(K_X + B)^2 \geq 0$ and $(K_X + B) \cdot G \geq 0$. These imply

$$r - 4 \leq K_X^2 \leq r.$$

Furthermore, $(K_X + B) \cdot \ell \geq 0$ and $(2K_X + B) \cdot \ell < 0$. Hence one of the following two cases occurs:

C1: $B \cdot \ell = 2$;

C2: $B \cdot \ell = 3$.

§3.3.1. Proof in Case C1.

In this case, we have $(K_X + B) \cdot \ell = 0$. Thus there is a divisor A on C such that $K_X + B \sim p^*A$. Then $(K_X + B)^2 = 0$ and $K_X^2 = r - 4$ hold. Hence, $(K_X + B) \cdot G = r - K_X^2 = 4$. This implies $\deg A = 2$.

Now $p : B \to C$ is a double-covering. Hence C is not rational since B is non-hyperelliptic. We have $K_X^2 = 8(1 - g(C))$, since $p : X \to C$ is a $\boldsymbol{P}^1$-bundle. Thus $K_X^2 = r - 4 \geq -4$ implies that $g(C) = 1$ and $r = 4$. In particular, C is an elliptic curve.

We may assume that X is isomorphic to $\boldsymbol{P}_C(\mathscr{E})$ for one of the following locally free sheaves $\mathscr{E}$ of rank 2:

(α): $\mathscr{E} \simeq \mathscr{O}_C \oplus \mathscr{L}$ for an invertible sheaf $\mathscr{L}$ of $\deg \mathscr{L} \leq 0$;

(β): $\mathscr{E}$ has a non-splitting exact sequence:

$$0 \to \mathscr{O}_C \to \mathscr{E} \to \mathscr{O}_C \to 0;$$

(γ): $\mathscr{E}$ has a non-splitting exact sequence:

$$0 \to \mathscr{O}_C \to \mathscr{E} \to \mathscr{O}_C(x) \to 0$$

for a point $x \in C$.

The case (γ) does not occur by the following:

Lemma 3.10. *In the case* (γ), $|-K_X| = \varnothing$.

Proof. Let G be a member of $|-K_X|$. Then $G = G_1 + G_2$ for a horizontal prime divisor G_1 and a non-zero effective divisor G_2 by Corollary 1.2. Then G_1 and G_2 are nef, since $\mathscr{E}$ is stable. Thus we have $G_1^2 = G_2^2 = G_1 \cdot G_2 = 0$ from $K_X^2 = 0$. The divisor G_1 is not a section, since $\mathscr{E}$ is stable. Hence $G_1 \to C$ is a double-covering and G_2 is contained in fibers. Since some ample divisor of X is written as a combination of G_1 and G_2, we infer that $G_2 = 0$ by the Hodge index theorem. This is a contradiction. □

Case (β): We have a unique member C_0 of the linear system $|H_{\mathscr{E}}|$ which corresponds to the injection $\mathscr{O}_C \to \mathscr{E}$. Therefore $K_X \sim -2C_0$. Since the exact sequence

$$0 \to \mathscr{E} \to \mathrm{Sym}^2(\mathscr{E}) \to \mathscr{O}_C \to 0$$

does not split, $2C_0$ is a unique member of $|-K_X|$. Thus $G = 2C_0$. Now B is a member of $|p^*A - K_X|$. Hence $B|_{C_0} \sim p^*A|_{C_0}$. Let us take a point $x \in C_0 \cap B$ and let $\mu : X' \to X$ be the blowing-up at x. Let ℓ' be the proper transform of the fiber $p^{-1}(p(x))$, B' the proper transform of B, Ξ the μ-exceptional curve $\mu^{-1}(x)$, and $G' := \mu^*G - \Xi$. Then the basic triplet (X', B', G') satisfies the condition $\mathscr{C}_3$ and ℓ' is also a (-1)-curve with $B' \cdot \ell' = 1$. We can contract ℓ' and obtain another triplet (X'', B'', G''). The separations of (X, B, G) and (X'', B'', G'') are identical. Therefore, we can reduce to the *Case* ($\boldsymbol{a}$) with $\deg \mathscr{L} = -1$.

Case ($\boldsymbol{a}$): If $\mathscr{L} \simeq \mathscr{O}_C(-A)$, then (X, B, G) is constructed as Type C1 in §2. Thus it is enough to show the following:

LEMMA 3.11. *In the case* ($\boldsymbol{a}$), *we can reduce to the case*: $\mathscr{L} \simeq \mathscr{O}_C(-A)$.

PROOF. *Step* 1: Suppose that $\mathscr{L} \simeq \mathscr{O}_C$. Then $\mathscr{O}_X(-K_X) \simeq \pi^*\mathscr{O}_{\boldsymbol{P}^1}(2)$ for the first projection $\pi : X \simeq \boldsymbol{P}^1 \times C \to \boldsymbol{P}^1$. Thus $G = \pi^*(b_1 + b_2)$ for some points $b_1, b_2 \in \boldsymbol{P}^1$. We have $B \cdot \pi^*(b_1) = 2$ by $B \sim p^*A - K_X$. Let us take a point q in $B \cap \pi^*(b_1)$ and let $\nu : Y \to X$ be the blowing-up at q. Here we consider the separation $(M, D, E) \to (X, B, G)$. Then the morphism $M \to X$ factors through the blowing-up $Y \to X$. Let Γ be the exceptional curve $\nu^{-1}(q)$, B_Y the proper transform of B, $G_Y := \nu^*G - \Gamma$, and ℓ the proper transform of the fiber $p^{-1}(p(q))$. Then ℓ is a (-1)-curve with $B_Y \cdot \ell = 1$. Let $(Y, B_Y, G_Y) \to (X', B', G')$ be the contraction of ℓ. Then (M, D, E) is also the separation of (X', B', G'). Hence we can reduce to the case $\deg \mathscr{L} < 0$.

Step 2: Suppose that $\mathscr{L} \not\simeq \mathscr{O}_C$ but $\deg \mathscr{L} = 0$. Then we have two mutually disjoint sections C_0 and C_1 of the ruling $p : X \to C$ such that $C_0 \sim H_{\mathscr{E}}$ and $C_1 \sim H_{\mathscr{E}} - p^*\mathscr{L}$. Since $\mathscr{L}$ is not trivial, $C_0 + C_1$ is a unique member of $|-K_X|$. Hence $G = C_0 + C_1$. Now $B \cdot C_0 = 2$. Let us take a point q in $B \cap C_0$ and consider the elementary transformation of X at q. Then as in the previous argument, we can reduce to the case $\deg \mathscr{L} < 0$.

Step 3: Suppose that $\deg \mathscr{L} = -1$. Then we have two mutually disjoint sections C_0 and C_1 such that $C_0 \sim H_{\mathscr{E}}$ and $C_1 \sim H_{\mathscr{E}} - p^*\mathscr{L}$. Then $G = C_0 + G_1$ for an effective divisor G_1 with $G_1 \sim C_1$. Now $B \cdot C_0 = 1$. Let q be the point $B \cap C_0$. By taking the elementary transformation of X at q, we can reduce to the case $\deg \mathscr{L} \leq -2$.

Step 4: Suppose finally that $\deg \mathscr{L} \leq -2$. We have two mutually disjoint sections C_0 and C_1 such that $C_0 \sim H_{\mathscr{E}}$ and $C_1 \sim H_{\mathscr{E}} - p^*\mathscr{L}$. We have $B|_{C_0} \sim p^*(A + \mathscr{L})|_{C_0}$. The inequality $B \cdot C_0 \geq 0$ implies that $\deg \mathscr{L} = -2$ and $B \cap C_0 = \emptyset$. In particular, $\mathscr{L} \simeq \mathscr{O}_C(-A)$. □

§3.3.2. Proof in Case C2.

In this case, we have $(K_X + B) \cdot \ell = 1$ and $(2K_X + B) \cdot \ell = -1$. Thus $X \simeq \boldsymbol{P}_C(\mathscr{E})$ for $\mathscr{E} := p_*\mathscr{O}_X(K_X + B)$. Since $K_X^2 \geq -4$, the genus $g(C)$ is 0 or 1.

Case $g(C) = 0$: Now X is isomorphic to the Hirzebruch surface Σ_d for some $d \geq 0$. Thus $K_X^2 = 8$ and hence $8 \leq r \leq 12$. Let C_0 be the minimal section. Then we can write $K_X + B \sim C_0 + m\ell$ for some $m \geq d$, since $K_X + B$ is nef. We have $-K_X \sim 2C_0 + (d+2)\ell$. Thus $B \sim 3C_0 + (m+d+2)\ell$. Hence

$$4 = (K_X + B) \cdot B = (C_0 + m\ell) \cdot (3C_0 + (m+d+2)\ell) = -2d + 4m + 2.$$

Thus $2m = 1 + d$, which implies $m = d = 1$. Therefore X is the Hirzebruch surface Σ_1 and C_0 is the unique (-1)-curve in which $B \cdot C_0 = 1$. Let $X \to \boldsymbol{P}^2$ be the blowing-down of C_0 and let B' and G' be the image of B and G, respectively. Then the separation of (X, B, G) is also that of $(\boldsymbol{P}^2, B', G')$. Thus we are reduced to the Type D.

Case $g(C) = 1$: We have $-K_X \sim 2H_{\mathscr{E}} - p^*(\det \mathscr{E})$ and $B \sim H_{\mathscr{E}} - K_X$. Thus $r = B \cdot G = (H_{\mathscr{E}} - K_X) \cdot (-K_X) = \deg \mathscr{E}$. Hence $4 = (K_X + B) \cdot B = H_{\mathscr{E}} \cdot (H_{\mathscr{E}} - K_X) = 2 \deg \mathscr{E}$. Therefore $\deg \mathscr{E} = r = 2$. The locally free sheaf $\mathscr{E}$ is one of the following:

C2-0: $\mathscr{E} \simeq \mathscr{O}_C \oplus \mathscr{A}$ for an invertible sheaf $\mathscr{A}$ on C of $\deg \mathscr{A} = 2$;

C2-1: $\mathscr{E} \simeq \mathscr{O}_C(q_1) \oplus \mathscr{O}_C(q_2)$ for two points q_1, q_2 of C;

C2-2: There is a non-split exact sequence

$$0 \to \mathscr{O}_C(q) \to \mathscr{E} \to \mathscr{O}_C(q) \to 0$$

for a point $q \in C$.

Case C2-0: Let C_0 be the negative section of $p : X \to C$. Then $C_0 \sim H_{\mathscr{E}} - p^*\mathscr{A}$ and $-K_X \sim 2C_0 + p^*\mathscr{A}$. Thus $B \sim 3C_0 + 2p^*\mathscr{A}$. Hence $B \cdot C_0 = -\deg \mathscr{A} = -2$. This is a contradiction.

Case C2-1: First assume that $q_1 = q_2$. Then $X \simeq \boldsymbol{P}^1 \times C$. Let $\pi : X \to \boldsymbol{P}^1$ be the first projection and let ℓ be the fiber $p^{-1}(q_1) = p^{-1}(q_2)$. Then $B \sim H_{\mathscr{E}} - K_X \sim \pi^*\mathscr{O}_{\boldsymbol{P}^1}(3) + \ell$. Thus

$$p_*\mathscr{O}_X(B) \simeq p_*(\pi^*\mathscr{O}_{\boldsymbol{P}^1}(3) \otimes \mathscr{O}_X(\ell)) \simeq \mathscr{O}_C^{\oplus 4} \otimes \mathscr{O}_C(q_1).$$

Therefore, ℓ is a fixed component of the linear system $|B|$. This is a contradiction. Thus $q_1 \neq q_2$.

Let ℓ_i be the fiber $p^{-1}(q_i)$ for $i = 1, 2$. Let C_1 and C_2 be the minimal sections of $p : X \to C$ with $C_1 \sim H_{\mathscr{E}} - \ell_1$ and $C_2 \sim H_{\mathscr{E}} - \ell_2$, respectively. Then $C_1 \cap C_2 = \varnothing$, $\mathscr{O}_{C_1}(C_1) \simeq \mathscr{O}_{C_1}(\ell_2 - \ell_1)$, $\mathscr{O}_{C_2}(C_2) \simeq \mathscr{O}_{C_2}(\ell_1 - \ell_2)$, and $-K_X \sim C_1 + C_2$. Since $G|_{C_1} \sim (\ell_2 - \ell_1)|_{C_1}$ is non-trivial, G contains C_1 and also C_2. Thus $G = C_1 + C_2$.

Suppose that $2q_1 \sim 2q_2$ on C. Then $2C_1 \sim 2C_2$. Thus the base point free linear system $|2C_1|$ defines a morphism $\phi : X \to \boldsymbol{P}^1$. Now $B \sim H_{\mathscr{E}} + G$ and $B \cdot C_1 = 1$. Thus ϕ induces a double-covering $B \to \boldsymbol{P}^1$. This is a contradiction. Hence $2q_1 \not\sim 2q_2$. Therefore the (X, B, G) is obtained as Type C2-1 in §2.

Case C2-2: We have $K_X + B \sim C_0 + \ell$ and $-K_X \sim 2C_0$ for the minimal section C_0. Hence $G = 2C_0$ and $B \sim 3C_0 + \ell$. Thus the (X, B, G) is constructed as Type C2-2 in §2.

§3.4. Proof in the case of Type D.

Let (X, B, G) be a minimal basic triplet satisfying $\mathscr{C}_r$ such that $X \simeq \boldsymbol{P}^2$ and that $-(2K_X + B)$ is ample. Then B is a smooth quartic curve, since $g(B) = 3$. Therefore $r = 12$. Hence (X, B, G) is obtained as Type D in §2. This completes the proof of Main Theorem.

References

[1] V. I. Arnol'd, S. M. Guseĭn-Zade and A. N. Varchenko, Singularities of differentiable maps. Vol. I, Nauka, Moscow, 1982; Vol. II, Nauka, Moscow, 1984; English transl. Monogr. Math., **82** (1985), **83** (1988), Birkhäuser.

[2] A. I. Degtyarev, Classification of surfaces of degree four having a nonsimple singular point, Izv. Akad. Nauk SSSR, **53** (1989), 1269–1290; English transl. Math. USSR Izvestiya, **35** (1990), 607–627.

[3] M. Demazure, Surfaces de del Pezzo II–V, In: Séminaire sur les Singularités des Surfaces (eds. M. Demazure, H. Pinkham and B. Teissier), Lecture Notes in Math., **777** (1980), pp. 23–69.

[4] Y. Ishii, Classification of normal quartic surfaces, Master's thesis, Dept. of Math. Sci. Univ. Tokyo, 1993.

[5] E. Looijenga, Rational surfaces with an anticanonical cycle, Ann. of Math., **114** (1981), 267–322.

[6] S. Mori, Threefolds whose canonical bundles are not numerically effective, Ann. of Math., **116** (1982), 133–176.

[7] Y. Umezu, On normal projective surfaces with trivial dualizing sheaf, Tokyo J. Math., **4** (1981), 343–354.

[8] Y. Umezu, Quartic surfaces of elliptic ruled type, Trans. Amer. Math. Soc., **283** (1984), 127–143.

[9] T. Urabe, On quartic surfaces and sextic curves with singularities of type $\tilde{E}_8, T_{2,3,7}, E_{12}$. Publ. Res. Inst. Math. Sci., **20** (1984), 1185–1245.

[10] T. Urabe, Singularities in a certain class of quartic surfaces and sextic curves and Dynkin graphs, In: Proceedings of the 1984 Vancouver conference in algebraic geometry, CMS Conf. Proc., **6**, Amer. Math. Soc., 1986, pp. 477–497.

Yuji Ishii

The Bank of Tokyo-Mitsubishi, Ltd.
Tokyo
Japan

Noboru Nakayama

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502
Japan
E-mail: nakayama@kurims.kyoto-u.ac.jp