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Spectral multipliers for Markov chains

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Abstract. We prove an analog to the classical Mikhlin-Hörmander multiplier theorem for Markov chains.

1. Introduction.

Let $m(\xi)$ be a bounded measurable function in $\boldsymbol{R}^n$ and let T_m be the operator defined by $\widehat{T_m f}(\xi) = m(\xi)\hat{f}(\xi)$ (where $\hat{f}$ denotes the Fourier transform of the function f). Then, by the Plancherel formula, T_m is an operator bounded on L^2. The Mikhlin-Hörmander multiplier theorem (cf. [**Hö**]) asserts that if

$$\sup_{\xi \in \boldsymbol{R}^n} |\xi|^k |\nabla^k m(\xi)| < +\infty, \quad k = 0, 1, \dots, \left[\frac{n}{2}\right] + 1,$$

then T_m extends to an operator bounded on L^p, $1 < p < \infty$ and from L^1 to weak-L^1. This result has had many generalisations to abstract contexts (see for example [**A1**], [**An**], [**He**], [**CS**] and the refferences therein). In this article we shall prove an analogous result for Markov chains.

More precisely, let X be a measurable space endowed with a positive σ-finite measure dx and a measurable distance $d(.,.)$ and let us denote by $B(x, r)$, $x \in X$, $r > 0$ the ball of center x and radius r. If A is a measurable subset of X, then we shall set $|A| = dx\text{-measure}(A)$.

We shall assume that X has the doubling volume property, i.e. there is $c > 0$ such that for all $x \in X$ and $r > 0$

$$|B(x, 2r)| \leq c|B(x, r)|.$$

This implies that there is $D \geq 0$ such that

$$\frac{|B(x, r)|}{|B(x, s)|} \leq c\left(\frac{r}{s}\right)^D, \quad r \geq s > 0,\ x \in X. \tag{1.1}$$

Note that X is a space of homogeneous type in the sense of Coifman and Weiss [**CW**].

Let $P(x, y)$ be a bounded symmetric Markov kernel on X and let us set $P_0(x, y) = \delta_x$, where δ_x is the Dirac mass at x, $P_1(x, y) = P(x, y)$ and $P_n(x, y) = \int P_{n-1}(x, z) P(z, y)\, dz$, for $n \geq 2$.

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We shall assume that there is $c > 0$ such that

$$P_n(x, y) \le \frac{c}{|B(x, \sqrt{n})|} \exp\left(-\frac{d(x, y)^2}{cn}\right), \quad x, y \in X,\ n \in \boldsymbol{N}. \tag{1.2}$$

Note that the above estimate is satisfied by the transition probabilities of random walks on discrete groups of polynomial volume growth (cf. [**HS**]). It is also satisfied by a large class of random walks on graphs (see for example [**De**]).

We shall denote also by P_n the operator

$$P_n f(x) = \int P_n(x, y) f(y)\, dy.$$

The operator $I - P$ is symmetric. It is also positive, since for all $f \in L^2$

$$\langle (I - P)f, f\rangle = \frac{1}{2}\int\int (f(x) - f(y))^2 P(x, y)\, dxdy \ge 0.$$

Furthermore,

$$\|(I - P)f\|_2 \le \|f\|_2 + \|Pf\|_2 \le 2\|f\|_2.$$

So, $I - P$ admits the spectral decomposition $I - P = \int_0^2 \lambda\, dE_\lambda$ (cf. [**Yo**]). Let m be a bounded Borel measurable function. Then, by the spectral theorem we can define the operator

$$m(I - P) = \int_0^2 m(\lambda)\, dE_\lambda.$$

Note that $m(I - P)$ is bounded on L^2: $\|m(I - P)\|_{2\to 2} \le \|m\|_\infty$.

Let us consider a function $0 \le \varphi \in C^\infty(\boldsymbol{R})$ and let us assume that $\varphi(t) = 1$ for $t \in [1, 2]$ and that $\varphi(t) = 0$ for $t \notin [1/2, 4]$.

In this article, we shall prove the following analog to the Mikhlin-Hörmander multiplier theorem.

THEOREM 1.1. *Let φ be as above and let us assume that* $\operatorname{supp}(m) \subseteq [0, 1/2]$ *and that, for some $\varepsilon > 0$,*

$$\sup_{0<t\le 1} \|\varphi(.)m(t.)\|_{C^{(D/2)+\varepsilon}(\boldsymbol{R})} < \infty. \tag{1.3}$$

Then, $m(I - P)$ extends to an operator bounded on L^p, $1 < p < \infty$ and from L^1 to weak-L^1.

The part of the spectrum of $I - P$ which lies in the interval $[1/2, 2]$ can be treated by making use of the following:

PROPOSITION 1.2. *Let us assume that* $\operatorname{supp}(m)$ *is a compact subset of $(0, \infty)$ and that, for some $\varepsilon > 0$, one of the following three conditions is satisfied:*

(1) $m \in C^{(D/2)+1+\varepsilon}(\boldsymbol{R})$.

(2) $1 \notin \operatorname{supp}(m)$ *and* $m \in C^{(D/2)+\varepsilon}(\boldsymbol{R})$.

(3) *The space X has the discrete topology and* $m \in C^{(D/2)+\varepsilon}(\boldsymbol{R})$.

Then the operator $m(I - P)$ is bounded on L^p, $1 \le p \le \infty$.

In the case of random walks with finite range on discrete groups of polynomial volume growth, the above results have been proved in [**A3**]. The proofs in [**A3**] though rely on the assumption that the random walk has finite range. For example the proof of Proposition 1.2 given in [**A3**] does not make use of the estimate (1.2).

The operator $m(I-P)$ in the above theorem is, in general, a singular integral operator. It is enough to prove that it is bounded from L^1 to weak-L^1. Then by interpolating with the L^2 result, it will follow that it is bounded on L^p, for $1<p<2$ and by duality that it is bounded on L^p, for $2<p<\infty$. In order to prove that $m(I-P)$ is bounded from L^1 to weak-L^1 we shall perform a Calderon-Zygmund decomposition.

Let us denote by $K(x,y)$ the kernel of the operator $m(I-P)$. The standard way to proceed is to divide the multiplier $m(\lambda)$ into pieces of compactly supported ones $m_j(\lambda)$ (cf. [**Hö**]). Thus, we get operators $m_j(L)$ with kernels $K_j(x,y)$. The goal now is to obtain good enough estimates for the kernels $K_j(x,y)$ which will imply that their sum, the kernel $K(x,y)$, satisfies for example the Hörmander integral condition. This condition translates into some regularity assumption for the kernels $P_n(x,y)$. We avoid making such assumptions by using an adaptation in the semigroup context (see for example [**CD**], [**DO**], [**Ru**], [**SW**]) of an argument originally due to [**Fe**].

A technical aspect of the proofs is that we must consider the operator $\sqrt{I-P}$. If we set $f_j(s)=m_j(s^2)$ then we have $m_j(I-P)=f_j(\sqrt{I-P})$. Since the function f_j is even we have

$$m_j(I-P)=(1/\sqrt{2\pi})\int_{-\infty}^{\infty}\hat{f}_j(t)\cos(t\sqrt{I-P})\,dt. \tag{1.4}$$

Now we can observe that the function $\cos x$ developped in a power series involves only even powers of x. So the operator $\cos(t\sqrt{I-P})$ can be written as a series involving only integral powers of the operator $I-P$. In this article we shall exploit these observations.

Let us now assume that L is a symmetric differential operator (for example the Laplace-Beltrami operator on a Riemannian manifold, or a sub-Laplacian on a connected Lie group of polynomial volume growth) and that the associated heat kernel $P_t(x,y)$ (or else the kernel of the semigroup $T_t=e^{-tL}$) satisfies a Gaussian estimate similar to (1.2). In the last section, we shall explain how one can adapt the arguments given in this article in order to prove analogous results for the operator L.

Throughout this article, the different constants will always be denoted with the same letter c. When their dependence or independence is significant, it will be clearly stated.

If $K(x,y)$ and $S(x,y)$ are kernels on X then we shall denote by KS their convolution product defined by $KS(x,y)=\int K(x,z)S(z,y)\,dz$.

We shall set

$$A_p(x)=B(x,2^{(p+1)/2})\backslash B(x,2^{p/2}),\quad p\in\boldsymbol{N}.$$

Note that by (1.1)

$$\frac{|A_p(x)|}{|B(x,1)|}\le c2^{Dp/2},\quad p\in\boldsymbol{N},\ x\in X. \tag{1.5}$$

2. An approximation lemma.

We shall need the following:

LEMMA 2.1. *Assume that the function* $f(x) \in C(\boldsymbol{R})$ *has compact support and that it possesses n continuous derivatives* $f'(x), f''(x), \ldots, f^{(n)}(x)$ *and let* $M_\alpha = \sup\{|f^{(n)}(x+t) - f^{(n)}(x)|/t^\alpha, t > 0, x \in \boldsymbol{R}\}$, $0 < \alpha \leq 1$. *Then, for every* $\lambda > 0$, *there is an even bounded integrable function* $\psi_\lambda(x) \in C(\boldsymbol{R})$ *and a constant* $c > 0$, *independent of* λ *and* f, *such that*

$$\|\hat{\psi}_\lambda\|_\infty \leq c \tag{2.1}$$

$$\operatorname{supp}(\hat{\psi}_\lambda) \subseteq [-\lambda, \lambda]$$

$$|f(x) - f * \psi_\lambda(x)| \leq c\frac{M_\alpha}{\lambda^{n+\alpha}}, \quad x \in \boldsymbol{R}.$$

The above lemma can be proved by induction on n, in the same way as in the periodic case (cf. [**Lo**, p.57], [**Na**, p.88]). The only change is that instead of using Jackson's kernel $U_n(x) = k_n[\sin(nx/2)/\sin(x/2)]^4$, $n \in \boldsymbol{N}$, (k_n is a constant making the integral $\int_0^{2\pi} U_n(x)\,dx = 1$) we must use its analog for $\boldsymbol{R}$, $H_\lambda(x) = K_\lambda[\sin(\lambda x/2)/(x/2)]^4$, $\lambda > 0$ (again K_λ is such that $\int_{-\infty}^{\infty} H_\lambda(x)\,dx = 1$).

3. Auxiliary estimates.

We shall also need the following:

LEMMA 3.1. *There is* $\eta \in (0,1)$ *and* $c > 0$ *such that for all* $p \in \boldsymbol{N}$, $x \in A_p(y)$, $|t| \leq \eta 2^{p/2}$ *and all* $1 \leq k \leq 2^{p/2}$

$$|e^{itP}P_k(x,y)| \leq \frac{c}{|B(y,\sqrt{k})|}e^{-d(x,y)/c} \tag{3.1}$$

$$|(\cos(t\sqrt{I-P}))P_k(x,y)| \leq \frac{c}{|B(y,\sqrt{k})|}e^{-d(x,y)/c}. \tag{3.2}$$

PROOF. We shall only prove (3.2). The proof of (3.1) is similar.
We have

$$\cos(t\sqrt{I-P}) = \sum_{n\geq 0}\frac{t^{2n}}{(2n)!}(\sqrt{I-P})^{2n} = \sum_{n\geq 0}\frac{t^{2n}}{(2n)!}(I-P)^n.$$

Since, $(I-P)^n P_k(x,y) \leq (I+P)^n P_k(x,y)$, we have

$$|(\cos(t\sqrt{I-P}))P_k(x,y)| \leq \sum_{n\geq 0}\frac{t^{2n}}{(2n)!}(I+P)^n P_k(x,y), \quad x, y \in X.$$

By (1.2) and the binomial formula

$$(I+P)^n P_k(x,y) \leq 2^n\frac{c}{|B(y,\sqrt{k})|}\exp\left(-\frac{d(x,y)^2}{c(n+k)}\right), \quad x, y \in X, n \in \boldsymbol{N}.$$

It follows that if $x \in A_p(y)$ and $|t| \le \eta 2^{p/2}$, then

$$(3.3)\qquad \sum_{0 \le n \le 2^{p/2}} \frac{t^{2n}}{(2n)!}(I+P)^n P_k(x,y) \le \sum_{0 \le n \le 2^{p/2}} \frac{(t)^{2n}}{(2n)!} 2^n \frac{c}{|B(y,\sqrt{k})|} \exp\left(-\frac{d(x,y)^2}{c2^{p/2}}\right)$$
$$\le \sum_{0 \le n \le 2^{p/2}} \frac{(\eta 2^{p/2}\sqrt{2})^{2n}}{(2n)!} \frac{c}{|B(y,\sqrt{k})|} e^{-2^{p/2}/c}$$
$$\le \frac{c}{|B(y,\sqrt{k})|} e^{-2^{p/2}/c} \sum_{n \ge 0} \frac{(\eta 2^{p/2}\sqrt{2})^n}{n!}$$
$$= \frac{c}{|B(y,\sqrt{k})|} e^{-2^{p/2}/c} e^{\eta\sqrt{2}2^{p/2}}$$
$$= \frac{c}{|B(y,\sqrt{k})|} e^{-((1/c)-\eta\sqrt{2})2^{p/2}}.$$

Also by Stirling's formula $\Gamma(1+n) = n! \ge \sqrt{\pi}\sqrt{n} n^n e^{-n}$, $n \in \boldsymbol{N}$. So, by choosing η small enough, we have

$$(3.4)\qquad \sum_{n \ge 2^{p/2}} \frac{t^{2n}}{(2n)!}(I+P)^n P_k(x,y) \le \sum_{n \ge 2^{p/2}} \frac{(t)^{2n}}{(2n)!} 2^n \frac{c}{|B(y,\sqrt{k})|}$$
$$\le \frac{c}{|B(y,\sqrt{k})|} \sum_{n \ge 2^{p/2}} \frac{(\eta 2^{p/2}\sqrt{2})^{2n}}{(2n)!}$$
$$\le \frac{c}{|B(y,\sqrt{k})|} \sum_{n \ge 2^{p/2}} \frac{(\eta 2^{p/2}\sqrt{2})^{2n}}{\sqrt{\pi}\sqrt{2n}(2n)^{2n} e^{-2n}}$$
$$\le \frac{c}{|B(y,\sqrt{k})|} \sum_{n > 2^{p/2}} \left(\frac{\eta 2^{p/2} e}{\sqrt{2}n}\right)^n$$
$$\le \frac{c}{|B(y,\sqrt{k})|} \sum_{n > 2^{p/2}} \left(\frac{\eta 2^{p/2} e}{\sqrt{2}2^{p/2}}\right)^n$$
$$\le \frac{c}{|B(y,\sqrt{k})|} \sum_{n > 2^{p/2}} \left(\frac{\eta e}{\sqrt{2}}\right)^n$$
$$\le \frac{c}{|B(y,\sqrt{k})|} e^{-2^{p/2}/c}.$$

(3.2) follows from (3.3) and (3.4). □

4. Proof of proposition 1.2.

Let us assume that the condition (1) is satisfied by the multiplier $m(\lambda)$.

Let $K(x,y)$ denote the kernel of the operator $m(I-P) - m(1)I$. Then to prove the proposition it is enough to prove that there is $c > 0$ such that for all $y \in X$

$$(4.1)\qquad \|K(.\,,y)\|_1 \le c < \infty.$$

Let $h(\lambda) = (m(\lambda) - m(1))(1-\lambda)^{-1}$. Then $h \in C^{((D/2)+\varepsilon)}(\boldsymbol{R})$ and $K = h(I-P)P$. Hence

$$\begin{aligned}(4.2)\qquad \|K(.\,,y)\|_1 &= \|h(I-P)P(.\,,y)\|_1\\ &= \|h(I-P)P(.\,,y)\|_{L^1(B(y,1))} + \sum_{p\ge 0} \|h(I-P)P(.\,,y)\|_{L^1(A_p(y))}\\ &\le |B(y,1)|^{1/2}\|h(I-P)P(.\,,y)\|_{L^2(B(y,1))}\\ &\quad + \sum_{p\ge 0}\|h(I-P)P(.\,,y)\|_{L^1(A_p(y))}.\end{aligned}$$

We have

$$\begin{aligned}(4.3)\qquad |B(y,1)|^{1/2}\|h(I-P)P(.\,,y)\|_{L^2(B(y,1))} &\le |B(y,1)|^{1/2}\|h\|_\infty \|P(.\,,y)\|_2\\ &\le |B(y,1)|^{1/2}\|h\|_\infty |B(y,\sqrt{2})|^{-1/2} < \infty.\end{aligned}$$

To estimate the remaining terms in (4.2), we observe that

$$h(I-P) = \frac{1}{\sqrt{2\pi}}\int \hat{m}(t)e^{it(I-P)}\,dt$$

and hence

$$K(x,y) = \frac{1}{\sqrt{2\pi}}\int \hat{m}(t)e^{it(I-P)}P\,dt.$$

Making use of (2.1) let us consider, for all $p \ge 0$, a function ψ_p such that

$$\operatorname{supp}\hat{\psi}_p \subseteq [-\eta 2^{p/2}, \eta 2^{p/2}]$$

$$\|m - m * \psi_p\|_\infty \le c2^{-((D/2)+\varepsilon)p/2}.$$

Then

$$\begin{aligned}(4.4)\qquad \|h(I-P)P(.\,,y)\|_{L^1(A_p(y))} &\le |A_p(y)|^{1/2}\|((m - m*\psi_p)(I-P))P(.\,,y)\|_{L^2(A_p(y))}\\ &\quad + \|m*\psi_p(I-P)P(.\,,y)\|_{L^1(A_p(y))}\\ &\le |A_p(y)|^{1/2}\|P(.\,,y)\|_2\|m - m*\psi_p\|_\infty\\ &\quad + |A_p(y)|\,\|m*\psi_p(I-P)P(.\,,y)\|_{L^\infty(A_p(y))}.\end{aligned}$$

If $x \in A_p(y)$ then by (3.1)

$$\begin{aligned}(4.5)\qquad |(m*\psi_p(I-P)P(.\,,y))(x)| &= \left|\frac{1}{\sqrt{2\pi}}\int \hat{m}(t)\hat{\psi}_p(t)e^{it(I-P)}P(x,y)\,dt\right|\\ &\le c\int_{|t|\le \eta 2^{p/2}} |e^{-itP}P(x,y)|\,dt\\ &\le c\eta 2^{p/2}\frac{1}{|B(y,1)|}e^{-2^{p/2}/c}\\ &\le \frac{c}{|B(y,1)|}e^{-2^{p/2}/c}.\end{aligned}$$

By (4.4) and (4.5) we have

$$\|h(I-P)P(.\,,y)\|_{L^1(A_p(y))} \le c2^{Dp/4}2^{-((D/2)+\varepsilon)p/2} + c2^{Dp/2}e^{-2^{p/2}/c} \le c2^{-\varepsilon p/2}. \tag{4.6}$$

It follows from (4.2), (4.3) and (4.6) that

$$\|K(.\,,y)\|_1 \le c + c\sum_{p\ge 0} 2^{-\varepsilon p/2} < \infty$$

which proves (4.1).

If the condition (2) is satisfied by $m(\lambda)$, then we set $h(\lambda) = m(\lambda)(1-\lambda)^{-1}$. Then $h \in C^{(D/2)+\varepsilon}(\boldsymbol{R})$ and the kernel $K(x,y)$ of the operator $m(I-P)$ can be written as $K = h(I-P)P$. So, arguing again in the same way, we can prove that $K(x,y)$ satisfies (4.1).

Finally, if the condition (3) is satisfied and if $|\{y\}| = 0$ then $m(I-P)\mathbf{1}_{\{y\}} = 0$. If $|\{y\}| \ne 0$ then the kernel $K(x,y)$ of the operator $m(I-P)$ can be written as

$$K(x,y) = \left(m(I-P)\mathbf{1}_{\{y\}}\frac{1}{|\{y\}|}\right)(x)$$

and arguing in the same way, we can prove again that $K(x,y)$ satisfies (4.1).

5. Proof of theorem 1.1.

It is enough to prove that $m(I-P)$ is bounded from L^1 to week-L^1. Then by interpolation and symmetry, we can conclude that $m(I-P)$ is also bounded on L^p, $1 < p < \infty$.

5.1. Preliminary considerations.

Let us observe that

$$\left\|\int_0^{1/n} dE_\lambda(f)\right\|_2 = \left\|\int_0^{1/n}(1-\lambda)^{-n}\,dE_\lambda(P_n f)\right\|_2 \le \|P_n f\|_2 \sup\{|(1-\lambda)^{-n}|; [0,1/n]\}.$$

Since, by (1.2), $\lim_{n\to\infty}\|P_n f\|_2 = 0$, we conclude that

$$\lim_{n\to\infty}\left\|\int_0^{1/n} dE_\lambda(f)\right\|_2 = 0.$$

This shows that the point $\lambda = 0$, in the spectral resolution of $I-P$, may be neglected.

Let us consider a C^∞ function ϕ satisfying

$$\operatorname{supp}\phi \subseteq (1/8, 3/2), \quad \sum_{j\ge 1}\phi(2^j t) = 1, \quad t \in (0, 1/2]$$

and set $m_j(\lambda) = m(\lambda)\phi(2^j\lambda)$, $j \ge 1$. Then

$$m(\lambda) = \sum_{j\ge 1} m_j(\lambda).$$

Let us denote by $K(x,y)$ and $K_j(x,y)$, $j \geq 1$ the kernel of the operators $m(I-P)$ and $m_j(I-P)$, $j \geq 1$ respectively. Then

$$K(x,y) = \sum_{j\geq 1} K_j(x,y).$$

We have

$$m_j(\lambda) = m_j(\lambda)(1-\lambda)^{-2^j}(1-\lambda)^{2^j}.$$

Let

$$h_j(s) = m_j(s^2)(1-s^2)^{-2^j}$$

$$h_{j,\tau}(s) = m_j(s^2)(1-s^2)^{-2^j}(1-s^2)^{2^\tau}$$

$$\zeta_j(s) = s^2 m_j(s^2)(1-s^2)^{-2^j}.$$

Let us recall that $\mathrm{supp}(m_j) \subset (2^{-j}/8, 2^{-j}3/2)$ and hence the above defined functions are supported in $(2^{-j/2}/\sqrt{8}, 2^{-j/2}\sqrt{3}/\sqrt{2})$. Also, if $s \in (2^{-j/2}/\sqrt{8}, 2^{-j/2}\sqrt{3}/\sqrt{2})$, then

$$(1-s^2)^{2^\tau} = e^{2^\tau \log(1-s^2)}, \quad 2^\tau \log(1-s^2) \sim 2^\tau(-s^2) \sim -2^{\tau-j}.$$

Thus the functions $h_j, h_{j,\tau}$ and ζ_j satisfy

(5.1.1)
$$\|h_j\|_{C^{(D/2)+\varepsilon}(\boldsymbol{R})} \leq c2^{((D/2)+\varepsilon)j/2}$$

$$\|h_{j,\tau}\|_{C^{(D/2)+\varepsilon}(\boldsymbol{R})} \leq c2^{((D/2)+\varepsilon)\tau/2} e^{-2^{\tau-j}/c}, \quad \tau \geq j$$

$$\|\zeta_j\|_{C^{(D/2)+\varepsilon}(\boldsymbol{R})} \leq c2^{-j}2^{((D/2)+\varepsilon)j/2}.$$

Finally, since the functions $h_j, h_{j,\tau}$ and ζ_j are even, we have

$$m_j(I-P) = h_j(\sqrt{I-P})P_{2^j} = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} \hat{h}_j(t)\cos(t\sqrt{I-P})\,dtP_{2^j}$$

$$m_j(I-P)P_{2^\tau} = h_{j,\tau}(\sqrt{I-P})P_{2^j} = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} \hat{h}_{j,\tau}(t)\cos(t\sqrt{I-P})\,dtP_{2^j}$$

$$(I-P)m_j(I-P) = \zeta_j(\sqrt{I-P})P_{2^j} = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} \hat{\zeta}_j(t)\cos(t\sqrt{I-P})\,dtP_{2^j}.$$

5.2. Calderon-Zygmund decomposition.

Let us consider a function $0 \leq f \in L^1 \cap L^2$ with bounded support and let $a > 0$. Then, following [**CW**, pp. 73–75], there are constants $C = C(X) > 0$ and $k = k(X) > 0$ and a sequence of balls $B(x_i, r_i)$, such that

(1) $f(x) \leq Ca$, for almost all $x \in X \setminus \bigcup_i B(x_i, r_i)$,

(2) $(1/|B(x_i,r_i)|)\int_{B(x_i,r_i)} f(x)\,dx \leq Ca$,

(3) $\sum_i |B(x_i,r_i)| \leq (C/a)\int_X f(x)\,dx$ and

(4) each point $x \in X$ belongs to at most k balls $B(x_i, r_i)$.

Let

$$\eta_i(x) = \begin{cases} \frac{\mathbf{1}_{B(x_i,r_i)}(x)}{\sum_j \mathbf{1}_{B(x_j,r_j)}(x)}, & x \in B(x_i, r_i) \\ 0, & x \notin B(x_i, r_i) \end{cases}$$

and set $w_i = \eta_i f$.

Now, arguing as in [**Fe**] (see [**CD**], [**Ru**], [**SW**] for an adaptation of this argument in the semigroup context) we set

$$b_i = \begin{cases} P_{[r_i^2]} w_i, & r_i \geq 1 \\ P w_i, & 0 < r_i < 1 \end{cases}$$

where $[t] = n$, for $n \leq t < n+1$, $n \in \boldsymbol{Z}$.

We observe that for all $x \in X$

$$\begin{aligned} \sum_{0<r_i<1} b_i(x) = \sum_{0<r_i<1} P w_i(x) &\leq \frac{c}{|B(x,1)|} \sum_{0<r_i<1} e^{-d(x,x_i)^2/c} \|w_i\|_1 \\ &\leq ca \frac{1}{|B(x,1)|} \sum_{0<r_i<1} e^{-d(x,x_i)^2/c} |B(x_i, r_i)| \\ &\leq ca \left\| \frac{1}{|B(x,1)|} e^{-d(x,.)^2/c} \right\|_1 \\ &\leq ca. \end{aligned}$$

Let

$$\theta_i = w_i - b_i$$

$$g = \mathbf{1}_{X \setminus \bigcup_i B(x_i,r_i)} f + \sum_{0<r_i<1} b_i.$$

Then

$$f = g + \sum_{r_i \geq 1} b_i + \sum_i \theta_i \tag{5.2.1}$$

$$|g(x)| \leq ca, \quad x \in X.$$

5.3. Kernel estimates.

Lemma 5.3.1. *There is $c > 0$ such that for all $p, j \in \boldsymbol{N}$, $1 \leq j \leq p$ and $y \in X$*

$$\|K_j(\,.\,, y)\|_{L^1(A_p(y))} \leq c 2^{-\varepsilon(p-j)/2}. \tag{5.3.1}$$

Proof. We have

$$K_j = h_j(\sqrt{I-P}) P_{2^j}.$$

By (2.1) and (5.1.1), there is a function $\psi_{j,p}$ such that

$$\operatorname{supp} \hat{\psi}_{j,p} \subseteq [-\eta 2^{p/2}, \eta 2^{p/2}] \tag{5.3.2}$$

$$\|h_j - h_j * \psi_{j,p}\|_\infty \leq c 2^{((D/2)+\varepsilon)j/2} 2^{-((D/2)+\varepsilon)p/2}.$$

By (3.2), if $x \in A_p(y)$, then

$$|h_j * \psi_{j,p}(\sqrt{I-P})P_{2^j}(x,y)| \tag{5.3.3}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |\hat{\psi}_{j,p}(t)\hat{h}_j(t)\cos(t\sqrt{I-P})P_{2^j}(x,y)|\,dt$$
$$\leq c\int_{|t|\leq \eta 2^{p/2}} |\cos(t\sqrt{I-P})P_{2^j}(x,y)|\,dt$$
$$\leq c\eta 2^{p/2}\frac{1}{|B(y,2^{j/2})|}e^{-2^{p/2}/c} \leq \frac{c}{|B(y,2^{j/2})|}e^{-2^{p/2}/c}.$$

By (5.3.2) and (5.3.3)

$$\begin{aligned}
\|K_j(.\,,y)\|_{L^1(A_p(y))} &= \|h_j(\sqrt{I-P})P_{2^j}(.\,,y)\|_{L^1(A_p(y))} \\
&\leq \|(h_j - h_j * \psi_{j,p})(\sqrt{I-P})P_{2^j}(.\,,y)\|_{L^1(A_p(y))} \\
&\quad + \|h_j * \psi_{j,p}(\sqrt{I-P})P_{2^j}(.\,,y)\|_{L^1(yA_p)} \\
&\leq |A_p(y)|^{1/2}\|(h_j - h_j * \psi_{j,p})(\sqrt{I-P})P_{2^j}(.\,,y)\|_2 \\
&\quad + |A_p(y)|\,\|h_j * \psi_{j,p}(\sqrt{I-P})P_{2^j}(.\,,y)\|_{L^\infty(A_p(y))} \\
&\leq |A_p(y)|^{1/2}\|h_j - h_j * \psi_{j,p}\|_\infty \|P_{2^j}(.\,,y)\|_2 \\
&\quad + |A_p(y)|\frac{c}{|B(y,2^{j/2})|}e^{-2^{p/2}/c} \\
&\leq |A_p(y)|^{1/2}2^{((D/2)+\varepsilon)j/2}2^{-((D/2)+\varepsilon)p/2}\|P_{22^j}(.\,,y)\|_\infty^{1/2} \\
&\quad + c2^{Dp/2}2^{-Dj/2}e^{-2^{p/2}/c} \\
&\leq c2^{-\varepsilon(p-j)/2} + c2^{-Dj/2}2^{Dp/2}e^{-2^{p/2}/c} \\
&\leq c2^{-\varepsilon(p-j)/2}
\end{aligned}$$

which proves the lemma. □

LEMMA 5.3.2. *There is $c > 0$ such that for all $\tau \in \mathbf{N}$, $0 \leq j \leq \tau$ and $y \in X$*

$$\|K_j P_{2^\tau}(.\,,y)\|_1 \leq ce^{-2^{\tau-j}/c}. \tag{5.3.4}$$

PROOF. We have

$$K_j P_{2^\tau} = h_{j,\tau}(\sqrt{I-P})P_{2^j}.$$

Let

$$I(j,p) = \|h_{j,\tau}(\sqrt{I-P})P_{2^j}(.\,,y)\|_{L^1(A_p(y))}.$$

Then

$$\|K_j P_{2^\tau}(.\,,y)\|_1 \leq \|h_{j,\tau}(\sqrt{I-P})P_{2^j}(.\,,y)\|_{L^1(B(y,42^{(j+1)/2}))} + \sum_{p\geq j+4} I(j,p). \tag{5.3.5}$$

We have

$$\|h_{j,\tau}(\sqrt{I-P})P_{2^j}(\,.\,,y)\|_{L^1(B(y,42^{(j+1)/2}))} \tag{5.3.6}$$
$$\leq |B(y,42^{(j+1)/2})|^{1/2}\|h_{j,\tau}(\sqrt{I-P})P_{2^j}(\,.\,,y)\|_2$$
$$\leq |B(y,42^{(j+1)/2})|^{1/2}\|h_{j,\tau}\|_\infty\|P_{2^j}(\,.\,,y)\|_2$$
$$\leq ce^{-2^{\tau-j}/c}.$$

On the other hand, by (2.1) and (5.1.1), there is a function $\psi_{j,\tau,p}$ such that

$$\operatorname{supp}\hat{\psi}_{j,\tau,p} \subseteq [-\eta 2^{p/2}, \eta 2^{p/2}] \tag{5.3.7}$$
$$\|h_{j,\tau} - h_{j,\tau} * \psi_{j,\tau,p}\|_\infty \leq c2^{((D/2)+\varepsilon)\tau/2}2^{-((D/2)+\varepsilon)p/2}e^{-2^{\tau-j}/c}.$$

By (3.2), if $x \in A_p(y)$, then

$$|h_{j,\tau} * \psi_{j,\tau,p}(\sqrt{I-P})P_{2^j}(x,y)| \tag{5.3.8}$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} |\hat{\psi}_{j,\tau,p}(t)\hat{h}_{j,\tau}(t)\cos(t\sqrt{I-P})P_{2^j}(x,y)|\,dt$$
$$\leq ce^{-2^{\tau-j}/c}\int_{|t|\leq\eta 2^{p/2}} |\cos(t\sqrt{I-P})P_{2^j}(x,y)|\,dt$$
$$\leq \frac{c}{|B(y,2^{j/2})|}e^{-2^{p/2}/c}e^{-2^{\tau-j}/c}.$$

If $p \geq j+4$, then by (5.3.7) and (5.3.8)

$$I(j,p) \leq \|(h_{j,\tau} - h_{j,\tau} * \psi_{j,\tau,p})(\sqrt{I-P})P_{2^j}(\,.\,,y)\|_{L^1(A_p(y))} \tag{5.3.9}$$
$$+ \|h_{j,\tau} * \psi_{j,\tau,p}(\sqrt{I-P})P_{2^j}(\,.\,,y)\|_{L^1(yA_p)}$$
$$\leq |A_p(y)|^{1/2}\|(h_{j,\tau} - h_{j,\tau} * \psi_{j,\tau,p})(\sqrt{I-P})P_{2^j}(\,.\,,y)\|_2$$
$$+ |A_p(y)|\,\|h_{j,\tau} * \psi_{j,\tau,p}(\sqrt{I-P})P_{2^j}(\,.\,,y)\|_{L^\infty(A_p(y))}$$
$$\leq |A_p(y)|^{1/2}\|h_{j,\tau} - h_{j,\tau} * \psi_{j,\tau,p}\|_\infty\|P_{2^j}(\,.\,,y)\|_2$$
$$+ |A_p(y)|\frac{c}{|B(y,2^{j/2})|}e^{-2^{p/2}/c}e^{-2^{\tau-j}/c}$$
$$\leq |A_p(y)|^{1/2}2^{((D/2)+\varepsilon)\tau/2}2^{-((D/2)+\varepsilon)p/2}e^{-2^{\tau-j}/c}\|P_{22^j}(\,.\,,y)\|_\infty^{1/2}$$
$$+ c2^{Dp/2}2^{-Dj/2}e^{-2^{p/2}/c}e^{-2^{\tau-j}/c}$$
$$\leq c2^{-\varepsilon(p-j)/2}2^{\varepsilon(\tau-j)/2}2^{-D(\tau-j)/4}e^{-2^{\tau-j}/c} + c2^{D(p-j)/2}e^{-2^{p/2}/c}e^{-2^{\tau-j}/c}$$
$$\leq c2^{-\varepsilon(p-j)/2}e^{-2^{\tau-j}/c}.$$

The lemma follows from (5.3.5), (5.3.6) and (5.3.9). □

An immediate consequence of (5.3.1) and (5.3.4) is the following:

COROLLARY 5.3.3. *There is* $c > 0$ *such that for all* $\tau \in \mathbf{N}$, $1 \le j \le \tau$ *and* $y \in X$

$$\|K_j(I - P_{2^\tau})(.\,, y)\|_{L^1(\{d(x,y) \ge 2^{\tau/2}\})} \le c 2^{-\varepsilon(\tau - j)/2}. \tag{5.3.10}$$

LEMMA 5.3.4. *There is* $c > 0$ *such that for all* $j \ge 1$ *and* $y \in X$

$$\|K_j(I - P)(.\,, y)\|_1 < c 2^{-j}. \tag{5.3.11}$$

PROOF. The proof of (5.3.11) is similar to the proof of (5.3.4). We have

$$K_j(I - P) = \zeta_j(\sqrt{I - P}) P_{2^j}.$$

Let again

$$I(j, p) = \|\zeta_j(\sqrt{I - P}) P_{2^j}(.\,, y)\|_{L^1(A_p(y))}.$$

Then

$$\|K_j(I - P)(.y)\|_1 \le \|\zeta_j(\sqrt{I - P}) P_{2^j}(.\,, y)\|_{L^1(B(y, 4 2^{(j+1)/2}))} + \sum_{p \ge j+4} I(j, p). \tag{5.3.12}$$

We have

$$\begin{aligned} &\|\zeta_j(\sqrt{I - P}) P_{2^j}(.\,, y)\|_{L^1(B(y, 4 2^{(j+1)/2}))} \\ &\quad \le |B(y, 4 2^{(j+1)/2})|^{1/2} \|\zeta_j(\sqrt{I - P}) P_{2^j}(.\,, y)\|_2 \\ &\quad \le |B(y, 4 2^{(j+1)/2})|^{1/2} \|\zeta_j\|_\infty \|P_{2^j}(.\,, y)\|_2 \le c 2^{-j}. \end{aligned} \tag{5.3.13}$$

By (2.1) and (5.1.1), there is a function $\psi_{j,p}$ such that

$$\begin{aligned} &\operatorname{supp} \hat{\psi}_{j,p} \subseteq [-\eta 2^{p/2}, \eta 2^{p/2}] \\ &\|\zeta_j - \zeta_j * \psi_{j,p}\|_\infty \le c 2^{-j} 2^{((D/2)+\varepsilon) j/2} 2^{-((D/2)+\varepsilon) p/2}. \end{aligned} \tag{5.3.14}$$

By (3.2), if $x \in A_p(y)$, then

$$\begin{aligned} &|\zeta_j * \psi_{j,p}(\sqrt{I - P}) P_{2^j}(x, y)| \\ &\quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |\hat{\psi}_{j,p}(t) \hat{\zeta}_j(t) \cos(t\sqrt{I - P}) P_{2^j}(x, y)|\, dt \\ &\quad \le c 2^{-j} \int_{|t| \le \eta 2^{p/2}} |\cos(t\sqrt{I - P}) P_{2^j}(x, y)|\, dt \\ &\quad \le c 2^{-j} \frac{1}{|B(y, 2^{j/2})|} e^{-2^{p/2}/c}. \end{aligned} \tag{5.3.15}$$

If $p \ge j + 4$, then by (5.3.14) and (5.3.15)

$$\begin{aligned} I(j, p) &\le \|(\zeta_j - \zeta_j * \psi_{j,p})(\sqrt{I - P}) P_{2^j}(.\,, y)\|_{L^1(A_p(y))} \\ &\quad + \|\zeta_j * \psi_{j,p}(\sqrt{I - P}) P_{2^j}(.\,, y)\|_{L^1(A_p(y))} \end{aligned} \tag{5.3.16}$$

$$\leq |A_p(y)|^{1/2}\|\zeta_j - \zeta_j * \psi_{j,p}\|_\infty \|P_{2^j}(\,.\,, y)\|_2$$
$$+ |A_p(y)|\, \|\zeta_j * \psi_{j,p}(\sqrt{I-P})P_{2^j}(\,.\,, y)\|_{L^\infty(A_p(y))}$$
$$\leq c2^{-j}2^{\varepsilon j/2}2^{-\varepsilon p/2} + c2^{-j}2^{D(p-j)/2}e^{-2^{p/2}/c}.$$

The lemma follows from (5.3.12), (5.3.13) and (5.3.16). □

An immediate consequence of the previous lemma is the following:

Corollary 5.3.5. *There is $c > 0$ such that for all $y \in X$*

$$\|(I-P)K(\,.\,, y)\|_1 < c. \tag{5.3.17}$$

Another consequence of lemma 5.3.4 is the following:

Corollary 5.3.6. *There is $c > 0$ such that for all $j \geq \tau \geq 0$ and all $y \in X$*

$$\|K_j(I - P_{2^\tau})(\,.\,, y)\|_1 < c2^\tau 2^{-j}. \tag{5.3.18}$$

Proof. We have

$$K_j(I - P_{2^\tau}) = \sum_{0 \leq i < 2^\tau} K_j(I-P)P_i.$$

So (5.3.18) follows from (5.3.11). □

Combining the (5.3.18) and (5.3.10) we have the following:

Corollary 5.3.7. *There is $c > 0$ such that for all $\tau \in \boldsymbol{N}$, $j \geq 0$ and $y \in X$*

$$\|K(I - P_{2^\tau})(\,.\,, y)\|_{L^1(\{d(x,y) \geq 2^{\tau/2}\})} < c. \tag{5.3.19}$$

Lemma 5.3.8. *Let $b \in L^1$ and let us assume that* $\operatorname{supp}(b) \subseteq B(z, r)$ *for some $z \in X$ and $r > 0$. Then there is $c > 0$, independent of b such that for all $u \in L^2$ and all $n \geq r^2$*

$$\langle u, P_n b\rangle \leq c\frac{\|b\|_1}{|B(z,r)|}\langle Mu, \mathbf{1}_{B(z,r)}\rangle \tag{5.3.20}$$

where $Mu(y) = \sup_{r>0}(1/|B(y,r)|)\int_{B(y,r)}|u(x)|\,dx$ *is the Hardy-Littlewood maximal function.*

The above lemma is implicit in [**CD**, pp. 1158–1160]. For reasons of completeness, we reproduce below its proof.

Proof. It follows from (1.1) and (1.2) that for all $x \in X$

$$\begin{aligned} P_n b(x) &\leq \int P_n(x,y)|b(y)|\,dy \\ &\leq \frac{c}{|B(z,\sqrt{n})|}\int_{B(z,r)} e^{-d(x,y)^2/cn}|b(y)|\,dy \\ &\leq \frac{c}{|B(z,\sqrt{n})|}e^{-d(x,z)^2/cn}\|b\|_1 \end{aligned} \tag{5.3.21}$$

$$\le \frac{c}{|B(z,\sqrt{n})|} e^{-d(x,z)^2/cn} \frac{\|b\|_1}{|B(z,r)|} \int \mathbf{1}_{B(z,r)}(y)\,dy$$

$$\le c \frac{\|b\|_1}{|B(z,r)|} \int \frac{1}{|B(y,\sqrt{n})|} e^{-d(x,y)^2/cn} \mathbf{1}_{B(z,r)}(y)\,dy.$$

On the other hand

$$\int \frac{1}{|B(y,\sqrt{n})|} e^{-d(x,y)^2/cn} |u(x)|\,dx \tag{5.3.22}$$

$$= \frac{1}{|B(y,\sqrt{n})|} \left(\int_{B(y,\sqrt{n})} e^{-d(x,y)^2/cn} |u(x)|\,dx + \sum_{p\ge 0} \int_{\sqrt{n}2^p \le d(x,y) \le \sqrt{n}2^{p+1}} e^{-d(x,y)^2/cn} |u(x)|\,dx \right)$$

$$\le \frac{c}{|B(y,\sqrt{n})|} \left(\int_{B(y,\sqrt{n})} |u(x)|\,dx + \sum_{p\ge 0} e^{-2^{2p}/c} \int_{B(y,\sqrt{n}2^{p+1})} |u(x)|\,dx \right)$$

$$\le c\left(1 + \sum_{p\ge 0} \frac{|B(y,\sqrt{n}2^{p+1})|}{|B(y,\sqrt{n})|} e^{-2^{2p}/c} \right) Mu(y)$$

$$\le c\left(1 + \sum_{p\ge 0} 2^{Dp} e^{-2^{2p}/c} \right) Mu(y) \le cMu(y).$$

Making use of (5.3.21) and (5.3.22), we have

$$\langle u, P_n b\rangle = \int u(x) P_n b(x)\,dx$$

$$\le c \frac{\|b\|_1}{|B(z,r)|} \int |u(x)| \left(\int \frac{1}{|B(y,\sqrt{n})|} e^{-d(x,y)^2/cn} \mathbf{1}_{B(z,r)}(y)\,dy \right) dx$$

$$= c \frac{\|b\|_1}{|B(z,r)|} \int \left(\int |u(x)| \frac{1}{|B(y,\sqrt{n})|} e^{-d(x,y)^2/cn}\,dx \right) \mathbf{1}_{B(z,r)}(y)\,dy$$

$$\le c \frac{\|b\|_1}{|B(z,r)|} \int Mu(y) \mathbf{1}_{B(z,r)}(y)\,dy = c \frac{\|b\|_1}{|B(z,r)|} \langle Mu, \mathbf{1}_{B(z,r)}\rangle$$

which proves the lemma. □

COROLLARY 5.3.9. *Let the functions b_i be as in section 5.2. Then there is $c > 0$, independent of f such that*

$$\left\| \sum_{r_i \ge 1} b_i \right\|_2^2 \le ca\|f\|_1. \tag{5.3.23}$$

PROOF. Let $u \in L^2$. Then, by (5.3.20)

$$\left\langle u, \sum_{r_i \geq 1} b_i \right\rangle \leq c \sum_{r_i \geq 1} \frac{\|w_i\|_1}{|B(x_i, r_i)|} \langle Mu, \mathbf{1}_{B(x_i,r_i)} \rangle \tag{5.3.24}$$

$$\leq ca \left\langle Mu, \sum_{r_i \geq 1} \mathbf{1}_{B(x_i,r_i)} \right\rangle$$

$$\leq ca \|Mu\|_2 \left\| \sum_{r_i \geq 1} \mathbf{1}_{B(x_i,r_i)} \right\|_2$$

$$\leq ca \|u\|_2 \left\| \sum_{r_i \geq 1} \mathbf{1}_{B(x_i,r_i)} \right\|_2 .$$

On the other hand, it follows from the properties (3) and (4) of the Calderon-Zygmund decomposition that

$$\left\| \sum_{r_i \geq 1} \mathbf{1}_{B(x_i,r_i)} \right\|_2^2 \leq c \left\| \sum_{r_i \geq 1} \mathbf{1}_{B(x_i,r_i)} \right\|_1 \leq c \sum_{r_i \geq 1} |B(x_i, r_i)| \leq (c/a)\|f\|_1. \tag{5.3.25}$$

(5.3.23) follows from (5.3.24) and (5.3.25). □

5.4. End of the proof of theorem 1.1.

We want to prove that there is $c > 0$ such that for all $f \in L^1$

$$|\{|m(I-P)f| > a\}| \leq c \frac{\|f\|_1}{a}. \tag{5.4.1}$$

With the notation of section 5.2 we have

$$|\{|m(I-P)f| > a\}| \leq |\{|m(I-P)g| > a/4\}| + \left| \left\{ \left| m(I-P) \sum_{r_i \geq 1} b_i \right| > a/4 \right\} \right| \tag{5.4.2}$$

$$+ \left| \left\{ \left| m(I-P) \sum_{r_i \leq 1} \theta_i \right| > a/4 \right\} \right|$$

$$+ \left| \left\{ \left| \sum_{r_i > 1} m(I-P)\theta_i \right| > a/4 \right\} \right|.$$

We have

$$|\{|m(I-P)g| > a/4\}| \leq \frac{16}{a^2} \|m(I-P)g\|_2^2 \leq \frac{c}{a^2} \|g\|_2^2 \tag{5.4.3}$$

$$\leq \frac{c}{a^2} a \left\| \frac{g}{a} g \right\|_1 \leq \frac{c}{a} \|g\|_1 \leq \frac{c}{a} \|f\|_1.$$

By (5.3.23)

$$\left|\left\{\left|m(I-P)\sum_{r_i\geq 1} b_i\right| > a/4\right\}\right| \leq \frac{16}{a^2}\left\|m(I-P)\sum_{r_i\geq 1} b_i\right\|_2^2 \tag{5.4.4}$$

$$\leq \frac{c}{a^2}\left\|\sum_{r_i\geq 1} b_i\right\|_2^2 \leq \frac{c}{a}\|f\|_1.$$

By (5.3.17)

$$\left\|\sum_{0<r_i\leq 1} m(I-P)\theta_i\right\|_1 = \left\|\sum_{r_i\leq 1}(I-P)m(I-P)w_i\right\|_1 \leq c\left\|\sum_{r_i\leq 1} w_i\right\|_1 \leq c\|f\|_1. \tag{5.4.5}$$

Finally, if $r_i > 1$, then it follows from (5.3.19) that

$$\|m(I-P)\theta_i(x)\|_{L^1(X\setminus B(x_i, cr_i))} \leq c\|w_i\|_1.$$

Hence

$$\left|\left\{x \notin \bigcup_i B(x_i, cr_i) : \left|\sum_{r_i>1} m(I-P)\theta_i(x)\right| > \frac{a}{4}\right\}\right| \leq \frac{c}{a}\sum_{r_i>1}\|w_i\|_1 \leq \frac{c}{a}\|f\|_1. \tag{5.4.6}$$

Since by construction

$$\left|\bigcup_i B(x_i, cr_i)\right| \leq \frac{c}{a}\|f\|_1,$$

we can conclude that

$$\left|\left\{\left|\sum_{r_i>1} m(I-P)\theta_i\right| > \frac{a}{4}\right\}\right| \leq \frac{c}{a}\|f\|_1. \tag{5.4.7}$$

(5.4.2), (5.4.3), (5.4.4), (5.4.5) and (5.4.7) prove (5.4.1) and the theorem follows.

6. Differential operators.

Let X be a differentiable manifold endowed with a distance $d(.,.)$ and a σ-finite measure dx satisfying (1.1) and let L be a formally symmetric and non-negative second order differentiable operator on X. Let $P_t(x, y)$ be the associated heat kernel, i.e. the kernel of the heat semigroup e^{-tL} and let us assume that there is $c > 0$ such that

$$P_t(x, y) \leq \frac{c}{|B(x, \sqrt{t})|}\exp\left(-\frac{d(x, y)^2}{ct}\right), \quad x, y \in X,\ t > 0. \tag{6.1}$$

Examples of such operators are the left invariant sub-Laplacians on connected Lie groups of polynomial volume growth and the Laplace-Beltrami operator on Riemannian manifolds with non-negative Ricci curvature.

As before, the operator L admits a spectral resolution $L = \int_0^\infty \lambda \, dE_\lambda$ and given a bounded Borel measurable function m we can define the operator

$$m(L) = \int_0^\infty m(\lambda) \, dE_\lambda.$$

We have the following analog to the Mikhlin-Hörmander multiplier theorem:

THEOREM 6.1. *Let φ be as in Theorem* 1.1 *and let us assume that for some $\varepsilon > 0$*

$$\sup_{t>0} \|\varphi(.)m(t.)\|_{C^{(D/2)+\varepsilon}(\boldsymbol{R})} < \infty. \tag{6.2}$$

Then, $m(L)$ extends to an operator bounded on L^p, $1 < p < \infty$ and from L^1 to weak-L^1.

SKETCH OF THE PROOF. The above result can be proved by arguing in a similar way as in the proof of Theorem 1.1. We shall sketch below the main modifications that one has to make. The details are given in [**A2**].

If f_j is a compactly supported function, then instead of (1.4), we must use the expression

$$f_j(P_{2^j}) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} \hat{f}_j(t) e^{itP_{2^j}} \, dt. \tag{1.4'}$$

Arguing as in the proof of Lemma 3.1, we can prove that there is $c > 0$ and $\eta \in (0,1)$ such that for all $p, j \in \boldsymbol{Z}$, $p \geq j$, $x \in A_p(y)$ and $|t| \leq \eta 2^{(p-j)/2}$

$$|e^{itP_{2^j}} P_{2^j}(x,y)| \leq \frac{c}{|B(y, 2^{j/2})|} e^{-2^{(p-j)/2}/c}. \tag{3.2'}$$

We break, in the same way, the multiplier $m(\lambda)$ into compactly supported multipliers $m_j(\lambda)$ and denote by $K_j(x,y)$ the kernels of the operators $m_j(L)$. The only difference is that now $j \in \boldsymbol{Z}$.

We observe that $m_j(\lambda) = m_j(-2^{-j} \log e^{-2^j\lambda}) e^{2^j\lambda} e^{-2^j\lambda}$ and set

$$h_j(s) = m_j(-2^{-j} \log s) s^{-1}$$

$$\zeta_{j,\tau}(s) = (1 - s^{2^{\tau-j}}) h_j(s), \quad \tau \leq j$$

$$\xi_{j,\tau}(s) = h_j(s) s^{2^{\tau-j}}, \quad \tau \geq j.$$

Then, there is $c > 0$ such that

$$\|h_j\|_{C^{(D/2)+\varepsilon}(\boldsymbol{R})} \leq c \tag{5.1.1'}$$

$$\|\zeta_{j,\tau}\|_{C^{(D/2)+\varepsilon}(\boldsymbol{R})} \leq c 2^{\tau-j}$$

$$\|\xi_{j,\tau}\|_{C^{(D/2)+\varepsilon}(\boldsymbol{R})} \leq c e^{-2^{\tau-j}/c}.$$

Also

$$m_j(L) = h_j(P_{2^j}) P_{2^j}$$

$$(I - P_{2^\tau}) m_j(L) = \zeta_{j,\tau}(P_{2^j}) P_{2^j}$$

$$m_j(L) P_{2^\tau} = \xi_{\tau,j}(P_{2^j}) P_{2^j}.$$

Arguing in a similar way as in the Section 5.3, we can prove that there is $c > 0$ such that for all $p, j \in \boldsymbol{Z}$, $p, \tau \geq j$ and $y \in X$

$$\|K_j(.\,,y)\|_{L^1(A_p(y))} \leq c2^{-\varepsilon(p-j)/2}, \tag{5.3.1$'$}$$

$$\|K_j P_{2^\tau}(.\,,y)\|_1 \leq ce^{-2^{\tau-j}/c}, \tag{5.3.4$'$}$$

which imply that there is $c > 0$ such that for all $\tau, j \in \boldsymbol{Z}$, $j \leq \tau$ and $y \in X$

$$\|K_j(I - P_{2^\tau})(.\,,y)\|_{L^1(\{d(x,y) \geq 2^{\tau/2}\})} \leq c2^{-\varepsilon(\tau-j)/2}. \tag{5.3.10$'$}$$

Finally, we prove that there is $c > 0$ such that for all $j, \tau \in \boldsymbol{Z}$, $j \geq \tau$ and $y \in X$

$$\|K_j(I - P_{2^\tau})(.\,,y)\|_1 < c2^{\tau-j}. \tag{5.3.18$'$}$$

(5.3.10′) and (5.3.18′) imply that there is $c > 0$ such that for all $j, \tau \in \boldsymbol{N}$ and $y \in X$

$$\|K(I - P_{2^\tau})(.\,,y)\|_{L^1(\{d(x,y) \geq 2^{\tau/2}\})} < c. \tag{5.3.19$'$}$$

Once we have proved these estimates, we proceed by performing a Calderon-Zygmund decomposition as before. We define the functions w_i exactly in the same way and we set

$$b_i = P_{r_i^2} w_i$$

$$\theta_i = w_i - b_i$$

$$g = \mathbf{1}_{X \setminus \bigcup_i B(x_i, r_i)} f.$$

We finish the proof by arguing in the same way as in Section 5.4.

6.1. Examples of spectral multipliers.

Particularly interesting examples of spectral multipliers are the Riesz means and the oscillating multipliers (see for example [**DC**], [**Fe**], [**FeS**]).

The Riesz means of order $a > 0$, are defined as the family of operators

$$m_{a,R}(L) = \int_0^\infty \left(1 - \frac{\lambda}{R}\right)_+^a dE_\lambda, \quad R > 0.$$

Oscillating multipliers are multipliers of the type

$$m_{a,b}(\lambda) = \psi(|\lambda|)|\lambda|^{-b/2} e^{i|\lambda|^{a/2}}, \quad a, b > 0,$$

where $\psi(\lambda)$ is a C^∞ function, such that $\psi(\lambda) = 0$, for $|\lambda| \leq 1$ and $\psi(\lambda) = 1$, for $|\lambda| \geq 2$. These multipliers provide examples of strongly singular operators. They are intimately related to the Cauchy problem for the wave and Schrödinger operators. They are also related to the Riesz means for the Schrödinger operator, which are defined by

$$I_{k,a}(L) = kt^{-k} \int_0^t (t-s)^{k-1} e^{isL^{a/2}}\, ds, \quad k, a > 0.$$

In [**A4**], [**AL**], we have studied the above operators, using the finite propagation speed for the wave operator. Our results can also be reproduced by using the previous arguments.

6.2. Higher order differential operators.

Let us assume that L is a higher order differential operator, of order $2m$ and that the associated heat kernel $P_t(x, y)$ satisfies the estimate

$$|P_t(x,y)| \leq \frac{c}{|B(x,t^{1/2m})|} \exp\left(-\frac{d(x,y)^{2m/(2m-1)}}{ct^{1/(2m-1)}}\right), \quad x,y \in X,\ t>0.$$

Such operators have been studied, for example, by [**BD**], [**Da**], in the context of $\boldsymbol{R}^n$ and by [**Ro**], in the context of Lie groups of polynomial volume growth. Theorem 6.1 is still valid for such operators and the proof is essentially the same (we just have to make the obvious modification in the definition of the sets $A_p(x)$). An extension of Theorem 6.1 to this family of operators has been obtained in [**DO**], by means of the Davies-Helffer-Sjöstrand functional calculus. The order of differentiability required is $(D/2)+2$. So using our arguments we can slightly improve that result.

6.3. Homogeneous dimensions.

Let us assume that there are constants $D_0, D_\infty \geq 0$ and $c > 0$ such that for all $x \in X$

$$\frac{1}{c} r^{D_0} \leq |B(x,r)| \leq c r^{D_0}, \quad 0<r\leq 1, \quad \frac{1}{c} r^{D_\infty} \leq |B(x,r)| \leq c r^{D_\infty}, \quad r \geq 1.$$

This is the case for example when L is a left invariant sub-Laplacian on a nilpotent Lie group. D_0 and $D_\infty \geq 0$ are called, respectively, the local homogeneous dimension and the homogeneous dimension at infinity.

The condition (6.2) can be modified to reflect the existence of these two different dimensions (cf. [**A1**]). For example, if $D_0 > D_\infty$, then (6.2) can be replaced by

$$\sup_{0<t\leq 1} \|\varphi(.)m(t.)\|_{C^{(D_0/2)+\varepsilon}(\boldsymbol{R})} < \infty, \quad \sup_{t>1} \|\varphi(.)m(t.)\|_{C^{(D_\infty/2)+\varepsilon}(\boldsymbol{R})} < \infty.$$

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