J. Math. Soc. Japan
Vol. 56, No. 3, 2004

Socle deformations of selfinjective algebras of tubular type

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(Received Jan. 22, 2003)

Abstract. We classify all selfinjective finite dimensional algebras over an algebraically closed field which are socle equivalent to the tame selfinjective algebras which admit simply connected Galois coverings and whose Auslander-Reiten quiver consists only of stable tubes.

1. Introduction.

Throughout the paper K will denote a fixed algebraically closed field. By an algebra we mean a finite dimensional K-algebra with an identity, which we shall assume (without loss of generality) to be basic and connected. For an algebra Λ, we denote by $\operatorname{mod}\Lambda$ the category of finite dimensional right Λ-modules and by D the standard duality $\operatorname{Hom}_K(-,K)$ on $\operatorname{mod}\Lambda$. An algebra Λ is called *selfinjective* if $\Lambda \cong D(\Lambda)$ in $\operatorname{mod}\Lambda$, that is, the projective Λ-modules are injective.

We are concerned with the problem of classification of all selfinjective algebras whose stable Auslander-Reiten quiver consists only of stable tubes. A large class of such algebras is provided by the selfinjective algebras of tubular type. By [**3**], a *selfinjective algebra of tubular type* is a tame selfinjective having a simply connected Galois covering and the stable Auslander-Reiten quiver consisting only of stable tubes (see Section 2 for more details). We would like to mention that there are also wild selfinjective algebras whose stable Auslander-Reiten quiver consists only of stable tubes (see [**1**], [**9**]).

If Λ is a selfinjective algebra, then the left and the right socle of Λ coincide, and we denote them by $\operatorname{soc}\Lambda$. Two selfinjective algebras Λ and A are said to be *socle equivalent* if the factor algebras $\Lambda/\operatorname{soc}\Lambda$ and $A/\operatorname{soc}A$ are isomorphic. Frequently, selfinjective algebras are socle equivalent to (socle deformations of) selfinjective algebras having simply connected Galois coverings, and then we may reduce the study of such algebras and their representations to that for the corresponding algebras of finite global dimension. This is the case for all selfinjective algebras of finite representation type (see [**6**], [**15**], [**16**]). We also note that if a selfinjective algebra Λ is socle equivalent to a selfinjective algebra A of tubular type then Λ is tame and the stable Auslander-Reiten quiver of Λ consists only of stable tubes.

The main aim of this paper is to prove the following theorem.

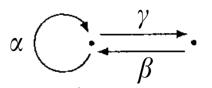
Theorem 1.1. *Let Λ be a basic connected selfinjective K-algebra. Then Λ is socle equivalent to a selfinjective algebra of tubular type if and only if exactly one of the following cases holds*:

2000 *Mathematics Subject Classification*. Primary 16D50, 16G60, 16G70.

Key Words and Phrases. selfinjective algebra, repetitive algebra, tubular algebra, socle equivalence.

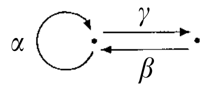
Supported by the Foundation for Polish Science and the Polish Scientific Grant KBN No. 5 P03A 008 21.

(1) *Λ is of tubular type,*
(2) *K is of characteristic 3 and Λ is isomorphic to one of the bound quiver algebras*



$\alpha^2 = \beta\gamma$
$\gamma\alpha\beta = \gamma\alpha^2\beta$
$\beta\gamma\alpha\beta = 0$
$\gamma\alpha\beta\gamma = 0$

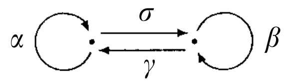
Λ_1



$\gamma\alpha^2 = 0, \quad \alpha^2\beta = 0$
$\gamma\beta\gamma = 0, \quad \beta\gamma\beta = 0$
$\gamma\beta = \gamma\alpha\beta$
$\alpha^3 = \beta\gamma$

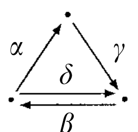
Λ_2

(3) *K is of characteristic 2 and Λ is isomorphic to one of the bound quiver algebras*



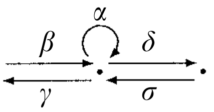
$\alpha^4 = 0, \quad \alpha^2\gamma = 0, \quad \sigma\alpha^2 = 0$
$\alpha^2 = \gamma\sigma + \alpha^3, \quad \lambda\beta^2 = \sigma\gamma$
$\alpha\gamma = \gamma\beta, \quad \beta\sigma = \sigma\alpha$

$\Lambda_3(\lambda)$
$\lambda \in K\backslash\{0,1\}$



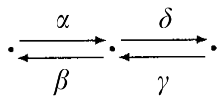
$\delta\beta\delta = \gamma\alpha, \quad (\beta\delta)^3\beta = 0$
$\gamma\alpha\beta\gamma = 0, \quad \alpha\beta\gamma\alpha = 0$
$\alpha\beta\gamma = \alpha\beta\delta\beta\gamma$

Λ_4



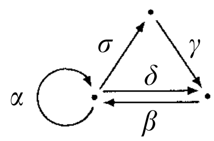
$\alpha^2 = \beta\gamma, \quad \alpha^3 = \sigma\delta, \quad \delta\beta = 0$
$\gamma\sigma = 0, \quad \delta\alpha = 0, \quad \alpha\sigma = 0$
$\gamma\beta\gamma = 0, \quad \beta\gamma\beta = 0, \quad \gamma\beta = \gamma\alpha\beta$

Λ_5



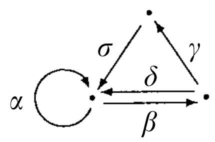
$\delta\gamma\delta\alpha = 0, \quad \beta\gamma\delta\gamma = 0$
$\alpha\beta\alpha = 0, \quad \beta\alpha\beta = 0$
$\beta\alpha = \beta\gamma\delta\alpha$
$\alpha\beta = \gamma\delta\gamma\delta$

Λ_6



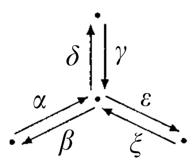
$\delta\beta = \delta\alpha\beta, \quad \sigma\alpha = 0, \quad \delta\alpha = \gamma\sigma$
$\alpha\beta\gamma = 0, \quad \alpha^2 = \beta\delta, \quad \delta\beta\gamma = 0$
$\beta\delta\beta = 0, \quad \delta\beta\delta = 0$

Λ_7



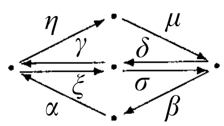
$\beta\delta = \beta\alpha\delta, \quad \alpha\sigma = 0, \quad \alpha\delta = \sigma\gamma$
$\gamma\beta\alpha = 0, \quad \alpha^2 = \delta\beta, \quad \gamma\beta\delta = 0$
$\beta\delta\beta = 0, \quad \delta\beta\delta = 0$

Λ_8



$\alpha\beta + \gamma\delta + \xi\varepsilon = 0$
$\delta\gamma = 0, \quad \varepsilon\xi = 0, \quad \alpha\beta\alpha = 0$
$\beta\alpha\beta = 0, \quad \beta\alpha = \beta\gamma\delta\alpha$

Λ_9



$\beta\mu = 0, \quad \eta\alpha = 0, \quad \alpha\beta = \gamma\delta$
$\sigma\xi = \mu\eta, \quad \delta\sigma = \xi\gamma + \delta\sigma\delta\sigma$
$\sigma\delta\sigma\delta = 0, \quad \gamma\xi\gamma\xi = 0$

Λ_{10}

The algebras Λ_1 and Λ_2 have been already discovered in [**18**], [**19**]. We refer to [**3**], [**4**] and [**19**] for a description of selfinjective algebras of tubular type. A prominent role in the proof of the above theorem is also played by results one socle deformations of selfinjective algebras established in [**22**] and [**23**].

For basic background on the representation theory of algebras we refer to [**2**], [**17**], and on selfinjective algebras to [**8**], [**24**].

2. Selfinjective algebras of tubular type.

An important class of selfinjective algebras is formed by the algebras of the form $\hat{B}/G$ where $\hat{B}$ is the repetitive algebra [**11**] (locally finite dimensional, without identity)

$$\hat{B} = \bigoplus_{m \in \mathbf{Z}} (B_m \oplus Q_m)$$

of an algebra B, where $B_m = B$ and $Q_m = D(B)$ for all $m \in \mathbf{Z}$, the multiplication in $\hat{B}$ is defined by

$$(a_m, f_m)_m \cdot (b_m, g_m)_m = (a_m b_m, a_m g_m + f_m b_{m-1})_m$$

for $a_m, b_m \in B_m$, $f_m, g_m \in Q_m$, and G is an admissible group of automorphisms of $\hat{B}$.

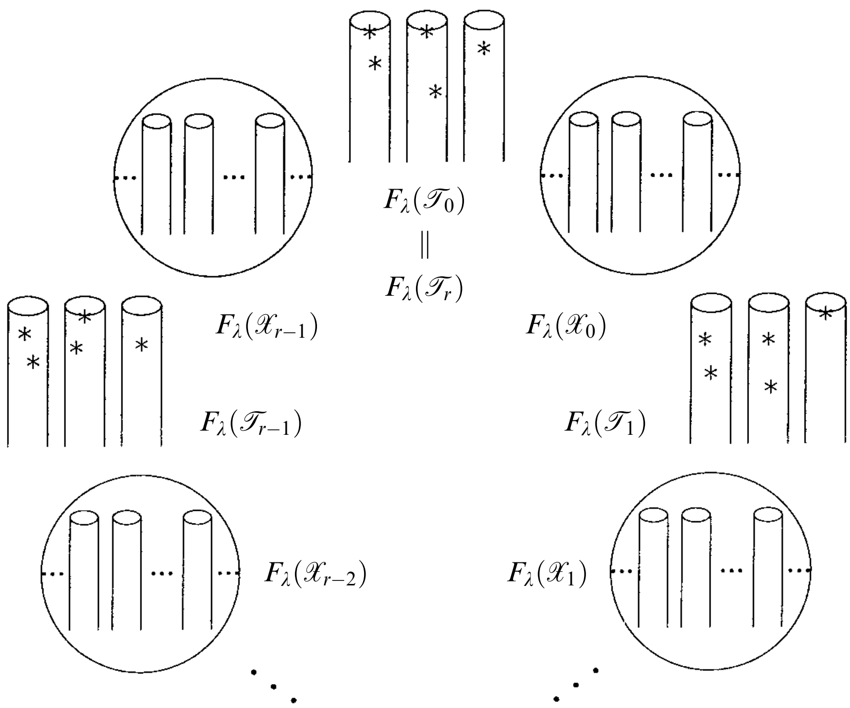
Let B be an algebra and $\mathscr{E} = \{e_i \mid 1 \le i \le m\}$ a fixed set of primitive orthogonal idempotents of B with $1_B = e_1 + \cdots + e_n$. Then we have the canonical set $\hat{\mathscr{E}} = \{e_{j,k} \mid 1 \le j \le n, k \in \mathbf{Z}\}$ of primitive orthogonal idempotents of $\hat{B}$ such that $e_{j,k}\hat{B} = (e_j B)_k \oplus (e_j D(B))_k$ for $1 \le j \le n$ and $k \in \mathbf{Z}$. By an *automorphism* of $\hat{B}$ we mean a K-algebra automorphism of $\hat{B}$ which fixes the chosen set $\hat{\mathscr{E}}$ of primitive orthogonal idempotents of $\hat{B}$. *A* group G of automorphisms of $\hat{B}$ is said to be *admissible* if the induced action of G on $\hat{\mathscr{E}}$ is free and has finitely many orbits. Then the orbit algebra $\hat{B}/G$ is a selfinjective algebra and the G-orbits in $\hat{\mathscr{E}}$ form a canonical set of primitive orthogonal idempotents of $\hat{B}/G$ whose sum is the identity of $\hat{B}/G$ ([**10**]). Moreover, there are a *Galois covering* $F : \hat{B} \to \hat{B}/G$ and the associated *push-down functor* $F_\lambda : \operatorname{mod} \hat{B} \to \operatorname{mod} \hat{B}/G$ ([**5**]). We denote by $\nu_{\hat{B}}$ the *Nakayama automorphism* of $\hat{B}$ such that $\nu_{\hat{B}}(e_{j,k}) = e_{j,k+1}$ for all $1 \le j \le n$, $k \in \mathbf{Z}$. Then the infinite cyclic group $(\nu_{\hat{B}})$ generated by $\nu_{\hat{B}}$ is admissible and $\hat{B}/(\nu_{\hat{B}})$ is the trivial extension of $B \ltimes D(B)$ of B by $D(B)$. An automorphism φ of $\hat{B}$ is said to be *positive* (respectively, *rigid*) when, for each $j \in \{1, \ldots, n\}$, $k \in \mathbf{Z}$, we have $\varphi(e_{j,k}) = e_{m,r}$ for some $m \in \{1, \ldots, n\}$, and $r \ge k$ (respectively, $\varphi(e_{j,k}) = e_{m,k}$ for some $m \in \{1, \ldots, n\}$). We refer to [**14**] for some results on the structure of automorphisms of repetitive algebras, and to [**23**] for results on the presentations of selfinjective algebras A in the form $A \cong \hat{B}/(\varphi \nu_{\hat{B}})$ with B an algebra and φ a positive automorphism of $\hat{B}$.

Following [**17**] by a *tubular algebra* we mean a tubular extension (equivalently, tubular coextension) B of a tame concealed algebra C of tubular type $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$. Then the rank of the Grothendieck group $K_0(B)$ of B is equal to 6, 8, 9, or 10, respectively. By a *selfinjective algebra of tubular type* we mean an algebra of the form $\hat{B}/G$, where B is a tubular algebra and G is an admissible group of automorphisms of $\hat{B}$. This is the class of all nondomestic polynomial growth algebras having simply connected Galois coverings [**19**]. Moreover, it has been recently shown [**3**] that a selfinjective algebra A is of tubular type if and only if A is tame, admits a simply connected Galois covering, and the stable Auslander-Reiten quiver of A consists only of tubes. We shall exhibit here basic facts on the repetitive algebras of tubular algebras and selfinjective algebras of tubular type, established in [**13**] and [**19**], needed in our further considerations.

Let B be a tubular algebra of tubular type $n_B = (n_\lambda)_{\lambda \in \mathbf{P}_1(K)}$ consisting of positive integers n_λ, $\lambda \in \mathbf{P}_1(K)$, and all but finitely many equal 1. We shall write instead of $(n_\lambda)_{\lambda \in \mathbf{P}_1(K)}$ the finite sequence consisting of all n_λ which are larger than 1, and arranged in nondecreasing order. Then n_B is one of the types $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$. It follows from [**13**, Section 3] that the Auslander-Reiten quiver $\Gamma_{\hat{B}}$ of $\hat{B}$ is of the form

$$\Gamma_{\hat{B}} = \bigvee_{p \in \mathbf{Z}} \mathscr{T}_p \vee \mathscr{X}_p$$

where, for each $p \in \boldsymbol{Z}$, $\mathscr{T}_p$ is a nonstable $\boldsymbol{P}_1(K)$-family of quasi-tubes (in the sense of [**20**, (1.2)]) whose stable part $\mathscr{T}_p^s$ is a $\boldsymbol{P}_1(K)$-family of stable tubes of tubular type n_B, $\mathscr{X}_p = \bigvee_{\gamma \in \boldsymbol{Q}_{p+1}^p} \mathscr{T}_\gamma$, $\boldsymbol{Q}_{p+1}^p = \boldsymbol{Q} \cap (p, p+1)$, and, for each $\gamma \in \boldsymbol{Q}_{p+1}^p$, $\mathscr{T}_\gamma$ is a $\boldsymbol{P}_1(K)$-family of stable tubes of tubular type n_B. Further, there exists $s \geq 3$ such that $\nu_{\hat{B}}(\mathscr{T}_p) = \mathscr{T}_{p+s}$ and $\nu_{\hat{B}}(\mathscr{X}_p) = \mathscr{X}_{p+s}$ for all $p \in \boldsymbol{Z}$. In particular, the stable Auslander-Reiten quiver $\Gamma_{\hat{B}}^s$ of $\hat{B}$ consists of the rational family of $\boldsymbol{P}_1(K)$-families of stable tubes, all of them are of tubular type n_B. Let G be an admissible group of automorphisms of $\hat{B}$ and $A = \hat{B}/G$ the associated selfinjective algebra (of tubular type n_B). Since G is, by [**19**, Proposition 2.2], infinite cyclic (hence torsion-free) the push-down functor $F_\lambda : \operatorname{mod} \hat{B} \to \operatorname{mod} \hat{B}/G = \operatorname{mod} A$ associated to the Galois covering $F : \hat{B} \to \hat{B}/G = A$ preserves the indecomposable modules and Auslander-Reiten sequences [**10**, Section 3]. Moreover, $\hat{B}$ is locally support-finite [**13**, Section 3], and hence invoking the main result of [**7**] we conclude that $F_\lambda : \operatorname{mod} \hat{B} \to \operatorname{mod} A$ is dense. As a consequence, the Auslander-Reiten quiver Γ_A of A is the orbit quiver $\Gamma_{\hat{B}}/G$, and so is obtained from $\Gamma_{\hat{B}}$ by identifying (via F_λ) $\mathscr{T}_p$ with $\mathscr{T}_{p+r}$ and $\mathscr{X}_p$ with $\mathscr{X}_{p+r}$ for some $r \geq 1$ and all $p \in \boldsymbol{Z}$. Thus Γ_A has the following "clock structure":



3. Selfinjective algebras with standard stable tubes.

The following known fact gives a characterization of selfinjective algebras of tubular type whose Auslander-Reiten quiver admits a (generalized) standard stable tube (in the sense of [**17**], [**21**]).

Proposition 3.1. *The Auslander-Reiten quiver of a selfinjective algebra A of tubular type admits a (generalized) standard stable tube if and only if $A \cong \hat{B}/(\psi\nu_{\hat{B}})$, for a tubular algebra B and a positive automorphism ψ of $\hat{B}$.*

Proof. See [**13**], [**17**, Section 5], and [**19**, Section 3]. □

The main aim of this section is to prove that this class of selfinjective algebras is invariant with respect to socle equivalences.

Proposition 3.2. *Let A be a selfinjective algebra of the form $\hat{B}/(\psi\nu_{\hat{B}})$, for a tubular algebra B and a positive automorphism ψ of $\hat{B}$, and Λ be a selfinjective algebra socle equivalent to A. Then Λ is isomorphic to an algebra of the form $\hat{B}/(\varphi\nu_{\hat{B}})$ for a positive automorphism φ of $\hat{B}$. In particular, Λ is of tubular type.*

Proof. It is known [**2**, (V.5.5)] that if P is an indecomposable projective Λ-module then we have in $\operatorname{mod}\Lambda$ an Auslander-Reiten sequence of the form

$$0 \to \operatorname{rad} P \to P \oplus \operatorname{rad} P/\operatorname{soc} P \to P/\operatorname{soc} P \to 0.$$

Since by our assumption $\Lambda/\operatorname{soc}\Lambda \cong A/\operatorname{soc} A$ we conclude that the Auslander-Reiten quivers Γ_Λ and Γ_A are isomorphic. Further, since $A = \hat{B}/(\psi\nu_{\hat{B}})$ with ψ a positive automorphism of $\hat{B}$, B is a factor algebra of $A/\operatorname{soc} A$, and consequently there is an ideal I in Λ such that Λ/I is isomorphic to B. Without loss of generality we may assume that $\Lambda/I = B$. We may choose a complete set $\{e_i, 1 \le i \le s\}$ of primitive orthogonal idempotents of Λ such that $1 = e_1 + \cdots + e_s$ and $\{e_i, 1 \le i \le t\}$, for some $t \le s$, is the subset of $\{e_i, 1 \le i \le s\}$ consisting of all idempotents e_i which are not in I. Then $e = e_1 + \cdots + e_t$ is an idempotent of Λ such that $e + I$ is the identity of $B = \Lambda/I$, called a residual identity of B. We note that such an idempotent e is uniquely determined by I up to an inner automorphism of Λ, $B \cong e\Lambda e/eIe$ naturally and $1 - e \in I$. Since Γ_Λ and Γ_A are isomorphic, it follows from the description of the Auslander-Reiten quivers of selfinjective algebras of tubular type, presented in Section 2, that Γ_Λ has the following "clock structure"

$$\begin{array}{ccccc} & & \mathscr{T}_0^* & & \\ & \mathscr{X}_{r-1}^* & & \mathscr{X}_0^* & \\ \mathscr{T}_{r-1}^* & & & & \mathscr{T}_1^* \\ & \mathscr{X}_{r-1}^* & & \mathscr{X}_1^* & \\ & \ddots & & \iddots & \end{array}$$

where, for each $p \in \{0, \dots, r-1\}$, $\mathscr{T}_p^*$ is a nonstable $\boldsymbol{P}_1(K)$-family of quasi-tubes and $\mathscr{X}_p^* = \bigvee_{\gamma \in \boldsymbol{Q}_{p+1}^p} \mathscr{T}_\gamma^*$, $\boldsymbol{Q}_{p+1}^p = \boldsymbol{Q} \cap (p, p+1)$, and, for $\gamma \in \boldsymbol{Q}_{p+1}^p$, $\mathscr{T}_\gamma^*$ is a $\boldsymbol{P}_1(K)$-family of stable tubes. In fact, since $A = \hat{B}/(\psi\nu_{\hat{B}})$ with ψ a positive automorphism of $\hat{B}$, we have $r \ge 3$ (see [**13**]). Further, since Γ_Λ and Γ_A are isomorphic, it follows from [**13**, Section 3] that there is $m \in \{2, \dots, r-1\}$ such that the (indecomposable) projective modules in the $\boldsymbol{P}_1(K)$-families $\mathscr{T}_0^*, \mathscr{T}_1^*, \dots, \mathscr{T}_m^*$ are the indecomposable projective Λ-modules of the form $e_i\Lambda$ with $i \in \{1, \dots, t\}$, and $\mathscr{X}_m^* = \bigvee_{\gamma \in \boldsymbol{Q}_{m+1}^m} \mathscr{T}_\gamma^*$ consists of all sincere stable tubes of the Auslander-Reiten quiver Γ_B of B (see [**17**, Section 5]). In fact, by [**17**, (5.2)(2)], every $\boldsymbol{P}_1(K)$-family $\mathscr{T}_\gamma^*$ is a separating family of stable tubes of Γ_B (in the sense of [**17**, (3.1)]), and consequently all tubes in $\mathscr{T}_\gamma^*$ are faithful stable tubes of Γ_B. Fix $\gamma \in \boldsymbol{Q}_{m+1}^m$ and put $\mathscr{T} = \mathscr{T}_\gamma^*$. Since $B = \Lambda/I$, then I is the annihilator $\operatorname{ann}_\Lambda(\mathscr{T})$ of $\mathscr{T}$ in Λ, that is, the intersection of annihilators $\operatorname{ann}_\Lambda(M)$ of all indecomposable modules M in $\mathscr{T}$. Denote by J the trace ideal of $\mathscr{T}$ in Λ, that is, the two-sided ideal of Λ generated by the images of all Λ-homomorphisms from modules in $\mathscr{T}$ to Λ. We claim that $J \subseteq I$. Take $\lambda \in J$ and a Λ-homomorphism $f : \Lambda \to X$ with X in $\mathscr{T}$. We show that $f(\lambda) = 0$. Suppose that $f(\lambda) \neq 0$. Since $\lambda \in J$, there is a Λ-homomorphism $g : N \to \Lambda$ such that N is a direct sum of a finite number of indecomposable modules from $\mathscr{T}$ and $\lambda = g(n)$ for some $n \in N$. Then $fg : N \to X$ is a nonzero morphism which factorizes through the projective Λ-module Λ_Λ, and consequently belongs to the infinite radical $\operatorname{rad}^\infty(\operatorname{mod}\Lambda)$ of $\operatorname{mod}\Lambda$, because $\mathscr{T}$ does not contain projective modules. But then fg is a nonzero morphism from the infinite radical $\operatorname{rad}^\infty(\operatorname{mod} B)$ of $\operatorname{mod} B$ (see [**2**, (V.7)]). This contradicts the fact that $\mathscr{T}$ is a (generalized) standard family of stable tubes of Γ_B (see [**17**, (5.2)]). Therefore, every $\lambda \in J$ belongs to the intersection of kernels of all Λ-homomorphisms $f : \Lambda \to M$ with $M \in \mathscr{T}$, and so $J \subseteq I$. Since $I = \operatorname{ann}\mathscr{T}$ and J is the trace ideal of $\mathscr{T}$ in Λ, we have also $JI = 0$, and hence J is a B-submodule of Λ. We shall prove now that J is isomorphic to the injective cogenerator $D(B)_B$ of $\operatorname{mod} B$. For each $i \in \{1, \dots, t\}$, we have a commutative diagram of monomorphisms

$$\begin{array}{ccc} e_i\Lambda/e_i(\operatorname{rad}\Lambda) = e_iB/e_i(\operatorname{rad} B) & \hookrightarrow & D(Be_i) \\ \downarrow & \swarrow & \\ \Lambda & & \end{array}$$

and hence Λ contains a B-submodule Q_i which is isomorphic to the indecomposable injective B-module $D(Be_i)$. On the other hand, $\mathscr{T}$ is a faithful family of tubes in Γ_B, and hence there is an epimorphism $N_i^{r_i} \to Q_i$ for some $r_i \geq 1$ and a direct sum N_i of finitely many indecomposable modules from $\mathscr{T}$. Hence $Q_i \subseteq J$. Finally, observe that the largest B-submodule of Λ_Λ is a minimal injective cogenerator in $\operatorname{mod} B$. Therefore, we have $J = \bigoplus_{1 \leq i \leq t} Q_i \cong D(B)_B$. Our next aim is to prove the equality $Ie = J$. Observe that $J = Je \subseteq Ie$ because J is a B-module and $J \subseteq I$. Since $\operatorname{soc} J \cong \operatorname{soc} D(B)_B$ and $\operatorname{soc} Ie = (\operatorname{soc} I)e \cong \operatorname{Hom}_\Lambda(e\Lambda, \operatorname{soc} I)$ is a B-module, we have $\operatorname{soc} Ie = \operatorname{soc} J$. Therefore, it is enough to show that $J/\operatorname{soc} J = Ie/\operatorname{soc} Ie$. Observe also that $I/\operatorname{soc} I \cong \operatorname{ann}_{\Lambda/\operatorname{soc}\Lambda}(\mathscr{T})$, $Ie/\operatorname{soc} Ie = (I/\operatorname{soc} I)e \cong \operatorname{Hom}_{\Lambda/\operatorname{soc}\Lambda}(e(\Lambda/\operatorname{soc}\Lambda), I/\operatorname{soc} I)$, and $e(\Lambda/\operatorname{soc}\Lambda) = e\Lambda/e \operatorname{soc}\Lambda$. Take a nonzero homomorphism $f : e\Lambda/e\operatorname{soc}\Lambda \to I/\operatorname{soc} I$ in $\operatorname{mod}\Lambda/\operatorname{soc}\Lambda$. Since $\Lambda/\operatorname{soc}\Lambda \cong A/\operatorname{soc} A$, $A = \hat{B}/(\psi\nu_{\hat{B}})$, ψ is a positive automorphism of $\hat{B}$ and there is a Galois covering of module categories $F_\lambda : \operatorname{mod}\hat{B} \to \operatorname{mod}\hat{B}/(\psi\nu_{\hat{B}})$, it follows from the structure of $\operatorname{mod}\hat{B}$ described in [**13**, Section 3] that f factorizes through a direct sum of modules lying in $\mathscr{T}$, and consequently its image is contained in $J/\operatorname{soc} J$. Then $Ie/\operatorname{soc} Ie \subseteq J/\operatorname{soc} J$, and so $Ie/\operatorname{soc} Ie = J/\operatorname{soc} I$. Hence $J = Ie$. In particular, we have $IeI = JI = 0$. We also note that the ordinary quiver Q_B of B has no oriented cycles, because B is a tubular algebra. Summing up our considerations above, we have proved that the following conditions are satisfied:

(1) The ordinary quiver Q_B of B has no oriented cycles,

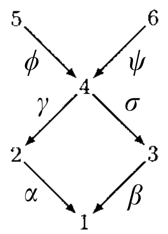
(2) $IeI = 0$,

(3) Ie is an injective cogenerator in $\operatorname{mod} B$.

We note that then I is a deforming ideal in the sense of [**22**, (2.1)]. Applying [**23**, Theorem 4.1] we obtain that Λ is isomorphic to $\hat{B}/(\varphi\nu_{\hat{B}})$, for a positive automorphism φ of $\hat{B}$. In particular, Λ is a selfinjective algebra of tubular type. □

4. Selfinjective algebras without standard stable tubes.

It follows from Proposition 3.1 and [**19**, Section 3] that the Auslander-Reiten quiver of a selfinjective algebra A of tubular type has no (generalized) standard stable tubes if and only if A is of the form $A = \hat{B}/(\sigma\varphi^k)$, where B is a tubular algebra, σ is a rigid automorphism of $\hat{B}$, φ is an automorphism of $\hat{B}$ such that $\varphi^l = \varrho\nu_{\hat{B}}$ for some $l \geq 2$ and a rigid automorphism ϱ of $\hat{B}$, and $1 \leq k < l$. We call such a selfinjective algebra of tubular type *exceptional*, and *normal* otherwise. Following [**19**] a tubular algebra B is said to be *exceptional* if there is an automorphism φ of $\hat{B}$ such that $\varphi^d = \varrho\nu_{\hat{B}}$ for some $d \geq 2$ and a rigid automorphism ϱ of $\hat{B}$, and *normal* otherwise. Consider the following family of bound quiver algebras (where a dotted line means that the sum of paths indicated by this line is zero if it indicates exactly three parallel paths, the commutativity of paths if it indicates exactly two parallel paths, and the zero path if it indicates only one path):

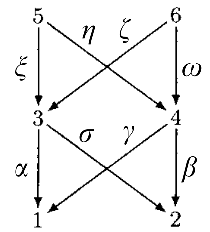


$\alpha\gamma\phi = \beta\sigma\phi$

$\alpha\gamma\psi = \lambda\beta\sigma\psi$

$B_1(\lambda)$

$\lambda \in K\backslash\{0, 1\}$

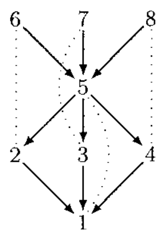


$\alpha\xi = \gamma\eta, \quad \alpha\zeta = \gamma\omega$

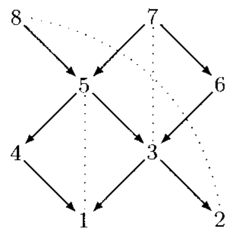
$\sigma\xi = \beta\eta, \quad \sigma\zeta = \lambda\beta\omega$

$B_2(\lambda)$

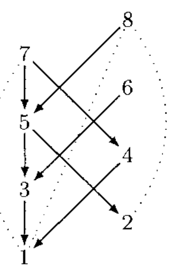
$\lambda \in K\backslash\{0, 1\}$



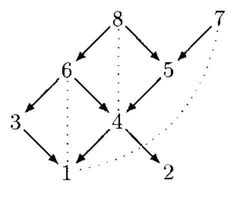
B_3



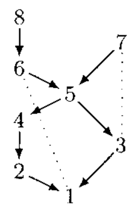
B_4



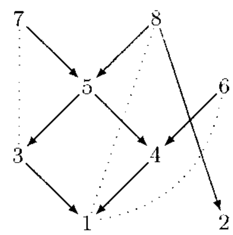
B_5



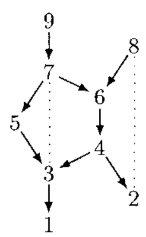
B_6



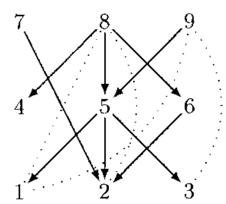
B_7



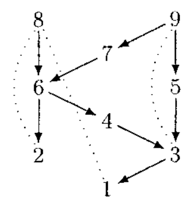
B_8



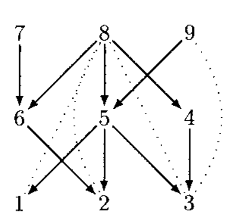
B_9



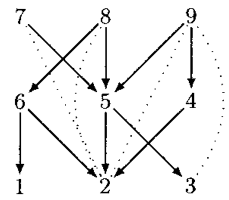
B_{10}



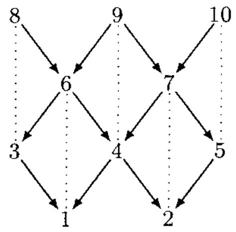
B_{11}



B_{12}



B_{13}



B_{14}

Then we have the following description of exceptional tubular algebras and their repetitive algebras.

Theorem 4.1. *Let B be a tubular algebra. Then the following equivalences hold*:

(i) *B is exceptional of tubular type $(2,2,2,2)$ if and only if $\hat{B}$ is isomorphic to $\widehat{B_1(\lambda)}$ or $\widehat{B_2(\lambda)}$, for some $\lambda \in K \backslash \{0,1\}$.*

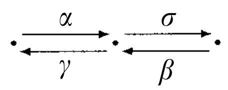
(ii) *B is exceptional of tubular type $(3,3,3)$ if and only if $\hat{B}$ is isomorphic to $\hat{B}_3$, $\hat{B}_4$, $\hat{B}_5$, $\hat{B}_6$, $\hat{B}_7$, or $\hat{B}_8$.*

(iii) *B is exceptional of tubular type $(2,4,4)$ if and only if $\hat{B}$ is isomorphic to $\hat{B}_9$, $\hat{B}_{10}$, $\hat{B}_{11}$, $\hat{B}_{12}$, or $\hat{B}_{13}$.*

(iv) *B is exceptional of tubular type $(2,3,6)$ if and only if $\hat{B}$ is isomorphic to $\hat{B}_{14}$.*

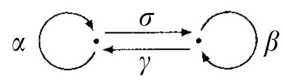
Proof. For tubular types $(2,2,2,2)$, $(3,3,3)$ and $(2,4,4)$ it is proved in [**3**, (4.1),(5.1),(6.1)], and for tubular type $(2,3,6)$ in [**12**, (5.3)]. □

In order to describe the exceptional selfinjective algebras of tubular type, consider the following family of bound quiver algebras:



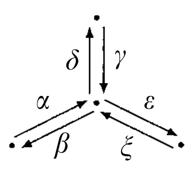
$\alpha\gamma\alpha = \beta\sigma\alpha$
$\alpha\gamma\beta = \lambda\beta\sigma\beta$
$\gamma\alpha\gamma = \gamma\beta\sigma$
$\sigma\alpha\gamma = \lambda\sigma\beta\sigma$

$A_1(\lambda)$
$\lambda \in K \backslash \{0,1\}$



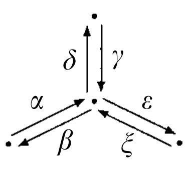
$\alpha^2 = \gamma\sigma$
$\lambda\beta^2 = \sigma\gamma$
$\alpha\gamma = \gamma\beta$
$\beta\sigma = \sigma\alpha$

$A_2(\lambda)$
$\lambda \in K \backslash \{0,1\}$



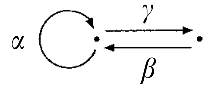
$\alpha\beta + \gamma\delta + \xi\varepsilon = 0$
$\beta\alpha = 0, \quad \varepsilon\xi = 0$
$\delta\gamma = 0$

A_3



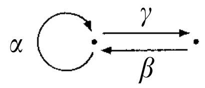
$\alpha\beta + \gamma\delta + \xi\varepsilon = 0$
$\beta\alpha = 0, \quad \varepsilon\gamma = 0$
$\delta\xi = 0$

A_4



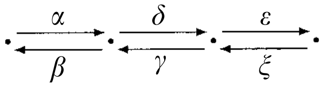
$\alpha^2 = \beta\gamma$
$\gamma\alpha\beta = 0$

A_5



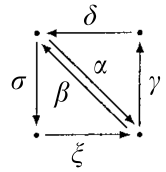
$\alpha^3 = \beta\gamma$
$\gamma\beta = 0$
$\alpha^2\beta = 0$
$\gamma\alpha^2 = 0$

A_6



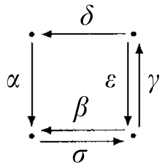
$\alpha\beta = \gamma\delta$
$\delta\gamma = \xi\varepsilon$
$\varepsilon\delta\alpha = 0$
$\beta\gamma\xi = 0$

A_7



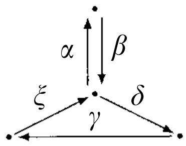
$\alpha\beta\xi = 0$
$\beta\alpha\delta = 0$
$\gamma\alpha\beta = 0$
$\sigma\beta\alpha = 0$
$\alpha\beta\alpha = \xi\sigma, \quad \gamma\xi = 0$
$\beta\alpha\beta = \delta\gamma, \quad \sigma\delta = 0$

A_8



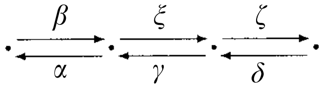
$\alpha\delta = \beta\varepsilon, \quad \varepsilon\gamma = \sigma\beta$
$\beta\sigma\alpha = 0, \quad \delta\gamma\varepsilon = 0, \quad \gamma\varepsilon\gamma\sigma = 0$

A_9



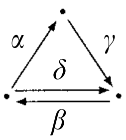
$\beta\alpha\xi = \xi\gamma\delta\xi$
$\delta\beta\alpha = \delta\xi\gamma\delta$
$\alpha\beta = 0, \quad (\gamma\delta\xi)^2\gamma = 0$

A_{10}



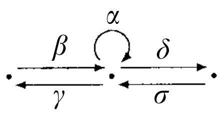
$\beta\alpha\gamma = \gamma\xi\gamma$
$\xi\beta\alpha = \xi\gamma\xi$
$\alpha\beta = 0, \quad \gamma\delta = 0$
$\zeta\xi = 0, \quad (\xi\gamma)^2 = \delta\zeta$

A_{11}



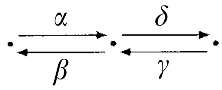
$\delta\beta\delta = \gamma\alpha$
$\alpha\beta\gamma = 0, \quad \beta(\delta\beta)^3 = 0$

A_{12}



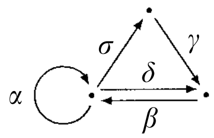
$\alpha^2 = \beta\gamma, \quad \delta\beta = 0, \quad \gamma\beta = 0$
$\gamma\sigma = 0, \quad \delta\alpha = 0, \quad \alpha\sigma = 0$
$\alpha^3 = \sigma\delta$

A_{13}



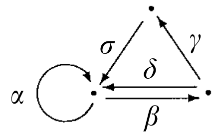
$\alpha\beta = \gamma\delta\gamma\delta$
$\delta\gamma\delta\alpha = 0$
$\beta\gamma\delta\gamma = 0$
$\beta\alpha = 0$

A_{14}



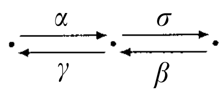
$\alpha\beta\gamma = 0, \quad \alpha^2 = \beta\delta$
$\delta\beta = 0, \quad \sigma\alpha = 0, \quad \delta\alpha = \gamma\sigma$

A_{15}



$\gamma\beta\alpha = 0, \quad \alpha^2 = \delta\beta$
$\beta\delta = 0, \quad \alpha\sigma = 0, \quad \alpha\delta = \sigma\gamma$

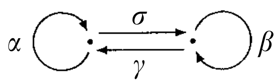
A_{16}



$\alpha\gamma\beta = \beta\sigma\beta$
$\alpha\gamma\alpha + \beta\sigma\alpha = 0$
$\gamma\alpha\gamma = \gamma\beta\sigma$
$\sigma\alpha\gamma + \sigma\beta\sigma = 0$

A_{17}

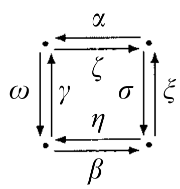
char $K \neq 2$



$\alpha\gamma = \gamma\beta$
$\alpha^2 = \gamma\sigma$
$\sigma\gamma = \beta^2$
$\sigma\alpha + \beta\sigma = 0$

A_{18}

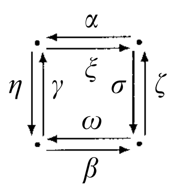
char $K \neq 2$



$\alpha\xi = \gamma\eta, \quad \alpha\zeta = \gamma\omega, \quad \sigma\xi = \beta\eta$
$\xi\sigma = \zeta\alpha, \quad \xi\beta = \zeta\gamma, \quad \eta\sigma = \omega\alpha$
$\sigma\zeta + \beta\omega = 0, \quad \eta\beta + \omega\gamma = 0$

A_{19}

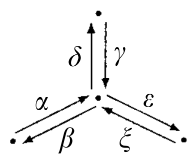
char $K \neq 2$



$\alpha\xi = \gamma\eta, \quad \alpha\zeta = \gamma\omega, \quad \sigma\xi = \beta\eta$
$\xi\alpha = \zeta\sigma, \quad \xi\gamma = \zeta\beta, \quad \eta\alpha = \omega\sigma$
$\sigma\zeta = \lambda\beta\omega, \quad \eta\gamma = \lambda\omega\beta$

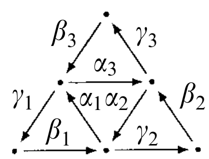
$A_{20}(\lambda)$

$\lambda \in K\backslash\{0, 1\}$



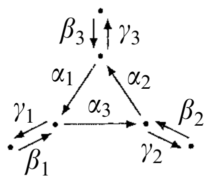
$\alpha\beta + \gamma\delta + \xi\varepsilon = 0$
$\delta\alpha = 0, \quad \beta\xi = 0$
$\varepsilon\gamma = 0$

A_{21}



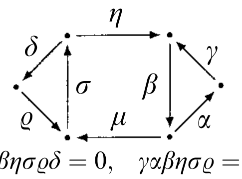
$\gamma_i\alpha_i\beta_i = 0, \quad \beta_i\gamma_i = \alpha_{i+1}\alpha_{i+2}$
for $i \in \{1, 2, 3\}, \quad \alpha_4 = \alpha_1, \quad \alpha_5 = \alpha_2$

A_{22}



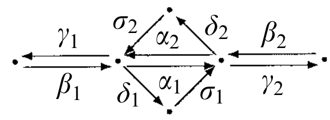
$\gamma_i\beta_i = 0, \quad \beta_i\gamma_i = \alpha_i\alpha_{i+1}\alpha_{i+2}$
$\gamma_i\alpha_i\alpha_{i+1} = 0, \quad \alpha_{i+1}\alpha_{i+2}\beta_i = 0$
for $i \in \{1,2,3\}$, $\alpha_4 = \alpha_1$, $\alpha_5 = \alpha_2$

A_{23}



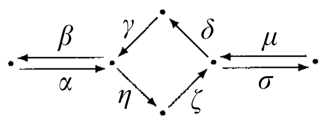
$\alpha\beta\eta\sigma\varrho\delta = 0, \quad \gamma\alpha\beta\eta\sigma\varrho = 0$
$\delta\sigma\mu\beta\gamma\alpha = 0, \quad \varrho\delta\sigma\mu\beta\gamma = 0$
$\gamma\alpha = \eta\sigma\mu, \quad \varrho\delta = \mu\beta\eta$
$\alpha\beta\gamma = 0, \quad \delta\sigma\varrho = 0$

A_{24}



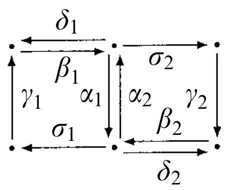
$\alpha_{i+1}\alpha_i = \beta_i\gamma_i, \quad \alpha_i\alpha_{i+1}\alpha_i = \sigma_i\delta_i$
$\alpha_i\sigma_{i+1} = 0, \quad \gamma_i\beta_i = 0, \quad \gamma_i\sigma_{i+1} = 0$
$\delta_i\alpha_{i+1} = 0, \quad \delta_i\beta_i = 0$, for $i \in \{1,2\}$
$\alpha_3 = \alpha_1, \quad \sigma_3 = \sigma_1$

A_{25}



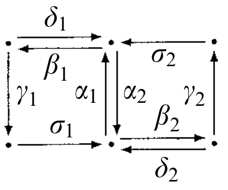
$\alpha\beta = \gamma\delta\zeta\eta, \quad \mu\sigma = \zeta\eta\gamma\delta$
$\beta\alpha = 0, \quad \sigma\mu = 0, \quad \delta\zeta\eta\alpha = 0$
$\eta\gamma\delta\mu = 0, \quad \beta\gamma\delta\zeta = 0, \quad \sigma\zeta\eta\gamma = 0$

A_{26}



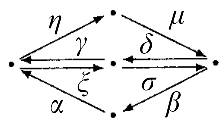
$\delta_i\alpha_{i+1} = \gamma_i\sigma_i, \quad \delta_i\beta_i = 0, \quad \sigma_i\alpha_i = 0$
$\alpha_{i+1}\alpha_i = \beta_i\delta_i, \quad \alpha_i\beta_i\gamma_i = 0$, for
$i \in \{1,2\}$, $\alpha_3 = \alpha_1$

A_{27}



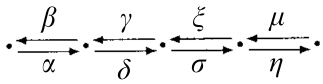
$\alpha_{i+1}\delta_i = \sigma_i\gamma_i, \quad \beta_i\delta_i = 0, \quad \alpha_i\sigma_i = 0$
$\alpha_i\alpha_{i+1} = \delta_i\beta_i, \quad \gamma_i\beta_i\alpha_i = 0$, for
$i \in \{1,2\}$, $\alpha_3 = \alpha_1$

A_{28}



$\beta\mu = 0, \quad \eta\alpha = 0, \quad \delta\sigma = \xi\gamma$
$\alpha\beta = \gamma\delta, \quad \sigma\xi = \mu\eta$

A_{29}



$\beta\alpha = 0, \quad \eta\mu = 0, \quad \delta\gamma = \xi\sigma$
$\alpha\beta = \gamma\delta, \quad \sigma\xi = \mu\eta$

A_{30}

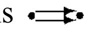
Theorem 4.2. *A selfinjective algebra A of tubular type is exceptional if and only if A is isomorphic to one of the algebras $A_1(\lambda)$, $A_2(\lambda)$, $A_{20}(\lambda)$, $\lambda \in K\backslash\{0,1\}$, A_i, $17 \le i \le 19$ (if $\operatorname{char} K \neq 2$), or A_i, for $i \in \{3,\dots,16,21,\dots,30\}$, listed above.*

Proof. It is a direct consequence of [**3**, (4.2),(5.2),(6.2)] and [**12**, (5.4)]. □

5. Proof of the main result.

We know from Proposition 3.2 that the class of normal selfinjective algebras of tubular type is closed under socle equivalences. Hence, in order to prove Theorem 1.1, it is sufficient to show that the class of all selfinjective algebras which are socle equivalent to the exceptional selfinjective algebras of tubular type (presented in Section 4) but nonisomorphic to these algebras coincides with the class of algebras Λ_1, Λ_2, $\Lambda_3(\lambda)$, $\lambda \in K\backslash\{0,1\}$, Λ_4, Λ_5, Λ_6, Λ_7, Λ_8 and Λ_{10}, for the corresponding characteristic of K. The proof of this fact will be a combination of several facts established bellow. We start with some general observations.

Let A be an exceptional selfinjective algebra of tubular type. Then $A = KQ/I$ where Q is a finite connected Gabriel quiver of A and I is an admissible ideal in the path algebra KQ of Q, generated by a finite system of linear combinations (called relations) $u_i = \lambda_{i1}u_{i1} + \lambda_{i2}u_{i2} + \cdots + \lambda_{it_i}u_{it_i}$, $1 \le i \le m = m_A$, and, for each $i \in \{1,\dots,m\}$, $\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{it_i}$ are elements of $K\backslash\{0\}$ and $u_{i1}, u_{i2}, \dots, u_{it_i}$ are paths in Q having common source and common target. Moreover, for the systems of relations generating I chosen in Section 4, the following facts hold:

(a) The quiver Q does not contain a subquiver of one of the forms  or .

(b) If $A \neq A_{11}, A_{13}, A_{25}$, and i is an element of $\{1, \dots, m\}$ with $t_i \geq 2$, then $u_{ij} + I \notin \operatorname{soc} A$ for any $j \in \{1, \dots, t_i\}$.

(c) If $A = A_{11}, A_{13}$, or A_{25}, there are elements $i \in \{1, \dots, m\}$ such that $u_i = u_{i1} - u_{i2}$ with $u_{i1} + I \in \operatorname{soc} A$ and $u_{i2} + I \in \operatorname{soc} A$.

Further, we have the following lemma.

Lemma 5.1. *Let A be an exceptional selfinjective algebra of tubular type. Then the canonical epimorphism $A \to A/\operatorname{soc} A$ induces an isomorphism $\operatorname{Aut}(A) \to \operatorname{Aut}(A/\operatorname{soc} A)$ of automorphism groups.*

Proof. For A different from A_{11}, A_{13}, and A_{25}, it follows from the above property (b). A direct checking shows that the algebras A_{11}, $A_{11}/\operatorname{soc} A_{11}$, A_{13}, and $A_{13}/\operatorname{soc} A_{13}$ have only trivial automorphisms groups. Finally, the automorphisms groups $\operatorname{Aut}(A_{25})$ and $\operatorname{Aut}(A_{25}/\operatorname{soc} A_{25})$ consist of the identity homomorphism and the canonical automorphism of order 2 given by the automorphism of the quiver Q exchanging the arrows α_1 and α_2, β_1 and β_2, γ_1 and γ_2, δ_1 and δ_2, σ_1 and σ_2. □

Let A' be a selfinjective algebra which is socle equivalent to an exceptional selfinjective algebra $A = KQ/I$ of tubular type. Observe that then Q is the Gabriel quiver of A', and consequently $A' \cong KQ/I'$ for an admissible ideal I' of KQ, generated by a finite system $u'_1, u'_2, \dots, u'_n$ of relations. Without loss of generality, we may assume that $A' = KQ/I'$. Invoking Lemma 5.1 and the property (a), we conclude that there is an algebra isomorphism $f : A/\operatorname{soc} A \to A'/\operatorname{soc} A'$ induced by a K-linear map $f^* : KQ \to KQ$ such that $f^*(e_i) = e_i$ for all primitive idempotents e_i associated to the vertices i of Q and $f^*(\alpha) = a_\alpha \alpha + w_\alpha$ for all arrows α of Q, with $a_\alpha \in K \backslash \{0\}$ and w_α a linear combination of paths (with coefficients in K) of length ≥ 2 having the same source and target as α. We also note that if w is an element of KQ such that $w + I \in \operatorname{soc} A$, then $w\xi$ and $\eta w \in I'$ for all arrows ξ and η of Q. In particular, it is the case for all relations $u_1, \dots, u_n$ generating the ideal I. Therefore, the relations $u'_1, u'_2, \dots, u'_n$ generating the ideal I' can be obtained from the relations $u_1, u_2, \dots, u_m$ generating the ideal I by:

(1) replacing some relations u_i by relations of the form $u_i - \Theta_i v_i$, $\Theta_i \in K$, for any path v_i in Q with $0 \neq v_i + I \in \operatorname{soc} A$ having the same source and target as the paths u_{ij} occuring in u_i;

(2) replacing, for A equal A_{11}, A_{13} or A_{25}, the relations $u_i = u_{i1} - u_{i2}$ with $u_{i1} + I, u_{i2} + I \in \operatorname{soc} I$, by relations of the forms $\Theta_{i1} u_{i1} - \Theta_{i2} u_{i2}$, for $\Theta_{i1}, \Theta_{i2} \in K$;

(3) keeping all the remaining relations u_i unchanged;

(4) adding the relations $u_i \xi$ and ηu_i, for all relations u_i replaced in (1), and arrows ξ (respectively, η) having the same target (respectively, source) as the paths u_{ij} occuring in u_i;

(5) adding, for A equal A_{11}, A_{13}, or A_{25}, the relations $u_{ij}\xi$ and ηu_{ij}, for all paths u_{ij} in Q, $j \in \{1, 2\}$, occuring in the relations $u_i = u_{i1} - u_{i2}$ replaced in (2), and arrows ξ (respectively, η) having the same target (respectively, source) as u_{ij}.

Clearly, all algebras $A' = KQ/I'$ for I' obtained from I by applying the procedures (1), (3), (4) are socle equivalent to A. We note that the procedure (4) can be replaced by:

(4′) adding the relations $v_i\xi$ and ηv_i, for all paths v_i in Q with $0 \neq v_i + I \in \operatorname{soc} A$ used in (1).

We may also use the procedure (4) for some relations u_i which have been replaced in (1), and the procedure (4′) for the remaining i for which the relations u_i were replaced in (1).

We will prove in Lemmas 5.6, 5.14, and 5.15 that Θ_{i1} and Θ_{i2} in (2) have to be both nonzero. In those lemmas we also prove that any algebra obtained by the procedures (1)–(5) is isomorphic to some algebra obtained by the procedures (1), (3) and (4). Finally, we note that many from the relations added in (4) (respectively, in (4′)) and (5) follow from the other relations and hence can be omitted. For example, it is the case in the lemma below.

In order to simplify notation, we will identify below the elements from KQ with their residue classes in KQ/I and KQ/I'.

Lemma 5.2. *Let Λ be a selfinjective algebra which is socle equivalent to A_5 but nonisomorphic to A_5. Then* $\operatorname{char} K = 3$ *and Λ is isomorphic to Λ_1.*

Proof. It follows from the above remarks that Λ is isomorphic to an algebra A_5' given by the quiver of A_5 and bound by relations

$$\beta\gamma\alpha\beta = 0, \quad \gamma\alpha\beta\gamma = 0, \quad \alpha^5 = 0, \quad \gamma\alpha\beta = \Theta_1\gamma\alpha^2\beta, \quad \alpha^2 - \beta\gamma = \Theta_2\alpha^4,$$

for some parameters $\Theta_1, \Theta_2 \in K$. Note that we may omit the relations $\alpha^4\beta = 0$ and $\gamma\alpha^4 = 0$ of type (4′), because $\alpha^3\beta = \beta\gamma\alpha\beta + \Theta_2\alpha^5\beta = 0$ and $\gamma\alpha^3 = \gamma\alpha\beta\gamma + \Theta_2\gamma\alpha^5 = 0$.

Assume that A_5 and A_5' are isomorphic, and let $f : A_5 \to A_5'$ be an algebra isomorphism. Then f is given by

$$f(\alpha) = a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_4\alpha^4,$$

$$f(\beta) = b_1\beta + b_2\alpha\beta + b_3\alpha^2\beta, \quad f(\gamma) = c_1\gamma + c_2\gamma\alpha + c_3\gamma\alpha^2,$$

for some parameters $a_1, b_1, c_1 \in K\backslash\{0\}$, $a_i \in K$, $2 \le i \le 4$, $b_i, c_i \in K$, $2 \le i \le 3$. Denote $a = a_1^{-1}a_2$, $b = b_1^{-1}b_2$, $c = c_1^{-1}c_2$. We have the following equalities:

$$f(\alpha^2 - \beta\gamma) = a_1^2\alpha^2 + 2a_1a_2\alpha^3 + (2a_1a_3 + a_2^2)\alpha^4$$

$$- (b_1c_1\beta\gamma + (b_1c_2 + b_2c_1)\alpha^3 + (b_1c_3 + b_2c_2 + b_3c_1)\alpha^4),$$

$$f(\gamma\alpha\beta) = a_1b_1c_1\gamma\alpha\beta + (a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1)\gamma\alpha^2\beta.$$

Hence, we obtain $a_1^2 = b_1c_1$, $2a = b + c$ and $3a = a + b + c = -\Theta_1$. Therefore, if K is of characteristic 3, then we have $\Theta_1 = 0$. Thus, for $\Theta_1 \neq 0$ and $\operatorname{char} K = 3$, the algebras A_5 and A_5' are not isomorphic, but clearly A_5 and A_5' are socle equivalent. We will prove now that, if $\operatorname{char} K = 3$ and $\Theta_1 \neq 0$, then A_5' is the unique (up to isomorphism) selfinjective algebra socle equivalent to A_5 but nonisomorphic to A_5.

Observe first that if $\operatorname{char} K \neq 3$ then there is an algebra isomorphism $f : A_5 \to A_5'$ given by

$$f(\alpha) = \alpha - \frac{\Theta_1}{3}\alpha^2, \quad f(\beta) = \beta - \frac{\Theta_1}{3}\alpha\beta, \quad f(\gamma) = \gamma - \frac{\Theta_1}{3}\gamma\alpha + \Theta_2\gamma\alpha^2,$$

whose inverse $f^{-1}: A_5' \to A_5$ is given by

$$f^{-1}(\alpha) = \alpha + \frac{\Theta_1}{3}\alpha^2 + \frac{2\Theta_1^2}{9}\alpha^3 + \frac{5\Theta_1^3}{27}\alpha^4,$$

$$f^{-1}(\beta) = \beta + \frac{\Theta_1}{3}\alpha\beta + \frac{2\Theta_1^2}{9}\alpha^2\beta, \quad f^{-1}(\gamma) = \gamma + \frac{\Theta_1}{3}\gamma\alpha + \left(\frac{2\Theta_1^2}{9} - \Theta_2\right)\gamma\alpha^2,$$

and, if $\operatorname{char} K = 3$ and $\Theta_1 = 0$, then there is an algebra isomorphism $f: A_5 \to A_5'$ given by

$$f(\alpha) = \alpha, \quad f(\beta) = \beta, \quad f(\gamma) = \gamma + \Theta_2\gamma\alpha^2,$$

whose inverse $f^{-1}: A_5' \to A_5$ is given by

$$f^{-1}(\alpha) = \alpha, \quad f^{-1}(\beta) = \beta, \quad f^{-1}(\gamma) = \gamma - \Theta_2\gamma\alpha^2.$$

Assume now that $\operatorname{char} K = 3$. Let A_5' be as above with $\Theta_1 \neq 0$ and A_5'' be an algebra given by the quiver of A_5 and bound by relations

$$\beta\gamma\alpha\beta = 0, \quad \gamma\alpha\beta\gamma = 0, \quad \alpha^5 = 0, \quad \gamma\alpha\beta = \Theta_1'\gamma\alpha^2\beta, \quad \alpha^2 - \beta\gamma = \Theta_2'\alpha^4,$$

for some parameters $\Theta_1', \Theta_2' \in K$ with $\Theta_1' \neq 0$. Denote $\vartheta = \Theta_1^{-1}\Theta_1'$. Then we have an algebra isomorphism $g: A_5' \to A_5''$ given by

$$g(\alpha) = \vartheta\alpha, \quad g(\beta) = \beta + (\Theta_2' - \vartheta^2\Theta_2)\alpha^2\beta, \quad g(\gamma) = \vartheta^2\gamma,$$

and its inverse $g^{-1}: A_5'' \to A_5'$ is given by

$$g^{-1}(\alpha) = \vartheta^{-1}\alpha, \quad g^{-1}(\beta) = \beta + (\Theta_2 - \vartheta^{-2}\Theta_2')\alpha^2\beta, \quad g^{-1}(\gamma) = \vartheta^{-2}\gamma.$$

This ends the proof, because Λ_1 is equal to A_5' for $\Theta_1 = 1$, $\Theta_2 = 0$. □

Lemma 5.3. *Let A be a selfinjective algebra which is socle equivalent to A_6 but nonisomorphic to A_6. Then $\operatorname{char} K = 3$ and A is isomorphic to Λ_2.*

Proof. The algebra A is isomorphic to an algebra A_6' given by the quiver of A_6 and relations

$$\beta\gamma\beta = 0, \quad \gamma\beta\gamma = 0, \quad \gamma\beta = \Theta_1\gamma\alpha\beta,$$

$$\gamma\alpha^2 = 0, \quad \alpha^2\beta = 0, \quad \alpha^5 = 0, \quad \alpha^3 - \beta\gamma = \Theta_2\alpha^4,$$

for some parameters $\Theta_1, \Theta_2 \in K$. We note that Λ_2 is equal to A_6' with $\Theta_1 = 1$ and $\Theta_2 = 0$.

Assume that the algebras A_6 and A_6' are isomorphic, and let $f: A_6 \to A_6'$ be an algebra isomorphism. Then f is given by

$$f(\alpha) = a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_4\alpha^4,$$

$$f(\beta) = b_1\beta + b_2\alpha\beta, \quad f(\gamma) = c_1\gamma + c_2\gamma\alpha,$$

for some parameters $a_1, b_1, c_1 \in K \setminus \{0\}$, $b_2, c_2 \in K$, $a_i \in K$, $2 \le i \le 4$. Denote $a = a_1^{-1}a_2$, $b = b_1^{-1}b_2$, $c = c_1^{-1}c_2$. Then the following equalities hold:

$$f(\gamma\beta) = b_1c_1\gamma\beta + (b_1c_2 + b_2c_1)\gamma\alpha\beta,$$

$$f(\alpha^3 - \beta\gamma) = a_1^3\alpha^3 + 3a_1^2a_2\alpha^4 - (b_1c_1\beta\gamma + (b_1c_2 + b_2c_1)\alpha^4).$$

Therefore, we obtain $b + c = -\Theta_1$, $a_1^3 = b_1c_1$ and $\Theta_2 + 3a = b + c = -\Theta_1$. Thus, if K is of characteristic 3, then we have $\Theta_1 + \Theta_2 = 0$. Hence, for $\Theta_1 + \Theta_2 \neq 0$ and $\operatorname{char} K = 3$, the algebras A_6 and A_6' are not isomorphic, but clearly A_6 and A_6' are socle equivalent. We will prove now that, if $\operatorname{char} K = 3$ and $\Theta_1 + \Theta_2 \neq 0$, then A_6' is the unique (up to isomorphism) selfinjective algebra socle equivalent to A_6 but nonisomorphic to A_6.

If $\operatorname{char} K \neq 3$ then there is an algebra isomorphism $f : A_6 \to A_6'$ given by

$$f(\alpha) = \alpha - \frac{\Theta_1 + \Theta_2}{3}\alpha^2, \quad f(\beta) = \beta - \Theta_1\alpha\beta, \quad f(\gamma) = \gamma,$$

whose inverse $f^{-1} : A_6' \to A_6$ is given by

$$f^{-1}(\alpha) = \alpha + \frac{\Theta_1 + \Theta_2}{3}\alpha^2 + \frac{2(\Theta_1 + \Theta_2)^2}{9}\alpha^3 + \frac{5(\Theta_1 + \Theta_2)^3}{27}\alpha^4,$$

$$f^{-1}(\beta) = \beta + \Theta_1\alpha\beta, \quad f^{-1}(\gamma) = \gamma,$$

and, if $\operatorname{char} K = 3$ and $\Theta_1 + \Theta_2 = 0$, then there is an algebra isomorphism $f : A_6 \to A_6'$ given by

$$f(\alpha) = \alpha, \quad f(\beta) = \beta - \Theta_1\alpha\beta, \quad f(\gamma) = \gamma,$$

whose inverse $f^{-1} : A_6' \to A_6$ is given by

$$f^{-1}(\alpha) = \alpha, \quad f^{-1}(\beta) = \beta + \Theta_1\alpha\beta, \quad f^{-1}(\gamma) = \gamma.$$

Assume now that $\operatorname{char} K = 3$. Let A_6' be as above with $\Theta_1 + \Theta_2 \neq 0$ and A_6'' be an algebra given by the quiver of A_6 bound by relations

$$\beta\gamma\beta = 0, \quad \gamma\beta\gamma = 0, \quad \gamma\beta = \Theta_1'\gamma\alpha\beta,$$

$$\gamma\alpha^2 = 0, \quad \alpha^2\beta = 0, \quad \alpha^5 = 0, \quad \alpha^3 - \beta\gamma = \Theta_2'\alpha^4,$$

for some parameters $\Theta_1', \Theta_2' \in K$ with $\Theta_1' + \Theta_2' \neq 0$. Denote $\vartheta = (\Theta_1 + \Theta_2)^{-1}(\Theta_1' + \Theta_2')$. Then we have an algebra isomorphism $g : A_6' \to A_6''$ given by

$$g(\alpha) = \vartheta\alpha, \quad g(\beta) = \vartheta^3\beta, \quad g(\gamma) = \gamma + (\vartheta\Theta_1 - \Theta_1')\gamma\alpha$$

(note that $\vartheta\Theta_1 - \Theta_1' = \Theta_2' - \vartheta\Theta_2$), whose inverse $g^{-1} : A_6'' \to A_6'$ is given by

$$g^{-1}(\alpha) = \vartheta^{-1}\alpha, \quad g^{-1}(\beta) = \vartheta^{-3}\beta, \quad g^{-1}(\gamma) = \gamma + (\vartheta^{-1}\Theta_1' - \Theta_1)\gamma\alpha.$$

This ends the proof, because A_2 is equal to A_6' for $\Theta_1 = 1$ and $\Theta_2 = 0$. □

Lemma 5.4. *Let Λ be a selfinjective algebra which is socle equivalent to $A_2(\lambda)$ but nonisomorphic to $A_2(\lambda)$. Then* $\operatorname{char} K = 2$ *and Λ is isomorphic to $A_3(\lambda)$.*

Proof. It follows from the above remarks that Λ is isomorphic to an algebra $A_2'(\lambda)$ given by the quiver of $A_2(\lambda)$ and bound by relations

$$\alpha\gamma\sigma = \alpha^3 = \gamma\sigma\alpha, \quad \alpha\gamma = \gamma\beta, \quad \lambda\gamma\beta^2 = \gamma\sigma\gamma = \alpha^2\gamma,$$

$$\beta\sigma\gamma = \lambda\beta^3 = \sigma\gamma\beta, \quad \beta\sigma = \sigma\alpha, \quad \lambda\beta^2\sigma = \sigma\gamma\sigma = \sigma\alpha^2,$$

$$\alpha^2 - \gamma\sigma = \Theta_1\alpha^3, \quad \lambda\beta^2 - \sigma\gamma = \Theta_2\lambda\beta^3,$$

for some parameters $\Theta_1, \Theta_2 \in K$. Note that we have $\gamma\beta^2 = 0$, $\beta^2\sigma = 0$, $\alpha^4 = 0$, and $\beta^4 = 0$, because $\lambda\gamma\beta^2 = \gamma\sigma\gamma = \alpha^2\gamma = \alpha\gamma\beta = \gamma\beta^2$, $\lambda\beta^2\sigma = \sigma\gamma\sigma = \sigma\alpha^2 = \beta\sigma\alpha = \beta^2\sigma$, and $1 \neq \lambda \in K$.

Assume that $A_2(\lambda)$ and $A_2'(\lambda)$ are isomorphic. Then there exists an algebra isomorphism $f : A_2(\lambda) \to A_2'(\lambda)$ given by

$$f(\alpha) = a_1\alpha + a_2\alpha^2 + a_3\alpha^3, \quad f(\beta) = b_1\beta + b_2\beta^2 + b_3\beta^3,$$

$$f(\gamma) = c_1\gamma + c_2\alpha\gamma, \quad f(\sigma) = d_1\sigma + d_2\sigma\alpha,$$

for some parameters $a_1, b_1, c_1, d_1 \in K\backslash\{0\}$, $a_i, b_i, c_i, d_i \in K$, $2 \leq i \leq 3$. Denote $a = a_1^{-1}a_2$, $b = b_1^{-1}b_2$, $c = c_1^{-1}c_2$, $d = d_1^{-1}d_2$. We obtain the following equalities:

$$\begin{aligned} f(\alpha^2 - \gamma\sigma) &= a_1^2\alpha^2 + 2a_1a_2\alpha^3 - (c_1d_1\gamma\sigma + (c_1d_2 + c_2d_1)\gamma\sigma\alpha) \\ &= a_1^2\alpha^2 - c_1d_1\gamma\sigma - (c_1d_2 + c_2d_1 - 2a_1a_2)\alpha^3, \\ f(\lambda\beta^2 - \sigma\gamma) &= \lambda(b_1^2\beta^2 + 2b_1b_2\beta^3) - (c_1d_1\sigma\gamma + (c_1d_2 + c_2d_1)\sigma\alpha\gamma) \\ &= b_1^2\lambda\beta^2 - c_1d_1\sigma\gamma - (c_1d_2 + c_2d_1 - 2b_1b_2)\lambda\beta^3. \end{aligned}$$

Hence, we get $a_1^2 = c_1d_1$, $\Theta_1 = c + d - 2a$, $b_1^2 = c_1d_1$ and $\Theta_2 = c + d - 2b$. Therefore, if K is of characteristic 2, then we have $\Theta_1 = c + d = \Theta_2$. Thus, for $\Theta_1 \neq \Theta_2$ and $\operatorname{char} K = 2$, the algebras $A_2(\lambda)$ and $A_2'(\lambda)$ are not isomorphic, but clearly $A_2(\lambda)$ and $A_2'(\lambda)$ are socle equivalent. We will prove now that, if $\operatorname{char} K = 2$, then $A_2'(\lambda)$ is the unique (up to isomorphism) selfinjective algebra socle equivalent to $A_2(\lambda)$ but non-isomorphic to $A_2(\lambda)$.

Observe first that if $\operatorname{char} K \neq 2$ then there is an algebra isomorphism $f : A_2(\lambda) \to A_2'(\lambda)$ given by

$$f(\alpha) = \alpha - \frac{\Theta_1}{2}\alpha^2, \quad f(\beta) = \beta - \frac{\Theta_2}{2}\beta^2, \quad f(\gamma) = \gamma, \quad f(\delta) = \delta,$$

whose inverse $f^{-1} : A_2'(\lambda) \to A_2(\lambda)$ is given by

$$f^{-1}(\alpha) = \alpha + \frac{\Theta_1}{2}\alpha^2 + \frac{\Theta_1^2}{2}\alpha^3, \quad f^{-1}(\beta) = \beta + \frac{\Theta_2}{2}\beta^2 + \frac{\Theta_2^2}{2}\beta^3,$$

$$f^{-1}(\gamma) = \gamma, \quad f^{-1}(\delta) = \delta,$$

and, if $\operatorname{char} K = 2$ and $\Theta_1 = \Theta_2$, then there is an algebra isomorphism $f : A_2(\lambda) \to A_2'(\lambda)$ given by

$$f(\alpha) = \alpha, \quad f(\beta) = \beta, \quad f(\gamma) = \gamma + \Theta_1\alpha\gamma, \quad f(\delta) = \delta,$$

whose inverse $f^{-1} : A_2'(\lambda) \to A_2(\lambda)$ is given by

$$f^{-1}(\alpha) = \alpha, \quad f^{-1}(\beta) = \beta, \quad f^{-1}(\gamma) = \gamma - \Theta_1 \alpha\gamma, \quad f^{-1}(\delta) = \delta.$$

Finally, assume that $\operatorname{char} K = 2$. Let $A_2'(\lambda)$ be as above with $\Theta_1 \neq \Theta_2$ and let $A_2''(\lambda)$ be an algebra given by the quiver of A_2 bound by relations

$$\alpha\gamma\sigma = \alpha^3 = \gamma\sigma\alpha, \quad \alpha\gamma = \gamma\beta, \quad \lambda\gamma\beta^2 = \gamma\sigma\gamma = \alpha^2\gamma,$$

$$\beta\sigma\gamma = \lambda\beta^3 = \sigma\gamma\beta, \quad \beta\sigma = \sigma\alpha, \quad \lambda\beta^2\sigma = \sigma\gamma\sigma = \sigma\alpha^2,$$

$$\alpha^2 - \gamma\sigma = \Theta_1'\alpha^3, \quad \lambda\beta^2 - \sigma\gamma = \Theta_2'\lambda\beta^3,$$

for some parameters $\Theta_1', \Theta_2' \in K$, $\Theta_1' \neq \Theta_2'$. Denote $\vartheta = (\Theta_1 - \Theta_2)^{-1}(\Theta_1' - \Theta_2')$. Then we have an algebra isomorphism $g : A_2'(\lambda) \to A_2''(\lambda)$ given by

$$g(\alpha) = \vartheta\alpha, \quad g(\beta) = \vartheta\beta, \quad g(\gamma) = \vartheta\gamma, \quad g(\sigma) = \vartheta\sigma + \vartheta(\Theta_1' - \vartheta\Theta_1)\sigma\alpha$$

(note that $\Theta_1' - \vartheta\Theta_1 = \Theta_2' - \vartheta\Theta_2$), and its inverse $g^{-1} : A_2''(\lambda) \to A_2'(\lambda)$ is given by

$$g^{-1}(\alpha) = \vartheta^{-1}\alpha, \quad g^{-1}(\beta) = \vartheta^{-1}\beta, \quad g^{-1}(\gamma) = \vartheta^{-1}\gamma,$$

$$g^{-1}(\sigma) = \vartheta^{-1}\sigma + \vartheta^{-1}(\Theta_1 - \vartheta^{-1}\Theta_1')\sigma\alpha.$$

This ends the proof, because $\Lambda_3(\lambda)$ is equal to $A_2'(\lambda)$ with $\Theta_1 = 1$ and $\Theta_2 = 0$. □

Lemma 5.5. *Let Λ be a selfinjective algebra which is socle equivalent to $A_{12}(\lambda)$ but nonisomorphic to $A_{12}(\lambda)$. Then* $\operatorname{char} K = 2$ *and Λ is isomorphic to $\Lambda_4(\lambda)$.*

Proof. The algebra Λ is isomorphic to an algebra A_{12}' given by the quiver of A_{12} and relations

$$\alpha\beta\gamma = \Theta\alpha\beta\delta\beta\gamma, \quad \alpha\beta\gamma\alpha = 0, \quad \gamma\alpha\beta\gamma = 0, \quad \delta\beta\delta = \gamma\alpha, \quad \beta(\delta\beta)^3 = 0,$$

for some parameter $\Theta \in K$.

Assume that A_{12} and A_{12}' are isomorphic. Then there exists an algebra isomorphism $f : A_{12} \to A_{12}'$ given by

$$f(\alpha) = a_1\alpha + a_2\alpha\beta\delta, \quad f(\beta) = b_1\beta + b_2\beta\delta\beta + b_3\beta\delta\beta\delta\beta,$$

$$f(\gamma) = c_1\gamma + c_2\delta\beta\gamma, \quad f(\delta) = d_1\delta + d_2\delta\beta\delta + d_3\delta\beta\delta\beta\delta,$$

for some parameters $a_1, b_1, c_1, d_1 \in K \setminus \{0\}$, $a_2, b_2, b_3, c_2, d_2, d_3 \in K$. Denote $a = a_1^{-1}a_2$, $b = b_1^{-1}b_2$, $c = c_1^{-1}c_2$, $d = d_1^{-1}d_2$. We have the following equalities:

$$f(\alpha\beta\gamma) = a_1b_1c_1\alpha\beta\gamma + (a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2)\alpha\beta\delta\beta\gamma,$$

$$f(\delta\beta\delta - \gamma\alpha) = b_1d_1^2\delta\beta\delta + (2b_1d_1d_2 + b_2d_1^2 - a_1c_2 - a_2c_1)\delta\beta\delta\beta\delta - a_1c_1\gamma\alpha.$$

Hence, we obtain the relations $a + b + c + \Theta = 0$, $a_1c_1 = b_1d_1^2$ and $2d + b = a + c$. Therefore, if K is of characteristic 2, we have $\Theta = -(a + b + c) = -2(b + d) = 0$. Then, for $\Theta \neq 0$ and $\operatorname{char} K = 2$, the algebras A_{12} and A_{12}' are nonisomorphic, so Λ_4 is socle equivalent but nonisomorphic to A_{12}. Now we prove that there is only one algebra (up to isomorphism) with this property.

Observe that $A_{12} = A'_{12}$ if $\Theta = 0$. Moreover, if $\operatorname{char} K \neq 2$ then there exists an algebra isomorphism $f : A_{12} \to A'_{12}$ given by

$$f(\alpha) = \alpha, \quad f(\beta) = \beta - \Theta\beta\delta\beta, \quad f(\gamma) = \gamma, \quad f(\delta) = \delta + \frac{\Theta}{2}\delta\beta\delta,$$

whose inverse $f^{-1} : A'_{12} \to A_{12}$ is given by

$$f^{-1}(\alpha) = \alpha, \quad f^{-1}(\beta) = \beta + \Theta\beta\delta\beta + \frac{3}{2}\Theta\delta\beta\delta\beta\delta,$$

$$f^{-1}(\gamma) = \gamma, \quad f^{-1}(\delta) = \delta - \frac{\Theta}{2}\delta\beta\delta.$$

Assume now that $\operatorname{char} K = 2$. Let A'_{12} be as above with $\Theta \neq 0$ and A''_{12} be an algebra given by the quiver of A_{12} bound by the relations

$$\alpha\beta\gamma = \Theta'\alpha\beta\delta\beta\gamma, \quad \alpha\beta\gamma\alpha = 0, \quad \gamma\alpha\beta\gamma = 0, \quad \delta\beta\delta = \gamma\alpha, \quad \beta(\delta\beta)^3 = 0,$$

for some parameter $\Theta' \in K\backslash\{0\}$. Denote $\vartheta = \Theta^{-1}\Theta'$. Then we have an algebra isomorphism $g : A'_{12} \to A''_{12}$ given by

$$g(\alpha) = \vartheta\alpha, \quad g(\beta) = \vartheta\beta, \quad g(\gamma) = \gamma, \quad g(\delta) = \delta.$$

This ends the proof, because, for $\Theta = 1$, A'_{12} is equal to Λ_4. □

Lemma 5.6. *Let Λ be a selfinjective algebra which is socle equivalent to A_{13} but nonisomorphic to A_{13}. Then* $\operatorname{char} K = 2$ *and Λ is isomorphic to Λ_5.*

Proof. Let Λ be a selfinjective algebra socle equivalent to A_{13} and let A'_{13} be an algebra isomorphic to Λ of the form described at the beginning of this section. We claim that in A'_{13} we have $\alpha^3 = \Theta\sigma\delta \neq 0$ for some nonzero parameter $\Theta \in K$.

In fact, since $\alpha^3, \sigma\delta \in \operatorname{soc} A'_{13}$ and the socle of any indecomposable projective A'_{13}-module is one-dimensional, we have either $\alpha^3 = 0$, or $\sigma\delta = 0$, or $\alpha^3 = \Theta\sigma\delta (\neq 0)$ for some $\Theta \in K\backslash\{0\}$. If $\sigma\delta = 0$, then δ is left-maximal in A'_{13}, but $\delta \in A_{13}\backslash\operatorname{soc} A_{13}$, a contradiction. Suppose that $\alpha^3 = 0$. Then $\alpha^2 \neq 0$, because $\alpha^2 \notin \operatorname{soc} A_{13}$. Let $\alpha^2\omega$ be a nonzero element of $\operatorname{soc} A'_{13}$. We may assume that $\omega = \alpha\omega_1 + \beta\gamma\omega_2 + \sigma\omega_3$. Since $\alpha^2\alpha = \alpha^3 = 0$, $\alpha^2\beta\gamma = \beta(\gamma\beta)\gamma = 0$ and $\alpha^2\sigma = \alpha(\alpha\sigma) = 0$, we get $\alpha^2\omega = 0$, a contradiction.

Therefore $\alpha^3 = \Theta\sigma\delta \neq 0$ and the algebra A'_{13} is given by the quiver of A_{13} bound by relations

$$\gamma\beta = \Theta_1\gamma\alpha\beta, \quad \delta\beta = 0, \quad \gamma\sigma = 0, \quad \delta\alpha = 0,$$

$$\alpha^2 = \beta\gamma + \Theta_2\alpha^3, \quad \alpha\sigma = 0, \quad \delta\sigma\delta = 0, \quad \sigma\delta\sigma = 0,$$

$$\alpha^3 = \Theta\sigma\delta, \quad \alpha^4 = 0, \quad \gamma\beta\gamma = 0, \quad \beta\gamma\beta = 0,$$

for some parameters $\Theta_1, \Theta_2 \in K$, $\Theta \in K\backslash\{0\}$. Note that, for $\Theta_1 = 1$ and $\Theta_2 = 0$, A'_{13} is equal to Λ_5.

Assume that A_{13} and A'_{13} are isomorphic, and let $f : A_{13} \to A'_{13}$ be an algebra isomorphism. Then f is given by

$$f(\alpha) = a_1\alpha + a_2\alpha^2 + a_3\alpha^3, \quad f(\beta) = b_1\beta + b_2\alpha\beta,$$
$$f(\gamma) = c_1\gamma + c_2\gamma\alpha, \quad f(\delta) = d\delta, \quad f(\sigma) = s\sigma,$$

for some parameters $a_1, b_1, c_1, d, s \in K \setminus \{0\}$, $a_2, a_3, b_2, c_2 \in K$. Denote $a = a_1^{-1}a_2$, $b = b_1^{-1}b_2$, $c = c_1^{-1}c_2$. Then we obtain the following equalities:

$$f(\gamma\beta) = b_1c_1\gamma\beta + (b_1c_2 + b_2c_1)\gamma\alpha\beta,$$
$$f(\alpha^2 - \beta\gamma) = a_1^2\alpha^2 - b_1c_1\beta\gamma - (b_1c_2 + b_2c_1 - 2a_1a_2)\alpha^3,$$
$$f(\alpha^3 - \sigma\delta) = a_1^3\alpha^3 - ds\sigma\delta.$$

Hence, we have the relations $a_1^2 = b_1c_1$, $b + c + \Theta_1 = 0$, $\Theta_2 = b + c - 2a$, $a_1^3\Theta = ds$. Therefore $\Theta_1 + \Theta_2 = -2a$, and $\Theta_1 + \Theta_2 = 0$ if $\operatorname{char} K = 2$. In particular, if $\Theta_1 + \Theta_2 \neq 0$ and $\operatorname{char} K = 2$, then the algebras A_{13} and A'_{13} are nonisomorphic. On the other hand, if $\operatorname{char} K \neq 2$, then there is an algebra isomorphism $f : A_{13} \to A'_{13}$ given by

$$f(\alpha) = \alpha - \frac{\Theta_1 + \Theta_2}{2}\alpha^2, \quad f(\beta) = \beta - \Theta_1\alpha\beta,$$
$$f(\gamma) = \gamma, \quad f(\delta) = \Theta\delta, \quad f(\sigma) = \sigma,$$

whose inverse $f^{-1} : A'_{13} \to A_{13}$ is given by

$$f^{-1}(\alpha) = \alpha + \frac{\Theta_1 + \Theta_2}{2}\alpha^2 + \frac{(\Theta_1 + \Theta_2)^2}{2}\alpha^3, \quad f^{-1}(\beta) = \beta + \Theta_1\alpha\beta,$$
$$f^{-1}(\gamma) = \gamma, \quad f^{-1}(\delta) = \Theta^{-1}\delta, \quad f^{-1}(\sigma) = \sigma,$$

and, if $\Theta_1 + \Theta_2 = 0$ and K is of characteristic 2, then there is an algebra isomorphism $f : A_{13} \to A'_{13}$ given by

$$f(\alpha) = \alpha, \quad f(\beta) = \beta - \Theta_1\alpha\beta, \quad f(\gamma) = \gamma, \quad f(\delta) = \Theta\delta, \quad f(\sigma) = \sigma.$$

Assume now that $\operatorname{char} K = 2$. Let A'_{13} be as above with $\theta = \Theta_1 + \Theta_2 \neq 0$, and A''_{13} be an algebra given by the quiver of A_{13} bound by relations

$$\gamma\beta = \Theta'_1\gamma\alpha\beta, \quad \delta\beta = 0, \quad \gamma\sigma = 0, \quad \delta\alpha = 0,$$
$$\alpha^2 = \beta\gamma + \Theta'_2\alpha^3, \quad \alpha\sigma = 0, \quad \delta\sigma\delta = 0, \quad \sigma\delta\sigma = 0,$$
$$\alpha^3 = \Theta'\sigma\delta, \quad \alpha^4 = 0, \quad \gamma\beta\gamma = 0, \quad \beta\gamma\beta = 0,$$

for some parameters $\Theta'_1, \Theta'_2 \in K$, $\Theta' \in K \setminus \{0\}$ with $\theta' = \Theta'_1 + \Theta'_2 \neq 0$. We will show that A'_{13} and A''_{13} are isomorphic. Denote $\vartheta = \theta^{-1}\theta'$. Then there exists an algebra isomorphism $g : A'_{13} \to A''_{13}$ given by

$$g(\alpha) = \vartheta\alpha, \quad g(\beta) = \vartheta\beta + \vartheta(\vartheta\Theta'_1 - \Theta_1)\alpha\beta,$$
$$g(\gamma) = \vartheta\gamma, \quad g(\delta) = \delta, \quad g(\sigma) = \vartheta^3\Theta^{-1}\Theta'\sigma,$$

and its inverse $g^{-1} : A''_{13}(\lambda) \to A'_{13}(\lambda)$ is given by

$$g^{-1}(\alpha) = \vartheta^{-1}\alpha, \quad g^{-1}(\beta) = \vartheta^{-1}\beta - \vartheta^{-1}(\vartheta^{-1}\Theta_1 + \Theta_1')\alpha\beta,$$

$$g^{-1}(\gamma) = \vartheta^{-1}\gamma, \quad g^{-1}(\delta) = \delta, \quad g^{-1}(\sigma) = \vartheta^{-3}\Theta'^{-1}\Theta\sigma. \qquad \square$$

Lemma 5.7. *Let Λ be a selfinjective algebra which is socle equivalent to A_{14} but nonisomorphic to A_{14}. Then* $\operatorname{char} K = 2$ *and Λ is isomorphic to Λ_6.*

Proof. The algebra Λ is isomorphic to an algebra A'_{14} given by the quiver of A_{14} and relations

$$\delta\gamma\delta\alpha = 0, \quad \beta\gamma\delta\gamma = 0, \quad (\gamma\delta)^3\gamma = 0, \quad \delta(\gamma\delta)^3 = 0,$$

$$\beta\alpha = \Theta_1\beta\gamma\delta\alpha, \quad \alpha\beta = (\gamma\delta)^2 + \Theta_2(\gamma\delta)^3, \quad \alpha\beta\alpha = 0, \quad \beta\alpha\beta = 0,$$

for some parameters $\Theta_1, \Theta_2 \in K$.

Assume that A_{14} and A'_{14} are isomorphic, and let $f : A_{14} \to A'_{14}$ be an algebra isomorphism. Then f is given by

$$f(\alpha) = a_1\alpha + a_2\gamma\delta\alpha, \quad f(\gamma) = c_1\gamma + c_2\gamma\delta\gamma + c_3\gamma\delta\gamma\delta\gamma,$$

$$f(\beta) = b_1\beta + b_2\beta\gamma\delta, \quad f(\delta) = d_1\delta + d_2\delta\gamma\delta + d_3\delta\gamma\delta\gamma\delta,$$

for some parameters $a_1, b_1, c_1, d_1 \in K\backslash\{0\}$, $a_2, b_2, c_2, c_3, d_2, d_3 \in K$. Denote $a = a_1^{-1}a_2$, $b = b_1^{-1}b_2$, $c = c_1^{-1}c_2$, $d = d_1^{-1}d_2$. Then the following equalities hold:

$$f(\beta\alpha) = a_1b_1\beta\alpha + (a_1b_2 + a_2b_1)\beta\gamma\delta\alpha,$$

$$f(\alpha\beta - \delta\gamma\delta\gamma) = a_1b_1\alpha\beta - c_1^2d_1^2(\gamma\delta)^2 + (a_1b_2 + a_2b_1 - 2(c_1c_2d_1^2 + c_1^2d_1d_2))(\gamma\delta)^3.$$

Therefore, we have the relations $a_1b_1 = c_1^2d_1^2$, $a + b + \Theta_1 = 0$, $\Theta_2 = 2(d + c) - (a + b)$. Hence $\Theta_2 - \Theta_1 = 2(c + d)$, and $\Theta_1 = \Theta_2$ if $\operatorname{char} K = 2$. In particular, if $\Theta_1 \neq \Theta_2$ and $\operatorname{char} K = 2$, then the algebras A_{14} and A'_{14} are nonisomorphic. Observe also that, if $\operatorname{char} K \neq 2$, then there is an algebra isomorphism $f : A_{14} \to A'_{14}$ given by

$$f(\alpha) = \alpha - \Theta_1\gamma\delta\alpha, \quad f(\beta) = \beta, \quad f(\gamma) = \gamma + \frac{\Theta_2 - \Theta_1}{2}\gamma\delta\gamma, \quad f(\delta) = \delta,$$

whose inverse $f^{-1} : A'_{14} \to A_{14}$ is given by

$$f^{-1}(\alpha) = \alpha + \Theta_1\gamma\delta\alpha, \quad f^{-1}(\beta) = \beta,$$

$$f^{-1}(\gamma) = \gamma + \frac{\Theta_1 - \Theta_2}{2}\gamma\delta\gamma + \frac{(\Theta_1 - \Theta_2)^2}{2}\gamma\delta\gamma\delta\gamma, \quad f^{-1}(\delta) = \delta,$$

and, if $\Theta_1 = \Theta_2$ and K is of characteristic 2, then there is an algebra isomorphism $f : A_{14} \to A'_{14}$ given by

$$f(\alpha) = \alpha - \Theta_1\gamma\delta\alpha, \quad f(\beta) = \beta, \quad f(\gamma) = \gamma, \quad f(\delta) = \delta.$$

Assume now that $\operatorname{char} K = 2$. Let A'_{14} be as above with $\theta = \Theta_2 - \Theta_1 \neq 0$, and A''_{14} be an algebra given by the quiver of A_{14} bound by relations

$$\delta\gamma\delta\alpha = 0, \quad \beta\gamma\delta\gamma = 0, \quad (\gamma\delta)^3\gamma = 0, \quad \delta(\gamma\delta)^3 = 0,$$

$$\beta\alpha = \Theta_1\beta\gamma\delta\alpha, \quad \alpha\beta = (\gamma\delta)^2 + \Theta_2(\gamma\delta)^3, \quad \alpha\beta\alpha = 0, \quad \beta\alpha\beta = 0,$$

for some parameters $\Theta_1', \Theta_2' \in K$ with $\theta' = \Theta_1' - \Theta_2' \neq 0$. We will show that A_{14}' and A_{14}'' are isomorphic. Denote $\vartheta = \theta^{-1}\theta'$. Then there exists an algebra isomorphism $g : A_{14}' \to A_{14}''$ given by

$$g(\alpha) = \vartheta\alpha + \vartheta(\vartheta\Theta_1 - \Theta_1')\gamma\delta\alpha, \quad g(\beta) = \vartheta\beta, \quad g(\gamma) = \vartheta\gamma, \quad g(\delta) = \delta,$$

and whose inverse $g^{-1} : A_{14}''(\lambda) \to A_{14}'(\lambda)$ is given by

$$g^{-1}(\alpha) = \vartheta^{-1}\alpha + \vartheta^{-1}(\vartheta^{-1}\Theta_1' - \Theta_1)\gamma\delta\alpha,$$

$$g^{-1}(\beta) = \vartheta^{-1}\beta, \quad g^{-1}(\gamma) = \vartheta^{-1}\gamma, \quad g^{-1}(\delta) = \delta.$$

This finishes the proof, because Λ_6 is equal to A_{14}' for $\Theta_1 = 1$, $\Theta_2 = 0$. □

Lemma 5.8. *Let Λ be a selfinjective algebra which is socle equivalent to A_{15} but nonisomorphic to A_{15}. Then* $\operatorname{char} K = 2$ *and Λ is isomorphic to Λ_7.*

Proof. The algebra Λ is isomorphic to an algebra A_{15}' given by the quiver of A_{15} and relations

$$\delta\beta = \Theta_1\delta\alpha\beta, \quad \sigma\alpha = 0, \quad \delta\alpha = \gamma\sigma, \quad \alpha\beta\gamma = 0,$$

$$\alpha^2 = \beta\delta + \Theta_2\alpha^3, \quad \delta\beta\gamma = 0, \quad \beta\delta\beta = 0, \quad \delta\beta\delta = 0, \quad \alpha^4 = 0,$$

for some parameters $\Theta_1, \Theta_2 \in K$. Clearly, for $\Theta_1 = 1$ and $\Theta_2 = 0$, A_{15}' is equal to Λ_7. Assume that A_{15} and A_{15}' are isomorphic, and let $f : A_{15} \to A_{15}'$ be an algebra isomorphism. Then f is given by

$$f(\alpha) = a_1\alpha + a_2\alpha^2 + a_3\alpha^3, \quad f(\beta) = b_1\beta + b_2\alpha\beta,$$

$$f(\gamma) = c\gamma, \quad f(\delta) = d_1\delta + d_2\delta\alpha, \quad f(\sigma) = s\sigma,$$

for some parameters $a_1, b_1, c, d_1, s \in K\backslash\{0\}$, $a_2, a_3, b_2, d_2 \in K$. Denote $a = a_1^{-1}a_2$, $b = b_1^{-1}b_2$, $d = d_1^{-1}d_2$. Then we have the following equalities:

$$f(\delta\alpha - \gamma\sigma) = a_1d_1\delta\alpha - cs\gamma\sigma,$$

$$f(\delta\beta) = b_1d_1\delta\beta + (b_1d_2 + b_2d_1)\delta\alpha\beta,$$

$$f(\alpha^2 - \beta\delta) = a_1^2\alpha^2 - b_1d_1\beta\delta - (b_1d_2 + b_2d_1 - 2a_1a_2)\alpha^3.$$

Hence we obtain the relations $a_1d_1 = cs$, $a_1^2 = b_1d_1$, $b + d + \Theta_1 = 0$, $\Theta_2 = b + d - 2a$. Therefore $\Theta_1 + \Theta_2 = -2a$, and $\Theta_1 + \Theta_2 = 0$ if $\operatorname{char} K = 2$. In particular, if $\Theta_1 + \Theta_2 \neq 0$ and $\operatorname{char} K = 2$, then the algebras A_{15} and A_{15}' are nonisomorphic. On the other hand, if $\operatorname{char} K \neq 2$, then there is an algebra isomorphism $f : A_{15} \to A_{15}'$ given by

$$f(\alpha) = \alpha - \frac{\Theta_1 + \Theta_2}{2}\alpha^2, \quad f(\beta) = \beta - \Theta_1\alpha\beta,$$

$$f(\gamma) = \gamma, \quad f(\delta) = \delta, \quad f(\sigma) = \sigma,$$

whose inverse $f^{-1} : A'_{15} \to A_{15}$ is given by

$$f^{-1}(\alpha) = \alpha + \frac{\Theta_1 + \Theta_2}{2}\alpha^2 + \frac{(\Theta_1 + \Theta_2)^2}{2}\alpha^3, \quad f^{-1}(\beta) = \beta + \Theta_1\alpha\beta,$$

$$f^{-1}(\gamma) = \gamma, \quad f^{-1}(\delta) = \delta, \quad f^{-1}(\sigma) = \sigma,$$

and, if $\Theta_1 + \Theta_2 = 0$ and K is of characteristic 2, then there is an algebra isomorphism $f : A_{15} \to A'_{15}$ given by

$$f(\alpha) = \alpha, \quad f(\beta) = \beta - \Theta_1\alpha\beta, \quad f(\gamma) = \gamma, \quad f(\delta) = \delta, \quad f(\sigma) = \sigma.$$

Assume now that $\operatorname{char} K = 2$. Let A'_{15} be as above with $\theta = \Theta_1 + \Theta_2 \neq 0$, and let A''_{15} be an algebra given by the quiver of A_{15} bound by relations

$$\delta\beta = \Theta'_1\delta\alpha\beta, \quad \sigma\alpha = 0, \quad \delta\alpha = \gamma\sigma, \quad \alpha\beta\gamma = 0,$$

$$\alpha^2 = \beta\delta + \Theta'_2\alpha^3, \quad \delta\beta\gamma = 0, \quad \beta\delta\beta = 0, \quad \delta\beta\delta = 0, \quad \alpha^4 = 0,$$

for some parameters $\Theta'_1, \Theta'_2 \in K$ with $\theta' = \Theta'_1 + \Theta'_2 \neq 0$. We will show that A'_{15} and A''_{15} are isomorphic. Denote $\vartheta = \theta^{-1}\theta'$. Then there exists an algebra isomorphism $g : A'_{15} \to A''_{15}$ given by

$$g(\alpha) = \vartheta\alpha, \quad g(\beta) = \vartheta\beta + \vartheta(\vartheta\Theta_1 - \Theta'_1)\alpha\beta,$$

$$g(\gamma) = \vartheta\gamma, \quad g(\delta) = \vartheta\delta, \quad g(\sigma) = \vartheta\sigma,$$

and whose inverse $g^{-1} : A''_{15}(\lambda) \to A'_{15}(\lambda)$ is given by

$$g^{-1}(\alpha) = \vartheta^{-1}\alpha, \quad g^{-1}(\beta) = \vartheta^{-1}\beta + \vartheta^{-1}(\vartheta^{-1}\Theta'_1 - \Theta_1)\alpha\beta,$$

$$g^{-1}(\gamma) = \vartheta^{-1}\gamma, \quad g^{-1}(\delta) = \vartheta^{-1}\delta, \quad g^{-1}(\sigma) = \vartheta^{-1}\sigma. \qquad \square$$

LEMMA 5.9. *Let Λ be a selfinjective algebra which is socle equivalent to A_{16} but nonisomorphic to A_{16}. Then* $\operatorname{char} K = 2$ *and Λ is isomorphic to Λ_8.*

PROOF. This is a consequence of the above lemma, because $A_{16} \cong A_{15}^{\mathrm{op}}$ and $\Lambda_8 \cong \Lambda_7^{\mathrm{op}}$. □

LEMMA 5.10. *Let Λ be a selfinjective algebra which is socle equivalent to A_3 but nonisomorphic to A_3. Then* $\operatorname{char} K = 2$ *and Λ is isomorphic to Λ_9.*

PROOF. The algebra Λ is isomorphic to an algebra A'_3 given by the quiver of A_3 and relations

$$\alpha\beta + \gamma\delta + \xi\varepsilon = \Theta_1\alpha\beta\gamma\delta, \quad \beta\alpha = \Theta_2\beta\gamma\delta\alpha, \quad \delta\gamma = \Theta_3\delta\xi\varepsilon\gamma, \quad \varepsilon\xi = \Theta_4\varepsilon\alpha\beta\xi,$$

$$\delta\alpha\beta\gamma\delta = 0, \quad \varepsilon\alpha\beta\gamma\delta = 0, \quad \alpha\beta\gamma\delta\alpha = 0, \quad \alpha\beta\gamma\delta\xi = 0,$$

$$\beta\alpha\beta = 0, \quad \delta\gamma\delta = 0, \quad \gamma\delta\gamma = 0, \quad \varepsilon\xi\varepsilon = 0, \quad \xi\varepsilon\xi = 0,$$

for some parameters $\Theta_1, \Theta_2, \Theta_3, \Theta_4 \in K$. Assume that A_3 and A'_3 are isomorphic, and let $f : A_3 \to A'_3$ be an algebra isomorphism. Then f is given by

$$f(\alpha) = a_1\alpha + a_2\gamma\delta\alpha + a_3\xi\varepsilon\alpha, \quad f(\beta) = b_1\beta + b_2\beta\gamma\delta + b_3\beta\xi\varepsilon,$$
$$f(\gamma) = c_1\gamma + c_2\xi\varepsilon\gamma + c_3\alpha\beta\gamma, \quad f(\delta) = d_1\delta + d_2\delta\xi\varepsilon + d_3\delta\alpha\beta,$$
$$f(\xi) = e_1\xi + e_2\alpha\beta\xi + e_3\gamma\delta\xi, \quad f(\varepsilon) = z_1\varepsilon + z_2\varepsilon\alpha\beta + z_3\varepsilon\gamma\delta,$$

for some parameters $a_1, b_1, c_1, d_1, e_1, z_1 \in K\backslash\{0\}$, $a_i, b_i, c_i, d_i, e_i, z_i \in K$ with $i \in \{2,3\}$. Denote $a = a_1^{-1}(a_2 - a_3)$, $b = b_1^{-1}(b_2 - b_3)$, $c = c_1^{-1}(c_2 - c_3)$, $d = d_1^{-1}(d_2 - d_3)$, $e = e_1^{-1}(e_2 - e_3)$, $z = z_1^{-1}(z_2 - z_3)$. Then the following equalities hold:

$$f(\beta\alpha) = a_1b_1\beta\alpha + (a_1(b_2 - b_3) + (a_2 - a_3)b_1)\beta\gamma\delta\alpha,$$
$$f(\delta\gamma) = c_1d_1\delta\gamma + (c_1(d_2 - d_3) + (c_2 - c_3)d_1)\delta\xi\varepsilon\gamma,$$
$$f(\varepsilon\xi) = e_1z_1\varepsilon\xi + (e_1(z_2 - z_3) + (e_2 - e_3)z_1)\varepsilon\alpha\beta\xi,$$
$$\begin{aligned} f(\alpha\beta + \gamma\delta + \xi\varepsilon) &= a_1b_1\alpha\beta + c_1d_1\gamma\delta + e_1z_1\xi\varepsilon \\ &\quad + (a_1(b_2 - b_3) - (a_2 - a_3)b_1 + c_1(d_2 - d_3) \\ &\qquad - (c_2 - c_3)d_1 + e_1(z_2 - z_3) - (e_2 - e_3)z_1)\alpha\beta\gamma\delta. \end{aligned}$$

Hence, we have the relations $a + b + \Theta_2 = 0$, $c + d + \Theta_3 = 0$, $e + z + \Theta_4 = 0$, $a_1b_1 = c_1d_1 = e_1z_1$, $-a + b - c + d - e + z + \Theta_1 = 0$. Therefore, $2(b + d + z) + \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 = 0$, and $\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 = 0$ if $\operatorname{char} K = 2$. In particular, if $\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 \neq 0$ and $\operatorname{char} K = 2$, then the algebras A_3 and A_3' are nonisomorphic. Observe also that, if K is of characteristic 2 and $\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 = 0$, then there is an algebra isomorphism $f : A_3 \to A_3'$ given by

$$f(\alpha) = \alpha - \Theta_2\gamma\delta\alpha, \quad f(\beta) = \beta, \quad f(\gamma) = \gamma - \Theta_3\xi\varepsilon\gamma,$$
$$f(\delta) = \delta, \quad f(\xi) = \xi - \Theta_4\alpha\beta\xi, \quad f(\varepsilon) = \varepsilon,$$

and, if $\operatorname{char} K \neq 2$, then there is an algebra isomorphism $f : A_3 \to A_3'$ given by

$$f(\alpha) = \alpha - \Theta_2\gamma\delta\alpha, \quad f(\beta) = \beta, \quad f(\xi) = \xi + \frac{\Theta_1 + \Theta_2 + \Theta_3 - \Theta_4}{2}\alpha\beta\xi,$$
$$f(\gamma) = \gamma - \Theta_3\xi\varepsilon\gamma, \quad f(\delta) = \delta, \quad f(\varepsilon) = \varepsilon - \frac{\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4}{2}\varepsilon\alpha\beta.$$

Assume now that $\operatorname{char} K = 2$. Let A_3' be as above with $\theta = \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 \neq 0$, and A_3'' be an algebra given by the quiver of A_3 bound by relations

$$\alpha\beta + \gamma\delta + \xi\varepsilon = \Theta_1'\alpha\beta\gamma\delta, \quad \beta\alpha = \Theta_2'\beta\gamma\delta\alpha, \quad \delta\gamma = \Theta_3'\delta\xi\varepsilon\gamma, \quad \varepsilon\xi = \Theta_4'\varepsilon\alpha\beta\xi,$$
$$\delta\alpha\beta\gamma\delta = 0, \quad \varepsilon\alpha\beta\gamma\delta = 0, \quad \alpha\beta\gamma\delta\alpha = 0, \quad \alpha\beta\gamma\delta\xi = 0,$$
$$\beta\alpha\beta = 0, \quad \delta\gamma\delta = 0, \quad \gamma\delta\gamma = 0, \quad \varepsilon\xi\varepsilon = 0, \quad \xi\varepsilon\xi = 0,$$

for some (different) parameters $\Theta_1', \Theta_2', \Theta_3', \Theta_4' \in K$ with $\theta' = \Theta_1' + \Theta_2' + \Theta_3' + \Theta_4' \neq 0$. We will show that the algebras A_3' and A_3'' are isomorphic. Denote $\vartheta = \theta^{-1}\theta'$. Then we have an algebra isomorphism $g : A_3' \to A_3''$ given by

$$g(\alpha) = \alpha + (\vartheta\Theta_2 - \Theta_2')\gamma\delta\alpha, \quad g(\beta) = \vartheta\beta,$$
$$g(\gamma) = \gamma + (\vartheta\Theta_3 - \Theta_3')\xi\varepsilon\gamma, \quad g(\delta) = \vartheta\delta,$$
$$g(\xi) = \xi + (\vartheta\Theta_4 - \Theta_4')\alpha\beta\xi, \quad g(\varepsilon) = \vartheta\varepsilon,$$

and whose inverse $g^{-1} : A_3'' \to A_3'$ is given by

$$g^{-1}(\alpha) = \alpha + (\vartheta^{-1}\Theta_2' - \Theta_2)\gamma\delta\alpha, \quad g^{-1}(\beta) = \vartheta^{-1}\beta,$$
$$g^{-1}(\gamma) = \gamma + (\vartheta^{-1}\Theta_3' - \Theta_3)\xi\varepsilon\gamma, \quad g^{-1}(\delta) = \vartheta^{-1}\delta,$$
$$g^{-1}(\xi) = \xi + (\vartheta^{-1}\Theta_4' - \Theta_4)\alpha\beta\xi, \quad g^{-1}(\varepsilon) = \vartheta^{-1}\varepsilon.$$

This ends the proof, because Λ_9 is equal to A_3' for $\Theta_1 = 1$, $\Theta_2 = \Theta_3 = \Theta_4 = 0$. □

Lemma 5.11. *Let Λ be a selfinjective algebra which is socle equivalent to A_{29} but nonisomorphic to A_{29}. Then* $\operatorname{char} K = 2$ *and Λ is isomorphic to Λ_{10}.*

Proof. The algebra Λ is isomorphic to an algebra A_{29}' given by the quiver of A_{29} and relations

$$\beta\mu = \Theta_1\beta\sigma\delta\mu, \quad \eta\alpha = \Theta_2\eta\gamma\xi\alpha, \quad \alpha\beta = \gamma\delta + \Theta_3\gamma\delta\sigma\delta,$$
$$\delta\sigma = \xi\gamma + \Theta_4\delta\sigma\delta\sigma, \quad \sigma\xi = \mu\eta + \Theta_5\sigma\delta\sigma\xi,$$
$$\beta\mu\eta = 0, \quad \alpha\beta\mu = 0, \quad \eta\alpha\beta = 0, \quad \mu\eta\alpha = 0,$$
$$\delta\sigma\delta\sigma\delta = 0, \quad \sigma\delta\sigma\delta\sigma = 0, \quad \delta\sigma\delta\sigma\xi = 0, \quad \gamma\delta\sigma\delta\sigma = 0,$$
$$\gamma\delta\sigma\delta\mu = 0, \quad \gamma\delta\sigma\delta\sigma = 0, \quad \xi\gamma\delta\sigma\delta = 0, \quad \eta\gamma\delta\sigma\delta = 0,$$
$$\sigma\delta\sigma\xi\alpha = 0, \quad \sigma\delta\sigma\xi\gamma = 0, \quad \beta\sigma\delta\sigma\xi = 0, \quad \delta\sigma\delta\sigma\xi = 0,$$

for some parameters $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5 \in K$. Clearly, Λ_{10} is equal to A_{29}' with $\Theta_4 = 1$, $\Theta_1 = \Theta_2 = \Theta_3 = \Theta_5 = 0$. Assume that the algebras A_{29} and A_{29}' are isomorphic, and let $f : A_{29} \to A_{29}'$ be an algebra isomorphism. Then f is given by

$$f(\alpha) = a_1\alpha + a_2\gamma\xi\alpha, \quad f(\beta) = b_1\beta + b_2\beta\sigma\delta, \quad f(\gamma) = c_1\gamma + c_2\gamma\xi\gamma,$$
$$f(\delta) = d_1\delta + d_2\delta\sigma\delta, \quad f(\eta) = n_1\eta + n_2\eta\gamma\xi, \quad f(\mu) = m_1\mu + m_2\sigma\delta\mu,$$
$$f(\xi) = z_1\xi + z_2\xi\gamma\xi, \quad f(\sigma) = s_1\sigma + s_2\sigma\delta\sigma,$$

for some parameters $a_1, b_1, c_1, d_1, m_1, n_1, s_1, z_1 \in K\backslash\{0\}$, $a_2, b_2, c_2, d_2, m_2, n_2, s_2, z_2 \in K$. Denote $a = a_1^{-1}a_2$, $b = b_1^{-1}b_2$, $c = c_1^{-1}c_2$, $d = d_1^{-1}d_2$, $m = m_1^{-1}m_2$, $n = n_1^{-1}n_2$, $s = s_1^{-1}s_2$, $z = z_1^{-1}z_2$. We have then the following equalities:

$$f(\beta\mu) = b_1m_1\beta\mu + (b_1m_2 + b_2m_1)\beta\sigma\delta\mu,$$
$$f(\eta\alpha) = a_1n_1\eta\alpha + (a_1n_2 + a_2n_1)\eta\gamma\xi\alpha,$$

$$f(\alpha\beta - \gamma\delta) = a_1b_1\alpha\beta - c_1d_1\gamma\delta + (a_1b_2 + a_2b_1 - c_1d_2 - c_2d_1)\gamma\delta\sigma\delta,$$

$$f(\delta\sigma - \xi\gamma) = d_1s_1\delta\sigma - c_1z_1\xi\gamma + (d_1s_2 + d_2s_1 - c_1z_2 - c_2z_1)\delta\sigma\delta\sigma,$$

$$f(\sigma\xi - \mu\eta) = s_1z_1\sigma\xi - m_1n_1\mu\eta + (s_1z_2 + s_2z_1 - m_1n_2 - m_2n_1)\sigma\delta\sigma\xi.$$

Hence, we obtain the relations $d_1s_1 = c_1z_1$, $a_1b_1 = c_1d_1$, $s_1z_1 = m_1n_1$, $b + m + \Theta_1 = 0$, $a + n + \Theta_2 = 0$, $a + b + \Theta_3 = c + d$, $c + z = d + s + \Theta_4$, $m + n = s + z + \Theta_5$. Therefore, $\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_5 = -2(a + b - c + s)$, and $\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_5 = 0$ if $\operatorname{char} K = 2$. In particular, if $\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_5 \neq 0$ and $\operatorname{char} K = 2$, then the algebras A_{29} and A'_{29} are nonisomorphic. On the other hand, if $\operatorname{char} K \neq 2$, then there is an algebra isomorphism $f : A_{29} \to A'_{29}$ given by

$$f(\alpha) = \alpha, \quad f(\beta) = \beta, \quad f(\gamma) = \gamma + \Theta_3\gamma\xi\gamma,$$

$$f(\delta) = \delta, \quad f(\eta) = \eta - \Theta_2\eta\gamma\xi, \quad f(\mu) = \mu - \Theta_1\sigma\delta\mu,$$

$$f(\xi) = \xi - \frac{\Theta_1 + \Theta_2 + \Theta_3 - \Theta_4 + \Theta_5}{2}\xi\gamma\xi, \quad f(\sigma) = \sigma - \frac{\Theta_1 + \Theta_2 - \Theta_3 + \Theta_4 + \Theta_5}{2}\sigma\delta\sigma,$$

and, if $\operatorname{char} K = 2$ and $\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_5 = 0$, then there is an algebra isomorphism $f : A_{29} \to A'_{29}$ given by

$$f(\alpha) = \alpha, \quad f(\beta) = \beta, \quad f(\gamma) = \gamma + \Theta_3\gamma\xi\gamma, \quad f(\delta) = \delta, \quad f(\sigma) = \sigma,$$

$$f(\eta) = \eta + \Theta_2\eta\gamma\xi, \quad f(\mu) = \mu + \Theta_1\sigma\delta\mu, \quad f(\xi) = \xi + (\Theta_3 + \Theta_4)\xi\gamma\xi.$$

Assume now that $\operatorname{char} K = 2$. Let A'_{29} be as above with $\theta = \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_5 \neq 0$, and A''_{29} be an algebra given by the quiver of A_{29} bound by relations

$$\beta\mu = \Theta'_1\beta\sigma\delta\mu, \quad \eta\alpha = \Theta'_2\eta\gamma\xi\alpha, \quad \alpha\beta = \gamma\delta + \Theta'_3\gamma\delta\sigma\delta,$$

$$\delta\sigma = \xi\gamma + \Theta'_4\delta\sigma\delta\sigma, \quad \sigma\xi = \mu\eta + \Theta'_5\sigma\delta\sigma\xi,$$

$$\beta\mu\eta = 0, \quad \alpha\beta\mu = 0, \quad \eta\alpha\beta = 0, \quad \mu\eta\alpha = 0,$$

$$\delta\sigma\delta\sigma\delta = 0, \quad \sigma\delta\sigma\delta\sigma = 0, \quad \delta\sigma\delta\sigma\xi = 0, \quad \gamma\delta\sigma\delta\sigma = 0,$$

$$\gamma\delta\sigma\delta\mu = 0, \quad \gamma\delta\sigma\delta\sigma = 0, \quad \xi\gamma\delta\sigma\delta = 0, \quad \eta\gamma\delta\sigma\delta = 0,$$

$$\sigma\delta\sigma\xi\alpha = 0, \quad \sigma\delta\sigma\xi\gamma = 0, \quad \beta\sigma\delta\sigma\xi = 0, \quad \delta\sigma\delta\sigma\xi = 0,$$

for some (different) parameters $\Theta'_1, \Theta'_2, \Theta'_3, \Theta'_4, \Theta'_5 \in K$ with $\theta' = \Theta'_1 + \Theta'_2 + \Theta'_3 + \Theta'_4 + \Theta'_5 \neq 0$. We will show that the algebras A'_{29} and A''_{29} are isomorphic. Denote $\vartheta = \theta^{-1}\theta'$. Then we have an algebra isomorphism $g : A'_{29} \to A''_{29}$ given by

$$g(\alpha) = \alpha, \quad g(\beta) = \vartheta\beta, \quad g(\gamma) = \vartheta\gamma + \vartheta(\vartheta\Theta_3 + \Theta'_3)\gamma\xi\gamma,$$

$$g(\delta) = \delta, \quad g(\eta) = \vartheta\eta + \vartheta(\vartheta\Theta_2 + \Theta'_2)\eta\gamma\xi, \quad g(\mu) = \mu + (\vartheta\Theta_1 + \Theta'_1)\sigma\delta\mu,$$

$$g(\xi) = \xi + (\vartheta(\Theta_3 + \Theta_4) + (\Theta'_3 + \Theta'_4))\xi\gamma\xi, \quad g(\sigma) = \vartheta\sigma,$$

and its inverse $g^{-1} : A''_{29} \to A'_{29}$ is given by

$$g^{-1}(\alpha) = \alpha, \quad g^{-1}(\beta) = \vartheta^{-1}\beta, \quad g^{-1}(\gamma) = \vartheta^{-1}\gamma + \vartheta^{-1}(\vartheta^{-1}\Theta_3' + \Theta_3)\gamma\xi\gamma,$$

$$g^{-1}(\delta) = \delta, \quad g^{-1}(\xi) = \xi + (\vartheta^{-1}(\Theta_3' + \Theta_4') + (\Theta_3 + \Theta_4))\xi\gamma\xi, \quad g^{-1}(\sigma) = \vartheta^{-1}\sigma,$$

$$g^{-1}(\eta) = \vartheta^{-1}\eta + \vartheta^{-1}(\vartheta^{-1}\Theta_2' + \Theta_2)\eta\gamma\xi, \quad g^{-1}(\mu) = \mu + (\vartheta^{-1}\Theta_1' + \Theta_1)\sigma\delta\mu. \qquad \square$$

LEMMA 5.12. *Let A be one of the algebras $A_1(\lambda), A_{20}(\lambda)$, $\lambda \in K\backslash\{0,1\}$, A_i, for $i \in \{17, 19\}$ (if $\operatorname{char} K \neq 2$), or A_i for $i \in \{8, 22, 23, 24, 26, 27, 28\}$. Then every selfinjective algebra socle equivalent to A is isomorphic to A.*

PROOF. It follows directly from the remarks at the beginning of this section, because for all these algebras, in the chosen sets of generators of I, we have no relations which can be replaced by the procedures (1) or (2). $\square$

LEMMA 5.13. *Let K be of characteristic different from 2. There are no selfinjective algebras which are socle equivalent to A_{18} but nonisomorphic to A_{18}.*

PROOF. Assume that Λ is a selfinjective algebra socle equivalent to A_{18}. Then Λ is isomorphic to an algebra A_{18}' given by the quiver of A_{18} bound by relations

$$\alpha^2 = \gamma\sigma, \quad \alpha\gamma = \gamma\beta + \Theta_1\alpha\gamma\beta, \quad \alpha\gamma\beta^2 = \alpha^2\gamma\beta = 0, \quad \alpha\gamma\beta\sigma = \gamma\beta\sigma\alpha = 0,$$

$$\sigma\gamma = \beta^2, \quad \sigma\alpha + \beta\sigma = \Theta_2\beta\sigma\alpha, \quad \beta^2\sigma\alpha = \beta\sigma\alpha^2 = 0, \quad \beta\sigma\alpha\gamma = \sigma\alpha\gamma\beta = 0,$$

for some parameters $\Theta_1, \Theta_2 \in K$. Note that we have $\alpha^3 = 0$ and $\beta^3 = 0$, because $\alpha^3 = \alpha\gamma\sigma = \gamma\beta\sigma = -\gamma\sigma\alpha = -\alpha^3$ and $\beta^3 = \sigma\gamma\beta = \sigma\alpha\gamma = -\beta\sigma\gamma = -\beta^3$. Then there exists an algebra isomorphism $f : A_{18} \to A_{18}'$ given by

$$f(\alpha) = \alpha - \frac{\Theta_1}{2}\alpha^2, \quad f(\beta) = \beta + \frac{\Theta_1}{2}\beta^2, \quad f(\gamma) = \gamma, \quad f(\sigma) = \sigma + \frac{\Theta_2}{2}\sigma\alpha,$$

and whose inverse $f^{-1} : A_{18}' \to A_{18}$ is given by

$$f^{-1}(\alpha) = \alpha + \frac{\Theta_1}{2}\alpha^2, \quad f^{-1}(\beta) = \beta - \frac{\Theta_1}{2}\beta^2,$$

$$f^{-1}(\gamma) = \gamma, \quad f^{-1}(\sigma) = \sigma - \frac{\Theta_2}{2}\sigma\alpha + \frac{(\Theta_1 - \Theta_2)\Theta_2}{4}\sigma\alpha^2.$$

This finishes the proof. $\square$

LEMMA 5.14. *There are no selfinjective algebras which are socle equivalent to A_{25} but nonisomorphic to A_{25}.*

PROOF. Assume that there exists an algebra Λ which is socle equivalent to the algebra A_{25} but nonisomorphic to A_{25}. Then Λ is isomorphic to an algebra A_{25}' given by the quiver of A_{25} and relations

$$\alpha_{i+1}\alpha_i = \beta_i\gamma_i, \quad \alpha_i\sigma_{i+1} = 0, \quad \gamma_i\beta_i = 0, \quad \gamma_i\sigma_{i+1} = 0, \quad \delta_i\alpha_{i+1} = 0, \quad \delta_i\beta_i = 0,$$

$$(\alpha_i\alpha_{i+1})^2 = 0, \quad \gamma_i\alpha_{i+1}\alpha_i\alpha_{i+1} = 0, \quad \alpha_i\alpha_{i+1}\alpha_i\beta = 0, \quad \delta_i\sigma_{i+1}\delta_i = 0, \quad \sigma_{i+1}\delta_i\sigma_{i+1} = 0,$$

for $i \in \{1, 2\}$, $\alpha_3 = \alpha_1$, $\sigma_3 = \sigma_1$, exactly one of the relations

$$\alpha_1\alpha_2\alpha_1 = \Theta_1\sigma_1\delta_1, \quad \alpha_1\alpha_2\alpha_1 = 0, \quad \sigma_1\delta_1 = 0,$$

and exactly one of the relations

$$\alpha_2\alpha_1\alpha_2 = \Theta_2\sigma_2\delta_2, \quad \alpha_2\alpha_1\alpha_2 = 0, \quad \sigma_2\delta_2 = 0,$$

for some $\Theta_1, \Theta_2 \in K\backslash\{0\}$. We claim that in both cases the first relations from the above triples are satisfied. Since $\alpha_1\alpha_2\alpha_1, \sigma_1\delta_1 \in \operatorname{soc} A'_{25}$ and the socle of indecomposable projective module at each vertex is one-dimensional, we have either $\alpha_1\alpha_2\alpha_1 = 0$, or $\sigma_1\delta_1 = 0$, or $\alpha_1\alpha_2\alpha_1 = \Theta_1\sigma_1\delta_1(\neq 0)$ for some $\Theta_1 \in K\backslash\{0\}$. If $\sigma_1\delta_1 = 0$, then δ_1 is left-maximal in A'_{25}, but $\delta_1 \in A_{25}\backslash\operatorname{soc} A_{25}$, a contradiction. Assume that $\alpha_1\alpha_2\alpha_1 = 0$. Then $\alpha_1\alpha_2 \neq 0$, because $\alpha_1\alpha_2 \notin \operatorname{soc} A_{25}$. Let $\alpha_1\alpha_2\omega$ be a nonzero element of $\operatorname{soc} A'_{25}$. We may assume that $\omega = \alpha_1\omega_1 + \beta_2\gamma_2\omega_2 + \sigma_1\omega_3$. Then $\alpha_1\alpha_2\alpha_1 = 0$, $\alpha_1\alpha_2\beta_2\gamma_2 = \beta_2(\gamma_2\beta_2)\gamma_2 = 0$, and $\alpha_1\alpha_2\sigma_1 = \alpha_1(\alpha_2\sigma_1) = 0$. Therefore $\alpha_1\alpha_2\omega = 0$, a contradiction. Similarly we prove that $\sigma_2\delta_2 = 0$ and $\alpha_2\alpha_1\alpha_2 = 0$. Finally, A'_{25} is bound by relations

$$\alpha_{i+1}\alpha_i = \beta_i\gamma_i, \quad \alpha_i\alpha_{i+1}\alpha_i = \Theta_i\sigma_i\delta_i,$$

$$\alpha_i\sigma_{i+1} = 0, \quad \gamma_i\beta_i = 0, \quad \gamma_i\sigma_{i+1} = 0, \quad \delta\alpha_{i+1} = 0, \quad \delta_i\beta_i = 0,$$

for $i \in \{1,2\}$, $\alpha_3 = \alpha_1$, $\sigma_3 = \sigma_1$, and some $\Theta_1, \Theta_2 \in K\backslash\{0\}$. Therefore, we have an algebra isomorphism $f : A_{25} \to A'_{25}$ given by

$$f(\alpha_i) = \alpha_i, \quad f(\beta_i) = \beta_i, \quad f(\gamma_i) = \gamma_i, \quad f(\delta_i) = \delta_i, \quad f(\sigma_i) = \Theta_i\sigma_i,$$

for $i \in \{1,2\}$, and whose inverse $f^{-1} : A'_{25} \to A_{25}$ is given by

$$f^{-1}(\alpha_i) = \alpha_i, \quad f^{-1}(\beta_i) = \beta_i, \quad f^{-1}(\gamma_i) = \gamma_i, \quad f^{-1}(\delta_i) = \delta_i, \quad f^{-1}(\sigma_i) = \Theta_i^{-1}\sigma_i,$$

for $i \in \{1,2\}$. □

Lemma 5.15. *There are no selfinjective algebras which are socle equivalent to A_{11} but nonisomorphic to A_{11}.*

Proof. Assume that Λ is a selfinjective algebra socle equivalent to the algebra A_{11}. Then Λ is isomorphic to an algebra A'_{11} given by the quiver of A_{11} bound by relations

$$\beta\alpha\gamma = \gamma\xi\gamma, \quad \gamma\delta = 0, \quad \alpha\beta\alpha = 0, \quad \alpha\beta = \Theta_1\alpha\gamma\xi\beta,$$

$$\xi\beta\alpha = \xi\gamma\xi, \quad \zeta\xi = 0, \quad \beta\alpha\beta = 0, \quad (\xi\gamma)^2 = \Theta_2\delta\zeta,$$

for some parameters $\Theta_1 \in K$, $\Theta_2 \in K\backslash\{0\}$. Indeed, since the socle of the indecomposable projective module at each vertex is one-dimensional and $\delta\zeta, (\xi\gamma)^2 \in \operatorname{soc} A'_{11}$, then exactly one of the following equalities holds:

$$\delta\zeta = 0, \quad (\xi\gamma)^2 = 0, \quad (\xi\gamma)^2 = \Theta_2\delta\zeta \quad \text{for some } \Theta_2 \in K\backslash\{0\}.$$

If $\delta\zeta = 0$, then δ is right-maximal in A'_{11}, which contradicts $\delta \in A_{11}\backslash\operatorname{soc} A_{11}$. Assume that $(\xi\gamma)^2 = 0$. Then $\xi\gamma\xi$ is right-maximal, because $\xi\gamma\xi\gamma = 0$ and $\xi\gamma\xi\beta = \xi\beta\alpha\beta = 0$, a contradiction. Hence $(\xi\gamma)^2 = \Theta_2\delta\zeta$ for some $\Theta_2 \in K\backslash\{0\}$. Then there exists an algebra isomorphism $f : A_{11} \to A'_{11}$ given by

$$f(\alpha) = \alpha - \Theta_1\alpha\gamma\xi, \quad f(\beta) = \beta, \quad f(\gamma) = \gamma,$$
$$f(\delta) = \Theta_2\delta, \quad f(\zeta) = \zeta, \quad f(\xi) = \xi,$$

and whose inverse $f^{-1} : A'_{11} \to A_{11}$ is given by

$$f^{-1}(\alpha) = \alpha + \Theta_1\alpha\gamma\xi, \quad f^{-1}(\beta) = \beta, \quad f^{-1}(\gamma) = \gamma,$$
$$f^{-1}(\delta) = \Theta_2^{-1}\delta, \quad f^{-1}(\zeta) = \zeta, \quad f^{-1}(\xi) = \xi.$$

This finishes the proof. □

LEMMA 5.16. *Let A be one of the algebras A_4, A_{21}, and Λ be a selfinjective algebra socle equivalent to A. Then Λ is isomorphic to A.*

PROOF. Assume that Λ is a selfinjective algebra socle equivalent to A_4. Then Λ is isomorphic to an algebra A'_4 given by the quiver of A_4 bound by relations

$$\alpha\beta + \gamma\delta + \xi\varepsilon = \Theta_1\alpha\beta\gamma\delta, \quad \beta\alpha = \Theta_2\beta\gamma\delta\alpha, \quad \varepsilon\gamma = 0, \quad \delta\xi = 0,$$
$$\beta\alpha\beta = 0, \quad \delta\alpha\beta\gamma\delta = 0, \quad \varepsilon\alpha\beta\gamma\delta = 0, \quad \alpha\beta\gamma\delta\alpha = 0, \quad \alpha\beta\gamma\delta\gamma = 0,$$

for some parameters $\Theta_1, \Theta_2 \in K$. Then we have an algebra isomorphism $f : A_4 \to A'_4$ given by

$$f(\alpha) = \alpha - \Theta_2\gamma\delta\alpha, \quad f(\beta) = \beta, \quad f(\gamma) = \gamma,$$
$$f(\delta) = \delta + (\Theta_1 - \Theta_2)\delta\gamma\delta, \quad f(\xi) = \xi, \quad f(\varepsilon) = \varepsilon,$$

and with the inverse $f^{-1} : A'_4 \to A_4$ is given by

$$f^{-1}(\alpha) = \alpha + \Theta_2\gamma\delta\alpha, \quad f^{-1}(\beta) = \beta, \quad f^{-1}(\gamma) = \gamma,$$
$$f^{-1}(\delta) = \delta + (\Theta_2 - \Theta_1)\delta\gamma\delta, \quad f^{-1}(\xi) = \xi, \quad f^{-1}(\varepsilon) = \varepsilon.$$

Assume now that Λ is a selfinjective algebra socle equivalent to A_{21}. Then, by the facts mentioned at the beginning of this section, Λ is isomorphic to an algebra A'_{21} given by the quiver of A_{21} bound by relations

$$\alpha\beta + \gamma\delta + \xi\varepsilon = \Theta\alpha\beta\gamma\delta, \quad \delta\alpha = 0, \quad \beta\xi = 0, \quad \varepsilon\gamma = 0,$$
$$\beta\alpha\beta\gamma\delta = 0, \quad \varepsilon\alpha\beta\gamma\delta = 0, \quad \alpha\beta\gamma\delta\gamma = 0, \quad \alpha\beta\gamma\delta\xi = 0,$$

for some parameter $\Theta \in K$. Then there exists an algebra isomorphism $f : A_{21} \to A'_{21}$ given by

$$f(\alpha) = \alpha, \quad f(\beta) = \beta + \Theta\beta\alpha\beta, \quad f(\gamma) = \gamma,$$
$$f(\delta) = \delta, \quad f(\xi) = \xi, \quad f(\varepsilon) = \varepsilon,$$

and whose inverse $f^{-1} : A'_{21} \to A_{21}$ is given by

$$f^{-1}(\alpha) = \alpha, \quad f^{-1}(\beta) = \beta - \Theta\beta\alpha\beta, \quad f^{-1}(\gamma) = \gamma,$$
$$f^{-1}(\delta) = \delta, \quad f^{-1}(\xi) = \xi, \quad f^{-1}(\varepsilon) = \varepsilon.$$

This finishes the proof. □

Lemma 5.17. *There are no selfinjective algebras which are socle equivalent to A_7 but nonisomorphic to A_7.*

Proof. Let A_7' be a selfinjective algebra socle equivalent to the algebra A_7. By previous considerations we may assume that A_7' is isomorphic to an algebra given by the quiver of A_7 bound by relations

$$\alpha\beta = \gamma\delta + \Theta_1\gamma\delta\gamma\delta, \quad \delta\gamma = \xi\varepsilon + \Theta_2\xi\varepsilon\xi\varepsilon, \quad \varepsilon\delta\alpha = 0, \quad \beta\gamma\xi = 0,$$

$$\gamma\delta\gamma\delta\gamma = 0, \quad \delta\gamma\delta\gamma\delta = 0, \quad \gamma\delta\gamma\delta\alpha = 0, \quad \beta\gamma\delta\gamma\delta = 0,$$

$$\xi\varepsilon\xi\varepsilon\xi = 0, \quad \varepsilon\xi\varepsilon\xi\varepsilon = 0, \quad \gamma\xi\varepsilon\xi\varepsilon = 0, \quad \xi\varepsilon\xi\varepsilon\delta = 0,$$

for some parameters Θ_1, Θ_2. In this case, there exists an algebra isomorphism $f : A_7 \to A_7'$ given by

$$f(\alpha) = \alpha - \Theta_1\alpha\beta\alpha, \quad f(\beta) = \beta, \quad f(\gamma) = \gamma,$$

$$f(\delta) = \delta, \quad f(\xi) = \xi, \quad f(\varepsilon) = \varepsilon + \Theta_2\varepsilon\xi\varepsilon,$$

and with the inverse $f^{-1} : A_7' \to A_7$ given by

$$f^{-1}(\alpha) = \alpha + \Theta_1\alpha\beta\alpha, \quad f^{-1}(\beta) = \beta, \quad f^{-1}(\gamma) = \gamma,$$

$$f^{-1}(\delta) = \delta, \quad f^{-1}(\xi) = \xi, \quad f^{-1}(\varepsilon) = \varepsilon - \Theta_2\varepsilon\xi\varepsilon.$$

This finishes the proof. □

Lemma 5.18. *There are no selfinjective algebras which are socle equivalent to A_9 but nonisomorphic to A_9.*

Proof. Assume that Λ is a selfinjective algebra socle equivalent to the algebra A_9. Therefore, Λ is isomorphic to an algebra A_9' given by the quiver of A_9 bound by relations

$$\alpha\delta = \beta\varepsilon, \quad \varepsilon\gamma = \sigma\beta + \Theta\sigma\alpha\delta\gamma, \quad \beta\sigma\alpha = 0, \quad \delta\gamma\varepsilon = 0,$$

$$\gamma\varepsilon\gamma\sigma = 0, \quad \gamma\sigma\alpha\delta\gamma = 0, \quad \sigma\alpha\delta\gamma\sigma = 0,$$

for some parameter $\Theta \in K$. Note that we have $\sigma\alpha\delta\gamma = \sigma\beta\varepsilon\gamma = \sigma\beta\sigma\beta$. Then there exists an algebra isomorphism $f : A_9 \to A_9'$ given by

$$f(\alpha) = \alpha, \quad f(\beta) = \beta, \quad f(\gamma) = \gamma,$$

$$f(\delta) = \delta, \quad f(\sigma) = \sigma + \Theta\sigma\beta\sigma, \quad f(\varepsilon) = \varepsilon,$$

and whose inverse $f^{-1} : A_9' \to A_9$ is given by

$$f^{-1}(\alpha) = \alpha, \quad f^{-1}(\beta) = \beta, \quad f^{-1}(\gamma) = \gamma,$$

$$f^{-1}(\delta) = \delta, \quad f^{-1}(\sigma) = \sigma - \Theta\sigma\beta\sigma, \quad f^{-1}(\varepsilon) = \varepsilon.$$

This finishes the proof. □

Lemma 5.19. *There are no selfinjective algebras which are socle equivalent to* A_{10} *but nonisomorphic to* A_{10}.

Proof. Let A_{10}' be a selfinjective algebra socle equivalent to the algebra A_{10}. We may assume that A_{10}' is isomorphic to an algebra given by the quiver of A_{10} bound by relations

$$\beta\alpha\xi = \xi\gamma\delta\xi, \quad \delta\beta\alpha = \delta\xi\gamma\delta, \quad \alpha\beta = \Theta\alpha\xi\gamma\delta\beta,$$

$$(\gamma\delta\xi)^2\gamma = 0, \quad \alpha\beta\alpha = 0, \quad \beta\alpha\beta = 0,$$

for some parameter $\Theta \in K$. Then we have an algebra isomorphism $f : A_{10} \to A_{10}'$ given by

$$f(\alpha) = \alpha - \Theta\alpha\xi\gamma\delta, \quad f(\beta) = \beta, \quad f(\gamma) = \gamma, \quad f(\delta) = \delta, \quad f(\xi) = \xi,$$

and its inverse $f^{-1} : A_{10}' \to A_{10}$ is given by

$$f^{-1}(\alpha) = \alpha + \Theta\alpha\xi\gamma\delta, \quad f^{-1}(\beta) = \beta, \quad f^{-1}(\gamma) = \gamma, \quad f^{-1}(\delta) = \delta, \quad f^{-1}(\xi) = \xi. \qquad \square$$

Lemma 5.20. *There are no selfinjective algebras which are socle equivalent to* A_{30} *but nonisomorphic to* A_{30}.

Proof. Assume that Λ is a selfinjective algebra socle equivalent to the algebra A_{30}. Then Λ is isomorphic to an algebra A_{30}' given by the quiver of A_{30} bound by relations

$$\beta\alpha = 0, \quad \eta\mu = 0, \quad \alpha\beta = \gamma\delta, \quad \sigma\xi = \mu\eta,$$

$$\delta\gamma = \xi\sigma + \Theta\delta\gamma\delta\gamma, \quad \delta\gamma\delta\gamma\xi = 0, \quad \sigma\delta\gamma\delta\gamma = 0,$$

for some parameter $\Theta \in K$. We have an algebra isomorphism $f : A_{30} \to A_{30}'$ given by

$$f(\alpha) = \alpha, \quad f(\beta) = \beta, \quad f(\gamma) = \gamma, \quad f(\delta) = \delta - \Theta\delta\sigma\delta,$$

$$f(\eta) = \eta, \quad f(\mu) = \mu, \quad f(\xi) = \xi, \quad f(\sigma) = \sigma,$$

and whose inverse $f^{-1} : A_{30}' \to A_{30}$ is given by

$$f^{-1}(\alpha) = \alpha, \quad f^{-1}(\beta) = \beta, \quad f^{-1}(\gamma) = \gamma, \quad f^{-1}(\delta) = \delta + \Theta\delta\sigma\delta,$$

$$f^{-1}(\eta) = \eta, \quad f^{-1}(\mu) = \mu, \quad f^{-1}(\xi) = \xi, \quad f^{-1}(\sigma) = \sigma.$$

This finishes the proof. $\square$

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