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Laplacian comparison and sub-mean-value theorem for multiplier Hermitian manifolds

By Tomonori Noda and Masashi Oda

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Abstract. In this note, we study the Laplacian comparison theorem and the sub-mean-value theorem for a special type of Hermitian manifolds called multiplier Hermitian manifolds. By conformal change of the metrics, this covers much wider objects than in the case of ordinary Kähler manifolds.

1. Introduction.

The purpose of this paper is to show a sub-mean-value property for multiplier Hermitian manifolds (cf. Theorem B below), where a key of the proof lies in proving a Laplacian comparison result (cf. Theorem A below; see Greene-Wu [**3**] for Riemannian cases) for multiplier Hermitian manifolds.

A multiplier Hermitian manifold (cf. [**8**]) is a quantitive generalization of a Kähler-Ricci soliton [**11**] (see also a recent result of Wang and Zhu [**13**]). A multiplier Hermitian manifold can possibly be noncompact, while by the associated conformal changes of a Kähler metric, we can have a large varieties of Ricci forms, as in passing from the theory of projective algebraic surfaces, in algebraic geometry, to that of open algebraic surfaces.

Let (M, ω) be an n-dimensional connected complete Kähler manifold with complex structure J. For a system of holomorphic local coordinates $(z^1, z^2, \dots, z^n)$ on M, we write

$$\omega = \sqrt{-1} \sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\, dz^\alpha \wedge dz^{\bar{\beta}}.$$

Fix a holomorphic vector field $X \in H^0(M, \mathcal{O}(T^{1,0}M))$ on M, assuming that the corresponding real vector field $X_{\boldsymbol{R}} = X + \bar{X}$ is Hamiltonian, i.e. there exists a real-valued smooth function u on M satisfying $i(X_{\boldsymbol{R}})\omega = du$. Let I be the interval defined as the image of $u : M \to \boldsymbol{R}$. For a real-valued nonconstant smooth function σ on I, we put $\psi := \sigma(u)$. Let $\tilde{\omega}$ be the conformal change of ω defined by

$$\tilde{\omega} := \exp(-\psi/n)\omega,$$

and the pair $(M, \tilde{\omega})$ is called a *multiplier Hermitian manifold* (cf. [**10**]). The associated Ricci form is

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$$\mathrm{Ric}^{\sigma}(\omega) = \sqrt{-1}\bar{\partial}\partial \log(\tilde{\omega}^n) = \mathrm{Ric}(\omega) + \sqrt{-1}\partial\bar{\partial}\psi,$$

where $\mathrm{Ric}(\omega) = \sqrt{-1}\bar{\partial}\partial \log(\omega^n)$ is the Ricci form of ω. As an operator on functions on M, the Laplacian $\square_\sigma$ of the multiplier Hermitian manifold $(M, \tilde{\omega})$ is

$$\square_\sigma := \sum_{\alpha,\beta} g^{\bar{\beta}\alpha}(\partial^2/\partial z^\alpha \partial z^{\bar{\beta}}) - \sum_{\alpha,\beta} g^{\bar{\beta}\alpha}(\partial\psi/\partial z^\alpha)(\partial/\partial z^{\bar{\beta}}) = \square + \sqrt{-1}\dot{\sigma}(u)\bar{X}, \tag{1.1}$$

where $\square$ is the Laplacian for the Kähler manifold (M, ω). This operator $\square_\sigma$ plays an important role in the study of "Kähler-Einstein metrics" in the sense of [**7**]. Define the real part $\mathrm{Re}\,\square_\sigma$ of $\square_\sigma$ by $2\,\mathrm{Re}\,\square_\sigma := \square_\sigma + \overline{\square}_\sigma$.

Given a Riemannian manifold (K, g), a point p on K is called a *pole* if the exponential map $\exp_p : T_pK \to K$ is a diffeomorphism. It is easily seen that a manifold with a pole is always complete. For a geodesic γ joining p to a point q in $K \backslash \{p\}$, the vector field tangent to γ with unit speed is called a *radial vector field* and is denoted by $\dot{\gamma}$. A *radial curvature* is the restriction of the sectional curvature to a plane containing the radial vector field. For a pole p of K, the manifold K is called a *model* if every linear isometry φ of T_pK extends to Φ_* for some isometry Φ of K satisfying $\Phi(p) = p$ and $\Phi_{*,p} = \varphi$. Namely if K is a model, then the linear isotropy group at p is the full orthogonal group. For a manifold K with a pole, we always denote by ρ_K the distance function on K from the pole.

Let (N, ω_N) be a Kähler manifold with a pole p_N, and let $(N', \omega_{N'})$ be a Kähler manifold with a point $p_{N'}$ such that $\dim N = \dim N' = n$. Let $X_N, X_{N'}$ be holomorphic vector fields on N, N' vanishing at $p_N, p_{N'}$ respectively, so that

$$i((X_N)_{\boldsymbol{R}})\omega_N = du_N \quad \text{and} \quad i((X_{N'})_{\boldsymbol{R}})\omega_{N'} = du_{N'}$$

for some real-valued smooth functions $u_N, u_{N'}$ on N, N' respectively. Let $\rho_N, \rho_{N'}$ be distance functions on N, N' from $p_N, p_{N'}$ respectively. Set $\psi_N := \sigma_N(u_N)$ and $\psi_{N'} := \sigma_{N'}(u_{N'})$.

THEOREM A. *Assume that (N, p_N) is a model with non-positive radial curvature. Assume furthermore that for any $(q, q') \in (N\backslash\{p_N\}) \times (N'\backslash(\{p_{N'}\} \cup Cut(p_{N'})))$, the inequalities*

$$\mathrm{Ric}^{\sigma_{N'}}(\dot{\gamma}_{N'}, J\dot{\gamma}_{N'})(q') \geq \mathrm{Ric}^{\sigma_N}(\dot{\gamma}_N, J\dot{\gamma}_N)(q), \tag{1.2}$$

$$\sqrt{-1}\partial\bar{\partial}\psi_{N'}(\dot{\gamma}_{N'}, J\dot{\gamma}_{N'})(q') \geq \sqrt{-1}\partial\bar{\partial}\psi_N(\dot{\gamma}_N, J\dot{\gamma}_N)(q) \tag{1.3}$$

hold whenever $\rho_N(q) = \rho_{N'}(q')$, where $Cut(p_{N'})$ denotes the cut locus of $p_{N'}$ and $\gamma_N, \gamma_{N'}$ are the geodesics in N, N' joining $p_N, p_{N'}$ with q, q', respectively. Then

$$\{\square_{\sigma_{N'}} f(\rho_{N'})\}(q') \leq \{\square_{\sigma_N} f(\rho_N)\}(q) \tag{1.4}$$

for all (q, q') as above, if f is a non-decreasing smooth function on $[0, \infty)$.

Let $\mathrm{inj}_{p_{N'}}$ be the injectivity radius of $(N', \omega_{N'})$ at $p_{N'}$, and let $B = B(r)$, $B' = B'(r)$ be balls of radius r less than $\mathrm{inj}_{p_{N'}}$ centered at $p_N, p_{N'}$ in N, N', respectively.

Theorem B. *We assume that u_N is written as a function in ρ_N alone. Under the same assumption as in Theorem A, let h be a non-negative real-valued smooth function on N' such that* $\operatorname{Re}\Box_{\sigma_{N'}}h \le 0$. *Then*

$$\int_{B'} h\tilde{\omega}_{N'}^n/n! \le h(p_{N'})V, \tag{1.5}$$

where $\tilde{\omega}_N := \exp(-\psi_N/n)\omega_N$, $\tilde{\omega}_{N'} := \exp(-\psi_{N'}/n)\omega_{N'}$ *and* $V := \int_B \tilde{\omega}_N^n/n!$.

Next, we formulate special cases of the above theorems as a corollary.

Corollary. *Let $(N',\omega_{N'})$ be a multiplier Hermitian manifold with $\psi_{N'}$ such that $X_{N'}$ vanishes at $p_{N'}$ in N'.*

(i) *Assume that, for all $q' \in N' \backslash (p_{N'} \cup Cut(p_{N'}))$, the inequalities*

$$\mathrm{Ric}^{\sigma_{N'}}(\dot{\gamma}_{N'}, J\dot{\gamma}_{N'})(q') \ge 1, \tag{1.1a}$$

$$\sqrt{-1}\partial\bar{\partial}\psi_{N'}(\dot{\gamma}_{N'}, J\dot{\gamma}_{N'})(q') \ge 1, \tag{1.2a}$$

hold, then for any non-negative real-valued smooth function h satisfying $\operatorname{Re}\Box_{\sigma_{N'}}h \le 0$, *the following holds*:

$$\int_{B'(r)} h\tilde{\omega}_{N'}^n/n! \le h(p_{N'})\left(1 - \sum_{k=1}^{n} \frac{e^{-r^2} r^{2(n-k)}}{(n-k)!}\right)\pi^n. \tag{1.4a}$$

(ii) *Assume that, for all $q' \in N' \backslash (p_{N'} \cup Cut(p_{N'}))$, the inequalities*

$$\mathrm{Ric}^{\sigma_{N'}}(\dot{\gamma}_{N'}, J\dot{\gamma}_{N'})(q') \ge 0, \tag{1.1b}$$

$$\sqrt{-1}\partial\bar{\partial}\psi_{N'}(\dot{\gamma}_{N'}, J\dot{\gamma}_{N'})(q') \ge 1, \tag{1.2b}$$

hold, then for any non-negative real-valued smooth function h satisfying $\operatorname{Re}\Box_{\sigma_{N'}}h \le 0$,

$$\int_{B'(r)} h\tilde{\omega}_{N'}^n/n! \le h(p_{N'})\Omega_n, \tag{1.4b}$$

where Ω_n denotes the volume of the unit ball of hyperbolic n-space.

To see (i) above, let $N = \boldsymbol{C}^n$, $\omega_N = \sqrt{-1}\sum dz^\alpha \wedge dz^{\bar{\alpha}}$ and $\sigma_N = \mathrm{id}$ in Theorem A. Then for $X_N = -\sqrt{-1}\sum z^\alpha(\partial/\partial z^\alpha)$ and $\sigma_{N'} = \ell\,\mathrm{id}$, the conditions (1.2) and (1.3) in Theorem A reduce to (1.1a) and (1.2a). In additon, by taking $p_N = 0$ in Thoerem B, we obtain (1.4a). We also have $\int_{B'} e^{-\psi_{N'}}\omega_{N'}^n/n! \le \pi r^{2n}/n!$ by taking $X_N = 0$ and $h = 1$.

In the original comparison theorem as in Greene-Wu [**3**], the conditions (1.1a) and (1.2a) are replaced by the following condition on the Ricci curvature:

$$\mathrm{Ric}(\omega_{N'})(\dot{\gamma}_{N'}, J\dot{\gamma}_{N'})(q') \ge 0 \quad \text{for all } q' \in N'. \tag{1.6}$$

By letting $\ell = 0$, we obtain the ordinary Laplacian comparison theorem for Kähler manifolds. Moreover, in view of the equality $\mathrm{Ric}^{\sigma_{N'}}(\dot{\gamma}_{N'}, J\dot{\gamma}_{N'}) = \mathrm{Ric}(\omega_{N'})(\dot{\gamma}_{N'}, J\dot{\gamma}_{N'}) + \sqrt{-1}\partial\bar{\partial}\psi_{N'}(\dot{\gamma}_{N'}, J\dot{\gamma}_{N'})$, choosing $\sqrt{-1}\partial\bar{\partial}\psi_{N'}(q') \gg 1$, say by letting $\ell \gg 1$, we see that both (1.1a) and (1.2a) hold even if (1.6) does not hold. In this sense, Theorems A and B above give some generalization of the classical results of Greene-Wu and are applicable to many cases which the original comparison theorem in Greene-Wu [3] cannot cover.

We also obtain (ii) of the corollary by setting $N = \{z \in \boldsymbol{C}^n; \|z\| < 1\}$, $\omega_N = \sqrt{-1}\sum\{(1-\|z\|^2)\delta_{\alpha\beta} + z^\alpha z^{\bar\beta}\}(1-\|z\|^2)^{-2}\, dz^\alpha \wedge dz^{\bar\beta}$, $\sigma_N = \mathrm{id}$ and $u_N = \log(1-\|z\|^2)^{-1}$.

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2. Laplacian and star operators.

In this section, we define multiplier analogues of the star operator. For a multiplier Hermitian manifold $(M, \tilde\omega)$, where $\tilde\omega$ is as in Introduction, we put $\check{*} = e^{-\psi}*$ and $\hat{*} = e^{\psi}*$, where $*$ is the Hodge star operator of the Kähler manifold (M, ω). For a real-valued smooth function f,

$$\hat{*}\partial\check{*}\bar\partial f = e^{\psi} * \partial(e^{-\psi} * \bar\partial f) = e^{\psi} * (-e^{-\psi}\partial\psi \wedge *\bar\partial f + e^{-\psi}\partial * \bar\partial f)$$

$$= -\langle \partial\psi, \partial f\rangle + *\partial * \bar\partial f = -\sum_{\alpha,\beta} g^{\bar\beta\alpha}(\partial\psi/\partial z^\alpha)(\partial f/\partial z^{\bar\beta}) + \square f, \quad \text{i.e.} \quad \square_\sigma = \hat{*}\partial\check{*}\bar\partial.$$

Remark 2.1. Both $\hat{*}$ and $\check{*}$ are real operators. Moreover we have the identities $\check{*}\hat{*} = \hat{*}\check{*} = *^2$.

Lemma 2.2. *Let U be an open subset of M with smooth boundary ∂U. For any real-valued smooth functions h, h_0 on a neighborhood of U,*

$$\int_U (h\square_\sigma h_0 - h_0\overline{\square}_\sigma h)\tilde\omega^n/n! = \int_{\partial U}\{h(\check{*}\bar\partial h_0) - h_0(\check{*}\partial h)\}.$$

Proof. By $\bar\partial h \wedge \check{*}\partial h_0 = \partial h_0 \wedge \check{*}\bar\partial h$, we have

$$d\{h(\check{*}\bar\partial h_0) - h_0(\check{*}\partial h)\} = \partial h \wedge \check{*}\bar\partial h_0 + h(\partial\check{*}\bar\partial h_0) - \bar\partial h_0 \wedge \check{*}\partial h - h_0(\bar\partial\check{*}\partial h)$$

$$= h(\check{*}\hat{*}\partial\check{*}\bar\partial h_0) - h_0(\check{*}\hat{*}\bar\partial\check{*}\partial h) = \check{*}(h\square_\sigma h_0 - h_0\overline{\square}_\sigma h).$$

Hence, by Stokes' theorem and $\square_\sigma = \hat{*}\partial\check{*}\bar\partial$, we have the required equality. $\square$

3. Preliminaries.

In this section, we show a couple of lemmas peculiar to multiplier Hermitian manifolds. For M, ω, X, u, ψ as in Introduction, fix a point p in M. Let $\rho_M : M \to [0, \mathrm{inj}_p)$ be the distance function from p and let $\gamma : [0, \mathrm{inj}_p) \to M$ be the geodesic emanating from p such that $\dot\gamma$ coincides with the gradient vector field of ρ_M restricted to γ.

Lemma 3.1. *If X vanishes at $p \in M$, then $(X_{\boldsymbol{R}})\rho_M = (X + \bar X)\rho_M = 0$.*

Proof. We use a technique in Mabuchi [8]. For a point $q \in M$, let $b \in \boldsymbol{R}$ such that $q = \gamma(b)$. On a small neighborhood of q in M, we choose a local coordinates $(z^1, z^2, \ldots, z^n)$ centered at q such that

$$\dot\gamma(b) = (\partial/\partial x^1) \quad \text{and} \quad J\dot\gamma(b) = (\partial/\partial y^1).$$

Here $z^\alpha = x^\alpha + \sqrt{-1}y^\alpha$ for all α. We may assume that the local expression $g_{\alpha\bar{\beta}}$ of ω with respect to this holomorphic local coordinates satisfies $g_{\alpha\bar{\beta}}(q) = \delta_{\alpha\bar{\beta}}/2$ and $dg_{\alpha\bar{\beta}}(q) = 0$. A direct calculation gives

$$2(\bar{X}\rho_M)(q) = \sqrt{-1}(\partial u/\partial z^1)(q) \tag{3.1}$$

by $\bar{X} = \sqrt{-1}\sum(\partial u/\partial z^\alpha)(\partial/\partial z^{\bar{\alpha}})$. Consider the exponential map $\exp_q : T_qM \to M$ at q. Defining $\xi(s) := \exp_q(sJ\dot{\gamma}(b))$ on sufficiently small interval $-\varepsilon \le s \le \varepsilon$, we have

$$\begin{cases} \dot{\gamma}(t) = \gamma_*(\partial/\partial t) = (\partial/\partial x^1) + O(|t-b|^2), \\ \xi_*(\partial/\partial s) = (\partial/\partial y^1) + O(|s|^2) \end{cases} \tag{3.2}$$

in a neighborhood of q. Since X is holomorphic, we have $(\partial/\partial\bar{z}^1)^2(u)(q) = 0$, i.e. $(\partial/\partial x^1)^2(u)(q) = (\partial/\partial y^1)^2(u)(q) = 0$ and $(\partial^2/\partial x^1\partial y^1)(u)(q) = 0$ in the corresponding real coordinates. Now we consider a map F from $[-\varepsilon, \varepsilon] \times [0, b]$ to M defined by $F(s,t) := \exp_{\gamma(t)}(sJ\dot{\gamma}(t))$ and set $\tilde{u} := F^*u$ and $\tilde{\psi} := F^*\psi$. Obviously $\tilde{\psi} = \sigma(\tilde{u})$. It follows from (3.2) that

$$\begin{cases} (\partial/\partial t)(\tilde{u})|_{s=0} = \gamma^*\{(\partial/\partial x^1)u\} + O(|t-b|^2) \\ (\partial/\partial s)(\tilde{u})|_{t=b} = \xi^*\{(\partial/\partial y^1)u\} + O(|s|^2) \end{cases} \tag{3.3}$$

in a neighborhood of $(s,t) = (0,b)$. In (3.3), differentiating the upper equation with respect to t at $t = b$ and differentiating the lower equation with respect to s at $s = 0$, we have $(\partial/\partial t)^2(\tilde{u}) = (\partial/\partial s)^2(\tilde{u})$ on $\{0\} \times [0,b]$. From

$$\nabla_{\partial/\partial t}(\partial/\partial s)|_{(s,t)=(0,b)} = \nabla_{\partial/\partial s}(\partial/\partial t)|_{(s,t)=(0,b)} = 0 \quad \text{and} \quad F_*(\partial/\partial s)|_{(s,t)=(0,b)} = (\partial/\partial y^1),$$

we obtain $F_*(\partial/\partial s) = (\partial/\partial y^1) + O(|s|^2 + |t-b|^2)$. Together with (3.2) and $(\partial^2/\partial x^1\partial y^1)(u)(x) = 0$, we have $(\partial^2/\partial t\partial s)(\tilde{u}) = 0$ on $\{0\} \times [0,b]$. It follows that $\partial\tilde{u}/\partial s$ is constant on $\{0\} \times [0,b]$ and then for all t in $[0,b]$

$$(\partial\tilde{u}/\partial s)(0,0) = (\partial\tilde{u}/\partial s)(0,t) = 0,$$

because u is critical at p. This together with (3.1) and (3.2) completes the proof. □

LEMMA 3.2. *If X vanishes at p, then for $\gamma(t)$ as in the proof of Lemma* 3.1,

$$\int_0^b \sqrt{-1}\partial\bar{\partial}\psi(\dot{\gamma}, J\dot{\gamma})\,dt = -2\sqrt{-1}\dot{\sigma}(u)(\bar{X}\rho_M)(q).$$

PROOF. For the holomorphic coordinates as in Lemma 3.1, we have

$$\begin{aligned} \partial\bar{\partial}\psi(\dot{\gamma}, J\dot{\gamma}) &= \sum(\ddot{\sigma}(u)(\partial u/\partial z^\alpha)(\partial u/\partial z^{\bar{\beta}}) + \dot{\sigma}(u)(\partial^2 u/\partial z^\alpha\partial z^{\bar{\beta}}))(dz^\alpha \wedge dz^{\bar{\beta}})(\dot{\gamma}, J\dot{\gamma}) \\ &= -2\sqrt{-1}(\ddot{\sigma}(u)(\partial u/\partial z^1)(\partial u/\partial z^{\bar{1}}) + \dot{\sigma}(u)(\partial^2 u/\partial z^1\partial z^{\bar{1}})) \\ &= -2\sqrt{-1}(\partial/\partial z^{\bar{1}})(\dot{\sigma}(u)(\partial u/\partial z^1)). \end{aligned}$$

Hence, $(\dot\gamma+\sqrt{-1}J\dot\gamma)\{\sqrt{-1}\dot\sigma(u)\bar X\rho_M\} = -\sqrt{-1}\partial\bar\partial\psi(\dot\gamma, J\dot\gamma)/2 = -I(t)/2$. Using Lemma 3.1, we obtain

$$-\frac{1}{2}I(t) = (\dot\gamma+\sqrt{-1}J\dot\gamma)\{\sqrt{-1}\dot\sigma(u)\bar X\rho_M\} = \mathrm{Re}\{(\dot\gamma+\sqrt{-1}J\dot\gamma)(\sqrt{-1}\dot\sigma(u)\bar X\rho_M)\}$$
$$= \dot\gamma\{\sqrt{-1}\dot\sigma(u)\bar X\rho_M\} = (d/dt)\{\sqrt{-1}\dot\sigma(u)\bar X\rho_M\}.$$

Integrating this equalities, by our assumption $X(p_M) = 0$, we now complete the proof. □

4. Proof of Theorem A.

Let (K, g) be a Riemannian manifold with a fixed point p, and let q be a point in $B\backslash\{p\}$, where B is a ball centered at p with radius less than or equal to the injectivity radius at p. Let γ be the geodesic with unit speed such that $\gamma(0) = p$ and $\gamma(b) = q$ for a suitable $b > 0$. Choose an orthonormal basis $\{E_i^{\#}\}$, $2 \le i \le \dim K$, for the orthogonal complement of $\boldsymbol{R}\dot\gamma$ in the tangent space T_qK at q. For each $i \in \{2, \dots, \dim K\}$, choose a vector field $E_i(t)$, $0 \le t \le b$, along γ such that $E_i(0) = 0$, $E_i(b) = E_i^{\#}$ and that $\|E_i(t)\| = \|E_j(t)\|$ for all $t \in [0, b]$. We use the following fact in Greene-Wu [**3**, Proposition 2.15 and its proof]:

Fact 4.1. *For the Laplacian Δ of (K, g),*

$$\Delta\rho \le \int_0^b \left\{ \sum_{i=2}^{\dim K} \|\dot E_i\|^2 - \|E_2\|^2\,\mathrm{Ric}(\dot\gamma,\dot\gamma)\right\} dt.$$

The equality holds if and only if $E_i(t)$ is a Jacobi field along γ for all i.

Remark 4.2. In the case where K is the underlying Riemannian structure of (N, ω_N) in Theorem A, let $W_i(t)$, $t \in [0, b]$, be the Jacobi field defined by $W_i(0) = 0$ and $W_i(b) = E_i^{\#}$. Each $W_i(t)$ can be mapped to each $W_j(t)$ by an isometry of N fixing p_N, the orthogonality of $W_i(b)$, $2 \le i \le n$, shows $W_i(t)$, $2 \le i \le n$, are mutually orthogonal for every $t \in [0, b]$ (Greene-Wu [**3**, Corollary 2.14]). Hence if W_i's are chosen as E_i's, then the inequality in Fact 4.1 reduces to an equality.

Proof of Theorem A. Recall that $\Box_\sigma f(\rho) = (1/2)\ddot f(\rho) + \dot f\Box_\sigma\rho$ on N or N', according as (σ, ρ) is (σ_N, ρ_N) or $(\sigma_{N'}, \rho_{N'})$, respectively. Hence we may, without loss of generality, that $f = \mathrm{id}$ on $[0, \infty)$. It is now sufficient to show that $(\Box_{\sigma_{N'}}\rho_{N'})(q') \le (\Box_{\sigma_N}\rho_N)(q)$. By (1.2), Lemma 3.2 and Remark 4.2, $(\Box_{\sigma_N}\rho_N)(q)$ is

$$\frac{1}{2}\int_0^b \left\{ \sum_{i=2}^{2n} \|\dot W_i\|^2 - \|W_2\|^2\,\mathrm{Ric}(\dot\gamma_N, J\dot\gamma_N) - \sqrt{-1}\partial\bar\partial\psi(\dot\gamma_N, J\dot\gamma_N)\right\} dt.$$

For vector fields $\{E_i\}$, $2 \le i \le 2n$, along $\gamma_{N'}$ with valued in TN' satisfying $\|E_i\|(t) = \|W_i\|(t)$ and $\|\dot E_i\|(t) = \|\dot W_i\|(t)$ for all $t \in [0, b]$, we see that $(\Box_\sigma\rho_{N'})(q')$ does not exceed

$$\frac{1}{2}\int_0^b \left\{ \sum_{i=2}^{2n} \|\dot E_i\|^2 - \|E_2\|^2\,\mathrm{Ric}(\dot\gamma_{N'}, J\dot\gamma_{N'}) - \sqrt{-1}\partial\bar\partial\psi(\dot\gamma_{N'}, J\dot\gamma_{N'})\right\} dt,$$

by (1.2), Lemma 3.2 and Fact 4.1. Since $\|W_2\|^2(t)$ is a convex function in t because (N,ω_N) is of non-positive radius curvature and since $\|W_2\|^2(0)=0$ and $\|W_2\|^2(b)=1$ from our assumption, we have $0\le\|W_2\|^2\le 1$ for all $t\in[0,b]$. Since $\|E_i\|^2=\|W_i\|^2$ holds for all $t\in[0,b]$, we have

$$\begin{aligned}&(\Box_{\sigma_N}\rho_N)(x)-(\Box_{\sigma_{N'}}\rho_{N'})(x')\\&\ge\frac{1}{2}\int_0^b\{-\|W_2\|^2(\mathrm{Ric}(\dot\gamma_N,J\dot\gamma_N)-\mathrm{Ric}(\dot\gamma_{N'},J\dot\gamma_{N'}))\\&\quad-\sqrt{-1}\partial\bar\partial\psi_N(\dot\gamma_N,J\dot\gamma_N)+\sqrt{-1}\partial\bar\partial\psi_{N'}(\dot\gamma_{N'},J\dot\gamma_{N'})\}\,dt\\&\ge\frac{1}{2}\int_0^b-\|W_2\|^2(\mathrm{Ric}^{\sigma_N}(\dot\gamma_N,J\dot\gamma_N)-\mathrm{Ric}^{\sigma_{N'}}(\dot\gamma_{N'},J\dot\gamma_{N'}))\,dt,\end{aligned}$$

where the last inequality follows from (1.3) and $0\le\|W_2\|^2\le 1$. Finally by (1.2), we obtain the required inequality. □

5. Proof of Theorem B.

For (M,ω) and $\psi=\sigma(u)$ as in Introduction we first observe

LEMMA 5.1. *Let $S(r)$ be the sphere in M centered at p of radius r and let $v(r)$ be the volume of $S(r)$ with respect to the multiplier Hermitian metric $\tilde\omega$. If u is written as a function in ρ_M alone, then $dv/dr=2(\Box_\sigma\rho_M)v$.*

PROOF. The volume $v(r)$ is nothing but $v(r)=\int_{S(r)}e^{-\psi}\Omega_r$, where Ω_r is the volume form on $S(r)$ induced by Kähler metric ω on M. Let Y be a complex gradient vector field of ρ_M with respect to the Kähler form ω on M, i.e. $Y=\sum_{\alpha,\beta}g^{\bar\beta\alpha}(\partial\rho_M/\partial z^{\bar\beta})(\partial/\partial z^\alpha)$. By Lemma 3.1, $Y_{\boldsymbol{R}}\psi=-2\sqrt{-1}\dot\sigma(u)\bar X\rho_M$. By Lemma 3.2, $\Box\rho_M$ and $Y_{\boldsymbol{R}}\psi$ depends only on r, and so does $\Box_\sigma\rho_M$. Hence,

$$\begin{aligned}\frac{dv}{dr}&=\frac{d}{dr}\int_{S(r)}e^{-\psi}\Omega_r=\int_{S(r)}L_{Y_{\boldsymbol{R}}}(e^{-\psi}\Omega_r)=\int_{S(r)}\{(-Y_{\boldsymbol{R}}\psi_N)e^{-\psi}\Omega_r+e^{-\psi}L_{Y_{\boldsymbol{R}}}\Omega_r\}\\&=\int_{S(r)}\{(-Y_{\boldsymbol{R}}\psi)e^{-\psi}\Omega_r+(\Delta\rho_M)e^{-\psi}\Omega_r\}\quad(\text{cf. }[\mathbf{2},\text{p. }273\text{–}274])\\&=2\int_{S(r)}\{\Box\rho_M+\sqrt{-1}\dot\sigma(u)\bar X\rho_M\}e^{-\psi}\Omega_r=2\int_{S(r)}(\Box_\sigma\rho_M)e^{-\psi}\Omega_r\\&=2(\Box_\sigma\rho_M)\int_{S(r)}e^{-\psi}\Omega_r=2(\Box_\sigma\rho_M)v(r).\end{aligned}$$
□

PROOF OF THEOREM B. We define the real-valued function f on $[0,\infty)$ by

$$f(r)=\int_1^r v(t)^{-1}\,dt,$$

where $v(t)$ is the volume of a sphere $S(t)$ in N centered at p_N of radius t. Since $2\Box_{\sigma_N}f(\rho_N)=\ddot f(\rho_N)+2\dot f\Box_{\sigma_N}\rho_N$, it follows that $\Box_\sigma f(\rho_N)=0$ on $N\backslash\{p_N\}$ by Lemma

5.1. Next, we consider the real-valued function $f(\rho_{N'})$ on $N'\backslash\{p_{N'}\}$. By Theorem A, $\square_{\sigma_{N'}} f(\rho_{N'}) \le 0$ on $N'\backslash\{p_{N'}\}$.

Let Ω_t be the volume form of $S(t)$ in terms of the multiplier Hermitian metric $\tilde{\omega}_{N'}$, and let U be the open subset $B(r)\backslash\bar{B}(r_0)$ with $0 < r_0 < r$, where $\bar{B}(r_0)$ denotes the closure of $B(r_0)$ in N'. By fixing r, we define a function h_0 in $\rho_{N'}$ by $h_0(\rho_{N'}) := f(r) - f(\rho_{N'})$, so that $h_0(r) = 0$ if $\rho_{N'} = r$. We have that $\square_{\sigma_{N'}} h_0 = \overline{\square}_{\sigma_{N'}} h_0$ in view of Lemma 3.1. Since h and $\square_\sigma h_0$ are non-negative, Lemma 2.2 implies

$$\int_U h_0(\mathrm{Re}\,\square_\sigma h)\tilde{\omega}^n_{N'}/n! \ge \int_U (h_0\,\mathrm{Re}\,\square_\sigma h - h\square_\sigma h_0)\tilde{\omega}^n_{N'}/n!$$

$$= \int_{\partial U} \{h_0(\check{*}dh) - h(\check{*}dh_0)\} = P(r_0) + Q(r) - Q(r_0),$$

where $P(r_0) := \{f(r_0) - f(r)\}\int_{S(r_0)} \check{*}dh$ and $Q(t) := v(t)^{-1}\int_{S(t)} h\check{*}d\rho_{N'}$. Since h is smooth, there exists a positive real number M such that $\int_{S(r_0)} \check{*}dh \le \int_{S(r_0)} Me^{-\psi}\omega^n_{N'}/n!$. By the definiton of $f(r_0)$, the vanishing order of $\int_{S(r_0)} Me^{-\psi}\omega^n_{N'}/n!$ as $r_0 \to 0$ is definitely greater than that of $f(r_0)$. Hence we have $P(r_0) \to 0$ as $r_0 \to 0$. If $r_0 \to 0$, then the open set U approaches to $B'(r)$. Since $\check{*}d\rho_{N'}$ restricted to $S(t)$ is Ω_t, we have $Q(r_0) \to h(p_{N'})$ as $r_0 \to 0$. By passing to the limit, we have

$$0 \ge \int_{B'(r)} \{(\mathrm{Re}\,\square_{\sigma_{N'}} h)\int_{\rho_{N'}}^r v(t)^{-1}\,dt\} \ge -h(p_{N'}) + \frac{1}{v(r)}\int_{S(r)} h\Omega_r.$$

Hence, $\int_{S(r)} h\Omega_N \le v(r)h(p_{N'})$. We now conclude that

$$\int_{B'(r)} h\tilde{\omega}^n_{N'}/n! = \int_0^r dt\int_{S(t)} h\Omega_t \le h(p_{N'})\int_0^r v(t)\,dt = h(p_{N'})V(r),$$

as required. □

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Tomonori Noda
Department of Mathematics
Graduate school of science
Osaka University
1-1 Machikaneyama, Toyonaka
Osaka 560-0043
Japan

Masashi Oda
Department of Mathematics
Graduate school of science
Osaka University
1-1 Machikaneyama, Toyonaka
Osaka 560-0043
Japan