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Newton diagrams and equivalence of plane curve germs

Dedicated to Professor Bernard Teissier on his 60th birthday.

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Abstract. We introduce an equivalence of plane curve germs which is weaker than Zariski's equisingularity and prove that the set of all Newton diagrams of a germ is an invariant of this equivalence. Then we show how to construct all Newton diagrams of a plane many-branched singularity starting with some invariants of branches and their orders of contact.

Introduction.

Let C be a plane curve germ at a fixed point O of a complex nonsingular surface. For any chart (x, y) centered at O we consider the Newton diagram $\boldsymbol{\Delta}_{x,y}(C) \subset (\boldsymbol{R}_+)^2$. The aim of this paper is to study the set $\mathscr{N}(C)$ of all Newton diagrams $\boldsymbol{\Delta}_{x,y}(C)$ where (x, y) runs over all charts centered at O. It turns out that $\mathscr{N}(C)$ is an invariant of the germ C. To make this statement precise, we introduce an equivalence of germs (in symbols $C \equiv D$) based on the notion of reduced order of contact $d'(C, D)$ of germs C, D determined by the intersection numbers of their components with smooth branches (see Section 1 for the definitions). Multiplicity $m(C)$, number of tangents $t(C)$, contact exponent $d(C)$ (see [**H**]) are invariants of this equivalence. Two equisingular germs (see [**Z2**]) are equivalent. If all branches of the germs C, D are smooth then $C \equiv D$ if and only if C and D are equisingular. Two branches are equivalent if they have equal multiplicities and first Puiseux exponents.

Our first result (Theorem 1.5) improves M. Lejeune-Jalabert (see [**LJ**, Section 4]) and M. Oka theorems (see [**O**, Theorem 5.1]) on the stability of the Newton boundary: we prove that $C \equiv D$ implies $\mathscr{N}(C) = \mathscr{N}(D)$. To study the properties of $\mathscr{N}(C)$ we consider the set $\mathscr{N}(C)_{\mathrm{s}}$ of special Newton diagrams $\boldsymbol{\Delta}_{x,y}(C)$ such that C and $\{x = 0\}$ intersect transversally. Our main result (Theorem 1.6) is the complete description of the sets $\mathscr{N}(C)_{\mathrm{s}}$ and $\mathscr{N}(C)$ in geometric terms. Then we obtain invariant descriptions of the relations $\mathscr{N}(C)_{\mathrm{s}} = \mathscr{N}(D)_{\mathrm{s}}$ and $\mathscr{N}(C) = \mathscr{N}(D)$ (Corollaries 1.8 and 1.9) which allow us to construct two non-equivalent germs C, D with $\mathscr{N}(C) = \mathscr{N}(D)$. We give also an example of two germs C, D such that $\mathscr{N}(C)_{\mathrm{s}} = \mathscr{N}(D)_{\mathrm{s}}$ but $\mathscr{N}(C) \neq \mathscr{N}(D)$ (Example 1.11(c), (d)). The paper is organized as follows. In Section 0 (Preliminaries) we review some basic facts on the Newton diagrams using the notation proposed by Teissier (see [**T1**, pp. 616–621]). In Section 1 we present the main results and examples. In Sections 2, 3 and 5 we study the ultrametric space of plane curve germs and give auxilary

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results on the maximal contact and equivalence of germs. The proofs of the main results are given in Section 4 (Theorem 1.5) and in Section 6 (Theorem 1.6). Throughout this paper conventions about calculating with ∞ are usual.

0. Preliminaries.

Let $\boldsymbol{R}_+ = \{x \in \boldsymbol{R} : x \geq 0\}$. For any subsets A, B of the quarter $\boldsymbol{R}_+^2$ we consider the arithmetical sum $A + B = \{a + b : a \in A \text{ and } b \in B\}$. If $S \subset \boldsymbol{N}^2$ then $\boldsymbol{\Delta}(S)$ is the convex hull of the set $S + \boldsymbol{R}_+^2$. The subset $\boldsymbol{\Delta}$ of $\boldsymbol{R}_+^2$ is a *Newton diagram* if $\boldsymbol{\Delta} = \boldsymbol{\Delta}(S)$ for a set $S \subset \boldsymbol{N}^2$ (see [**K**]). According to Teissier we put $\{\frac{a}{b}\} = \boldsymbol{\Delta}(S)$ if $S = \{(a,0),(0,b)\}$, $\{\frac{a}{\infty}\} = (a,0) + \boldsymbol{R}_+^2$ and $\{\frac{\infty}{b}\} = (0,b) + \boldsymbol{R}_+^2$ for any $a, b > 0$ and call such diagrams *elementary Newton diagrams*. The Newton diagrams form the semigroup $\mathscr{N}$ with respect to the arithmetical sum. The elementary Newton diagrams generate $\mathscr{N}$. If $\boldsymbol{\Delta} = \sum_{i=1}^{r} \{\frac{a_i}{b_i}\}$ then a_i/b_i are the inclinations of edges of the diagram $\boldsymbol{\Delta}$ (by convention $\frac{a}{\infty} = 0$ and $\frac{\infty}{b} = \infty$ for $a, b > 0$). We put $\boldsymbol{i}(\boldsymbol{\Delta}) = \sup_i\{a_i/b_i\}$ and call $\boldsymbol{i}(\boldsymbol{\Delta})$ inclination of $\boldsymbol{\Delta}$.

A Newton diagram is *special* if it intersects the vertical axis and if all inclinations of its edges are ≥ 1. The special Newton diagrams form a subsemigroup $\mathscr{N}_{\mathrm{s}}$ of $\mathscr{N}$. The Newton diagram $\boldsymbol{\Delta}$ is *nearly convenient* if the distances of the diagram to the axes are ≤ 1 (the notion of convenient Newton diagram due to Kouchnirenko [**K**] is too restrictive for our purpose).

For any special Newton diagram $\boldsymbol{\Delta} = \sum\{\frac{a_i}{b_i}\}$ and for any integer $N > 0$ we consider

$$\boldsymbol{\Delta}^N = \sum_{i \in I(N)} \left\{\tfrac{a_i}{b_i}\right\} + \sum_{i \in I(N)^c} \left\{\tfrac{Nb_i}{b_i}\right\}$$

where $I(N) = \{i : a_i/b_i < N\}$ and $I(N)^c = \{i : a_i/b_i \geq N\}$. We put by convention $\boldsymbol{\Delta}^\infty = \boldsymbol{\Delta}$. Then $\boldsymbol{\Delta}^N \supset \boldsymbol{\Delta}$ with equality for $N \geq \boldsymbol{i}(\boldsymbol{\Delta})$. The diagrams $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}^N$ have the same part of the boundary formed by edges of inclination strictly less than N and the same vertex lying on the vertical axis. Moreover $\boldsymbol{\Delta}^1 = \{\frac{m}{m}\}$ where $m > 0$. The unique edge of $\boldsymbol{\Delta}^N$ whose inclination is $\geq N$ has inclination N.

Fix a complex nonsingular surface i.e. a complex holomorphic variety of dimension 2. In all this paper we consider *reduced* plane curve germs $C, D, \ldots$ centered at a fixed point O of this surface. We denote by (C,D) the intersection multiplicity of C and D and by $m(C)$ the multiplicity of C. We have $(C,D) \geq m(C)m(D)$; if $(C,D) = m(C)m(D)$ then we say that C and D intersect transversally. Let (x,y) be a chart centered at O. Then a plane curve germ C has a local equation $f(x,y) = \sum c_{\alpha\beta}x^\alpha y^\beta \in \boldsymbol{C}\{x,y\}$ without multiple factors. We put $\boldsymbol{\Delta}_{x,y}(C) = \boldsymbol{\Delta}(S)$ where $S = \{(\alpha,\beta) \in \boldsymbol{N}^2 : c_{\alpha\beta} \neq 0\}$. Clearly $\boldsymbol{\Delta}_{x,y}(C)$ is a nearly convenient Newton diagram which depends on C and (x,y). We have two fundamental properties of Newton diagrams:

(N_1) If (C_i) is a finite family of plane curve germs such that C_i and C_j $(i \neq j)$ have no common irreducible component, then

$$\boldsymbol{\Delta}_{x,y}\Big(\bigcup_i C_i\Big) = \sum_i \boldsymbol{\Delta}_{x,y}(C_i)\,.$$

(N_2) If C is an irreducible germ (a branch) then

$$\mathbf{\Delta}_{x,y}(C) = \left\{\frac{(C, y=0)}{(C, x=0)}\right\} .$$

For the proof we refer the reader to [**BK**, pp. 634–640].

1. Statement of the results.

For any reduced plane curve germs C and D with irreducible components (C_i) and (D_j) we put $d(C,D) = \inf_{i,j}\{(C_i, D_j)/(m(C_i)m(D_j))\}$ and call $d(C,D)$ the *order of contact* of germs C and D. We have for any C, D and E:

(d_1) $d(C,D) = \infty$ if and only if $C = D$ is a branch,
(d_2) $d(C,D) = d(D,C)$,
(d_3) $d(C,D) \geq \inf\{d(C,E), d(E,D)\}$.

The proof of (d_3) is given in [**ChP**] for the case of irreducible C, D, E which implies the general case. We call (d_3) the *Strong Triangle Inequality* (the STI for short). It is equivalent to the following: at least two of three numbers $d(C,D)$, $d(C,E)$, $d(E,D)$ are equal and the third is not smaller than the other two.

Remark 1.1. If (C_i) and (D_j) are finite families of plane curve germs (not necessarily irreducible) then $d(\bigcup C_i, \bigcup D_j) = \inf_{i,j}\{d(C_i, D_j)\}$.

For each germ C we define

$$d(C) = \sup\{d(C,L) : L \text{ runs over all smooth branches}\}$$

and call $d(C)$ the *contact exponent* of C (see [**H**, Definition 1.5] where the term characteristic exponent is used). Using the STI we check that $d(C) \leq d(C,C)$.

We say that a smooth germ L has *maximal contact* with C if $d(C,L) = d(C)$. Note that $d(C) = \infty$ if and only if C is a smooth branch. If C is singular then $d(C)$ is a rational number and there exists a smooth branch L which has maximal contact with C (see [**H**], [**BK**] and Section 2 of this paper).

For any germs C and D we define the *reduced order of contact* $d'(C,D)$ by putting

$$d'(C,D) = \inf\{d(C), d(C,D), d(D)\} .$$

It is easy to check that the STI holds for the reduced order of contact in the set of plane curve germs. We have $d'(C,C) = d(C)$ for any germ C.

Let Γ and C be plane curve germs. Recall that $\Gamma \subset C$ if and only if Γ is a sum of a finite number of branches of C.

Definition 1.2. Let Γ be a germ with irreducible components (Γ_i). We call Γ a *quasi-branch* if the function $(i,j) \mapsto d'(\Gamma_i, \Gamma_j)$ is constant. A quasi branch Γ is called a *quasi-component* of a germ C if $\Gamma \subset C$ and for every quasi-branch $\tilde{\Gamma}$ such that $\Gamma \subset \tilde{\Gamma} \subset C$ we have $\Gamma = \tilde{\Gamma}$.

Note that every branch is a quasi-branch and a smooth irreducible component of C is a quasi-component of C. Every germ C has a finite number $\rho(C)$ of quasi-components. If C has irreducible components (C_i) then C_i, C_j are contained in the same quasi-component of C if and only if $d'(C_i, C_j) = d(C_i) = d(C_j)$.

The following definition is basic for our purpose.

Definition 1.3. Let C and D be two plane curve germs with quasi-components (Γ_i) and (Δ_j) respectively. We call the germs C and D equivalent (in symbols $C \equiv D$) if

(1) $\rho(C) = \rho(D)$, and for a suitable arrangement of indices,
(2) $m(\Gamma_i) = m(\Delta_i)$ for all i,
(3) $d'(\Gamma_i, \Gamma_j) = d'(\Delta_i, \Delta_j)$ for all i, j.

Putting $i = j$ in (3) we get $d(\Gamma_i) = d(\Delta_i)$ for all i. If $C \equiv D$ then $m(C) = m(D)$ and $d(C) = d(D)$ (see Section 2, Proposition 2.6). The equivalence of C and D does not imply that C and D have the same number of branches.

Proposition 1.4. *Let C be a plane curve germ. Then C is a quasi-branch if and only if every Newton diagram $\mathbf{\Delta}_{x,y}(C)$ is elementary.*

The proof of the proposition is given in Section 4 of this paper. The following result is an improvement of the theorems on the stability of the Newton boundary (see Bibliographical Note) mentioned in Introduction.

Theorem 1.5. *Let C and D be equivalent plane curve germs. Then for every chart (x, y) there is a chart (z, w) such that*

$$\mathbf{\Delta}_{x,y}(C) = \mathbf{\Delta}_{z,w}(D) .$$

Let us put

$$\mathscr{N}(C) = \{\mathbf{\Delta}_{x,y}(C) : (x, y) \text{ runs over all charts centered at } O\} .$$

Then Theorem 1.5 may be stated as follows: if $C \equiv D$ then $\mathscr{N}(C) = \mathscr{N}(D)$. At the end of this section we construct two nonequivalent germs C and D such that $\mathscr{N}(C) = \mathscr{N}(D)$. The proof of Theorem 1.5 is given in Section 4.

Let C be a germ with quasi-components (Γ_i). We say that a quasi-component Γ_k is *dominating* if the following condition holds: for every quasi-component Γ_i such that $d'(\Gamma_k, \Gamma_i) = d(\Gamma_k)$ we have $d(\Gamma_k) = d(\Gamma_i)$. It is easy to see that the dominating quasi-components exist: if $d(\Gamma_k) = \sup\{d(\Gamma_i)\}$ then Γ_k is obviously dominating. For every dominating quasi-component Γ_k we consider the *Newton diagram associated with* Γ_k:

$$\mathbf{\Delta}_k(C) = \sum_i \left\{ \frac{m(\Gamma_i) d'(\Gamma_i, \Gamma_k)}{m(\Gamma_i)} \right\} .$$

Using the assumption about Γ_k one checks that the diagram $\mathbf{\Delta}_k(C)$ is well-defined: the numbers $m(\Gamma_i) d'(\Gamma_i, \Gamma_k)$ are integers for all i (see Remark 3.4).

Note that all Newton diagrams associated with dominating quasi-components of a germ C are special: they intersect the vertical axis at point $(0, m(C))$ and the inclinations of their edges are $d'(\Gamma_i, \Gamma_k) \geq 1$. In the sequel the diagrams $\mathbf{\Delta}_k(C)$ play an important part. Recall that according to the definition given in Introduction

$$\mathbf{\Delta}_k(C)^N = \sum_{i \in I(N)} \left\{ \frac{m(\Gamma_i) d'(\Gamma_i, \Gamma_k)}{m(\Gamma_i)} \right\} + \sum_{i \in I(N)^c} \left\{ \frac{m(\Gamma_i) N}{m(\Gamma_i)} \right\} \text{ for any } 0 < N \in \boldsymbol{N} \cup \{\infty\}$$

where $I(N) = \{i : d'(\Gamma_i, \Gamma_k) < N\}$, $I(N)^c = \{i : d'(\Gamma_i, \Gamma_k) \geq N\}$.

Let $\mathscr{N}(C)_{\mathrm{s}} = \{\mathbf{\Delta}_{x,y}(C) : \mathbf{\Delta}_{x,y}(C)$ is a special Newton diagram$\}$. Clearly $\mathbf{\Delta}_{x,y}(C) \in \mathscr{N}(C)_{\mathrm{s}}$ if and only if C and $\{x = 0\}$ intersect transversally. Let $\sigma(\mathscr{N}(C)_{\mathrm{s}}) = \{\sigma(\mathbf{\Delta}) : \mathbf{\Delta} \in \mathscr{N}(C)_{\mathrm{s}}\}$ where $\sigma : \boldsymbol{R}^2_+ \to \boldsymbol{R}^2_+$ is the symmetry defined by $\sigma(\alpha, \beta) = (\beta, \alpha)$ for $(\alpha, \beta) \in \boldsymbol{R}^2_+$.

Here is our main result.

THEOREM 1.6. *Let C be a plane curve germ with quasi-components (Γ_i). Set $K = \{k : \Gamma_k$ is a dominating quasi-component of $C\}$ and $\mathbf{\Delta}_k = \mathbf{\Delta}_k(C)$ for $k \in K$. Then*

(a) $\mathscr{N}(C)_{\mathrm{s}} = \bigcup_{N>0} \{\mathbf{\Delta}_k^N : k \in K\}$,

(b) $\mathscr{N}(C) = \mathscr{N}(C)_{\mathrm{s}} \cup \sigma(\mathscr{N}(C)_{\mathrm{s}}) \cup \bigcup_{N,N'>1} \{\sigma(\mathbf{\Delta}_k^N) \cap \mathbf{\Delta}_l^{N'} : k, l \in K,\ d'(\Gamma_k, \Gamma_l) = 1\}$.

In (a) and (b) we allow N, N' to be equal to ∞. We give the proof of Theorem 1.6 in Section 6. Recall that $\boldsymbol{i}(\mathbf{\Delta})$ denotes the inclination of a special diagram $\mathbf{\Delta}$.

COROLLARY 1.7. *Let C be a germ with quasi-components (Γ_i). For every special Newton diagram $\mathbf{\Delta}$ the following two conditions are equivalent*

(i) $\mathbf{\Delta} \in \mathscr{N}(C)_{\mathrm{s}}$ *and* $\boldsymbol{i}(\mathbf{\Delta}) \notin \boldsymbol{N}$,
(ii) $\mathbf{\Delta}$ *is associated with a dominating quasi-component of C.*

PROOF. From Theorem 1.6(a) it follows that $\mathbf{\Delta} \in \mathscr{N}(C)_{\mathrm{s}}$ if and only if $\mathbf{\Delta} = \mathbf{\Delta}_k^N$ for a dominating component Γ_k and an $N > 0$. It sufficies to observe that $\boldsymbol{i}(\mathbf{\Delta}_k) = \sup\{d'(\Gamma_i, \Gamma_k)\} = d(\Gamma_k) \notin \boldsymbol{N}$, $\mathbf{\Delta}_k^N = \mathbf{\Delta}_k$ for $N > d(\Gamma_k)$ and $\boldsymbol{i}(\mathbf{\Delta}_k^N) = N$ for $N < d(\Gamma_k)$. □

COROLLARY 1.8. *Let C and D be plane curve germs. Then $\mathscr{N}(C)_{\mathrm{s}} = \mathscr{N}(D)_{\mathrm{s}}$ if and only if the sets of the Newton diagrams associated with dominating quasi-components of germs C and D are equal.*

PROOF. Use Theorem 1.6(a). □

COROLLARY 1.9. *Let C and D be plane curve germs. Then $\mathscr{N}(C) = \mathscr{N}(D)$ if and only if*

(a) *the sets of the Newton diagrams associated with dominating quasi-components of germ C and D are equal,*

(b) *two Newton diagrams are associated with transversal dominating quasi-component of C if and only if they are associated with transversal dominating quasi-components of D.*

Proof. Observe that $d'(\Gamma, \Delta) = 1$ if and only if the quasi-branches Γ, Δ are transversal and use Theorem 1.6. □

Remark 1.10.

(a) If C is a quasi-branch then the Newton diagram associated with C is $\left\{\frac{m(C)d(C)}{m(C)}\right\}$.
(b) Let C be a germ which all branches C_i $(i = 1, \ldots, r)$ are smooth. Then C_i are quasi-components of C. Since $d(C_i) = \infty$ all are dominating. The Newton diagrams associated with C_i are

$$\sum_{i=1}^{r} \left\{\frac{(C_i, C_k)}{1}\right\}, \quad k = 1, \ldots, r\ .$$

Example 1.11.

(a) Let $C = \{x^a + y^b = 0\}$ where $0 < b < a$ are integers. Then there is only one Newton diagram $\boldsymbol{\Delta}$ associated with quasi-branches of C. We have $\boldsymbol{\Delta} = \{\frac{a}{b}\}$ if $\frac{a}{b} \notin \boldsymbol{N}$ and $\boldsymbol{\Delta} = \{\frac{(b-1)d}{b-1}\} + \{\frac{\infty}{1}\}$ if $d = \frac{a}{b} \in \boldsymbol{N}$.
(b) Let $C = \{xy(x^a + y^b) = 0\}$ where $0 < b < a$ are integers such that $\frac{a}{b} \notin \boldsymbol{N}$. Then $\Gamma_1 = \{x = 0\}$, $\Gamma_2 = \{y = 0\}$ and $\Gamma_3 = \{x^a + y^b = 0\}$ are quasi-components of C. We have $\boldsymbol{\Delta}_1(C) = \{\frac{b+1}{b+1}\} + \{\frac{\infty}{1}\}$, $\boldsymbol{\Delta}_2(C) = \{\frac{1}{1}\} + \{\frac{a}{b}\} + \{\frac{\infty}{1}\}$. Γ_3 is not a dominating component since $d'(\Gamma_3, \Gamma_2) = d(\Gamma_3) = \frac{a}{b}$ and $d(\Gamma_2) = \infty$.
(c) Take $C = \bigcup_{i=1}^{8} C_i$ and $D = \bigcup_{i=1}^{8} D_i$ such that $(C_i, C_j) = 1$ if $1 \le i < j \le 8$ for $(i,j) \neq (5,6), (7,8)$ and $(C_5, C_6) = (C_7, C_8) = 2$; and $(D_i, D_j) = 1$ if $1 \le i < j \le 8$ for $(i,j) \neq (3,4), (5,6), (7,8)$ and $(D_3, D_4) = (D_5, D_6) = (D_7, D_8) = 2$. To be more specific: let

$$C = \{(y-x)(y-2x)(y-3x)(y-4x)(y-5x)$$
$$(y-5x-x^2)(y-6x)(y-6x-x^2) = 0\}$$
$$D = \{(y-x)(y-2x)(y-3x)(y-3x-x^2)(y-4x)$$
$$(y-4x-x^2)(y-5x)(y-5x-x^2) = 0\}.$$

The germs C and D are not equivalent. However, it is easy to check that the diagrams associated with quasi-components of C are $\{\frac{7}{7}\} + \{\frac{\infty}{1}\}$ and $\{\frac{6}{6}\} + \{\frac{2}{1}\} + \{\frac{\infty}{1}\}$ and we get the same diagrams associated with quasi-components of D. It is easy to check that Condition (b) of Corollary 1.9 is satisfied. Thus $\mathcal{N}(C) = \mathcal{N}(D)$ by Corollary 1.9. Note that $t(C) = 6$ and $t(D) = 5$. Therefore we cannot calculate the number of tangents $t(C)$ from $\mathcal{N}(C)$.
(d) Take $C = \bigcup_{i=1}^{5} C_i$ and $D = \bigcup_{i=1}^{5} D_i$ with $(C_i, C_j) = 1$ if $i < j$, $(i,j) \neq (4,5)$ and

$(C_4, C_5) = 2$; and $(D_i, D_j) = 1$ if $i < j$ for $(i,j) \neq (2,3),(4,5)$ and $(D_2, D_3) = (D_4, D_5) = 2$. For example we may take

$$C = \{(y-x)(y-2x)(y-3x)(y-4x)(y-4x-x^2) = 0\}$$
$$D = \{(y-x)(y-2x)(y-2x-x^2)(y-3x)(y-3x-x^2) = 0\}\ .$$

Let $\mathbf{\Delta} = \{\frac{4}{4}\} + \{\frac{\infty}{1}\}$ and $\mathbf{\Delta}' = \{\frac{3}{3}\} + \{\frac{2}{1}\} + \{\frac{\infty}{1}\}$. It is easy to see that $\mathbf{\Delta}_1(C) = \mathbf{\Delta}_2(C) = \mathbf{\Delta}_3(C) = \mathbf{\Delta}$, $\mathbf{\Delta}_4(C) = \mathbf{\Delta}_5(C) = \mathbf{\Delta}'$ and $\mathbf{\Delta}_1(D) = \mathbf{\Delta}$, $\mathbf{\Delta}_2(D) = \mathbf{\Delta}_3(D) = \mathbf{\Delta}_4(D) = \mathbf{\Delta}_5(D) = \mathbf{\Delta}'$. Therefore we get $\mathscr{N}(C)_{\mathrm{s}} = \mathscr{N}(D)_{\mathrm{s}}$ by Corollary 1.8. We claim that $\mathscr{N}(C) \neq \mathscr{N}(D)$. Indeed, $\sigma(\mathbf{\Delta}) \cap \mathbf{\Delta} = \sigma(\mathbf{\Delta}_1(C)) \cap \mathbf{\Delta}_2(C) \in \mathscr{N}(C)$ since C_1 and C_2 intersect transversally and $\sigma(\mathbf{\Delta}) \cap \mathbf{\Delta} \notin \mathscr{N}(D)$ since for any transversal D_i and D_j $\sigma(\mathbf{\Delta}) \cap \mathbf{\Delta} \neq \sigma(\mathbf{\Delta}_i(D)) \cap \mathbf{\Delta}_j(D)$. We use Corollary 1.9(b).

Remark 1.12. Let us consider $\nu(C) = \sup\{\nu(\mathbf{\Delta}) : \mathbf{\Delta} \in \mathscr{N}(C)\}$ where $\nu(\mathbf{\Delta})$ is the Newton number of the diagram $\mathbf{\Delta}$ (see [**O**, Definition 2.1]). If $C \equiv D$ then $\nu(C) = \nu(D)$ by Theorem 1.5. If C is a unitangent germ then $\nu(C) = \sup\{\nu(\mathbf{\Delta}_k(C)) : \Gamma_k$ is a dominating quasi-component of $C\}$ by Theorem 1.6(a).

2. Contact exponent.

We use notation introduced in Section 1. In particular $C, D, \ldots$ are reduced plane curve germs centered at a fixed point of a given nonsingular surface, $d(C,D)$ is the order of contact of germs C, D and $d(C)$ the contact exponent of C. The following lemma is well-known (see [**H**] and [**BK**]).

Lemma 2.1. *For any plane curve germ C there is a smooth branch L which has maximal contact with C i.e. such that $d(C,L) = d(C)$.*

Note that $d(C) = \infty$ if and only if C is a smooth germ. If C is a singular germ then $d(C) \in \boldsymbol{Q}$ by Lemma 2.1 since $d(C,L) \in \boldsymbol{Q}$ if $C \neq L$ by the definition of the order of contact. Using the STI we will prove

Proposition 2.2. *Let C and D be two plane germs.*

(a) *If there exists a smooth branch which has maximal contact with C and D then $d(C,D) \geq \inf\{d(C), d(D)\}$ with equality if $d(C) \neq d(D)$.*
(b) *Suppose that there exists no smooth branch which has maximal contact with C and D. Let L and M be smooth branches such that $d(C,L) = d(C)$ and $d(D,M) = d(D)$. Then*
  (b_1) *$d(C,D) = d(L,D) = d(C,M) = d(L,M)$,*
  (b_2) *$d(C,D) < \inf\{d(C), d(D)\}$ and $d(C,D) \in \boldsymbol{N}$.*

Proof. If there exists a smooth branch L_0 such that $d(C, L_0) = d(C)$ and $d(D, L_0) = d(D)$ then to get (a) we apply the STI to the germs C, D and L_0.

To check (b) suppose that such a branch does not exist. By hypothesis $d(C,M) < d(C) = d(C,L)$ and by the STI $d(C,M) = d(L,M)$. Similarly from $d(D,L) < d(D) =$

$d(D,M)$ we get $d(D,L)=d(L,M)$. Therefore

$$d(C,M)=d(L,D)=d(L,M)\ . \tag{1}$$

We may suppose that $d(C)\le d(D)$. Thus $d(C,M)<d(D)=d(D,M)$ and

$$d(C,M)=d(C,D)\ . \tag{2}$$

From (1) and (2) we get (b_1). Property (b_2) follows from Property (b_1) since $d(L,D)<d(D)$, $d(C,M)<d(C)$ and $d(L,M)\in \boldsymbol{N}$. □

Recall that $d'(C,D)=\inf\{d(C),d(C,D),d(D)\}$. Using Proposition 2.2 we check easily

Proposition 2.3. *We have* $d'(C,D)=\inf\{d(C),d(C,D)\}=\inf\{d(C,D),d(D)\}$ *for any plane curve germs* C *and* D.

In particular if one of the germs C and D is smooth then $d'(C,D)=d(C,D)$.

Proposition 2.4. *Let* C *and* D *be plane curve germs and let* L *and* M *be smooth branches such that* $d(C,L)=d(C)$ *and* $d(D,M)=d(D)$. *Then* $d'(C,D)\le d(L,M)$.

Proof. If there exists no smooth branch which has maximal contact with C and D then $d'(C,D)=d(L,M)$ by Proposition 2.2. If there is a smooth branch L_0 such that $d(C,L_0)=d(C)$ and $d(D,L_0)=d(D)$ then $d(L,L_0)\ge\inf\{d(L,C),d(C,L_0)\}=d(C)$ and $d(L_0,M)\ge\inf\{d(L_0,D),d(D,M)\}=d(D)$ by the STI. Using the STI again we get $d(L,M)\ge\inf\{d(L,L_0),d(L_0,M)\}\ge\inf\{d(C),d(D)\}\ge d'(C,D)$. □

Proposition 2.5. *Let* (C_i) $i=1,\dots,s$ *be a family of plane curve germs. Then* $d(\bigcup C_i,L)\le\inf\{d'(C_i,C_j): i,j=1,\dots,s\}$ *for every smooth branch* L. *If* $d(\bigcup C_i,L)<\inf\{d'(C_i,C_j): i,j=1,\dots,s\}$ *then* $d(C_i,L)<d(C_i)$ *for all* $i=1,\dots,s$.

Proof. Let $\inf\{d'(C_i,C_j): i,j=1,\dots,s\}=d'(C_{i_0},C_{j_0})$. By the STI we get $d'(C_{i_0},C_{j_0})\ge\inf\{d(C_{i_0},L),d(C_{j_0},L)\}\ge\inf\{d(C_i,L): i=1,\dots,s\}=d(\bigcup C_i,L)$. This proves the first part of Proposition 2.5. To check the second part let $d(\bigcup C_i,L)=d(C_{i_0},L)$. Since $d(C_{i_0},L)<\inf\{d'(C_i,C_j): i,j=1,\dots,s\}$ we get by the STI $d(C_i,L)=d(C_{i_0},L)$ for $i=1,\dots,s$. Now $d(C_i,L)<d'(C_i,C_j)\le d(C_i)$ for $i=1,\dots,s$. □

Using Proposition 2.5 we get

Proposition 2.6. *For any family* (C_i), $i=1,\dots,s$ *of plane curve germs we have* $d(\bigcup C_i)=\inf\{d'(C_i,C_j): i,j=1,\dots,s\}$. *If a smooth branch has maximal contact with* C_{i_0} *for an* $i_0\in\{1,\dots,s\}$ *then it has maximal contact with* $\bigcup C_i$.

Proposition 2.7. *Let* (C_i), $i=1,\dots,s$ *be a family of plane curve germs and let* k *be an integer such that* $1\le k\le\inf\{d'(C_i,C_j)\}$. *Then there exists a smooth branch* L *such that* $d(C_i,L)=k$ *for* $i=1,\dots,s$.

PROOF. We omit the simple proof of the proposition in the case of smooth C_i. Let us consider the general case. Let L_i be a smooth branch such that $d(C_i, L_i) = d(C_i)$ and let $k \geq 1$ be an integer such that $k \leq \inf\{d'(C_i, C_j)\}$. By Proposition 2.4 we get $k \leq \inf\{d(L_i, L_j)\}$. Then applying the proposition to the family of smooth branches (L_i), $i = 1, \ldots, s$ we confirm that there exists a smooth branch L such that $d(L_i, L) = k$ for all $i = 1, \ldots, s$. Observe that $k \leq d'(C_i, C_i) = d(C_i)$. By the STI we get $d(C_i, L) \geq \inf\{d(C_i, L_i), d(L_i, L)\} = \inf\{d(C_i), k\} = k$. If $d(C_i) > k$ then $d(C_i, L) = k$. When $d(C_i) = k$ then $k = d(C_i) \geq d(C_i, L) \geq k$. Therefore $d(C_i, L) = k$. □

PROPOSITION 2.8. *Let C be a plane curve germ. Then*

(a) *if $d(C, L) \neq d(C)$ for a smooth branch L then $d(C, L) \in \boldsymbol{N}$.*
(b) *If k is an integer such that $1 \leq k \leq d(C)$ then there is a smooth branch L such that $d(C, L) = k$.*

PROOF. Let L_0 be a smooth branch such that $d(C, L_0) = d(C)$. From $d(C, L) < d(C, L_0)$ we get by the STI $d(C, L) = d(L_0, L) \in \boldsymbol{N}$. This proves (a). Part (b) follows from Proposition 2.7. □

PROPOSITION 2.9. *Let (C_i) and (D_i), $i = 1, \ldots, s$ be two families of plane curve germs such that $d'(C_i, C_j) = d'(D_i, D_j)$ for $i, j = 1, \ldots, s$. Then for every smooth branch L there is a smooth branch M such that $d(C_i, L) = d(D_i, M)$ for $i = 1, \ldots, s$.*

PROOF. Fix a smooth branch L and put $d^* = \sup\{d(C_i, L)\}$. Then for a suitable arrangement of indices we may assume that $d(C_1, L) = \cdots = d(C_{s^*}, L) = d^*$ and $d(C_i, L) < d^*$ for $i > s^* \in [1, s]$.

CLAIM 1. *There exists a smooth germ M such that $d(D_1, M) = \cdots = d(D_{s^*}, M) = d^*$.*

First let us assume that $d^* \in \boldsymbol{N}$. Applying Proposition 2.7 to the family of germs $(D_i : i = 1, \ldots, d^*)$ and to the integer $k = d^*$ we get a smooth branch M such that $d(D_i, M) = d^* = d(C_i, L)$ for $i = 1, \ldots, s^*$.

Now, let us suppose that $d^* \notin \boldsymbol{N}$. Then $d(C_i, L) = d(C_i) = d^*$ for $i \in [1, s^*]$ by Proposition 2.8(a). Let M be a smooth branch such that $d(D_1, M) = d(D_1) = d(C_1)$. For any $i \in [1, s^*]$ we get $d(D_i, M) \geq \inf\{d'(D_i, D_1), d(D_1, M)\} = d'(D_i, D_1)$ since $d(D_1, M) = d(D_1)$ and $d'(D_i, D_1) \leq d(D_1)$. On the other hand $d'(D_i, D_1) = d'(C_i, C_1) = \inf\{d(C_1), d(C_1, C_i)\} = d^*$. Summing up we get $d(D_i, M) \geq d^*$ for $i \in [1, s^*]$. In fact we have $d(D_i, M) = d^*$ since $d(D_i, M) \leq d(D_i) = d(C_i) = d^*$.

CLAIM 2. *Suppose that $d(C_i, L) = d(D_i, M) = d^*$ for $i = 1, \ldots, s^*$ and $d(C_i, L) < d^*$ for $i > s^*$. Then $d(C_i, L) = d(D_i, M)$ for all $i \in [1, s]$.*

To check Claim 2 fix $i \in [1, s]$, $i > s^*$. Then we get by (d_3) $d(C_i, L) = d'(C_i, C_1)$ since $d(C_i, L) < d(C_1, L)$. Let us consider the sequence $d(D_i, M)$, $d'(D_i, D_1)$, $d(D_1, M) = d^*$. We have $d'(D_i, D_1) = d'(C_i, C_1) = d(C_i, L) < d^*$. Therefore $d(D_i, M) = d'(D_i, D_1) = d'(C_i, C_1) = d(C_i, L)$ and we are done.

Claims 1 and 2 prove the proposition. □

3. Quasi-branches.

Let Γ be a germ with irreducible components (Γ_i).

Lemma 3.1. *Γ is a quasi-branch if and only if for every smooth branch L the function $i \mapsto d(\Gamma_i, L)$ is constant.*

Proof. Suppose that for every smooth L the function $i \mapsto d(\Gamma_i, L)$ is constant. Let L_1 be a smooth branch such that $d(\Gamma_1, L_1) = d(\Gamma_1)$. Therefore $d(\Gamma_i, L_1) = d(\Gamma_1, L_1) = d(\Gamma_1) \notin \boldsymbol{N}$ and $d(\Gamma_i, L_1) = d(\Gamma_i)$ by Proposition 2.8. Hence we get $d(\Gamma_i) = d(\Gamma_1)$ for all i. Consequently $d(\Gamma_i, \Gamma_j) \geq \inf\{d(\Gamma_i, L), d(\Gamma_j, L)\} = d(\Gamma_1)$ for all i and $d'(\Gamma_i, \Gamma_j) = d(\Gamma_1)$ for all i, j that is Γ is a quasi-branch.

Now suppose that there exists a smooth branch L such that the function $i \mapsto d(\Gamma_i, L)$ is nonconstant. We may assume that $d(\Gamma_1, L) < d(\Gamma_2, L)$. Hence $d'(\Gamma_1, \Gamma_2) = d(\Gamma_1, L) < d(\Gamma_2, L) \leq d(\Gamma_2) = d'(\Gamma_2, \Gamma_2)$ which shows that Γ is not a quasi-branch. □

Lemma 3.2. *Suppose that Γ is a quasi-branch with irreducible components (Γ_i). Then $d'(\Gamma_i, \Gamma_j) = d(\Gamma)$ and $d(\Gamma_i, L) = d(\Gamma, L)$ for all indices i, j and for every smooth branch L. Moreover the following three conditions are equivalent*:

(i) *L has maximal contact with Γ,*
(ii) *L has maximal contact with a branch of Γ,*
(iii) *L has maximal contact with every branch of Γ.*

Proof. The first part follows from Proposition 2.6 and from Lemma 3.1. We get the equivalence of conditions (i), (ii), (iii) from the first part. □

Lemma 3.3. *If Γ is a singular quasi-branch then $d(\Gamma) \notin \boldsymbol{N}$ and $m(\Gamma)d(\Gamma) \in \boldsymbol{N}$. For every smooth branch L we have $m(\Gamma)d(\Gamma, L) = (\Gamma, L)$.*

Proof. If Γ is a branch then the lemma is well-known. If Γ is a singular quasi-branch with components Γ_i then $d(\Gamma) \equiv d(\Gamma_i) \notin \boldsymbol{N}$ and $m(\Gamma)d(\Gamma) = \sum m(\Gamma_i)d(\Gamma) = \sum m(\Gamma_i)d(\Gamma_i) \in \boldsymbol{N}$. If L is smooth then $(\Gamma, L) = \sum m(\Gamma_i)d(\Gamma_i, L) = \sum m(\Gamma_i)d(\Gamma, L) = m(\Gamma)d(\Gamma, L)$. □

Remark 3.4. Let C be a germ with quasi-components (Γ_i). Suppose that Γ_k is a dominating quasi-component. Then $m(\Gamma_i)d'(\Gamma_i, \Gamma_k) \in \boldsymbol{N}$ for all i. Indeed, if $d'(\Gamma_i, \Gamma_k) < d(\Gamma_i)$ then $d'(\Gamma_i, \Gamma_k) < d(\Gamma_k)$ and $d'(\Gamma_i, \Gamma_k) \in \boldsymbol{N}$ by Proposition 2.2. If $d'(\Gamma_i, \Gamma_k) = d(\Gamma_i)$ then $m(\Gamma_i)d'(\Gamma_i, \Gamma_k) = m(\Gamma_i)d(\Gamma_i) \in \boldsymbol{N}$ by Lemma 3.3.

Remark 3.5. Let C be a germ with irreducible components C_1 and C_2. If C_1 and C_2 are smooth then $d(C) = (C_1, C_2) \in \boldsymbol{N}$ by Proposition 2.6. If C_1 is a singular branch and C_2 is a smooth branch which has maximal contact with C_1 then again by Proposition 2.6 we get $d(C) = d(C_1)$. Consequently $m(C)d(C) = (m(C_1) + 1)d(C_1) = m(C_1)d(C_1) + d(C_1) \notin \boldsymbol{N}$. Thus the assumption of Lemma 3.3 is necessary.

4. Stability of the Newton boundary.

In this section we prove Proposition 1.4 and Theorem 1.5. The proof of the following lemma is easy.

LEMMA 4.1. *Let $(\mathbf{\Delta}_i)$ be a finite family of elementary Newton diagrams. Then the diagram $\mathbf{\Delta} = \sum \mathbf{\Delta}_i$ is elementary if and only if $\mathbf{\Delta}_i$ have the same inclination.*

In the sequel we write (Γ, y) resp. (Γ, x) instead of $(\Gamma, y = 0)$ resp. $(\Gamma, x = 0)$.

LEMMA 4.2. *Suppose that Γ is a quasi-branch. Then for every chart (x, y) :*

$$\mathbf{\Delta}_{x,y}(\Gamma) = \left\{ \frac{(\Gamma, y)}{(\Gamma, x)} \right\} .$$

PROOF. Let (Γ_i) be irreducible components of Γ. Using (N_1) and (N_2) we get

$$\mathbf{\Delta}_{x,y}(\Gamma) = \sum_i \left\{ \frac{(\Gamma_i, y)}{(\Gamma_i, x)} \right\} .$$

Moreover

$$\frac{(\Gamma_i, y)}{(\Gamma_i, x)} = \frac{d(\Gamma_i, y)}{d(\Gamma_i, x)} = \frac{(\Gamma, y)}{(\Gamma, x)}$$

since $d(\Gamma_i, x) = d(\Gamma, x)$ and $d(\Gamma_i, y) = d(\Gamma, y)$ for all indices i by Lemma 3.1. By the first part of Lemma 4.1 the diagram $\mathbf{\Delta}_{x,y}(\Gamma)$ is elementary. Thus $\mathbf{\Delta}_{x,y}(\Gamma) = \left\{ \frac{(\Gamma,y)}{(\Gamma,x)} \right\}$.

LEMMA 4.3. *Let Γ be a singular germ. If all diagrams $\mathbf{\Delta}_{x,y}(\Gamma)$ are elementary then Γ is a quasi-branch.*

PROOF. Let (Γ_i) be irreducible components of Γ. By Lemma 3.1 it suffices to check that for any smooth branch L the function $i \mapsto d(\Gamma_i, L)$ is constant. Fix a smooth branch L and take a chart (x, y) such that $\{x = 0\}$ and Γ intersects transversally and $L = \{y = 0\}$. Then

$$\mathbf{\Delta}_{x,y}(\Gamma_i) = \left\{ \frac{m(\Gamma_i) d(\Gamma_i, L)}{m(\Gamma_i)} \right\}$$

and $\sum_i \mathbf{\Delta}_{x,y}(\Gamma_i) = \mathbf{\Delta}_{x,y}(\Gamma)$ is elementary by the assumption of the lemma. By Lemma 4.1 the inclinations of $\mathbf{\Delta}_{x,y}(\Gamma_i)$ equal to $d(\Gamma_i, L)$ do not depend on the index i. □

PROOF OF PROPOSITION 1.4. Use Lemmas 4.2 and 4.3. □

Now, we can pass to the proof of Theorem 1.5. Let C and D be equivalent plane curve germs with quasi-components (Γ_i) and (Δ_i) respectively ($i = 1, \ldots, \rho$, $\rho = \rho(C) = \rho(D)$).

We assume that

(i) $m(\Gamma_i) = m(\Delta_i)$ for $i = 1, \dots, \rho$,
(ii) $d'(\Gamma_i, \Gamma_j) = d'(\Delta_i, \Delta_j)$ for $i, j = 1, \dots, \rho$.

Let us fix a chart (x, y). Omitting the trivial case $\mathbf{\Delta}_{x,y}(C) = \left\{\frac{m(C)}{m(C)}\right\}$ we may assume that C and $\{y = 0\}$ do not intersect transversally. Using Lemma 4.2 we get

$$\mathbf{\Delta}_{x,y}(\Gamma) = \sum_{i=1}^{\rho} \left\{\frac{(\Gamma_i, y)}{(\Gamma_i, x)}\right\} = \sum_{i=1}^{\rho} \left\{\frac{m(\Gamma_i)d(\Gamma_i, y)}{m(\Gamma_i)d(\Gamma_i, x)}\right\} . \tag{3}$$

By Proposition 2.9 there exist smooth branches $\{z = 0\}$ and $\{w = 0\}$ such that

$$d(\Gamma_i, x) = d(\Delta_i, z), \quad d(\Gamma_i, y) = d(\Delta_i, w) \quad \text{for } i = 1, \dots, \rho . \tag{4}$$

We claim that $\{z = 0\}$ and $\{w = 0\}$ intersect transversally. Since Γ and $\{y = 0\}$ do not intersect transversally there exists an index $i_0 \in [1, \rho]$ such that $d(\Gamma_{i_0}, y) > 1$. Then $d(\Gamma_{i_0}, x) = 1$ since $\{x = 0\}$ and $\{y = 0\}$ are transversal and Γ_{i_0} is unitangent. From (4) we get $d(\Delta_{i_0}, w) > 1$ and $d(\Delta_{i_0}, z) = 1$. Applying the STI to germs $\{z = 0\}$ and $\{w = 0\}$ and Δ_{i_0} we confirm that $d(z, w) = 1$, that is, $\{z = 0\}$ and $\{w = 0\}$ intersect transversally. Now, we get

$$\mathbf{\Delta}_{z,w}(\Delta) = \sum_{i=1}^{\rho} \left\{\frac{(\Delta_i, w)}{(\Delta_i, z)}\right\} = \sum_{i=1}^{\rho} \left\{\frac{m(\Delta_i)d(\Delta_i, w)}{m(\Delta_i)d(\Delta_i, z)}\right\} \tag{5}$$

and the equality $\mathbf{\Delta}_{x,y}(\Gamma) = \mathbf{\Delta}_{z,w}(\Delta)$ follows by (3), (4) and (5). □

5. Dominating quasi-components.

Let C be a germ with quasi-components (Γ_i). Recall that a quasi-component Γ_k is dominating if for every quasi-component Γ_i such that $d'(\Gamma_k, \Gamma_i) = d(\Gamma_k)$ we have $d(\Gamma_i) = d(\Gamma_k)$.

Lemma 5.1. *For every quasi-component Γ_k there is a dominating quasi-component $\Gamma_{\tilde{k}}$ such that $d'(\Gamma_k, \Gamma_{\tilde{k}}) = d(\Gamma_k)$.*

Proof. Fix a quasi-component Γ_k. Let $I = \{i : d'(\Gamma_k, \Gamma_i) = \inf\{d(\Gamma_k), d(\Gamma_i)\}\}$ and let $\Gamma_{\tilde{k}}$ be such that $d(\Gamma_{\tilde{k}}) = \sup\{d(\Gamma_i) : i \in I\}$. Since $\tilde{k} \in I$ we get

$$d'\big(\Gamma_k, \Gamma_{\tilde{k}}\big) = d(\Gamma_k) . \tag{6}$$

To check that $\Gamma_{\tilde{k}}$ is dominating fix a quasi-component Γ_i such that

$$d'\big(\Gamma_{\tilde{k}}, \Gamma_i\big) = d\big(\Gamma_{\tilde{k}}\big) . \tag{7}$$

Using the STI we get by (6) and (7)

$$d'(\Gamma_k, \Gamma_i) \geq \inf\{d'(\Gamma_k, \Gamma_{\tilde{k}}), d'(\Gamma_{\tilde{k}}, \Gamma_i)\} = \inf\{d(\Gamma_k), d(\Gamma_{\tilde{k}})\} = d(\Gamma_k)\ .$$

Therefore $d'(\Gamma_k, \Gamma_i) = d(\Gamma_k)$ which implies $i \in I$. Thus we get $d(\Gamma_i) \leq d(\Gamma_{\tilde{k}})$ and by (7) $d(\Gamma_i) = d(\Gamma_{\tilde{k}})$. □

Lemma 5.2. *Let L be a smooth branch. Fix a quasi-component Γ_k of C such that $d(\Gamma_k, L) = \sup\{d(\Gamma_i, L)\}$. Then there exists a dominating quasi-component $\Gamma_{\tilde{k}}$ that $d(\Gamma_k, L) = d(\Gamma_{\tilde{k}}, L)$.*

Proof. By Lemma 5.1 there exists a dominating quasi-component $\Gamma_{\tilde{k}}$ such that $d'(\Gamma_k, \Gamma_{\tilde{k}}) = d(\Gamma_k)$. Then we get

$$d(\Gamma_k, L) \geq d\big(\Gamma_{\tilde{k}}, L\big) \geq \inf\{d'(\Gamma_{\tilde{k}}, \Gamma_k), d(\Gamma_k, L)\} = \inf\{d(\Gamma_k), d(\Gamma_k, L)\} = d(\Gamma_k, L)$$

and the lemma follows. □

If C is a germ with quasi-components (Γ_i) then we put for every smooth branch L:

$$\mathbf{\Delta}(C, L) = \sum_i \left\{\frac{(\Gamma_i, L)}{m(\Gamma_i)}\right\} = \sum_i \left\{\frac{m(\Gamma_i) d(\Gamma_i, L)}{m(\Gamma_i)}\right\}\ .$$

Note that $\boldsymbol{i}(\mathbf{\Delta}(C, L)) = \sup\{d(\Gamma_i, L)\}$.

Proposition 5.3. *Let Γ_k be a dominating quasi-component of C and let L be a smooth branch such that $d(\Gamma_k, L) = d(\Gamma_k)$. Then $\mathbf{\Delta}(C, L) = \mathbf{\Delta}_k(C)$.*

Proof. Let $I = \{i : d(\Gamma_i, L) < d(\Gamma_k, L)\}$ and $I^c = \{i : d(\Gamma_i, L) \geq d(\Gamma_k, L)\}$. If $i \in I$ then $d(\Gamma_i, L) = d'(\Gamma_i, \Gamma_k)$ by the STI. If $i \in I^c$ then $d(\Gamma_i, L) = d(\Gamma_k, L)$. Indeed, if we had $d(\Gamma_i, L) > d(\Gamma_k, L)$ then we would get $d(\Gamma_k, L) = d'(\Gamma_k, \Gamma_i)$ i.e. $d'(\Gamma_k, \Gamma_i) = d(\Gamma_k)$ and consequently $d(\Gamma_i) = d(\Gamma_k)$ since Γ_k is a dominating quasi-component. Contradiction since $d(\Gamma_k) = d(\Gamma_k, L) < d(\Gamma_i, L) \leq d(\Gamma_i)$. Now, we can write

$$\mathbf{\Delta}_k(C, L) = \sum_{i \in I} \left\{\frac{m(\Gamma_i) d'(\Gamma_i, \Gamma_k)}{m(\Gamma_i)}\right\} + \sum_{i \in I^c} \left\{\frac{m(\Gamma_i) d(\Gamma_k)}{m(\Gamma_i)}\right\} = \mathbf{\Delta}_k(C)$$

since $d(\Gamma_k) = d'(\Gamma_k, L) = d'(\Gamma_i, \Gamma_k)$ for all $i \in I^c$. □

Theorem 5.4. *Let C be a plane curve germ.*

(a) *If Γ_k is a dominating quasi-component of C and $N > 0$ is an integer or $N = \infty$ then there exists a smooth branch L such that $\mathbf{\Delta}_k(C)^N = \mathbf{\Delta}(C, L)$ and $d(\Gamma_k, L) = \inf\{N, d(\Gamma_k)\}$.*
(b) *If L is a smooth branch then there exists a dominating quasi-component Γ_k of C and $N > 0$ (integer or ∞) such that $\mathbf{\Delta}(C, L) = \mathbf{\Delta}_k(C)^N$ and $d(\Gamma_k, L) = \inf\{N, d(\Gamma_k)\}$.*

Proof of (a). If $d(\Gamma_k) \leq N$ then we take a smooth branch L such that $d(\Gamma_k, L) = d(\Gamma_k)$ and get $\boldsymbol{\Delta}_k(C)^N = \boldsymbol{\Delta}_k(C) = \boldsymbol{\Delta}_k(C, L)$ by Proposition 5.3. Suppose that $0 < N < d(\Gamma_k)$. We will prove that there exists a smooth branch L such that

(α) $d(\Gamma_k, L) = N$,
(β) if $i \in I(N)$ then $d(\Gamma_i, L) = d'(\Gamma_i, \Gamma_k)$,
(γ) if $i \in I(N)^c$ then $d(\Gamma_i, L) = N$.

Conditions (β) and (γ) imply that $\boldsymbol{\Delta}(C, L) = \boldsymbol{\Delta}_k(C)^N$ which proves the proposition.

To prove the existence of L we distinguish two cases.

Case 1. $N \neq d'(\Gamma_i, \Gamma_k)$ for all i that is $I(N)^c = \{i : d'(\Gamma_i, \Gamma_k) > N\}$. Since $0 < N < d(\Gamma_k)$ there exists a smooth branch L such that $d(\Gamma_k, L) = N$. If $i \in I(N)$ then $d'(\Gamma_k, \Gamma_i) < d(\Gamma_k, L)$ and by the STI we get $d(\Gamma_i, L) = d'(\Gamma_i, \Gamma_k)$ that is Condition (β) is fulfilled. If $i \in I(N)^c$ then $d(\Gamma_i, L) = \inf\{d'(\Gamma_i, \Gamma_k), d(\Gamma_k, L)\} = N$ since $d(\Gamma_k, L) = N < d'(\Gamma_i, \Gamma_k)$ for $i \in I(N)^c$.

Case 2. There is an index i such that $N = d'(\Gamma_i, \Gamma_k)$. Observe that $k \in I(N)^c$. It is easy to check that $\inf\{d'(\Gamma_i, \Gamma_j) : i, j \in I(N)^c \times I(N)^c\} = N$. Applying Proposition 2.7 to the family $\{\Gamma_i : i \in I(N)^c\}$ we get a smooth branch L such that $d(\Gamma_i, L) = N$ for all $i \in I(N)^c$. In particular $d(\Gamma_k, L) = N$. If $i \in I(N)$ then $d'(\Gamma_i, \Gamma_k) < N = d(\Gamma_k, L)$ and consequently $d(\Gamma_i, L) = d'(\Gamma_i, \Gamma_k)$ that is (β) holds. Conditions (α) and (β) are fulfilled by the definition of L. □

Proof of (b). Fix a smooth branch L. Suppose that $\boldsymbol{i}(\boldsymbol{\Delta}(C, L)) \notin \boldsymbol{N}$ and let Γ_k be a quasi-component such that $d(\Gamma_k, L) = \sup\{d(\Gamma_i, L)\} = \boldsymbol{i}(\boldsymbol{\Delta}(C, L))$. We claim that $d(\Gamma_k, L) = d(\Gamma_k)$ and Γ_k is a dominating quasi-component.

Since $d(\Gamma_k, L) \notin \boldsymbol{N}$ then $d(\Gamma_k, L) = d(\Gamma_k)$. To check that Γ_k is a dominating quasi-component suppose that $d'(\Gamma_i, \Gamma_k) = d(\Gamma_k)$. We have $d(\Gamma_k) = \inf\{d'(\Gamma_i, \Gamma_k), d(\Gamma_k, L)\} \leq d(\Gamma_i, L) \leq d(\Gamma_k, L) = d(\Gamma_k)$. Thus $d(\Gamma_i, L) = d(\Gamma_k)$ which implies $d(\Gamma_i) = d(\Gamma_k)$. Then $\boldsymbol{\Delta}(C, L) = \boldsymbol{\Delta}_k(C) = \boldsymbol{\Delta}_k(C)^N$ for every $N \geq d(\Gamma_k)$ by Proposition 5.3.

Now suppose that $\boldsymbol{i}(\boldsymbol{\Delta}(C, L)) = N$. We have to check that there exists a dominating quasi-component Γ_k such that $d(\Gamma_k, L) = N$ and $\boldsymbol{\Delta}(C, L) = \boldsymbol{\Delta}_k(C)^N$. By Lemma 5.2 there exists a dominating quasi-branch Γ_k of C such that $d(\Gamma_k, L) = N$. Clearly $N < d(\Gamma_k)$. Using the STI we check that $d(\Gamma_i, L) < d(\Gamma_k, L)$ if and only if $d'(\Gamma_i, \Gamma_k) < N$. Let $I = \{i : d(\Gamma_i, L) < d(\Gamma_k, L)\}$ and $I^c = \{i : d(\Gamma_i, L) \geq d(\Gamma_k, L)\}$. By the STI we get $d(\Gamma_i, L) = d'(\Gamma_i, \Gamma_k)$ for $i \in I$ and $d(\Gamma_i, L) = N$ for $i \in I^c$. Moreover we have $I = I(N)$ and $I^c = I(N)^c$. A simple calculation shows that $\boldsymbol{\Delta}(C, L) = \boldsymbol{\Delta}_k(C)^N$. □

6. Proof of the main result.

We keep the notation introduced in Section 1. Our aim is to prove Theorem 1.6.

Lemma 6.1. *Let C be a plane curve germ.*

(a) $\boldsymbol{\Delta} \in \mathcal{N}(C)_{\mathrm{s}}$ *if and only if* $\boldsymbol{\Delta} = \boldsymbol{\Delta}(C, L)$ *for a smooth branch* L.
(b) $\boldsymbol{\Delta} \in \mathcal{N}(C)$ *if and only if* $\boldsymbol{\Delta} \in \mathcal{N}(C)_{\mathrm{s}} \cup \sigma(\mathcal{N}(C)_{\mathrm{s}})$ *or* $\boldsymbol{\Delta} = \sigma(\boldsymbol{\Delta}(C, L)) \cap \boldsymbol{\Delta}(C, L')$ *where* L, L' *are transversal smooth branches such that* C, L *and* C, L' *do not intersect transversally.*

Proof. Let L be a smooth branch and let (x, y) be a chart such that $\{x = 0\}$ intersects C transversally and $L = \{y = 0\}$. Then $\mathbf{\Delta}(C, L) = \mathbf{\Delta}_{x,y}(C)$. The lemma follows from the observations:

(1) if $\{x = 0\}$ intersects C transversally then $\mathbf{\Delta}_{x,y}(C) \in \mathcal{N}(C)_{\mathrm{s}}$,
(2) if $\{y = 0\}$ intersects C transversally then $\mathbf{\Delta}_{x,y}(C) = \sigma(\mathbf{\Delta}_{y,x}(C)) \in \sigma(\mathcal{N}(C)_{\mathrm{s}})$,
(3) if neither $\{x = 0\}$ nor $\{y = 0\}$ intersects C transversally then $\mathbf{\Delta}_{x,y}(C) = \mathbf{\Delta}_{x,y'}(C) \cap \mathbf{\Delta}_{x',y}(C)$ for any chart (x', y') such that $\{x' = 0\}$ and $\{y' = 0\}$ intersect C transversally. □

Lemma 6.2. *Let $\Gamma_1, \Gamma_2, L_1, L_2$ be plane curve germs such that $d'(\Gamma_i, L_i) > 1$ for $i = 1, 2$. Then $d'(\Gamma_1, \Gamma_2) = 1$ if and only if $d'(L_1, L_2) = 1$.*

Proof. It suffices to check that $d'(\Gamma_1, \Gamma_2) = 1$ implies $d'(L_1, L_2) = 1$. Since $d'(\Gamma_1, L_1) > 1$ we get by the STI $d'(\Gamma_2, L_1) = d'(\Gamma_1, \Gamma_2) = 1$. From $d'(\Gamma_2, L_1) = 1$, $d'(\Gamma_2, L_2) > 1$ we get by the STI $d(L_1, L_2) = 1$. □

We are in a good position to prove Theorem 1.6. Recall that $\mathbf{\Delta}_k^N = \mathbf{\Delta}_k(C)^N$ and $K = \{k : \Gamma_k \text{ is a dominating quasi-component of } C\}$. From Theorem 5.4 we get

(δ) For any Newton diagram $\mathbf{\Delta}$ the following two conditions are equivalent
 (δ_1) there exists a smooth branch L such that $\mathbf{\Delta} = \mathbf{\Delta}(C, L)$,
 (δ_2) there exists $k \in K$ and an integer $N > 0$ or $N = \infty$ such that $\mathbf{\Delta} = \mathbf{\Delta}_k(C)^N$.

Using Theorem 5.4 and Lemma 6.2 we check easily

(ε) For any Newton diagram $\mathbf{\Delta}$ the following two conditions are equivalent
 (ε_1) there exists smooth transversal branches L, L' such that $\mathbf{\Delta} = \sigma(\mathbf{\Delta}(C, L)) \cap \mathbf{\Delta}(C, L')$ where C, L and C, L' are not transversal,
 (ε_2) there exists $k, l \in K$ and integers $N > 1$ or $N = \infty$ and $N' > 1$ or $N' = \infty$ such that $\mathbf{\Delta} = \sigma(\mathbf{\Delta}_k^N) \cap \mathbf{\Delta}_l^{N'}$ and $d'(\Gamma_k, \Gamma_l) = 1$.

Now, Theorem 1.6(a) follows from (δ) and Lemma 6.1(a) whereas Theorem 1.6(b) follows from Theorem 1.6(a), (ε) and Lemma 6.1(b). □

Bibliographical Note

M. Lejeune-Jalabert studied in her 1973 thesis [**LJ**] Zariski's (a)-equivalence of plane algebroid curves by using the quadratic transforms and Newton diagrams. She proved (in the case of any characteristic) that the set $\{\mathbf{\Delta} \in \mathcal{N}(C)_{\mathrm{s}} : \boldsymbol{i}(\mathbf{\Delta}) \notin \boldsymbol{N}\}$ is an invariant of (a)-equivalence (see [**LJ**, Lemma 4.1.2 and Remark 4.1.4]). Let $\mathbf{\Delta}_{x,y}(C)'$ be the part of $\mathbf{\Delta}_{x,y}(C)$ lying in the quarter $(1, 1) + \boldsymbol{R}_+^2$ and let

$$\mathcal{N}(C)' = \left\{\mathbf{\Delta}_{x,y}(C)' : \ (x, y) \text{ runs over all charts centered at } O\right\}.$$

M. Oka proved that $\mathcal{N}(C)'$ depends only on the (a)-equivalence class of C (see [**O**, Theorem 5.1]).

Clearly our Theorem 1.5 is an improvement of the above quoted results.

Let us also note that B. Teissier in [**T2**] asked if the configuration of all hyperplanes supporting the compact faces of all Newton diagrams of an isolated hypersurface singularity is a topological invariant and asserted that the answer is *yes* in the case of plane curves (see [**T2**, Remark on p. 206 and Note 2, p. 221]).

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