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Applications of controlled surgery in dimension 4: Examples

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Abstract. The validity of Freedman's disk theorem is known to depend only on the fundamental group. It was conjectured that it fails for nonabelian free fundamental groups. If this were true then surgery theory would work in dimension four. Recently, Krushkal and Lee proved a surprising result that surgery theory works for a large special class of 4-manifolds with free nonabelian fundamental groups. The goal of this paper is to show that this also holds for other fundamental groups which are not known to be good, and that it is best understood using controlled surgery theory of Pedersen-Quinn-Ranicki. We consider some examples of 4-manifolds which have the fundamental group either of a closed aspherical surface or of a 3-dimensional knot space. A more general theorem is stated in the appendix.

1. Introduction.

The purpose of this paper is to study 4-dimensional surgery problems by means of controlled surgery. The usual higher dimensional surgery procedure breaks down in dimension four since framed 2-spheres can generically only be immersed in a 4-manifold (whereas for surgery on them one would require embeddings). To get an embedding one uses the *Whitney trick*. Its basic ingredient is the existence of *Whitney disks* along which pairs of intersection points with opposite algebraic intersection number can be cancelled. If one finds these Whitney disks, surgery can be completed provided that the Wall obstruction vanishes. The celebrated Disk Theorem of Freedman asserts that (see [**Fr**]):

(1) The existence of Whitney disks in a 4-manifold M^4 depends only on the fundamental group of M^4. If they exist then $\pi_1(M^4)$ is called a *good* fundamental group.
(2) The (large) class of good fundamental groups includes the trivial group and $\boldsymbol{Z}$. (see also [**Fr-Qu**], [**Fr-Tei**], [**Kru-Qu**]).

It has been conjectured that nonabelian free groups are not good. Nevertheless, the following surprising result was proved by Krushkal and Lee ([**Kru-Lee**]):

THEOREM 1.1. *Let X be a 4-dimensional Poincaré complex with a free nonabelian fundamental group, and assume that the intersection form on X is extended from the integers. Let $f : M \to X$ be a degree one normal map, where M is a closed 4-manifold.*

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Then the vanishing of the Wall obstruction implies that f is normally bordant to a homotopy equivalence $f' : M' \to X$.

Whenever the intersection form is extended from the integers, it follows that X is homotopy equivalent to the connected sum $P \# M'$, where M' is simply connected and P can be assumed to be a finite sum $\overset{r}{\underset{1}{\#}} S^1 \times S^3$ (see [**He-Re-Sp**]). Recall that there exist Poincaré 4-complexes with free fundamental group and intersection form which is not extended from $\boldsymbol{Z}$ (see [**Ham-Tei**], [**He-Pic**]).

By means of this example we see that surgery can be completed for a large class of 4-manifolds (or Poincaré complexes) with fundamental group π, which is not good. This paper will confirm this fact for other fundamental groups. For instance we shall prove:

Theorem 1.2. *Let X be a spin Poincaré 4-complex and suppose that it has the fundamental group of a closed oriented aspherical surface and that the intersection form is extended from $\boldsymbol{Z}$. Then any degree one normal map $f : M \to X$ with vanishing Wall obstruction is normally bordant to a homotopy equivalence.*

We shall also give other examples, e.g. 4-manifolds having the fundamental group isomorphic to some special knot group. Moreover, we shall also recover Theorem 1.1.

The reason why this can happen is that one can divide the *global* surgery problem into smaller pieces for which the *local* fundamental groups are good, i.e. $\{1\}$ or $\boldsymbol{Z}$. One gets several local surgery obstructions which assemble to give the global surgery obstruction. This subdivision has to be done in such a way that the global surgery obstruction already determines the local ones. More precisely, the assembly map should be injective.

The subdivision is made according to a control map $p : X \to B$, where B is a finite-dimensional compact metric ANR. The map p must satisfy the following three conditions:

(i) p is a UV^1-map;
(ii) if X is a Poincaré complex then X must be a δ-controlled Poincaré complex over B, where $\delta > 0$ is smaller than some $\varepsilon_0 > 0$ which depends only on B; and
(iii) the assembly map $A : H_4(B, \boldsymbol{L}) \to L_4(\pi_1(B))$ is injective.

Definitions and more explanations will be given in Section 2. Once we have such a control map we can apply controlled surgery theory to obtain our results. Note that the extreme cases, i.e. when

(a) either $p = \mathrm{Id} : X \to X = B$;
(b) or $p = \mathrm{const} : X \to \{*\} = B$,

generically do not work since the case (a) does not satisfy condition (iii) whereas the case (b) does not satisfy condition (i) above. It is also obvious that $p : X \to B$ depends not only on $\pi_1(X)$ but also on the topology of X, so one gets solution of the surgery problem in individual cases.

Remark. The condition (i) can be weakened to:

(i') $UV^1(\delta)$, for every $0 < \delta < \varepsilon_0$.

Another example of a 4-manifold with a knot group fundamental group is stated in Theorem 3.2 below.

2. Controlled surgery theory.

To reach our goal we shall need to use the ε–δ surgery sequence and to compare it with the non-controlled one. Let $\boldsymbol{L}$ denote the 4-periodic simply connected surgery spectrum (see [**Qu1**] and [**Ni**] for geometric and [**Ra**] for algebraic definitions). For a space B we then have $\boldsymbol{L}$-homology (resp. cohomology) groups denoted by $H_p(B,\boldsymbol{L})$ (resp. $H^p(B,\boldsymbol{L})$). There is a well defined assembly map $A : H_p(B,\boldsymbol{L}) \to L_p(\pi_1(B))$, where $L_p(\pi_1(B))$ denotes the Wall group of obstructions to simple homotopy equivalences.

We shall only consider the oriented situation. Let X be an n-dimensional simple Poincaré complex. We suppose that X admits a degree one normal map $f_0 : M_0 \to X$, so fixing this we have an identification of all degree one normal maps into X, modulo a normal cobordism, with the homotopy set $[X, G/_{TOP}]$ (see [**Wa**]).

There is a well defined map

$$\Theta : [X, G/_{TOP}] \to H_n(X,\boldsymbol{L}),$$

which associates to a given degree one normal map into X its local surgery problems according to a small dissection of X (see [**Ra**]). Composition with the assembly map

$$\sigma : A \circ \Theta : [X, G/_{TOP}] \to L_n(\pi_1(X))$$

yields the classical surgery obstruction map of Wall.

The topological structure set of X consists of simple homotopy equivalences $f : M^n \to X$, where M^n is a closed manifold. Two such homotopy equivalences $f : M \to X$, $f' : M' \to X$ are equivalent if there is a homeomorphism $g : M \to M'$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\;g\;} & M' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & X & \end{array}$$

homotopy commutes. The set of equivalence classes will be denoted by $\mathscr{S}(X)$ and will be called the *topological structure* set of X.

Any homotopy equivalence determines a degree one normal map inducing $\mathscr{S}(X) \to [X, G/_{TOP}]$. For $n \geq 5$ and X a simple Poincaré complex, there is an exact ordinary surgery sequence

$$\mathscr{S}(X) \to [X, G/_{TOP}] \to L_n(\pi_1(X)).$$

This sequence can be extended to the left by the Wall realization of obstructions

$$L_{n+1}(\pi_1(X)) \to \mathscr{S}(X).$$

For this one has to fix a simple homotopy equivalence $f_0 : M_0 \to X$, where M_0 is a topological manifold and $\dim X \geq 6$. In the controlled concept there is a realization of elements in $H_{n+1}(B, \boldsymbol{L})$ giving a four–term exact sequence which also holds for $n = 4$. In this paper we do not consider this part of the sequence so we will not give more details.

Before we state the ε–δ surgery sequence we need some more definitions. Let $p : X \to B$ be a control map, B a (finite-dimensional) compact metric ANR. Then p is a $UV^1(\delta)$-map, $\delta > 0$, if every commutative diagram

$$\begin{array}{ccc} K_0 & \xrightarrow{\alpha_0} & X \\ \big\downarrow & & \big\downarrow p \\ K & \xrightarrow{\alpha} & B \end{array}$$

where K is a 2-complex and $K_0 \subset K$ is a subcomplex, can be completed by a map $\overline{\alpha} : K \to X$ such that $\overline{\alpha}|_{K_0} = \alpha_0$ and $d(p \circ \overline{\alpha}(u), \alpha(u)) < \delta$ for all $u \in K$. The map p is called a UV^1 map if it is a $UV^1(\delta)$ map for every $\delta > 0$. Here, $d : B \times B \to \boldsymbol{R}_+$ denotes the metric on B.

Suppose now that X is an n-dimensional Poincaré complex, and suppose that X has a simplicial structure. By the Borsuk theorem the simplicial structure hypothesis can often be obtained replacing X by a homotopy equivalent space and then working with it.

A space X is said to be an (oriented) *δ-Poincaré complex* with respect to $p : X \to B$ if

(i) for every simplex Δ of X the diameter of $p(\Delta) \subset B$ is less than δ; and
(ii) there exists a fundamental cocycle $\xi \in C_n(X)$ such that the cap product

$$\cap\xi : C^k(X) \to C_{n-k}(X)$$

is a δ-chain equivalence.

The second condition requires geometric module structure on the Λ-chain complex $\{C_k(X) = H_n(\tilde{X}^{(k)}, \tilde{X}^{(k-1)}) \mid k = 0, 1, 2, \dots\}$, where $\Lambda = \boldsymbol{Z}[\pi_1(X)]$. We shall not give any more details but will refer to the literature (see [**Ra-Ya1**], [**Ra-Ya2**]). A more geometric definition was given in [**Qu2**].

If X is a manifold then one obtains by barycentric subdivision a δ-Poincaré structure for every $\delta > 0$ with respect to $p = \mathrm{Id} : X \to X$, hence with respect to every $p : X \to B$.

Suppose that $f, g : Y \to X$ are given maps. Then f is said to be *δ-homotopic* to g if there is a homotopy $h : Y \times I \to X$ between f and g such that for any $y \in Y$ the diameter of $\{ph(y,t) \mid t \in I\} \subset B$ is less than δ.

Moreover, $f : Y \to X$ is called a *δ-homotopy equivalence* if there exists $g : X \to Y$ and homotopies $h : X \times I \to X$, $h' : Y \times I \to Y$ between $f \circ g$ and Id_X (resp. $g \circ f$ and Id_Y) such that the diameters of $\{ph(x,t) \mid t \in I\}$ and $\{pfh'(y,t) \mid t \in I\}$ are less than δ for all $x \in X$ (resp. $y \in Y$).

Lemma 2.1. *Suppose that $f : Y \to X$ is a δ-homotopy equivalence and Y a δ'-*

Poincaré complex with respect to $p \circ f$. Then X is a $(\delta' + 2\delta)$-Poincaré complex over B.

This is a useful observation. It follows easily from [**Ra-Ya1**, Proposition 2.3]. The following is the main theorem of [**Pe-Qu-Ra**]:

THEOREM 2.2. *Let B be a finite-dimensional compact ANR and X a closed topological n-manifold, where $n \geq 4$. Then there exists an $\varepsilon_0 > 0$, depending on B, such that for every $0 < \varepsilon < \varepsilon_0$ there exists $\delta > 0$ such that the following holds: If there is a map $p : X \to B$ satisfying the $UV^1(\delta)$ property, then we get the following controlled surgery exact sequence*

$$H_{n+1}(B, \boldsymbol{L}) \to \mathscr{S}_{\varepsilon,\delta}(X, p) \to [X, G/_{TOP}] \to H_n(B, \boldsymbol{L}).$$

In fact, the following supplement holds (however, we shall not need it):

SUPPLEMENT. *If the map, $p : X \to B$ is only assumed to be a sufficiently small controlled Poincaré complex over B (instead of assuming that X is a manifold), then the surgery sequence*

$$\mathscr{S}_{\varepsilon,\delta}(X, p) \to [X, G/_{TOP}] \to H_n(B, \boldsymbol{L})$$

is still exact.

The controlled structure set $\mathscr{S}_{\varepsilon,\delta}(X, p)$ is defined as follows: Its elements are represented by δ-homotopy equivalence $f : M \to X$ over B, where M is a closed topological n-manifold. Another δ-homotopy equivalence $g : N \to X$ is said to be ε-*related* to (M, f) if there exists a homeomorphism $h : M \to N$, so that

$$\begin{array}{ccc} M & \xrightarrow{\quad h \quad} & N \\ & {}_{f}\searrow \qquad \swarrow_{g} & \\ & X & \end{array}$$

is ε-homotopy commutative. This relation is reflexive and symmetric, so it gives rise to an equivalence relation. (However, the proof actually shows the transitivity of the relations above.) This theorem has a relative version which we shall not use.

As explained above, we now have the following commutative diagram for $n = 4$ (assuming the hypotheses of the theorem),

$$\begin{array}{ccccc} \mathscr{S}_{\varepsilon,\delta}(X, p) & \longrightarrow & [X, G/_{TOP}] & \longrightarrow & H_4(B, \boldsymbol{L}) \\ \downarrow & & \| & & \downarrow{\scriptstyle A} \\ \mathscr{S}(X) & \longrightarrow & [X, G/_{TOP}] & \longrightarrow & L_4(\pi_1(B)). \end{array}$$

Hence in order to solve the 4-dimensional surgery problem with target X one needs a control map $p : X \to B$ satisfying conditions (i), (ii) and (iii) stated in Section 1. Examples will be given in the following section.

3. Examples.

There is another characterization of UV^1-maps which is useful in the applications (see [**Dav**] or [**BFMW**]). A subset A of a space X is said to be UV^1 if for each neighborhood U of A in X there is another neighborhood V of A with $V \subset U$, such that the induced map $\pi_1(V) \to \pi_1(U)$ is zero for any base point in V, and any two points in V can be connected in U. The following is a special case of the Approximate lifting theorem ([**Dav**, p. 126]):

Theorem 3.1 ([**Dav**]). *Suppose X is a metric space and G is an upper semicontinuous UV^1-decomposition of X (i.e. each member $A \in G$ is a UV^1 subset). Let $B = X/_G$ and $p : X \to B$. Then p is a UV^1 map, i.e. p is a $UV^1(\delta)$ map for every $\delta > 0$.*

As the first example we consider X as a Poincaré 4-complex with free nonabelian fundamental group and $\boldsymbol{Z}$-extended Λ-intersection form, $\Lambda = \boldsymbol{Z}[\pi_1(X)]$. By results from [**He-Re-Sp**], X is homotopy equivalent to $\big(\overset{r}{\underset{1}{\#}} S^1 \times S^3\big)\#M' = M$. A homotopy equivalence $M \to X$ induces an "isomorphism" of the ordinary short exact surgery sequences, i.e. we transform a surgery problem with target X to a surgery problem with target M.

Lemma 3.2. *Let $M\#M'$ be the connected sum of two topological manifolds and $p : M\#M' \to M$ the map which collapses M' to a point. If M' is simply connected then p is a UV^1 map.*

Proof. The map p is the composition of the maps p_2 and p_1. First, $p_1 : M\#M' \to M \vee M'$ is the map which collapses the 3-sphere $\sum^3 \subset M\#M'$ to the base point of the wedge $M \vee M'$. More precisely, a bicollar $[-1, 1] \times \sum^3$ is radially smashed to $D^4 \vee D'^4 \subset M \vee M'$, fixing $\{\pm 1\} \times \sum^3$. The inverse images are points or a nicely embedded $\sum^3$. Hence by Theorem 3.1, p_1 is a UV^1 map.

Next, the map $p_2 : M \vee M' \to M$ is the projection. Since $\pi_1(M') = \{1\}$, Theorem 3.1 again implies that p_2 is a UV^1 map.

It remains to observes that the composition of UV^1-maps is again a UV^1-map. □

The proof of Lemma 3.2 shows the following:

Lemma 3.3. *If $M_1\#M_2$ is a connected sum of topological manifolds, then the smash map $p : M_1\#M_2 \to M_1 \vee M_2$, is UV^1.*

We now consider the following composition

$$p = p_3 \circ p_2 \circ p_1 : \left(\overset{r}{\underset{1}{\#}} S^1 \times S^3\right)\#M' \xrightarrow{p_1} \left(\overset{r}{\underset{1}{\vee}} S^1 \times S^3\right) \vee M'$$

$$\xrightarrow{p_2} \left(\overset{r}{\underset{1}{\vee}} S^1 \times S^3\right) \xrightarrow{p_3} \overset{r}{\underset{1}{\vee}} S^1 = B$$

where p_3 is induced by the projection $S^1 \times S^3 \to S^1$. Obviously, p_3 is a UV^1 map, hence we obtain the following:

Corollary 3.4. *The map* $p : \left(\overset{r}{\underset{1}{\#}} S^1 \times S^3\right) \# M' = M \to \overset{r}{\underset{1}{\vee}} S^1 = B$ *is* UV^1.

From the Atiyah–Hirzebruch spectral sequence

$$E^2_{rs} = H_r(B, \pi_s(\boldsymbol{L})) \Longrightarrow H_{r+s}(B, \boldsymbol{L})$$

we deduce the well–known fact that $A : H_4(B, \boldsymbol{L}) \to L_4(\pi_1(B))$ is an isomorphism. In particular $E^2_{rs} = 0$ for $r > 1$, so the spectral sequence collapses and $H_4(B, \boldsymbol{L}) = H_0(B, \boldsymbol{Z}) = \boldsymbol{Z} \cong L_4(\pi_1(B))$. Recall that:

$$\pi_s(\boldsymbol{L}) = \begin{cases} 0 & \text{if } s \text{ is odd} \\ \boldsymbol{Z}_2 & \text{if } s \equiv 2(4) \\ \boldsymbol{Z} & \text{if } s \equiv 0(4), \end{cases}$$

Since M is a manifold, the map $p : M \to \overset{r}{\underset{1}{\vee}} S^1$ satisfies conditions (i), (ii) and (iii). This proves Theorem 1.1. □

To prove Theorem 1.2 we consider a Poincaré complex X^4 with $w_2(X) = 0$ and Λ-intersection form extended from $\boldsymbol{Z}$. The fundamental group of X is that of some surface F. The construction of [**Cav-He-Rep**] applies to give a degree one normal map $X \xrightarrow{f} F \times S^2$. This splits the Λ-intersection form. Since it is extended from the $\boldsymbol{Z}$-intersection form one gets a homotopy equivalence $X \simeq F \times S^2 \# M' = M$, where M' is simply connected.

We get as above the following UV^1-map:

$$p = p_3 \circ p_2 \circ p_1 : M \to F \times S^2 \vee M' \to F \times S^2 \to F = B.$$

The Mayer–Vietoris technique can be applied to the L-functor (see [**Capp**]) and to $H_n(B, \boldsymbol{L})$ to show that $A : H_4(B, \boldsymbol{L}) \to L_4(\pi_1(B))$ is an isomorphism. In particular $L_4(\pi_1(B)) = \boldsymbol{Z} \oplus \boldsymbol{Z}_2$. So $p : M \to B$ satisfies the conditions (i), (ii) and (iii), which proves Theorem 1.2. □

Our next examples are 4-manifolds whose fundamental groups are knot groups. As the control space B we take a spine of the knot complement in S^3. It is well known that B is an aspherical space and that $H_p(B) = \boldsymbol{Z}$ for $p = 0, 1$ and trivial otherwise. The Atiyah–Hirzebruch spectral sequence then gives $H_4(B, \boldsymbol{L}) = \boldsymbol{Z}$, in fact $A : H_4(B, \boldsymbol{L}) \to L_4(\pi_1(B))$ is an isomorphism.

Theorem 3.5. *Let* $X = \partial\big((S^3 \setminus \overset{o}{N}(k)) \times D^2\big)$ *be the boundary of a regular neighborhood of the spine of the complement of a torus knot, embedded in* $\boldsymbol{R}^5$*. Then the surgery sequence*

$$\mathscr{S}(X) \to [X, G/_{TOP}] \to L_4(\pi_1(X))$$

is exact.

Note that X is a manifold. So it remains to verify only condition (i) of Section 1. We state it as follows:

Lemma 3.6. *Let $p : X \to B$ be the restriction of the neighborhood collapsing map. Then p is a UV^1 map.*

Proof. We shall show that the inverse images of points are UV^1 subsets of X and then we shall apply Theorem 3.1 to get the assertion. So let $k \subset S^1 \times S^1 \subset S^3$ be a torus knot in S^3 of type (a, b), where $(a, b) = 1$, and the torus $S^1 \times S^1$ divides S^3 into two solid tori T and T^* such that $k \subset T \cap T^* = \partial T = \partial T^* = S^1 \times S^1$.

Let $M^3 = S^3 \setminus \overset{o}{N}{}^3(k)$, where $N^3(k)$ is a small tubular neighborhood of the knot k in S^3. The spine of M^3 (i.e. a compact 2-polyhedron onto which the 3-manifold M^3 collapses) consists of 2-dimensional compact polyhedra $\Sigma \subset T$ and $\Sigma^* \subset T^*$, intersecting in an annulus $A = l \times [-1, 1]$ which lies on $T \cap T^* = S^1 \times S^1$ and where the curve l is parallel on $S^1 \times S^1$ to the knot k. (see Figure 1).

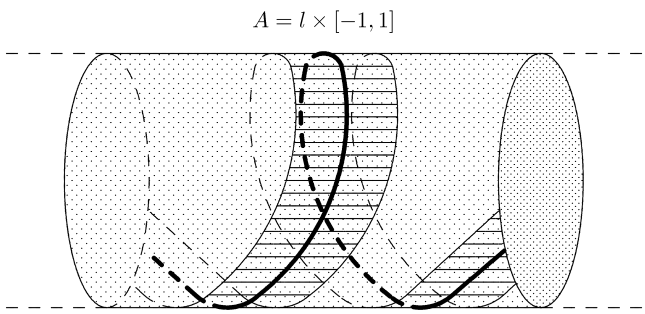

Figure 1.

If we look at the 2-disk cross sections C and C^* of the solid tori T and T^* (which are orthogonal to the longitudes of these solid tori), the pictures of Σ and Σ^*, respectively, are as in the Figure 2 below (for the case when $(a, b) = (3, 2)$, i.e. when k is the $(3, 2)$-torus knot): where Σ and Σ^* are depicted by bold lines.

The collapsing map $\rho : M^3 \to B = \Sigma \cup \Sigma^*$ is described on Figure 3. Let us consider the point inverses of the map ρ. There are essentially three different types of points to consider: $\rho^{-1}(a)$ (resp. $\rho^{-1}(a^*)$) is a bouquet of 3 (resp. 2) intervals, $\rho^{-1}(b)$ and $\rho^{-1}(c)$ (resp. $\rho^{-1}(b^*)$ and $\rho^{-1}(c^*)$) are bouquets of 2 intervals, and finally, $\rho^{-1}(d)$ (resp. $\rho^{-1}(d^*)$) is just a point.

We now consider the embedding $B \subset \boldsymbol{R}^5$ given by

$$B \subset M^3 \subset S^3 \subset S^3 \times D^1 \subset \boldsymbol{R}^4 \times \{0\} \subset \boldsymbol{R}^5.$$

Let X be the boundary of the regular neighborhood $M^3 \times D^2$ of $B \subset \boldsymbol{R}^5$:

$$X = \partial(M^3 \times D^2) = \partial M^3 \times D^2 \cup M^3 \times \partial D^2.$$

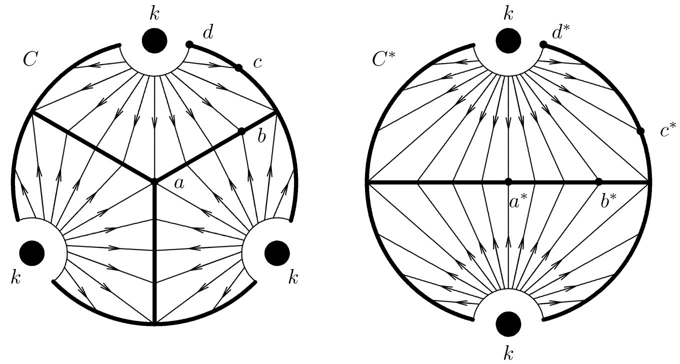

Figure 2.

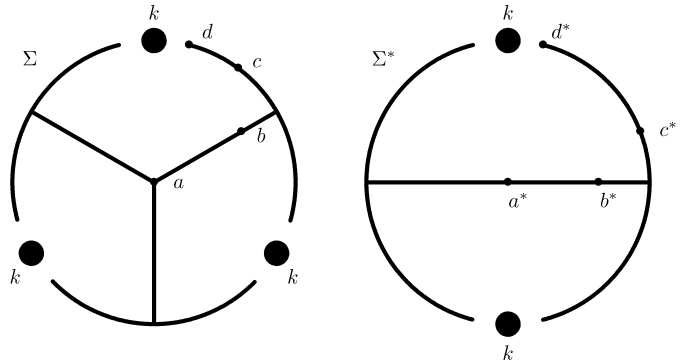

Figure 3.

Then the collapsing map $p : X \to B$ is the composition of the canonical projection $\pi : M^3 \times D^2 \to M^3$ followed by the collapsing map $\rho : M^3 \to B$ described above, that is $p = \rho \circ (\pi|_X)$.

It can now be easily verified that $p^{-1}(z) = (\pi|_X)^{-1}(\rho^{-1}(z))$ is indeed a UV^1 subset of X, for every point $z \in B$. We shall do this for interior points a and b (resp. a^* and b^*) of Σ (resp. Σ^*) and we leave the reader the verification for the boundary points c and d (resp. c^* and d^*). As we have seen above, in such a case, $\rho^{-1}(z)$ is a bouquet of arcs $\overset{m}{\underset{i=1}{\vee}} D_i^1$ with endpoints d_i. Therefore we can easily see that $p^{-1}(z)$ is in this case a 2-sphere with finitely many disks attached along the equator (see Figure 4). Therefore $p^{-1}(z)$ is certainly a UV^1 subset:

$$\boldsymbol{R}^3 \times \boldsymbol{R}^2 \supset \partial(M^3 \times D^2) \supset p^{-1}(z) = \partial\Big(\overset{m}{\underset{i=1}{\vee}} D_i^1 \times D^2\Big)$$

$$= \Big(\partial\Big(\overset{m}{\underset{i=1}{\vee}} D_i^1\Big) \times D^2\Big) \cup \Big(\overset{m}{\underset{i=1}{\vee}} D_i^1 \times \partial D^2\Big) \simeq \overset{m-1}{\underset{i=1}{\vee}} S^2.$$

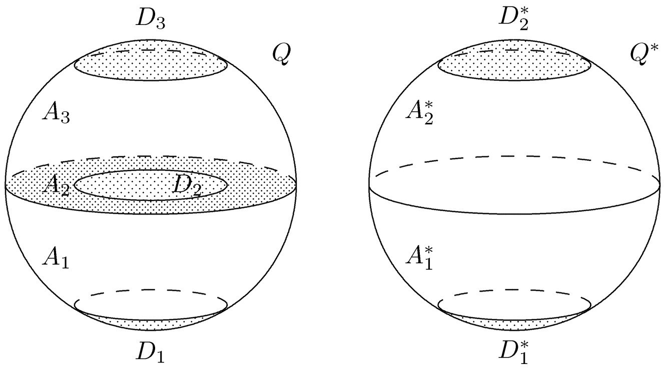

Figure 4.

It therefore follows by Theorem 3.2 that p is indeed a UV^1-map, as it was asserted. □

Remark. Yamasaki has recently proved that our strategy can also be used to prove Theorem 3.5 for hyperbolic knots [**Ya**].

4. Appendix

1. Using Daverman's theorem (see Theorem 3.1 above) one gets UV^1-maps $p : X \to B$ in the following way: Let $U \subset X$ be a compact 4-dimensional submanifold with boundary ∂U. If every component of U is simply connected then the projection $p_U : X \to X/_U$ is UV^1. We shall say that a 4-manifold X is a *good manifold* if there exists such a submanifold U that $A : H_4(X/_U; \boldsymbol{L}) \to L_4(\pi_1(X))$ is injective. Note that $\pi_1(X) \cong \pi_1(X/_U)$. Then one gets the following more general result:

Theorem 4.1. *If a 4-manifold M^4 is homotopy equivalent to a good 4-manifold X, then the surgery sequence*

$$\mathscr{S}(X) \to [X, G/_{TOP}] \to L_4(\pi_1)$$

is exact.

Note that "X has a good fundamental group" does not imply that "X is a good manifold", and vice–versa.

2. Let π be the fundamental group of an arbitrary knot $k \subset S^3$. There is a well-known procedure (see e.g. [**Ma**]) by which one can construct a (special) compact 2-polyhedron $K \subset S^3$, such that $\pi_1(K) \cong \pi = \pi_1(\overline{S^3 \setminus k})$. We briefly outline it below.

Let a knot k be given by its projection onto S^2, which is discontinued at the double points – in order to show which of the diagram's parts goes over the other. Glue a long closed strip (a tunnel) to S^2 and to itself along the knot projection, as it is shown in Figure 5.

We obtain a (special) spine K of the twice punctured knot complement, i.e. of the

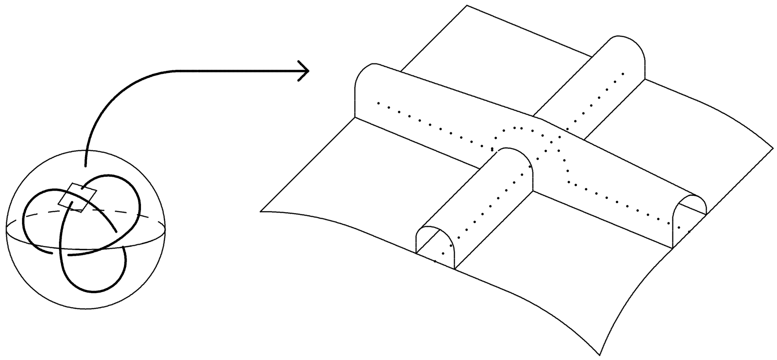

Figure 5.

compact 3-dimensional manifold $N^3 \subset S^3$ which collapses to K, $N^3 \searrow K$, and $\partial N = S^2 \cup (S^1 \times S^1) \cup S^2$. Unfortunately, the 2-polyhedron K contains a nontrivial 2-sphere, i.e. $H_2(K, \boldsymbol{Z}_2) \neq 0$. However, since $L_4(\pi) = \boldsymbol{Z}$, the control space $B = K$ should have trivial $H_2(K, \boldsymbol{Z})$, in order to guarantee that the assembly map $A : H_4(B, \boldsymbol{L}) \to L_4(\pi)$ is an isomorphism.

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