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# ‘Spindles’ in symmetric spaces

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**Abstract.** We study families of submanifolds in symmetric spaces of compact type arising as exponential images of $s$-orbits of variable radii. If the $s$-orbit is symmetric such submanifolds are the most important examples of adapted submanifolds, i.e. of submanifolds of symmetric spaces with curvature invariant tangent and normal spaces.

## 1. Introduction.

Among Riemannian manifolds the symmetric spaces form an important class of examples. On the one hand they are sufficiently general to serve as examples for a lot of geometric phenomena. On the other hand they enjoy enough algebraic structure to make explicit calculations possible. The most important algebraic structure on a symmetric space is the Lie triple product on its tangent spaces given by the curvature tensor. Thus it is natural to ask which submanifolds of a given Riemannian symmetric space have curvature invariant tangent and normal spaces at each point. Such a submanifold is called *adapted*. Naitoh classified locally these submanifolds using the theory of Grassmann geometries (cf. [**11**] and the references given there). It turns out that if the ambient symmetric space $M$ is irreducible and of rank greater than 1 then a full adapted submanifold, i.e. one not contained in a proper totally geodesic subspace, (also called a *normal curved Lie triple* (cf. [**4**])) is locally isomorphic to a symmetric $s$-orbit of $M$. An *$s$-orbit* is a connected component of an isotropy orbit of a symmetric space. A result of Ferus (cf. [**5**]) shows that symmetric $s$-orbits, also known as symmetric *$R$-spaces*, are exactly the closed extrinsically symmetric submanifolds of euclidean space. A submanifold $S$ of $M$ is called *extrinsically symmetric* if for each point $q \in S$ there exists an isometry $f_q$ of $M$ satisfying $f_q(q) = q$, $f_q(S) = S$ and $\mathrm{D}f_q(q)X = -X$ if $X$ is tangent to $S$ and $\mathrm{D}f_q(q)X = X$ if $X$ is normal to $S$. The isometry $f_q$ is called the *extrinsic symmetry* of $S$ at $q$, e.g. if $M$ is a euclidean space then $f_q$ is the reflection along the normal space of $S$ at $q$.

In this work we study more closely families of exponential images of $s$-orbits, in particular the cases where the $s$-orbits are symmetric. These families form *chains of spindles* in the ambient space. The number of knots in a chain of spindles is called the *spindle number*. The submanifolds occurring in such chains of spindles of symmetric $s$-orbits are described in [**1**, pp. 263, 264].

Given a Lie triple $(\mathfrak{p}, R)$ of compact type (i.e. a vector space $\mathfrak{p}$ equipped with a triple

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product $R$ satisfying the algebraic curvature identities with the additional property that $R(X,Y)$ is a derivation of $R$ for all $X, Y \in \mathfrak{p}$ and such that the orthogonal symmetric Lie algebra obtained by the Cartan construction is of compact type (cf. [**4**], [**7**])), there are two canonical symmetric spaces associated with it. On the one hand the associated simply connected space which is the universal Riemannian covering of all symmetric spaces associated with $(\mathfrak{p}, R)$. On the other hand the *adjoint space* which is covered by any symmetric space associated with $(\mathfrak{p}, R)$. We show that spindle numbers of adjoint spaces are 1 (see section 2.1) and describe the relation between the spindle number and the center of the isometry group (see section 2.2). Further we study the extrinsic geometry of the chains of spindles associated with symmetric $s$-orbits (see section 3) and determine the spindle numbers associated with symmetric $s$-orbits of the classical simply connected symmetric spaces of compact type (see section 4). It turns out that any natural number can be realized as the spindle number associated with a Veronese type embedding of some projective space.

We refer to [**7**] for the general theory of symmetric spaces and to [**1**] for a detailed description of $s$-orbits.

## 2. Chains of spindles and spindle numbers.

Let $M$ be a Riemannian symmetric space of compact type and let $p$ be some point in $M$. Denote by G its isometry group and by K the isotropy group of $p$. The geodesic symmetry of $M$ at $p$ will be denoted by $s_p$ and the corresponding involuting automorphism $\mathrm{Ad}(s_p)$ of the Lie algebra $\mathfrak{g}$ of G by $\sigma$. The Cartan decomposition of $\mathfrak{g}$ associated with these data is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where the $(+1)$-eigenspace $\mathfrak{k}$ of $\sigma$ is the Lie algebra of K and the $(-1)$-eigenspace $\mathfrak{p}$ of $\sigma$ is identified with $T_pM$ (cf. [**7**]).

Let $\gamma$ be a closed geodesic on $M$ emanating from $p$ and let $\xi \in \mathfrak{p}$ be its tangent vector at $p$. Considering a maximal torus (flat) in $M$ containing $\gamma$, one sees that, after some suitable change of the parametrization of $\gamma$, the imaginary parts of the (purely imaginary) eigenvalues of the anti-selfadjoint homomorphism $\mathrm{ad}(\xi)$ are integers. If these integers are relatively prime we call the parametrization of $\gamma$ and the (non-zero) element $\xi \in \mathfrak{p}$ *canonical*. In this article $\xi$ always denotes a canonical element of $\mathfrak{p}$ and $i$ always the imaginary unit $\sqrt{-1}$. It is well known that $i\nu$ is an eigenvalue of $\mathrm{ad}(\xi)$ if and only if $-i\nu$ is an eigenvalue of $\mathrm{ad}(\xi)$.

A point $q$ in $M$ is called an *antipode* of $p$ if $q$ is the midpoint of a closed geodesic emanating from $p$. The set $\mathscr{A}(p)$ of antipodes of $p$ is the fixed point set of $s_p$. Isolated points in $\mathscr{A}(p)$ are called *poles* of $p$ and connected components of $\mathscr{A}(p)$ having positive dimension are called *polars* of $p$. Polars are examples of adapted submanifolds: Let $q$ be a point of some polar $M^+(q)$ of $p$. On the one hand $M^+(q)$ is totally geodesic in $M$. On the other hand the connected component of the fixed point set $s_q \circ s_p$ containing $q$, called a *meridian*, is a totally geodesic submanifold whose tangent space at $q$ is the normal space of $M^+(q)$ (cf. [**2**]). Any connected component of $\mathscr{A}(p)$ is an orbit of $\mathrm{K}_0$, the identity component of K, in $M$ (cf. [**2**]). Hence let $q$ be a point of some polar $M^+(q)$ of $p$ and let $\gamma$ be the closed geodesic emanating from $p$ whose midpoint is $q = \gamma(t_0)$ then $M^+(q) = \mathrm{Exp}_p(t_0 \cdot M^\xi)$, where $M^\xi = \mathrm{Ad}_{\mathrm{G}}(\mathrm{K}_0)\xi \subset \mathfrak{p}$ is the $s$-orbit of $\xi = \gamma'(0)$ and $\mathrm{Exp}_p$ the Riemannian exponential map of $M$ at $p$. The objects introduced in this paragraph were extensively studied by Nagano and Tanaka in a series of subsequent papers with the same title (see the references given in [**9**]).

Let now $\xi$ be a canonical element in $\mathfrak{p}$ and let $M^\xi$ be its $s$-orbit. The family of submanifolds $\left(M_t^\xi\right)_{t\in\boldsymbol{R}}$, $M_t^\xi = \mathrm{Exp}_p(t\cdot M^\xi)$ is called the *chain of spindles* in $M$ associated with the canonical element $\xi$. A submanifold $M_t^\xi$ which is a point is called a *knot*, a family of submanifolds between two consecutive knots is called a *spindle* and a submanifold in the middle between two consecutive knots is called a *centriole* (cf. [**3**]) of the chain of spindles. Notice that the set of the geodesic symmetries at the knots acts transitively on the set of spindles of a given chain of spindles. Let $t_\xi$ be the period of the closed geodesic $\gamma_\xi$ defined by $\xi$, i.e. $\gamma_\xi(t_\xi) = \gamma_\xi(0) = p$ and $p \notin \gamma_\xi((0, t_\xi))$. The *spindle number* $\lambda(M,\xi)$ associated with the pair $(M,\xi)$ is the number of knots of the chain of spindles associated with $\xi$ the geodesic $\gamma_\xi(t)$ meets in the interval $[0, t_\xi)$. Since $p = \gamma_\xi(0)$ is a knot, the spindle number is at least 1. Observe that $\lambda(M, \mathrm{Ad}_{\mathrm{G}}(k)\xi) = \lambda(M,\xi)$ for $k \in \mathrm{K}$.

Example. The best known example of a chain of spindles is formed by the 'parallels of latitude' on a sphere: Let $M$ be the 2-dimensional standard sphere $S^2$, $p$ its north pole and $\xi$ any non-zero vector in $T_pS^2$. The corresponding $s$-orbit is the one dimensional sphere $S^1$ in $T_pS^2$. The spindle defined by $\xi$ is just the family of parallels of latitude. The knots are the north and the south pole and the spindle number is 2.

Given a canonical element $\xi$ of $\mathfrak{p}$, we consider the decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{g}_{-\nu_r} \oplus \mathfrak{g}_{-\nu_{r-1}} \oplus \cdots \oplus \mathfrak{g}_{-\nu_1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\nu_1} \oplus \cdots \oplus \mathfrak{g}_{\nu_{r-1}} \oplus \mathfrak{g}_{\nu_r},$$

where $\mathfrak{g}_\nu$ is the $(i\nu)$-eigenspace of $\mathrm{ad}(\xi)$. Then $\mathfrak{k}$ and $\mathfrak{p}$ can be decomposed as follows:

$$\mathfrak{k} = \mathfrak{k}_+ \oplus \mathfrak{k}_-, \quad \text{where} \quad \mathfrak{k}_- = \sum_{j=1}^r \mathfrak{k}_{\nu_j}, \quad \mathfrak{k}_{\nu_j} = \mathfrak{k} \cap \left(\mathfrak{g}_{-\nu_j} \oplus \mathfrak{g}_{\nu_j}\right) \quad \text{and} \quad \mathfrak{k}_+ = \mathfrak{k} \cap \mathfrak{g}_0;$$

$$\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-, \quad \text{where} \quad \mathfrak{p}_- = \sum_{j=1}^r \mathfrak{p}_{\nu_j}, \quad \mathfrak{p}_{\nu_j} = \mathfrak{p} \cap \left(\mathfrak{g}_{-\nu_j} \oplus \mathfrak{g}_{\nu_j}\right) \quad \text{and} \quad \mathfrak{p}_+ = \mathfrak{p} \cap \mathfrak{g}_0.$$

Observe that $\mathfrak{p}_- \cong T_\xi M^\xi$ and $\mathfrak{p}_+ \cong N_\xi M^\xi$. Let $X'$ be an element of $\mathfrak{k}_-$ and let $X = \mathrm{ad}(X')\xi = \sum_{j=1}^r X_{\nu_j} \in \mathfrak{p}_-$, $X_{\nu_j} \in \mathfrak{p}_{\nu_j}$. Consider the geodesic variation

$$\alpha : \boldsymbol{R} \times \boldsymbol{R} \longrightarrow M, \quad (s,t) \longmapsto \gamma_{\mathrm{Ad}_{\mathrm{G}}(\exp(sX'))\xi}(t).$$

Its variation vector field for $s = 0$ is the Jacobi field $J_X$ along $\gamma_\xi$ given by

$$J_X(t) = \sum_{j=1}^r \frac{1}{\nu_j} \sin(\nu_j t) X_{\nu_j}(t), \tag{$*$}$$

where $X_{\nu_j}(t)$ is the parallel vector field along $\gamma_\xi$ with $X_{\nu_j}(0) = X_{\nu_j}$. Assume $X_{\nu_j}(t) \neq 0$ for all $j \in \{1, \ldots, r\}$. Since the vector fields $X_{\nu_j}(t)$ are linearly independent and the eigenvalues $\{\nu_j\}$ relatively prime, $J_X$ vanishes if and only if $t \in \boldsymbol{Z}\pi$. Since symmetric spaces are variationally complete, this shows that $M_t^\xi$ is a knot if and only if $t \in \boldsymbol{Z}\pi$ and a centriole if and only if $t \in \left(\boldsymbol{Z} + \frac{1}{2}\right)\pi$. Moreover the length of $\gamma_\xi$ is $\lambda(M,\xi) \cdot \pi \cdot |\xi|$.

Two submanifolds $S_1$ and $S_2$ of $M$ are called *extrinsically isometric*, if there exists an isometry $f$ of $M$ mapping $S_1$ onto $S_2$, i.e. $f(S_1) = S_2$.

Lemma 1. *The submanifolds $M_{n\pi+t}^\xi$ and $M_{n\pi-t}^\xi$, $n \in \boldsymbol{Z}$ are extrinsically isometric.*

Proof. The geodesic symmetry at the knot $M_{n\pi}^\xi$ maps $M_{n\pi+t}^\xi$ onto $M_{n\pi-t}^\xi$. □

**2.1. Spindle numbers of adjoint spaces.**

Denote by $\overline{\mathrm{G}}$ the adjoint group $\mathrm{Ad}(\mathrm{G})$ of G. For an element $g \in \mathrm{G}$ we denote by $\overline{g}$ the corresponding element in $\overline{\mathrm{G}}$. Consider further the subgroup $\overline{\mathrm{K}}$ of $\overline{\mathrm{G}}$ formed by the elements of $\overline{\mathrm{G}}$ commuting with $\overline{s_p}$. Then $\overline{M} = \overline{\mathrm{G}}/\overline{\mathrm{K}}$ is a symmetric space, called the *adjoint space* of the orthogonal symmetric Lie algebra $(\mathfrak{g},\sigma)$. Notice that the isotropy actions of $\overline{\mathrm{K}}_0$ and $\mathrm{K}_0$ on $\mathfrak{p}$ have the same orbits. The adjoint space $\overline{M}$ is covered by all symmetric spaces associated with $(\mathfrak{g},\sigma)$ (cf. [**7**, p. 327]). We denote by $\pi$ the covering map $M \xrightarrow{\pi} \overline{M}$.

Proposition 2. *Spindle numbers of adjoint spaces are* 1.

Proof. Since $\xi$ is canonical, $\overline{\exp}(\pi\xi) = \mathrm{Ad}(\exp(\pi\xi))$ is an element of order 2, i.e. $\overline{\exp}(\pi\xi)^2 = \overline{\exp}(2\pi\xi) = \mathrm{I}$. Since $s_p \circ \exp(\pi\xi) \circ s_p = \exp(-\pi\xi)$, we observe that $\mathrm{Ad}(\exp(\pi\xi) \circ s_p) = \mathrm{Ad}(s_p \circ \exp(\pi\xi))$. Hence $\overline{\exp}(\pi\xi)$ is an element of $\overline{\mathrm{K}}$ and $\lambda(\overline{M},\xi)$ is 1. □

Proposition 2 shows that $\lambda(M,\xi)$ describes how many times the geodesic in $\overline{M}$ defined by $\xi$ is covered by the corresponding geodesic in $M$. More precisely $\pi$ maps the knots of the spindle chain $\left(M_t^\xi\right)_{t\in\boldsymbol{R}}$ in $M$ onto $\pi(p)$ and the centrioles onto the polar (which is not a pole, since the spindle number is 1) of the geodesic in $\overline{M}$ defined by $\xi$. This shows that centrioles of chains of spindles are totally geodesic in $M$ (cf. [**3**]).

**2.2. Spindle numbers and centers of isometry groups.**

Proposition 3. *The spindle number $\lambda(M,\xi)$ divides the double of the order of the center $\mathrm{Z}(\mathrm{G}_0)$ of the identity component $\mathrm{G}_0$ of G, i.e.*

$$\lambda(M,\xi) \;\mid\; 2 \cdot \mathrm{card}(\mathrm{Z}(\mathrm{G}_0)).$$

*If $\lambda(M,\xi)$ is odd then $\lambda(M,\xi)$ already divides the cardinality of $\mathrm{Z}(\mathrm{G}_0)$.*

Proof. Let $g = \exp(\pi\xi) \in \mathrm{G}_0$. Since $\xi$ is canonical, we have $\mathrm{Ad}(g^2) = e^{\mathrm{ad}(2\pi\xi)} = \mathrm{I}$. Thus $g^2$ lies in $\mathrm{Z}(\mathrm{G}_0)$. Since $g^{\lambda(M,\xi)} = \exp(\lambda(M,\xi)\pi\xi)$ is an element of K, we see that $g^{2\lambda(M,\xi)}$ lies in $\mathrm{Z}(\mathrm{G}_0) \cap \mathrm{K}$, i.e. $g^{2\lambda(M,\xi)} = \mathrm{I}$. Let $\mu$ be the smallest positive integer satisfying $g^{2\mu} = \mathrm{I}$. Then $\mu$ divides the cardinality of $\mathrm{Z}(\mathrm{G}_0)$ and $\lambda(M,\xi)$ is a multiple of $\mu$, i.e. $\lambda(M,\xi) = n\mu$ for some positive integer $n$. Since $p = g^{2\mu}p = \exp(2\pi\mu\xi)p$, we have $2\mu \geq \lambda(M,\xi)$ and hence $2 \geq n$. If $n = 1$ then $\lambda(M,\xi)$ divides already the cardinality of $\mathrm{Z}(\mathrm{G}_0)$. If $n = 2$ then $\lambda(M,\xi)$ divides $2 \cdot \mathrm{card}(\mathrm{Z}(\mathrm{G}_0))$. □

The next example shows that Proposition 3 is optimal: Let $M = \mathrm{SU}(6)/\mathrm{SO}(6)$ and

$\xi = i\begin{pmatrix} -\frac{5}{6} & 0 \\ 0 & \frac{1}{6}\mathrm{I}_5 \end{pmatrix}$. Then the spindle number $\lambda(\mathrm{SU}(6)/\mathrm{SO}(6), \xi)$ is 6 (see section 4) and the cardinality of the center of $\mathrm{G}_0 \cong \mathrm{SU}(6)/\boldsymbol{Z}_2$ is 3.

### 2.3. Spindle numbers of products.

Let $M_1$ and $M_2$ be two symmetric spaces of compact type and let $M = M_1 \times M_2$. Let $p = (p_1, p_2)$ be a point in $M$. By $\mathrm{G}_i$ we denote the isometry group of $M_i$ and by $\mathrm{K}_i$ the isotropy group of $p_i$, $i = 1, 2$. The Cartan decomposition of the Lie algebra of the isometry group G of $M$ is $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ is the Lie algebra of the isotropy group K of $p$ and $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \cong T_pM$ where $\mathfrak{p}_i \cong T_{p_i}M_i$, $i = 1, 2$. Assume $\xi_i \in \mathfrak{p}_i$, $i = 1, 2$ to be canonical. Then $\xi = (\xi_1, \xi_2) \in \mathfrak{p}$ is canonical and the spindle number $\lambda(M, \xi)$ is the smallest common multiple of the spindle numbers $\lambda(M_1, \xi_1)$ and $\lambda(M_2, \xi_2)$.

## 3. Chains of spindles of extrinsically symmetric type.

Among the canonical elements there might be some whose $s$-orbit $M^\xi$ is extrinsically symmetric in the euclidean space $\mathfrak{p}$. This happens exactly if

$$\mathrm{ad}(\xi)^3 = -\mathrm{ad}(\xi),$$

i.e. if the eigenvalues of $\mathrm{ad}(\xi)$ are $\pm i$ and 0 (cf. [**5**]). Such elements will be called of *extrinsically symmetric type*. In this section $\xi$ always denotes such an element. Considering again the geodesic variation field $(*)$, one sees that chains of spindles associated with elements of extrinsically symmetric type have the following characterizing property: all submanifolds $M_t^\xi$ which are not knots have the same dimension.

If $\xi$ is of extrinsically symmetric type the Jacobi field given in equation $(*)$ simplifies:

$$J_X(t) = \mathrm{DExp}_p|_{t\xi}(tX) = \sin(t)X(t).$$

Since parallel transports along curves are linear isometries, this shows that the tangent space $T_{\gamma_{\mathrm{Ad_G}(k)\xi}(t)}M_t^\xi$, $t \notin \pi\boldsymbol{Z}$ is just the parallel transport of $T_{\mathrm{Ad_G}(k)\xi}M^\xi \cong \mathrm{Ad_G}(k)\mathfrak{p}_-$ along $\gamma_{\mathrm{Ad_G}(k)\xi}$. Take now an element $Y \in \mathfrak{p}_+ \cong N_\xi(M^\xi)$. The Jacobi field $J_Y(t) = \mathrm{DExp}_p|_{t\xi}(tY)$ along $\gamma_\xi$ is $J_Y(t) = tY(t)$, where $Y(t)$ is the parallel vector field along $\gamma_\xi$ defined by $Y$. Hence the normal space $N_{\gamma_{\mathrm{Ad_G}(k)\xi}(t)}M_t^\xi$, $t \notin \pi\boldsymbol{Z}$ is the parallel transport of $N_{\mathrm{Ad_G}(k)\xi}M^\xi \cong \mathrm{Ad_G}(k)\mathfrak{p}_+$ along $\gamma_{\mathrm{Ad_G}(k)\xi}$, $k \in \mathrm{K}_0$. Since $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are curvature invariant if $\xi$ is of extrinsically symmetric type, and since the curvature tensor of $M$ is parallel, we get: *The submanifolds $M_t^\xi$, $t \notin \pi\boldsymbol{Z}$ are adapted.* In fact, these submanifolds form the most prominent class of adapted submanifolds in symmetric spaces of compact type (cf. [**4**], [**11**]).

A submanifold $S$ of $M$ is called *reflective* if there exists an isometry $r$ of $M$ fixing $S$ pointwise and reversing the geodesics of $M$ which intersect $S$ perpendicularly or, equivalently, if $S$ is a connected component of the fixed point set of an involuting isometry of $M$, e.g. polars and meridians are reflective. The involuting isometry $r$ is called the *reflection* of $M$ in $S$. A reflective submanifold in a symmetric space is totally geodesic and moreover extrinsically symmetric: Let $r$ be the reflection of $M$ in $S$ and let $q$ be any point in $S$. Then $r \circ s_q$ is the extrinsic symmetry of $S$ at $q$. Conversely, a totally geodesic extrinsically symmetric submanifold is reflective.

PROPOSITION 4 (cf. [**1**, pp. 256, 257], [**10**]). *Let $S$ be a totally geodesic submanifold of a simply connected symmetric space $M$. Then the following statements are equivalent*:

(1) *$S$ is extrinsically symmetric*;
(2) *$S$ is adapted*;
(3) *$S$ is reflective.*

LEMMA 5. *Let $\xi$ be of extrinsically symmetric type. Assume $\lambda(M,\xi)$ to be odd or $M$ to be simply connected. Then the submanifolds $M_t^\xi$, $t \notin \boldsymbol{Z}\pi$ are extrinsically symmetric.*

PROOF. Let $q = \mathrm{Exp}_p(t\mathrm{Ad}_{\mathrm{G}}(k)\xi)$, $k \in \mathrm{K}_0$ be a point in $M_t^\xi$. If $\lambda(M,\xi)$ is odd then $P_k^+ = \mathrm{Exp}_p(\mathrm{Ad}(k)\mathfrak{p}_+)$ is a meridian and hence reflective. If $M$ is simply connected $P_k^+$ is adapted and by Proposition 4 reflective. Moreover $T_qP_k^+ = N_qM_t^\xi$ and $N_qP_k^+ = T_qM_t^\xi$. Let $r$ be the reflection of $M$ in $P_k^+$ and let $\tau_k$ be the reflection of $\mathfrak{p}$ along $N_{\mathrm{Ad}_{\mathrm{G}}(k)\xi}M^\xi \cong \mathrm{Ad}(k)\mathfrak{p}_+$ then $\mathrm{D}r(p) = \tau_k$. Take a point $q_1 = \mathrm{Exp}_p(t\mathrm{Ad}_{\mathrm{G}}(k_1)\xi)$, $k_1 \in \mathrm{K}_0$ in $M_t^\xi$. Since $M^\xi$ is extrinsically symmetric in the euclidean space $\mathfrak{p}$, there is an element $k_2 \in \mathrm{K}_0$ such $\tau_k(\mathrm{Ad}_{\mathrm{G}}(k_1)\xi) = \mathrm{Ad}_{\mathrm{G}}(k_2)\xi$. Hence $r(q_1) = r(\mathrm{Exp}_p(t\mathrm{Ad}_{\mathrm{G}}(k_1)\xi)) = \mathrm{Exp}_p(\tau_k(t\mathrm{Ad}_{\mathrm{G}}(k_1)\xi)) = \mathrm{Exp}_p(t\mathrm{Ad}_{\mathrm{G}}(k_2)\xi)$ is an element of $M_t^\xi$. Thus $r$ leaves $M_t^\xi$ invariant and $\mathrm{D}r(q)X = X$ if $X$ is normal to $M_t^\xi$ and $\mathrm{D}r(q)X = -X$ if $X$ is tangent to $M_t^\xi$, i.e. $r$ is the extrinsic symmetry of $M_t^\xi$ at $q$. □

REMARK. Assume $M$ to be an adjoint space $M = \overline{M}$ and moreover $M$ to be an *inner symmetric space*, i.e. $s_p$ is contained in $\mathrm{K}_0$, then $X$ lies in $M^\xi$ if and only if $-X$ lies in $M^\xi$. Although all submanifold $M_t^\xi$, $t \in (0,\pi)$ are symmetric spaces associated with the same orthogonal symmetric Lie algebra, the centrosome (polar) $M_{\frac{\pi}{2}}^\xi$ is not isomorphic to the other submanifolds. In fact, let $X \in M^\xi$, then $\gamma_X(t) = \gamma_{-X}(t)$ if and only if $t \in \boldsymbol{Z}\frac{\pi}{2}$. Examples of this phenomenon are the spindles in real projective spaces, but also the spindle associated with $M^\xi = \mathrm{SO}(n)$ in the adjoint space of $\mathrm{SO}(2n)/(\mathrm{SO}(n)\times\mathrm{SO}(n))$ if $n$ is even.

COROLLARY 6. *Let $\xi$ be of extrinsically symmetric type. Assume $\lambda(M,\xi)$ to be odd or $M$ to be simply connected. Then centrioles of chains of spindles are reflective.*

COROLLARY 7. *Let $\xi$ be of extrinsically symmetric type. Assume $\lambda(M,\xi)$ to be odd or $M$ to be simply connected. Then the submanifolds $M^\xi_{(2n+1)\frac{\pi}{2}-t}$ and $M^\xi_{(2n+1)\frac{\pi}{2}+t}$, $n \in \boldsymbol{Z}$ are extrinsically isometric.*

PROOF. The reflection in $M^\xi_{(2n+1)\frac{\pi}{2}}$ maps $M^\xi_{(2n+1)\frac{\pi}{2}-t}$ onto $M^\xi_{(2n+1)\frac{\pi}{2}+t}$. □

## 4. Examples.

In this section we calculate the spindle numbers of all pairs $(M,\xi)$, up to isomorphisms, where $M$ is an indecomposable connected, simply connected classical symmetric spaces of compact type, but not a group, or $M$ is a simple matrix group and $\xi$ is of extrinsically symmetric type. We use Cartan's notation for the different types of the symmetric spaces $M$. The spaces $M$ are denoted as quotient spaces, but the given groups might not act effectively. The exact notations as quotients of the identity components of the isometry groups can be found in [**12**]. The classification of symmetric spaces and their symmetric $s$-orbits on the Lie algebra level is done in [**8**] and the elements $\xi$ of extrinsically symmetric type are given explicitly in the classical cases. The list of the corresponding pairs $(M, M^{\xi})$ is taken from the appendix of [**1**]. To shorten our notations we denote by $J_n$ the matrix $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.

Remark. The Veronese type imbeddings of projective spaces over the number fields $\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$ as symmetric $s$-orbits and their chains of spindles show that any natural number occurs as spindle number (see type A I, A II and $\mathfrak{a}$).

**4.1. The classical non-hermitian case.**

Type A I. Let $M = \mathrm{SU}(p+q)/\mathrm{SO}(p+q)$, $1 \le p \le q$ and let $\xi = i \cdot \begin{pmatrix} a \cdot I_p & 0 \\ 0 & b \cdot I_q \end{pmatrix}$ with $a = -\frac{q}{p+q}$ and $b = \frac{p}{p+q}$. Then $\exp(t\xi) = \begin{pmatrix} \exp(ita) \cdot I_p & 0 \\ 0 & \exp(itb) \cdot I_q \end{pmatrix}$ lies in $\mathrm{SO}(p+q)$ if and only if $ta$ and $tb$ are elements of $\pi\boldsymbol{Z}$. Hence the spindle number is the smallest positive integer $n$ such that $n\frac{p}{p+q}$ and $n\frac{q}{p+q}$ are integers.

Type A II. Let $M = \mathrm{SU}(2(p+q))/\mathrm{Sp}(p+q)$, $1 \le p \le q$ and let $\xi = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$ with $E = i\begin{pmatrix} a \cdot I_p & 0 \\ 0 & b \cdot I_q \end{pmatrix}$, $a = -\frac{q}{p+q}$ and $b = \frac{p}{p+q}$. Then $\exp(t\xi) = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$ with $M = \begin{pmatrix} \exp(ita) \cdot I_p & 0 \\ 0 & \exp(itb) \cdot I_q \end{pmatrix}$ lies in $\mathrm{Sp}(p+q)$ if and only if $ta$ and $tb$ are elements of $\pi\boldsymbol{Z}$. Hence the spindle number is the smallest positive integer $n$ such that $n\frac{p}{p+q}$ and $n\frac{q}{p+q}$ are integers.

Type A III. Let $M = \mathrm{SU}(2n)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(n))$ and let $\xi = \frac{i}{2} \cdot \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. Then $\exp(t\xi) = \cos\left(\frac{t}{2}\right) \cdot \mathrm{I}_{2n} + i \sin\left(\frac{t}{2}\right) \cdot \begin{pmatrix} 0 & \mathrm{I}_n \\ \mathrm{I}_n & 0 \end{pmatrix}$ lies in $\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(n))$ if and only if $t \in 2\pi\boldsymbol{Z}$. Hence the spindle number is 2.

Type BD I. We have to distinguish two cases:

(1) Let $M = \mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$, $1 \le p \le q$ and let $\xi = \begin{pmatrix} 0 & E \\ -E^T & 0 \end{pmatrix}$ where $E$ is the rank one matrix $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}(p \times q)$. An easy calculation shows that $\exp(t\xi)$ lies in $\mathrm{SO}(p) \times \mathrm{SO}(q)$ if and only if $\sin(t) = 0$ and $\cos(t) = 1$. Thus $\lambda(M, M^{\xi}) = 2$.

(2) Let $M = \mathrm{SO}(2n)/\mathrm{SO}(n) \times \mathrm{SO}(n)$ and let $\xi = \frac{1}{2} \cdot J_n$. Then $\exp(t\xi) = \cos\left(\frac{t}{2}\right) \cdot \mathrm{I}_{2n} + \sin\left(\frac{t}{2}\right) \cdot J_n$ lies in $\mathrm{SO}(n) \times \mathrm{SO}(n)$ if and only if $\sin\left(\frac{t}{2}\right) = 0$ and $\cos\left(\frac{t}{2}\right) \cdot \mathrm{I}_{2n} \in \mathrm{SO}(n) \times \mathrm{SO}(n)$. Thus $\lambda(M, \xi) = 2$ if $n$ is even and $\lambda(M, \xi) = 4$ if $n$ is odd.

Type D III. Let $M = \mathrm{SO}(4n)/\mathrm{U}(2n)$ and $\xi = \frac{1}{2} \cdot \begin{pmatrix} J_n & 0 \\ 0 & -J_n \end{pmatrix}$. Then $\exp(t\xi) = \cos\left(\frac{t}{2}\right) \cdot \mathrm{I}_{4n} + 2 \sin\left(\frac{t}{2}\right) \cdot \xi$ lies in $\mathrm{U}(2n) \cong \mathrm{SO}(4n) \cap \mathrm{Sp}(2n)$ if and only if $t \in 2\pi\boldsymbol{Z}$. Hence the spindle number is 2.

Type C I. Let $M = \mathrm{Sp}(n)/\mathrm{U}(n)$ and let $\xi = \frac{i}{2} \cdot \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$. Then $\exp(t\xi) = \cos\left(\frac{t}{2}\right) \cdot \mathrm{I}_{2n} + 2 \sin\left(\frac{t}{2}\right) \cdot \xi$ lies in $\mathrm{U}(n) \cong \mathrm{Sp}(n) \cap \mathrm{SO}(2n)$ if and only if $t \in 2\pi\boldsymbol{Z}$. Hence the spindle number is 2.

TYPE C II. Let $M = \mathrm{Sp}(2n)/\mathrm{Sp}(n) \times \mathrm{Sp}(n)$ and $\xi = \frac{i}{2} \cdot \begin{pmatrix} 0 & 0 & 0 & I_n \\ 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{pmatrix}$. Then $\exp(t\xi) = \cos\left(\frac{t}{2}\right) \cdot \mathrm{I}_{4n} + 2\sin\left(\frac{t}{2}\right) \cdot \xi$ lies in $\mathrm{Sp}(n) \times \mathrm{Sp}(n)$ if and only if $t \in 2\pi\boldsymbol{Z}$. Hence the spindle number is 2.

### 4.2. The classical hermitian case.

Hermitian symmetric $s$-orbits arise as isotropy orbits of simple Lie groups: a compact, connected, simple Lie group G endowed with a bi-invariant metric is a symmetric space. The geodesic symmetry $s_e$ at the identity $e$ is just the inversion. The product $\mathrm{G} \times \mathrm{G}$ acts on G by isometries: $(\mathrm{G} \times \mathrm{G}) \times \mathrm{G} \to \mathrm{G}$, $(g, h)p = gph^{-1}$. The involution on $\mathrm{G} \times \mathrm{G}$ given by the conjugation with $s_e$ interchanges both factors. Thus its fixed point set is the diagonal $\Delta^+\mathrm{G}$ in $\mathrm{G} \times \mathrm{G}$. The Cartan decomposition of the symmetric Lie algebra associated with the symmetric space $\mathrm{G} \cong \mathrm{G} \times \mathrm{G}/\Delta^+\mathrm{G}$ is $\mathfrak{g} \times \mathfrak{g} = \Delta^+\mathfrak{g} \oplus \Delta^-\mathfrak{g}$, where $\Delta^+\mathfrak{g}$ denotes the diagonal and $\Delta^-\mathfrak{g}$ the antidiagonal in $\mathfrak{g} \times \mathfrak{g}$. Let $\xi$ be an element of $\mathfrak{g}$ of extrinsically symmetric type then $(\xi, -\xi)$ is also of extrinsically symmetric type in $\mathfrak{g} \times \mathfrak{g}$. Moreover, the vector $(\xi, -\xi)$ defines a hermitian structure on its symmetric $s$-orbit $M^\xi$. To determine the spindle number of $(\mathrm{G}, (\xi, -\xi))$ we have to look for the smallest positive integer $n$ such that $\exp(n\pi \cdot \xi) = \exp(-n\pi \cdot \xi)$.

TYPE $\mathfrak{a}$. Let $\mathrm{G} = \mathrm{SU}(p+q)$, $1 \leq p \leq q$ and let $\xi = i \cdot \begin{pmatrix} a \cdot I_p & 0 \\ 0 & b \cdot I_q \end{pmatrix}$ with $a = -\frac{q}{p+q}$ and $b = \frac{p}{p+q}$. Thus $\exp(t\xi) = \exp(-t\xi)$ if and only if $\exp(ita) = \exp(-ita)$ and $\exp(itb) = \exp(-itb)$. Hence the spindle number is the smallest positive integer $n$ such that $n\frac{p}{p+q}$ and $n\frac{q}{p+q}$ are integers.

TYPE $\mathfrak{b}$ AND $\mathfrak{d}$. Let $\mathrm{G} = \mathrm{SO}(n)$. We calculate the spindle number of $\mathrm{SO}(n)$ with respect to $\xi = \begin{pmatrix} 0 & E \\ -E^T & 0 \end{pmatrix}$ where $E$ is the rank one matrix $E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{M}(p \times q)$ with $p+q = n$. Thus a straightforward calculation shows that $\exp(t\xi) = \exp(-t\xi)$ if and only if $\sin(t) = 0$. Hence the spindle number of $\mathrm{SO}(n)$ with respect to $\xi$ is 1.

TYPE $\mathfrak{c}$. Let $\mathrm{G} = \mathrm{Sp}(n)$ and $\xi = \frac{i}{2} \cdot \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$. Thus $\exp(t\xi) = \cos\left(\frac{t}{2}\right) \cdot \mathrm{I}_{2n} + 2\sin\left(\frac{t}{2}\right) \cdot \xi = \exp(-t\xi)$ if and only if $\sin\left(\frac{t}{2}\right) = 0$. Hence the spindle number is 2.

TYPE $\mathfrak{d}$. Let $\mathrm{G} = \mathrm{SO}(2n)$. We determine the spindle of $\mathrm{SO}(2n)$ with respect to $\xi = \frac{1}{2} \cdot J_n$. Thus $\exp(t\xi) = \cos\left(\frac{t}{2}\right) \cdot \mathrm{I}_{2n} + \sin\left(\frac{t}{2}\right) \cdot J_n = \exp(-t\xi)$ if and only if $\sin\left(\frac{t}{2}\right) = 0$. Hence the spindle number of $\mathrm{SO}(2n)$ with respect to $\xi$ is 2.

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| $M$ | $M^{\xi}$ | spindle number |
|---|---|---|
| $\mathrm{SU}(p+q)/\mathrm{SO}(p+q)$ | $\mathrm{SO}(p+q)/\mathrm{S}(\mathrm{O}(p)\times\mathrm{O}(q))$ | smallest $n \in \boldsymbol{N}^*$ s.t. $n\frac{p}{p+q} \in \boldsymbol{N}^*$ and $n\frac{q}{p+q} \in \boldsymbol{N}^*$ |
| $\mathrm{SU}(2(p+q))/\mathrm{Sp}(p+q)$ | $\mathrm{Sp}(p+q)/\mathrm{Sp}(p)\times\mathrm{Sp}(q)$ | smallest $n \in \boldsymbol{N}^*$ s.t. $n\frac{p}{p+q} \in \boldsymbol{N}^*$ and $n\frac{q}{p+q} \in \boldsymbol{N}^*$ |
| $\mathrm{SU}(2n)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(n))$ | $\mathrm{U}(n)$ | 2 |
| $\mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$ | $(S^{p-1}\times S^{q-1})/\boldsymbol{Z}_2$ | 2 |
| $\mathrm{SO}(2n)/\mathrm{SO}(n)\times\mathrm{SO}(n)$ | $\mathrm{SO}(n)$ | 2, if $n$ is even<br>4, if $n$ is odd |
| $\mathrm{SO}(4n)/\mathrm{U}(2n)$ | $\mathrm{U}(2n)/\mathrm{Sp}(n)$ | 2 |
| $\mathrm{Sp}(n)/\mathrm{U}(n)$ | $\mathrm{U}(n)/\mathrm{O}(n)$ | 2 |
| $\mathrm{Sp}(2n)/\mathrm{Sp}(n)\times\mathrm{Sp}(n)$ | $\mathrm{Sp}(n)$ | 2 |
| $\mathrm{SU}(p+q)$ | $\mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$ | smallest $n \in \boldsymbol{N}^*$ s.t. $n\frac{p}{p+q} \in \boldsymbol{N}^*$ and $n\frac{q}{p+q} \in \boldsymbol{N}^*$ |
| $\mathrm{SO}(n)$ | $\mathrm{SO}(n)/(\mathrm{SO}(2)\times\mathrm{SO}(n-2))$ | 1 |
| $\mathrm{Sp}(n)$ | $\mathrm{Sp}(n)/\mathrm{U}(n)$ | 2 |
| $\mathrm{SO}(2n)$ | $\mathrm{SO}(2n)/\mathrm{U}(n)$ | 2 |

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