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On the fundamental groups of the complements of plane singular sextics

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Abstract. Recently, Oka-Pho proved that the fundamental group of the complement of any plane irreducible tame torus sextic is not abelian. We compute here the fundamental groups of the complements of some plane irreducible sextics which are not of torus type. For all our examples, we obtain that the fundamental group is abelian.

Introduction.

In [**Z1**], Zariski proved th at if C is an irreducible sextic in the complex projective plane $\boldsymbol{CP}^2$ with 6 cusps situated on a conic, then the fundamental group $\pi_1(\boldsymbol{CP}^2 - C)$ is isomorphic to the free product $(\boldsymbol{Z}/2\boldsymbol{Z}) * (\boldsymbol{Z}/3\boldsymbol{Z})$. He also proved that if there exists an irreducible sextic C' in $\boldsymbol{CP}^2$ with 6 cusps not situated on a conic, then $\pi_1(\boldsymbol{CP}^2 - C')$ is not isomorphic to $(\boldsymbol{Z}/2\boldsymbol{Z}) * (\boldsymbol{Z}/3\boldsymbol{Z})$. In [**Z2**], he justified the existence of this second family of curves C', and asserts that $\pi_1(\boldsymbol{CP}^2 - C')$ is isomorphic to $\boldsymbol{Z}/6\boldsymbol{Z}$. In [**O3**], Oka gave the first explicit example of such a curve C'. A curve C as above (6 cusps on a conic) is an example of the so-called sextics of torus type[1]. On the contrary, the curve C' (6 cusps not situated on a conic) is not of torus type.

CONJECTURE 0.1 (Oka). *Let C be an irreducible sextic in $\boldsymbol{CP}^2$ which is not of torus type. Then, we have the three following conjectures.*

(i) *The generic Alexander polynomial of C is trivial.*

(ii) *If moreover C has only simple singularities, then the fundamental groups $\pi_1(\boldsymbol{CP}^2 - C)$ and $\pi_1(\boldsymbol{C}^2 - C)$ are abelian, isomorphic to $\boldsymbol{Z}/6\boldsymbol{Z}$ and $\boldsymbol{Z}$ respectively.*

(iii) *The fundamental groups $\pi_1(\boldsymbol{CP}^2 - C)$ and $\pi_1(\boldsymbol{C}^2 - C)$ are abelian, isomorphic to $\boldsymbol{Z}/6\boldsymbol{Z}$ and $\boldsymbol{Z}$ respectively* (*without assuming that the singularities are simple*).

Notice that (i) is true for curves having only simple singularities and satisfying the condition $\rho(5) \leq 6$ (cf. [**O7**]). Observe also that (iii) implies (i), while the reverse is not true (cf. [**O7**]).

In the present paper, we give a first step toward (ii). More precisely, for each configuration of singularities Ξ in the following list[2]:

$$\begin{gathered}\{2A_8\},\ \{A_{17}\},\ \{A_{11}+E_6\},\ \{A_{14}+A_2\},\ \{A_{11}+A_5\},\\ \{A_8+A_5+A_2\},\ \{A_8+E_6+A_2\},\end{gathered} \tag{0.2}$$

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[1] A sextic $\{(X:Y:Z) \in \boldsymbol{CP}^2\,;\, F(X,Y,Z)=0\}$ is said of *torus type* if there is an expression $F(X,Y,Z) = F_2(X,Y,Z)^3 + F_3(X,Y,Z)^2$, where F_2 and F_3 are homogeneous polynomials of degree 2 and 3 respectively.

[2] We recall that a point p of a curve $\mathfrak{C}$ is called a singularity of type A_n, where n is an integer ≥ 1, if the germ $(\mathfrak{C},p)$ is topologically equivalent to the germ $(\{x^2+y^{n+1}=0\},O)$ as embedded germs (for the definition of "topologically equivalent", see e.g. [**Di**, Definition 1.4]). It is called a singularity of type E_6 if $(\mathfrak{C},p)$ is topologically equivalent to $(\{x^3+y^4=0\},O)$.

we give an explicit example of an irreducible non-torus sextic $C \subset \boldsymbol{CP}^2$ with the configuration Ξ such that $\pi_1(\boldsymbol{CP}^2 - C)$ and $\pi_1(\boldsymbol{C}^2 - C)$ are abelian (isomorphic to $\boldsymbol{Z}/6\boldsymbol{Z}$ and $\boldsymbol{Z}$ respectively). Then, denoting by $\mathscr{M}(\Xi)$ the moduli space of reduced sextics in $\boldsymbol{CP}^2$ with the configuration Ξ, one deduces that, for any curve $\mathfrak{C}$ belonging to the connected component of $\mathscr{M}(\Xi)$ containing our example C, the fundamental groups $\pi_1(\boldsymbol{CP}^2 - \mathfrak{C})$ and $\pi_1(\boldsymbol{C}^2 - \mathfrak{C})$ are abelian too. Our mains results are stated in Theorem 2.1 and Corollary 2.2. For the proof, we use the Zariski-van Kampen pencils method (cf. Section 1 below). Notice that, in practice, the computation of the fundamental group is not so easy, since it is extremely difficult to read the monodromy relations for curves which are defined over $\boldsymbol{C}$. Nevertheless, when the curve has many *real* singular pencil lines, the computation becomes usually easier. Moreover, as our purpose is to show the commutativity of the fundamental group, it is not necessary to consider all the monodromy relations provided we can find a "good" curve. Hereafter, we have chosen curves so that we shall only need to consider the monodromy relations at the *real* singular pencil lines. But in general if we use an equation which is not "good enough" we have to use the other monodromy relations even to show a commutativity.

Notice that, in [**OP**], Oka-Pho showed that the fundamental group of the complement of any irreducible tame torus sextic[3] in $\boldsymbol{CP}^2$ is isomorphic to $(\boldsymbol{Z}/2\boldsymbol{Z}) * (\boldsymbol{Z}/3\boldsymbol{Z})$ except one class (the exceptional class has the configuration of singularities $\{C_{3,9} + 3A_2\}$ and the fundamental group in this case is bigger than $(\boldsymbol{Z}/2\boldsymbol{Z}) * (\boldsymbol{Z}/3\boldsymbol{Z})$). Concerning the proof, in the case of irreducible tame torus sextics the computation can be in fact reduced to the special case of *maximal* curves, and it thus becomes easier to check the property since there exist only 7 moduli of maximal reduced tame torus sextics in $\boldsymbol{CP}^2$.

Notice also that, in [**O7**], the second author proved that the generic Alexander polynomial of any irreducible torus sextic in $\boldsymbol{CP}^2$ (not necessarily tame) is not trivial; in particular, this implies that the fundamental group of the complement of such a curve is not abelian.

The paper is organized as follows. In Section 1, we recall the Zariski-van Kampen pencils method. In Section 2, we give the statements of our main results (Theorem 2.1 and Corollary 2.2). Sections 3 to 7 concern the proof of Theorem 2.1.

This paper has been written using the SCURVE program made by Pho Duc Tai for MAPLE 7.

1. Zariski-van Kampen pencils method.

Let $F(X,Y,Z)$ be a reduced homogeneous polynomial of degree d. We denote by

$$C := \left\{(X:Y:Z) \in \boldsymbol{CP}^2 \mid F(X,Y,Z) = 0\right\}$$

the corresponding projective curve in $\boldsymbol{CP}^2$. The most effective way to compute the fundamental group $\pi_1(\boldsymbol{CP}^2 - C)$ is the Zariski-van Kampen pencils method. This method can be briefly described as follows.

Let $l(X,Y,Z)$, $l'(X,Y,Z)$ be two independent linear forms. For every point $\tau := (S:T) \in \boldsymbol{CP}^1$, denote by L_τ the projective line of $\boldsymbol{CP}^2$ defined by

$$L_\tau := \left\{(X:Y:Z) \in \boldsymbol{CP}^2 \mid T\, l(X,Y,Z) - S\, l'(X,Y,Z) = 0\right\}.$$

[3] A sextic of torus type $\{(X:Y:Z) \in \boldsymbol{CP}^2; F_2(X,Y,Z)^3 + F_3(X,Y,Z)^2 = 0\}$ is said *tame* if its singularities are sitting only at the intersection of the conic and the cubic defined by $F_2(X,Y,Z) = 0$ and $F_3(X,Y,Z) = 0$ respectively.

The family of lines $\mathscr{L} := (L_\tau)_{\tau\in\boldsymbol{CP}^1}$ is called the pencil generated by l and l'. The point $B_0 := L_{(0:1)} \cap L_{(1:0)}$ belongs to every line of the pencil; it is called the axis of $\mathscr{L}$. We assume that $B_0 \notin C$. A member L_τ of $\mathscr{L}$ is called a *generic* line, with respect to C, if it avoids the singularities of C and if it is transverse to the non-singular part of C; otherwise, it is called a *singular* line. If L_τ is generic, then it intersects C at exactly d points. If L_τ is singular, then it intersects C at a singular point or it is tangent to C at some simple point. Notice that the set of singular lines is finite. If necessary, one may consider some generic lines of $\mathscr{L}$ as "singular" ones. Let Σ the set of parameters $\tau \in \boldsymbol{CP}^1$ corresponding to the singular lines, and let L_{τ_0} and L_{τ_∞} be two generic lines (which we have not decided to consider as "singular"). Without loss of generality, we can assume that τ_∞ is the point at infinity of $\boldsymbol{CP}^1$ (i.e., $\tau_\infty = (1:0)$). Hereafter, we identify $\boldsymbol{CP}^2 - L_{\tau_\infty}$ with the affine space $\boldsymbol{C}^2$, and we denote by L^a_τ the affine line $L_\tau - L_{\tau_\infty} = L_\tau - B_0$. Notice that L^a_τ naturally identifies to $\boldsymbol{C}$. The complement $L_{\tau_0} - C$ (resp. $L^a_{\tau_0} - C$) is topologically the 2-sphere $\boldsymbol{S}^2$ minus d (resp. $d+1$) points. We take $b_0 = B_0$ as the base point in the case of $\pi_1(\boldsymbol{CP}^2 - C)$. In the affine case $\pi_1(\boldsymbol{C}^2 - C)$, we take the base point b_0 on L_{τ_0} sufficiently close to B_0 but $b_0 \neq B_0$.

It is well-known that there is a canonical action of $\pi_1(\boldsymbol{CP}^1 - \Sigma, \tau_0)$ on $\pi_1(L_{\tau_0} - C, b_0)$ and a canonical action of $\pi_1(\boldsymbol{CP}^1 - \Sigma^a, \tau_0)$ on $\pi_1(L^a_{\tau_0} - C, b_0)$, where $\Sigma^a = \Sigma \cup \tau_\infty$ (cf. e.g. [**O4**], [**O8**]). These actions are called the *monodromy actions*. For any $\sigma \in \pi_1(\boldsymbol{CP}^1 - \Sigma, \tau_0)$ and any ξ in $\pi_1(L_{\tau_0} - C, b_0)$, we denote by ξ^σ the image of (σ, ξ) by the monodromy action (of $\pi_1(\boldsymbol{CP}^1 - \Sigma, \tau_0)$ on $\pi_1(L_{\tau_0} - C, b_0)$). The relations

$$\xi = \xi^\sigma \quad \text{for} \quad \sigma \in \pi_1(\boldsymbol{CP}^1 - \Sigma, \tau_0) \text{ and } \xi \in \pi_1(L_{\tau_0} - C, b_0)$$

in the group $\pi_1(L_{\tau_0} - C, b_0)$ are called the *monodromy relations*. We use a similar notation and terminology in the affine case. We denote by N (resp. N^a) the normal subgroup of $\pi_1(L_{\tau_0} - C, b_0)$ (resp. $\pi_1(L^a_{\tau_0} - C, b_0)$) generated by

$$\left\{\xi^{-1}\xi^\sigma \mid \sigma \in \pi_1(\boldsymbol{CP}^1 - \Sigma, \tau_0),\ \xi \in \pi_1(L_{\tau_0} - C, b_0)\right\}$$
$$\left(\text{resp. } \left\{\xi^{-1}\xi^\sigma \mid \sigma \in \pi_1(\boldsymbol{CP}^1 - \Sigma^a, \tau_0),\ \xi \in \pi_1(L^a_{\tau_0} - C, b_0)\right\}\right).$$

THEOREM 1.1 (Zariski-van Kampen). (i) *The inclusion map $L_{\tau_0} - C \hookrightarrow \boldsymbol{CP}^2 - C$ induces an isomorphism*

$$\pi_1(L_{\tau_0} - C, b_0)/N \xrightarrow{\sim} \pi_1(\boldsymbol{CP}^2 - C, b_0).$$

(ii) *Similarly, the inclusion map $L^a_{\tau_0} - C \hookrightarrow \boldsymbol{C}^2 - C$ induces an isomorphism*

$$\pi_1(L^a_{\tau_0} - C, b_0)/N^a \xrightarrow{\sim} \pi_1(\boldsymbol{C}^2 - C, b_0).$$

Originally conjectured by Zariski [**Z1**], this theorem was proved by van Kampen [**vK**]. For a modern and complete proof, see Chéniot [**C**].

The relation between $\pi_1(\boldsymbol{CP}^2 - C, b_0)$ and $\pi_1(\boldsymbol{C}^2 - C, b_0)$ is described by the following result.

PROPOSITION 1.2 (cf. [**O1**], [**O2**]). (i) *Let $\imath : \boldsymbol{C}^2 - C \hookrightarrow \boldsymbol{CP}^2 - C$ be the inclusion map. We have the following central extension*:

$$1 \to \boldsymbol{Z} \to \pi_1(\boldsymbol{C}^2 - C, b_0) \xrightarrow{\imath_\sharp} \pi_1(\boldsymbol{CP}^2 - C, b_0) \to 1,$$

where, of course, $\imath_\sharp$ is induced by $\imath$. The generator for $\ker \imath_\sharp$ *is represented by a lasso for L_{τ_∞}.*

(ii) *The homomorphism $\imath_\sharp$ induces an isomorphism*

$$\mathscr{D}\big(\pi_1(\boldsymbol{C}^2 - C, b_0)\big) \xrightarrow{\sim} \mathscr{D}\big(\pi_1(\boldsymbol{CP}^2 - C, b_0)\big)$$

between the commutator subgroups $\mathscr{D}\big(\pi_1(\boldsymbol{C}^2 - C, b_0)\big)$ and $\mathscr{D}\big(\pi_1(\boldsymbol{CP}^2 - C, b_0)\big)$ of $\pi_1(\boldsymbol{C}^2 - C, b_0)$ and $\pi_1(\boldsymbol{CP}^2 - C, b_0)$ respectively.

We recall that a lasso is defined as follows. Let $\mathfrak{C} \subset \boldsymbol{CP}^2$ be a reduced curve and let $(\mathfrak{C}_i)_i$ be the irreducible components of $\mathfrak{C}$. An element $\zeta \in \pi_1(\boldsymbol{CP}^2 - \mathfrak{C}, *)$ is called a *lasso* oriented counter-clockwise for $\mathfrak{C}_i$ if it is represented by a loop written as $\rho\,\omega\rho^{-1}$, where ω is a loop running once counter-clockwise around the boundary circle of a small closed *normal* disk Δ of $\mathfrak{C}$ at a simple point such that Δ does not intersect with $\mathfrak{C}_j$ for $j \neq i$, and where ρ is a simple path connecting the base point $*$ and the loop ω such that $\operatorname{im}\rho \cap \Delta$ is reduced to a single point (cf. [**O4**]).

Of course, Proposition 1.2 implies that $\pi_1(\boldsymbol{C}^2 - C, b_0)$ is abelian if and only if $\pi_1(\boldsymbol{CP}^2 - C, b_0)$ is abelian. Moreover, if C is irreducible and if the fundamental groups $\pi_1(\boldsymbol{C}^2 - C, b_0)$ and $\pi_1(\boldsymbol{CP}^2 - C, b_0)$ are abelian, then we have the following isomorphisms (cf. [**O8**, Section 2.3]):

$$\pi_1(\boldsymbol{CP}^2 - C, b_0) \simeq \boldsymbol{Z}/d\boldsymbol{Z} \quad \text{and} \quad \pi_1(\boldsymbol{C}^2 - C, b_0) \simeq \boldsymbol{Z}.$$

NOTATION 1.3. (i) For our purpose, we shall use only the pencils $\mathscr{L}_{X,Z}$ and $\mathscr{L}_{Y,Z}$ generated by l_X, l_Z and l_Y, l_Z respectively, where

$$l_X(X,Y,Z) = X, \quad l_Y(X,Y,Z) = Y, \quad l_Z(X,Y,Z) = Z.$$

In these two special cases, L_{τ_∞} is just the line at infinity $L_\infty := \{(X:Y:Z) \in \boldsymbol{CP}^2 \mid Z = 0\}$ of $\boldsymbol{CP}^2$. Let $x := X/Z$ and $y := Y/Z$ be the affine coordinates on $\boldsymbol{C}^2 = \boldsymbol{CP}^2 - L_\infty$. Observe that, in $\boldsymbol{C}^2$, the pencils $\mathscr{L}_{X,Z}$ and $\mathscr{L}_{Y,Z}$ are given by $\{x = \eta\}_{\eta \in \boldsymbol{C}}$ and $\{y = \eta\}_{\eta \in \boldsymbol{C}}$ respectively. For any $\tau = (S:T) \in \boldsymbol{CP}^1 - \tau_\infty \simeq \boldsymbol{C}$, we shall also denote the line L_τ by L_η where $\eta = S/T$. Observe that, in $\boldsymbol{C}^2$, the line L_η is given by $x = \eta$ for the pencil $\mathscr{L}_{X,Z}$ and by $y = \eta$ for the pencil $\mathscr{L}_{Y,Z}$.

(ii) Hereafter, we shall consider the affine equation of C, that is the equation $f(x,y) = 0$ where $f(x,y) := F(x,y,1)$.

(iii) Everywhere, we shall always assume that ε is a sufficiently small strictly positive number.

(iv) In the figures, for simplicity of drawing pictures, we shall denote a lasso oriented counter-clockwise just by a path ending with a bullet ——• as in [**O5**], [**O6**] and [**OP**] (but of course this is a loop!).

2. Statements of the main results.

For each integer i, $1 \leq i \leq 7$, we consider the irreducible sextic C_i defined by the affine equation $f_i(x,y) = 0$, where

$$\begin{aligned} f_1(x,y) := {} & (1/4)x^6 + (3/2)x^5y + (26685/512)y^5x + (87/32)x^4y^2 + x^3y^3 + (589/1024)y^6 \\ & - (1/2)x^5 - (1667/32)y^5 - (79/32)x^3y^2 - (7743/1024)x^2y^4 - (25/16)x^2y^3 \\ & - 2x^4y + (13/2)xy^4 + (1/4)x^4 + (17/16)y^4 - (7/16)xy^3 - (9/4)x^2y^2 \\ & - (1/2)x^3y + x^2y + xy^2 + y^3 + y^2, \end{aligned}$$

$$\begin{aligned} f_2(x,y) :=\, & 360x^6 + (419/144)y^6 - 120x^5y + (295/216)y^5x + 25x^4y^2 - (1535/144)y^4x^2 \\ & + (373/6)x^3y^3 + 32x^5 + (7/4)y^5 + (373/3)x^4y + (145/12)y^4x - (59/36)x^3y^2 \\ & + (133/54)x^2y^3 + (1417/36)x^4 + (1/4)y^4 - (29/54)x^3y + 7xy^3 + (161/12)x^2y^2 \\ & + (16/9)x^3 + 7x^2y + xy^2 + x^2, \end{aligned}$$

$$\begin{aligned} f_3(x,y) :=\, & (-(9/8)x - 1)\,y^5 + \big(-(13/48)x^2 + (27/8)x + 3\big)\,y^4 \\ & + \big(-(83/32)x^3 - (35/24)x^2 - (27/8)x - 3\big)\,y^3 \\ & + \big((271/576)x^4 + (187/32)x^3 + (179/48)x^2 + (9/8)x + 1\big)\,y^2 \\ & + \big(-(61/48)x^5 - (17/12)x^4 - (13/4)x^3 - 2x^2\big)\,y \\ & + (15/16)x^6 + (17/8)x^5 + x^4, \end{aligned}$$

$$\begin{aligned} f_4(x,y) :=\, & \frac{13149}{141376}y^4 - \frac{10177}{6903125}x^5 + \frac{1}{625}x^4 - \frac{89779}{22090}y^4x - \frac{136993}{141376}y^6 + \frac{269603}{141376}y^5 \\ & + \frac{13885}{8836}y^5x - \frac{122147}{3534400}y^4x^2 - \frac{287135}{141376}y^3 + \frac{127723}{1767200}y^3x^2 + \frac{150841}{44180}y^3x \\ & + \frac{5207}{110450}y^3x^3 + y^2 + \frac{296909}{88360000}y^2x^4 + \frac{153509}{3534400}y^2x^2 - \frac{10177}{11045}y^2x - \frac{78261}{552250}y^2x^3 \\ & - \frac{11117}{88360000}x^4y + \frac{20354}{276125}x^3y + \frac{5681}{27612500}x^5y + \frac{144743}{2209000000}x^6 - \frac{2}{25}x^2y, \end{aligned}$$

$$\begin{aligned} f_5(x,y) :=\, & y^6 - 3y^5 + 3y^4x^2 + 2y^4x + 4y^4 - 2y^3x^3 - 13y^3x^2 - 6y^3x - 3y^3 + 9y^2x^4 + 12y^2x^3 \\ & + 13y^2x^2 + 4y^2x + y^2 - 6yx^5 - 17yx^4 - 8yx^3 - 2yx^2 + 7x^6 + 4x^5 + x^4, \end{aligned}$$

$$\begin{aligned} f_6(x,y) :=\, & (5/16)y^6 - (23/8)y^5x + (23/8)y^5 - (5/16)y^4x^2 + (31/8)y^4x - (123/16)y^4 \\ & + (15/8)y^3x^3 + (31/8)y^3x^2 - y^3x + (11/2)y^3 - (51/16)y^2x^4 - (13/4)y^2x^3 \\ & - (13/4)y^2x^2 - y^2 + (13/4)yx^5 + 2yx^3 - x^6, \end{aligned}$$

$$\begin{aligned} f_7(x,y) :=\, & (3/2)y^6 - (7/3)y^5x - 3y^5 - (71/18)y^4x^2 + 8y^4x + (1/2)y^4 + (76/9)y^3x^3 \\ & + (13/3)y^3x^2 - (29/3)y^3x + 2y^3 - 10y^2x^4 - (14/3)y^2x^3 - (1/6)y^2x^2 \\ & + 4y^2x - y^2 + (46/9)yx^5 + (16/3)yx^4 - (8/3)yx^3 - (16/9)x^6. \end{aligned}$$

For each i, the curve C_i is not of torus type. Let us prove this fact for example for the curve C_1. If C_1 was of torus type, then there would exist a conic D_1 meeting C_1 only at $(0,0)$ and $(1,0)$ (the two singular points of C_1) and such that $I(C_1,D_1;(0,0)) = I(C_1,D_1;(1,0)) = 6$ (cf. **[P]**), where $I(C_1,D_1;(0,0))$ and $I(C_1,D_1;(1,0))$ are the intersection multiplicity of C_1 with D_1 at $(0,0)$ and $(1,0)$ respectively; but we can easily check that there does not exist such a conic D_1. A similar argument can be used for the other curves $C_2,\dots,C_7$; the details are left to the reader.

For each i, we denote by Ξ_i the configuration of singularities of the curve C_i. We have:

$\Xi_1 = \{2A_8\}$; $\Xi_2 = \{A_{17}\}$;
$\Xi_3 = \{A_{11}+E_6\}$; $\Xi_4 = \{A_{14}+A_2\}$; $\Xi_5 = \{A_{11}+A_5\}$;
$\Xi_6 = \{A_8+A_5+A_2\}$; $\Xi_7 = \{A_8+E_6+A_2\}$.

The examples C_6 and C_7 are due to Tu Chanh Nguyen.
Our main result is as follows.

THEOREM 2.1. *For each i, $1 \leq i \leq 7$, we have the following isomorphisms*:

$$\pi_1(\boldsymbol{CP}^2 - C_i) \simeq \boldsymbol{Z}/6\boldsymbol{Z} \quad \textit{and} \quad \pi_1(\boldsymbol{C}^2 - C_i) \simeq \boldsymbol{Z}.$$

For each i, let $\mathscr{M}(\Xi_i)$ be the moduli space of reduced sextics in $\boldsymbol{CP}^2$ with the configuration of singularities Ξ_i, and let $\mathscr{M}_0(\Xi_i)$ be the connected component of $\mathscr{M}(\Xi_i)$ containing the curve C_i. Since the topology of the complements $\boldsymbol{CP}^2 - \mathfrak{C}_i$ or $\boldsymbol{C}^2 - \mathfrak{C}_i$ is independent on the choice of the curve $\mathfrak{C}_i$ in $\mathscr{M}_0(\Xi_i)$ (cf. [**Z3**], [**Z4**] and [**LR**]), Theorem 2.1 implies the following result.

COROLLARY 2.2. *For each i, $1 \leq i \leq 7$, and for any curve $\mathfrak{C}_i$ in $\mathscr{M}_0(\Xi_i)$, we have the following isomorphisms*:

$$\pi_1(\boldsymbol{CP}^2 - \mathfrak{C}_i) \simeq \boldsymbol{Z}/6\boldsymbol{Z} \quad \textit{and} \quad \pi_1(\boldsymbol{C}^2 - \mathfrak{C}_i) \simeq \boldsymbol{Z}.$$

REMARKS. (i) Let $\mathscr{M}_{00}(\Xi_1)$ be the set of non-torus irreducible curves $\mathfrak{C}_1$ in $\mathscr{M}(\Xi_1)$ such that, for at least one of the two singular points of $\mathfrak{C}_1$, the tangent cone to $\mathfrak{C}_1$ at this point passes through the second singularity. One can prove that $\mathscr{M}_{00}(\Xi_1)$ is a connected subspace of $\mathscr{M}(\Xi_1)$ (the proof is computational, very heavy, and cannot be presented here). On the other hand, it is easy to see that our curve C_1 belongs to this subspace. So, by Corollary 2.2, for any curve $\mathfrak{C}_1$ in $\mathscr{M}_{00}(\Xi_1)$, we have $\pi_1(\boldsymbol{CP}^2 - \mathfrak{C}_1) \simeq \boldsymbol{Z}/6\boldsymbol{Z}$ and $\pi_1(\boldsymbol{C}^2 - \mathfrak{C}_1) \simeq \boldsymbol{Z}$.

(ii) By [**INO**], the subset of $\mathscr{M}(\Xi_2)$ consisting of irreducible sextics which are not of torus type is a connected component of $\mathscr{M}(\Xi_2)$. Of course, this component is nothing but $\mathscr{M}_0(\Xi_2)$. So, Corollary 2.2 asserts, in particular, that for any irreducible non-torus sextic $\mathfrak{C}_2 \subset \boldsymbol{CP}^2$ with the configuration of singularities Ξ_2, we have $\pi_1(\boldsymbol{CP}^2 - \mathfrak{C}_2) \simeq \boldsymbol{Z}/6\boldsymbol{Z}$ and $\pi_1(\boldsymbol{C}^2 - \mathfrak{C}_2) \simeq \boldsymbol{Z}$.

It seems that for the other values of i (i.e., $i = 1,3,4,5,6,7$) the subset of $\mathscr{M}(\Xi_i)$ consisting of non-torus irreducible sextics is also a connected component of $\mathscr{M}(\Xi_i)$ (the proof would be computational and very heavy). If yes, then Corollary 2.2 would also provide a complete answer to point (ii) of Conjecture 0.1 for the configurations of singularities Ξ_i, $i = 1,3,4,5,6,7$.

(iii) Another step toward (ii) of Conjecture 0.1 is [**O3**, Theorem 5.8]. This theorem contains an example of a non-torus irreducible sextic $\mathfrak{C} \subset \boldsymbol{CP}^2$ with the configuration of singularities $\{6A_2\}$ such that $\pi_1(\boldsymbol{CP}^2 - \mathfrak{C})$ and $\pi_1(\boldsymbol{C}^2 - \mathfrak{C})$ are also abelian.

(iv) By [**OP**], for each i, $1 \leq i \leq 7$, and any irreducible *torus* sextic $\mathfrak{D}_i$ in $\mathscr{M}(\Xi_i)$, the fundamental group $\pi_1(\boldsymbol{CP}^2 - \mathfrak{D}_i)$ is isomorphic to the free product $(\boldsymbol{Z}/2\boldsymbol{Z}) * (\boldsymbol{Z}/3\boldsymbol{Z})$. So, if $\mathfrak{D}_i$ is such a curve and if $\mathfrak{C}_i$ is an element of $\mathscr{M}_0(\Xi_i)$, then $(\mathfrak{C}_i, \mathfrak{D}_i)$ is a Zariski pair[4].

Notice that the Zariski pairs found here were in fact already known. Indeed, it is well-known that the generic Alexander polynomial of any irreducible non-torus sextic $\mathfrak{C}_i \subset \boldsymbol{CP}^2$ with the

[4]We recall that a pair of irreducible curves $(\mathfrak{C},\mathfrak{D})$ in $\boldsymbol{CP}^2$ is called a *Zariski pair* if $\mathfrak{C}$ and $\mathfrak{D}$ have the same degree and if there exist regular neighbourhoods $T(\mathfrak{C})$ and $T(\mathfrak{D})$ of $\mathfrak{C}$ and $\mathfrak{D}$, respectively, such that the pairs $(T(\mathfrak{C}),\mathfrak{C})$ and $(T(\mathfrak{D}),\mathfrak{D})$ are homeomorphic, while the pairs $(\boldsymbol{CP}^2,\mathfrak{C})$ and $(\boldsymbol{CP}^2,\mathfrak{D})$ are not homeomorphic (cf. [**A**]).

configuration Ξ_i is trivial (cf. [**O7**]), while the generic Alexander polynomial of any irreducible torus sextic $\mathfrak{D}_i \subset \boldsymbol{CP}^2$ with the configuration Ξ_i is given by $\Delta(t) = t^2 - t + 1$ (cf. [**OP**] and [**O7**]). This directly implies that $(\mathfrak{C}_i, \mathfrak{D}_i)$ is a Zariski pair.

The remaining of the paper concerns the proof of Theorem 2.1. We prove successively that $\pi_1(\boldsymbol{CP}^2 - C_i)$ is abelian for $i = 1,2,3,4,5$. The proofs for $i = 6,7$ are essentially the same than for $1 \leq i \leq 5$ and will thus be omitted.

3. Proof of Theorem 2.1 for $i = 1$.

The curve C_1 has exactly two singularities of type A_8: one at the origin and one at $(1,0)$. Figure 1 shows the real plane section of C_1 (in the figures, we do not respect the numerical scale).

We use the Zariski-van Kampen pencils method. Consider the pencil $\mathscr{L}_{Y,Z}$ (cf. Notation 1.3); observe that the point B_0 (i.e., the axis of the pencil) does not belong to C_1 and that the line at infinity L_∞ is generic with respect to C_1. As explained in Section 1, it suffices to prove that the fundamental group $\pi_1(\boldsymbol{CP}^2 - C_1, b_0)$ is abelian. The pencil has 5 real singular lines $L_{\eta_1}, \ldots, L_{\eta_5}$, with respect to C_1, which correspond to the 5 real roots $\eta_1, \ldots, \eta_5$ of the discriminant $\Delta_x(f_1)$ of f_1 as a polynomial in x ($\Delta_x(f_1)$ is thus a polynomial in y):

$$\eta_1 = -0.022..., \ \eta_2 = 0, \ \eta_3 = 0.253..., \ \eta_4 = 0.326..., \ \eta_5 = 0.414...$$

We take generators $\xi_1, \ldots, \xi_6$ of the fundamental group $\pi_1(L_{\eta_5 - \varepsilon} - C_1, b_0)$ (which are also generators of $\pi_1(\boldsymbol{CP}^2 - C_1, b_0)$) as in Figure 2; $\xi_1, \ldots, \xi_6$ are lassos around the intersection points of $L_{\eta_5 - \varepsilon}$ with C_1.

The line L_{η_5} is tangent to the curve C_1 at the simple point p_0 (cf. Figure 1); the intersection multiplicity $I(L_{\eta_5}, C_1; p_0)$ of L_{η_5} with C_1 at p_0 is 2. So, by the implicit functions theorem, the germs (C_1, p_0) and $(\{y = -x^2\}, O)$ are topologically equivalent. The monodromy relations around L_{η_5} (obtained by moving y once counter-clockwise on the circle $|y - \eta_5| = \varepsilon$) thus give the relation

$$\xi_2 = \xi_3.$$

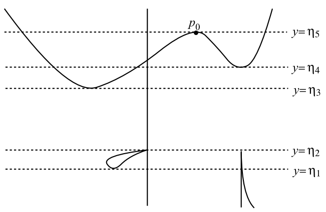

Figure 1. real plane section of C_1.

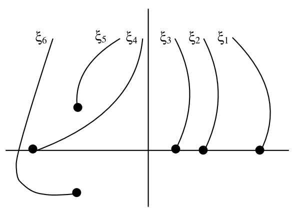

Figure 2. generators at $y = \eta_5 - \varepsilon$.

Similarly, we can see easily that the monodromy relations around L_{η_4} (obtained when y moves on the real axis from $y := \eta_5 - \varepsilon \longrightarrow \eta_4 + \varepsilon$, then runs once counter-clockwise on the circle $|y - \eta_4| = \varepsilon$, and then comes back on the real axis from $y := \eta_4 + \varepsilon \longrightarrow \eta_5 - \varepsilon$) give the relation

$$\xi_1 = \xi_2.$$

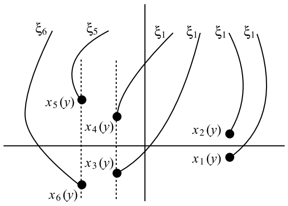

Figure 3. generators at $y = \eta_3 - \varepsilon$.

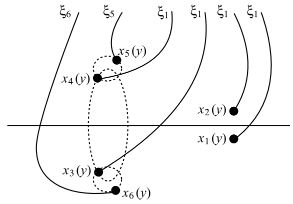

Figure 4. generators at $y = \eta_2 + \varepsilon$.

Similarly, the monodromy relations around L_{η_3} (obtained by moving y as follows: on the real axis from $y := \eta_5 - \varepsilon \longrightarrow \eta_4 + \varepsilon$; half-turn counter-clockwise on the circle $|y - \eta_4| = \varepsilon$; on the real axis from $y := \eta_4 - \varepsilon \longrightarrow \eta_3 + \varepsilon$; one turn counter-clockwise on the circle $|y - \eta_3| = \varepsilon$; on the real axis from $y := \eta_3 + \varepsilon \longrightarrow \eta_4 - \varepsilon$; half-turn clockwise on the circle $|y - \eta_4| = \varepsilon$; on the real axis from $y := \eta_4 + \varepsilon \longrightarrow \eta_5 - \varepsilon$) give the relation

$$\xi_3 = \xi_4.$$

To read the monodromy relations around L_{η_2}, we first show how the six roots $x_1(y), \dots, x_6(y)$ of the equation $f_1(x,y) = 0$ in x move when y moves on the real axis from $y := \eta_3 - \varepsilon \longrightarrow \eta_2 + \varepsilon$. Figure 3 shows the situation of the generators at $y = \eta_3 - \varepsilon$, and we have the following lemma.

LEMMA 3.1. *When y moves on the real axis from $\eta_3 - \varepsilon$ to $\eta_2 + \varepsilon$, the six roots $x_1(y), \dots, x_6(y)$ of the equation $f_1(x,y) = 0$ in x are deformed as in Figure* 4.

PROOF. We consider the polynomial

$$h(u,v,y) := f_1(u + iv, y)$$

for u, v, y real. We denote by $f_{1e}(u,v,y)$ and $f_{1o}(u,v,y)$ the real and the imaginary part of $h(u,v,y)$ respectively. They have degree 6 and 5 respectively in v. Suppose that there exists an $y_0 \in [\eta_2 + \varepsilon, \eta_3 - \varepsilon]$ such that four complex solutions of the equation (in x) $f_1(x, y_0) = 0$ are on a same vertical line $u = u_0$ in the complex plane $(\boldsymbol{C}, x = u + iv)$; in other words, assume that there are integers $1 \leq i_1 < i_2 < i_3 < i_4 \leq 6$ such that

$$\Re(x_{i_1}(y_0)) = \Re(x_{i_2}(y_0)) = \Re(x_{i_3}(y_0)) = \Re(x_{i_4}(y_0)) = u_0,$$

where of course $\Re(\cdot)$ is a notation for the real part. This implies that the equations (in v)

$$f_{1e}(u_0, v, y_0) = f_{1o}(u_0, v, y_0) = 0$$

have four common real solutions v_1, v_2, v_3, v_4. These solutions are not 0 since the equation (in y) $\Delta_x(f_1)(y) = 0$ has no solution on $[\eta_2 + \varepsilon, \eta_3 - \varepsilon]$. Thus, the equations (in v)

$$f_{1e}(u_0, v, y_0) = f_{1oo}(u_0, v, y_0) = 0,$$

where $f_{1oo}(u,v,y) := f_{1o}(u,v,y)/v$ (notice that v divides $f_{1o}(u,v,y)$, and thus $f_{1oo}(u,v,y)$ is a polynomial), have also v_1, v_2, v_3, v_4 as common solutions. As f_{1oo} has degree 4 in v, this implies that $f_{1oo}(u_0, v, y_0)$ divides $f_{1e}(u_0, v, y_0)$. Thus, the remainder $R(u,v,y)$ of f_{1e} by f_{1oo}, as a polynomial of v, must be identically 0 for $u = u_0$ and $y = y_0$ (of course, R is written as $R = R'/R''$, where R' is a polynomial in u, v, y, while R'' is a polynomial just depending on u and y). By an easy computation, we see that $R = (R_2'/R_2'')v^2 + (R_0'/R_0'')$, where R_2', R_2'', R_0' and R_0'' are polynomials in u and y. Thus, (u_0, y_0) is a common real solution of the equations

$$R_2'(u,y) = R_0'(u,y) = 0. \tag{3.2}$$

This implies that y_0 is a root of the resultant $\mathrm{Res}(y)$ of the polynomials $u \mapsto R_2'(u,y)$ and $u \mapsto R_0'(u,y)$. Note that the condition $\mathrm{Res}(y_0) = 0$ is necessary to have a real partner u_0 such that $R_2'(u_0,y_0) = R_0'(u_0,y_0) = 0$, but it is not sufficient since the possible partner u_0 might be not real. There are two real solutions y_0^1, y_0^2 of the equation $\mathrm{Res}(y) = 0$ on the interval $[\eta_2+\varepsilon, \eta_3-\varepsilon]$. Each of them gives a real number, say u_0^1 for y_0^1 and u_0^2 for y_0^2, such that (u_0^1, y_0^1) and (u_0^2, y_0^2) are two solutions of (3.2). We now have to check if these two solutions give four real roots v of the polynomial $v \mapsto f_{1oo}(u_0, v, y_0)$. Only the solution $(u_0, y_0) := (-0.18914..., 0.12557...)$ satisfies this requirement. Thus, we can have one (and only one) overcrossing. To check if it is the case, we look at the solutions x of the equation (in x) $f_1(x,y) = 0$ for some values of y near y_0. MAPLE actually gives an overcrossing. This completes the proof of Lemma 3.1. $\square$

Now, we look at the Puiseux parametrization of the curve at the origin (for details, see [**OP**, Section 2.2]):

$$\begin{cases} y = t^4 \\ x = i\sqrt{2}t^2 - \dfrac{3}{2}t^4 - \dfrac{5}{16}i\sqrt{2}t^6 - \dfrac{1}{8}\sqrt{210}\sqrt{i\sqrt{2}}t^7 + \text{higher terms.} \end{cases}$$

As explained in [**OP**, Section 4.1], when $y = \varepsilon\exp(i\theta)$ moves around the origin $\eta_2 = 0$ once counter-clockwise, the topological behavior of the four points $x_3(y)$, $x_4(y)$, $x_5(y)$, $x_6(y)$ looks like the movement of four satellites accompanying two planets, two satellites around each planet corresponding to $t = \varepsilon^{1/4}\exp(i\nu)$, $\nu = \theta/4,\ \theta/4+\pi/2,\ \theta/4+\pi,\ \theta/4+(3\pi)/2$. The movement of the planets is described by the term $i\sqrt{2}t^2$; each of them do $(1/2)$-turn around the sun ($\approx$ the origin). The movement of each satellite around its planet is described by the term $-(1/8)\sqrt{210}\sqrt{i\sqrt{2}}t^7$; each of them does $(7/4)$-turns around its planet. So, the monodromy relations around L_{η_2} give the relation

$$\xi_6 = (\omega\sigma)\xi_1(\omega\sigma)^{-1}, \tag{3.3}$$

where $\omega := \xi_6\xi_5\xi_1^2$ and $\sigma := \xi_5\xi_1\xi_5$.

On the other hand, we can see easily that the monodromy relations around L_{η_1} give the relation

$$\xi_1 = \xi_5.$$

The latter implies $\omega = \xi_6\xi_1^3$ and $\sigma = \xi_1^3$, and (3.3) then gives $\xi_6 = \xi_1$. So, the fundamental group $\pi_1(\boldsymbol{CP}^2 - C_1, b_0)$ is generated by a single generator, and it is thus abelian.

4. Proof of Theorem 2.1 for $i = 2$.

The curve C_2 has exactly one singularity of type A_{17} at the origin. Figure 5 shows the real plane section of C_2.

We consider the pencil $\mathscr{L}_{X,Z}$ (cf. Notation 1.3); observe that the point B_0 does not belong to C_2 and that L_∞ is generic with respect to C_2. Again, it suffices to prove that the fundamental group

$\pi_1(\boldsymbol{CP}^2 - C_2, b_0)$ is abelian. The pencil has 5 real singular lines $L_{\eta_1}, \ldots, L_{\eta_5}$, with respect to C_2, which correspond to the 5 real roots $\eta_1, \ldots, \eta_5$ of the discriminant $\Delta_y(f_2)$ of f_2 as a polynomial in y ($\Delta_y(f_2)$ is thus a polynomial in x):

$$\eta_1 = -0.191..., \ \eta_2 = -0.036..., \ \eta_3 = -0.027..., \ \eta_4 = -0.026..., \ \eta_5 = 0.$$

We take generators $\xi_1, \ldots, \xi_6$ of the fundamental group $\pi_1(L_{\eta_3-\varepsilon} - C_2, b_0)$ as in Figure 6; $\xi_1, \ldots, \xi_6$ are lassos around the intersection points of $L_{\eta_3-\varepsilon}$ with C_2.

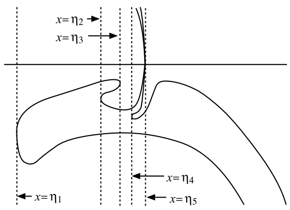

Figure 5. real plane section of C_2.

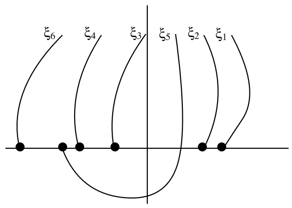

Figure 6. generators at $x = \eta_3 - \varepsilon$.

The monodromy relations around L_{η_3} and around L_{η_2} give the relations

$$\xi_3 = \xi_4 \quad \text{and} \quad \xi_5 = \xi_3^{-1}\xi_4\xi_3$$

respectively.

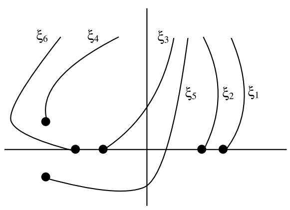

Figure 7. generators at $x = \eta_1 + \varepsilon$.

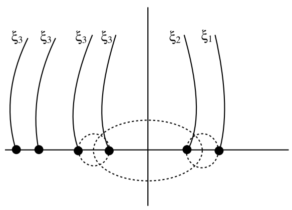

Figure 8. generators at $x = \eta_5 - \varepsilon$.

We show in Figure 7 how our generators at $x = \eta_2 + \varepsilon$ are deformed when x does half-turn counter-clockwise on the circle $|x - \eta_2| = \varepsilon$, and then moves on the real axis from $x := \eta_2 - \varepsilon \longrightarrow \eta_1 + \varepsilon$. The monodromy relations around L_{η_1} give the relation

$$\xi_3 = \xi_4^{-1}\xi_6\xi_4.$$

Combined with the foregoing, this shows that

$$\xi_3 = \xi_4 = \xi_5 = \xi_6.$$

To read the monodromy relations around L_{η_5}, we first show in Figure 8 how the generators at $x = \eta_3 - \varepsilon$ are deformed when x does half-turn counter-clockwise on the circle $|x - \eta_3| = \varepsilon$, then moves on the real axis from $x := \eta_3 + \varepsilon \longrightarrow \eta_4 - \varepsilon$, then does half-turn counter-clockwise on the circle $|x - \eta_4| = \varepsilon$, and finally moves on the real axis from $x := \eta_4 + \varepsilon \longrightarrow \eta_5 - \varepsilon$. Then we observe that, at the origin, the curve has two branches K_1 and K_2, given by

$$K_1: \ x = -\frac{1}{2}y^2 + \frac{3}{4}y^5 - \frac{1}{8}y^6 + \frac{7}{12}y^7 - \frac{1135}{288}y^8 + \frac{1}{1728}(4051 + 162\sqrt{22})y^9 + \text{higher terms},$$

$$K_2: \ x = -\frac{1}{2}y^2 + \frac{3}{4}y^5 - \frac{1}{8}y^6 + \frac{7}{12}y^7 - \frac{1135}{288}y^8 + \frac{1}{1728}(4051 - 162\sqrt{22})y^9 + \text{higher terms}.$$

An easy computation shows that the Puiseux parametrizations of K_1 and K_2 at the origin are given by

$$K_1: \quad x = t^2, \quad y = a_1 t + \ldots + a_7 t^7 + a_8 t^8 + \text{higher terms},$$

$$K_2: \quad x = t^2, \quad y = a'_1 t + \ldots + a'_7 t^7 + a'_8 t^8 + \text{higher terms},$$

for some complex numbers a_i and a'_i such that $a_i = a'_i$ for $1 \le i \le 7$, the number $a_1 = a'_1$ is non-zero, and $a_8 \neq a'_8$. These equations say us that the topological behavior of the four points which are closed to the origin $0 \in (\boldsymbol{C}, y)$ looks like the movement of four satellites accompanying two planets running around the sun ($\approx$ the origin), two satellites around each planet. Each planet does $(1/2)$-turn around the origin. Each satellite does 4-turns around its planet. So, the monodromy relations around L_{η_5} give the relations

$$\xi_1 = \xi_2 = \xi_3.$$

So, the fundamental group $\pi_1(\boldsymbol{CP}^2 - C_2, b_0)$ is generated by a single generator, and thus it is abelian.

5. Proof of Theorem 2.1 for $i = 3$.

The curve C_3 has exactly two singularities: one singularity of type A_{11} at the origin and one singularity of type E_6 at $(0,1)$. Figure 9 shows the real plane section of C_3.

We consider the pencil $\mathscr{L}_{Y,Z}$; observe that the point B_0 does not belong to C_3 and that L_∞ is generic with respect to C_3. Again, it suffices to prove that the fundamental group $\pi_1(\boldsymbol{CP}^2 - C_3, b_0)$ is abelian. The pencil has 5 real singular lines $L_{\eta_1}, \ldots, L_{\eta_5}$, with respect to C_3, which correspond to the 5 real roots $\eta_1, \ldots, \eta_5$ of the discriminant $\Delta_x(f_3)$ of f_3 as a polynomial in x:

$$\eta_1 = 0, \ \eta_2 = 0.297..., \ \eta_3 = 0.568..., \ \eta_4 = 1, \ \eta_5 = 1.001...$$

We take generators $\xi_1, \ldots, \xi_6$ of the fundamental group $\pi_1(L_{\eta_1+\varepsilon} - C_3, b_0)$ as in Figure 10; $\xi_1, \ldots, \xi_6$ are lassos around the intersection points of $L_{\eta_1+\varepsilon}$ with C_3.

To read the monodromy relations at the origin, we first observe that near $(0,0)$ the curve has two branches K_1 and K_2 given by

$$K_1: \quad y = x^2 + \frac{1}{2}x^3 + \frac{11}{12}x^4 + \frac{35}{24}x^5 + \frac{1}{144}(313 + 4i\sqrt{6})x^6 + \text{higher terms},$$

$$K_2: \quad y = x^2 + \frac{1}{2}x^3 + \frac{11}{12}x^4 + \frac{35}{24}x^5 + \frac{1}{144}(313 - 4i\sqrt{6})x^6 + \text{higher terms}.$$

An easy computation shows that the Puiseux parametrizations of K_1 and K_2 at the origin are given by

$$K_1: \quad y = t^2, \quad x = a_1 t + \ldots + a_4 t^4 + a_5 t^5 + \text{higher terms},$$

$$K_2: \quad y = t^2, \quad x = a'_1 t + \ldots + a'_4 t^4 + a'_5 t^5 + \text{higher terms},$$

for some complex numbers a_i and a'_i such that $a_i = a'_i$ for $1 \leq i \leq 4$, the number $a_1 = a'_1$ is non-zero, and $a_5 \neq a'_5$. As above, one deduces from these equations that the monodromy relations around L_{η_1} give the relations

$$\begin{aligned} \xi_1 &= (\sigma\xi_4)\xi_3(\sigma\xi_4)^{-1}, \\ \xi_2 &= \sigma^2\xi_4\sigma^{-2}, \end{aligned} \tag{5.1}$$

where $\sigma := \xi_4\xi_3$.

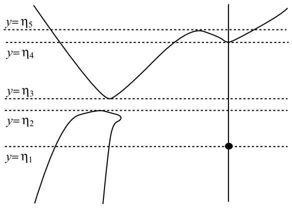

Figure 9. real plane section of C_3.

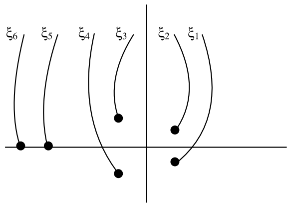

Figure 10. generators at $y = \eta_1 + \varepsilon$.

REMARK. After the analytic change of coordinates

$$(x,y) \mapsto \left(x, y + x^2 + \frac{1}{2}x^3 + \frac{11}{12}x^4 + \frac{35}{24}x^5\right),$$

the equation of C_3 near the origin takes the form

$$y^2 - \frac{313}{72}yx^6 + \frac{98065}{20736}x^{12} + \text{higher terms} = 0.$$

As the leading term $y^2 - (313/72)yx^6 + (98065/20736)x^{12}$ has no real factorization, the origin is an isolated point of the *real* plane section of C_3.

When y moves on the real axis from $y := \eta_1 + \varepsilon \longrightarrow \eta_2 - \varepsilon$, the situation of our generators at $y = \eta_2 - \varepsilon$ is again as in Figure 10. We see easily that the monodromy relations around L_{η_2} give the relation

$$\xi_5 = \xi_6.$$

To read the monodromy relations around L_{η_3}, we first show in Figure 11 how our generators at $y = \eta_2 - \varepsilon$ are deformed when y does half-turn counter-clockwise on the circle $|y - \eta_2| = \varepsilon$, then moves on the real axis from $y := \eta_2 + \varepsilon \longrightarrow \eta_3 - \varepsilon$. Then, it is easy to see that the monodromy relations around L_{η_3} give the relation

$$\xi_3 = \xi_4.$$

The latter, combined with (5.1), gives

$$\xi_1 = \xi_3 \quad \text{and} \quad \xi_2 = \xi_3.$$

To read the monodromy relations around L_{η_4}, we show in Figure 12 how our generators at $y = \eta_3 - \varepsilon$ are deformed when y does half-turn counter-clockwise on the circle $|y - \eta_3| = \varepsilon$, then moves on the real axis from $y := \eta_3 + \varepsilon \longrightarrow \eta_4 - \varepsilon$. Then we observe that, after the change of coordinates $(x,y) \mapsto (x, y+1)$, the Newton principal part of f_3 near $(0,1)$ (cf. [**K**]) is given by

$$-y^3 + \frac{31}{576}x^4.$$

We deduce that the monodromy relations around L_{η_4} give the relation

$$\xi_3 = \omega^{-2}\xi_5\omega^2, \tag{5.2}$$

where $\omega := \xi_5\xi_3\xi_5^{-1}$.

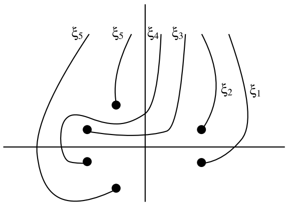

Figure 11. generators at $y = \eta_3 - \varepsilon$.

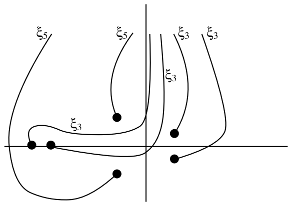

Figure 12. generators at $y = \eta_4 - \varepsilon$.

On the other hand, it is not difficult to see that the monodromy relations around L_{η_5} give the relation

$$\xi_3 = \omega.$$

The latter, combined with (5.2), implies

$$\xi_5 = \xi_3.$$

So, we have proved that the fundamental group $\pi_1(\boldsymbol{CP}^2 - C_3, b_0)$ is generated by a single generator. It is thus abelian.

6. Proof of Theorem 2.1 for $i = 4$.

The curve C_4 has exactly two singularities: one singularity of type A_{14} at the origin and one singularity of type A_2 at $(0,1)$. Figure 13 shows the real plane section of C_4.

We consider the pencil $\mathscr{L}_{X,Z}$; observe that the point B_0 does not belong to C_4 and that L_∞ is generic with respect to C_4. Again, it suffices to prove that the fundamental group $\pi_1(\boldsymbol{CP}^2 - C_4, b_0)$ is abelian. The pencil has 6 real singular lines $L_{\eta_1}, \ldots, L_{\eta_6}$, with respect to C_4, which correspond to the 6 real roots $\eta_1, \ldots, \eta_6$ of the discriminant $\Delta_y(f_4)$ of f_4 as a polynomial in y:

$$\eta_1 = -2.016..., \ \eta_2 = -1.973..., \ \eta_3 = -0.137..., \ \eta_4 = 0, \ \eta_5 = 0.050..., \ \eta_6 = 2.062...$$

We take generators $\xi_1, \ldots, \xi_6$ of the fundamental group $\pi_1(L_{\eta_3 - \varepsilon} - C_4, b_0)$ as in Figure 14; $\xi_1, \ldots, \xi_6$ are lassos around the intersection points of $L_{\eta_3-\varepsilon}$ with C_4.

The monodromy relations around L_{η_3} and around L_{η_2} give the relations

$$\xi_2 = \xi_3 \quad \text{and} \quad \xi_3 = \xi_4 \tag{6.1}$$

respectively.

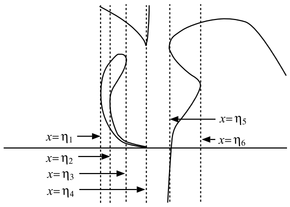

Figure 13. real plane section of C_4.

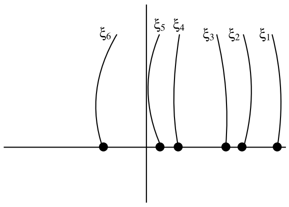

Figure 14. generators at $x=\eta_3-\varepsilon$.

In order to fix the ideas, we show in Figure 15 how our generators at $x=\eta_2+\varepsilon$ are deformed when x does half-turn counter-clockwise on the circle $|x-\eta_2|=\varepsilon$, and then moves on the real axis from $x:=\eta_2-\varepsilon \longrightarrow \eta_1+\varepsilon$. The monodromy relations around L_{η_1} give the relation

$$\xi_2=\xi_3^{-1}\xi_5\xi_3.$$

The latter, combined with (6.1), implies $\xi_5=\xi_3$. So, we already have

$$\xi_2=\xi_3=\xi_4=\xi_5.$$

We show in Figure 16 how our generators at $x=\eta_3-\varepsilon$ are deformed when x does half-turn counter-clockwise on the circle $|x-\eta_3|=\varepsilon$, then moves on the real axis from $x:=\eta_3+\varepsilon \longrightarrow \eta_4-\varepsilon$. On the other hand, after the change of coordinates $(x,y)\mapsto(x,y+1)$, we see that the Newton principal part of f_4 near $(0,1)$ (cf. [**K**]) is given by

$$-\frac{278369}{141376}y^3+\frac{507}{441800}x^2.$$

One deduces that the monodromy relations around L_{η_4} give the new relation

$$\xi_3=\xi_1.$$

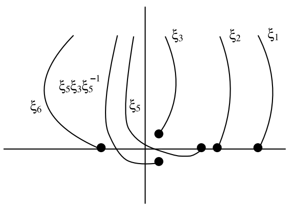

Figure 15. generators at $x=\eta_1+\varepsilon$.

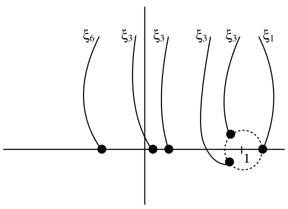

Figure 16. generators at $x=\eta_4-\varepsilon$.

Now, knowing that $\xi_1=\xi_2=\xi_3=\xi_4=\xi_5$, the big circle relation (i.e., the vanishing relation at infinity) obviously gives the new relation

$$\xi_6=\xi_1^{-5}.$$

So, the fundamental group $\pi_1(\boldsymbol{CP}^2-C_4,b_0)$ is generated by a single generator, and thus it is abelian.

7. Proof of Theorem 2.1 for $i = 5$.

The curve C_5 has exactly two singularities: one singularity of type A_{11} at the origin and one singularity of type A_5 at $(0,1)$. Figure 17 shows the real plane section of C_5.

We consider the pencil $\mathscr{L}_{Y,Z}$; observe that the point B_0 does not belong to C_5 and that L_∞ is generic with respect to C_5. Again, it suffices to prove that the fundamental group $\pi_1(\boldsymbol{CP}^2 - C_5, b_0)$ is abelian. The pencil has 6 real singular lines $L_{\eta_1}, \dots, L_{\eta_6}$, with respect to C_5, which correspond to the 6 real roots $\eta_1, \dots, \eta_6$ of the discriminant $\Delta_x(f_5)$ of f_5 as a polynomial in x:

$$\eta_1 = 0,\ \eta_2 = 0.847...,\ \eta_3 = 1,\ \eta_4 = 1.203...,\ \eta_5 = 1.286...,\ \eta_6 = 1.844...$$

We take generators $\xi_1, \dots, \xi_6$ of the fundamental group $\pi_1(L_{\eta_3+\varepsilon} - C_5, b_0)$ as in Figure 18; $\xi_1, \dots, \xi_6$ are lassos around the intersection points of $L_{\eta_3+\varepsilon}$ with C_5.

It is not difficult to see that the monodromy relations around L_{η_4}, L_{η_5} and L_{η_6} give the relations

$$\xi_3 = \xi_2, \quad \xi_1 = \xi_2^{-1}\xi_4\xi_2 \quad \text{and} \quad \xi_6 = \xi_5 \tag{7.1}$$

respectively.

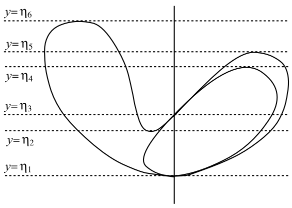

Figure 17. real plane section of C_5.

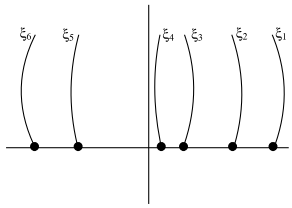

Figure 18. generators at $y = \eta_3 + \varepsilon$.

To read the monodromy relations around L_{η_3}, we first observe that near the point $(0,1)$ the curve has two branches K_1 and K_2 given by

$$K_1: \quad y = 1 + x - x^2 + (-1 + \sqrt{3})x^3 + \text{higher terms},$$
$$K_2: \quad y = 1 + x - x^2 + (-1 - \sqrt{3})x^3 + \text{higher terms}.$$

An easy computation shows that the Puiseux parametrizations of K_1 and K_2 near $(0,1)$ are given by

$$K_1: \quad y = 1 + t, \quad x = a_1 t + a_2 t^2 + a_3 t^3 + \text{higher terms},$$
$$K_2: \quad y = 1 + t, \quad x = a_1' t + a_2' t^2 + a_3' t^3 + \text{higher terms},$$

for some complex numbers a_i and a_i' such that $a_i = a_i'$ for $1 \le i \le 2$, the number $a_1 = a_1'$ is non-zero, and $a_3 \ne a_3'$. These equations show that the monodromy relations around L_{η_3} give the relation

$$\xi_3 = (\xi_4\xi_3)^2\xi_4\xi_3\xi_4^{-1}(\xi_4\xi_3)^{-2}.$$

To read the monodromy relations around L_{η_2}, we first show in Figure 19 how our generators at $y = \eta_3 + \varepsilon$ are deformed when y does half-turn counter-clockwise on the circle $|y - \eta_3| = \varepsilon$. Then we introduce, in the fibre $L_{\eta_3 - \varepsilon}$, the lassos μ and ν defined by

$$\mu := (\xi_4\xi_3)^{-1}\xi_3(\xi_4\xi_3),$$
$$\nu := (\xi_4\xi_3\mu)^{-1}\xi_4(\xi_4\xi_3\mu).$$

Lassos μ and ν are drawn in Figure 20. Owing to these new lassos, it is easy to see that the monodromy relations around L_{η_2} give the relation

$$\mu = \xi_5.$$

The latter, combined with (7.1), implies

$$\nu = \xi_5^{-1}\xi_1\xi_5. \tag{7.2}$$

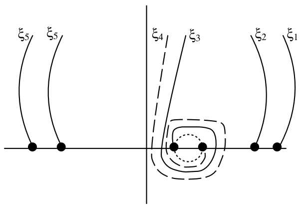

Figure 19. generators at $y = \eta_3 - \varepsilon$.

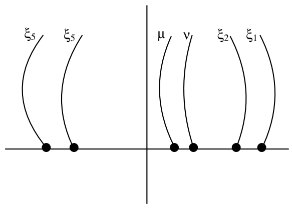

Figure 20. new generators at $y = \eta_3 - \varepsilon$.

To read the monodromy relations around L_{η_1}, we first show in Figure 21 how the generators at $y = \eta_2 + \varepsilon$ are deformed when y does half-turn counter-clockwise on the circle $|y - \eta_2| = \varepsilon$, then moves on the real axis from $y := \eta_2 - \varepsilon \longrightarrow \eta_1 + \varepsilon$. Then, we observe that at the origin the curve has two branches K_1' and K_2' given by

$$K_1' : \quad y = x^2 + \left(\frac{5}{2} + \frac{1}{2}\sqrt{21}\right) x^6 + \text{higher terms},$$

$$K_2' : \quad y = x^2 + \left(\frac{5}{2} - \frac{1}{2}\sqrt{21}\right) x^6 + \text{higher terms}.$$

An easy computation shows that the Puiseux parametrizations of K_1' and K_2' at the origin are given by

$$K_1' : \quad y = t^2, \quad x = a_1 t + \ldots + a_4 t^4 + a_5 t^5 + \text{higher terms},$$
$$K_2' : \quad y = t^2, \quad x = a_1' t + \ldots + a_4' t^4 + a_5' t^5 + \text{higher terms},$$

for some complex numbers a_i and a_i' such that $a_i = a_i'$ for $1 \le i \le 4$, the number $a_1 = a_1'$ is non-zero, and $a_5 \neq a_5'$. These equations show that the monodromy relations around L_{η_1} give the relation

$$\begin{aligned} \xi_1 &= (\xi_5\nu)^2\xi_5(\xi_5\nu)^{-2} \\ &= (\xi_1\xi_5)^2\xi_5(\xi_1\xi_5)^{-2} \quad \text{(by (7.2))}. \end{aligned} \tag{7.3}$$

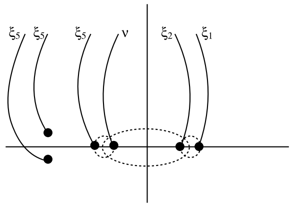

Figure 21. generators at $y = \eta_1 + \varepsilon$.

Now, we are ready to prove that $\pi_1(\boldsymbol{CP}^2 - C_5, b_0)$ is abelian. The big circle relation $\xi_6\xi_5\xi_4\xi_3\xi_2\xi_1 = 1$, combined with (7.1), gives

$$(\xi_2\xi_1)^2 = \xi_5^{-2}. \tag{7.4}$$

But, at $y = \eta_1 + \varepsilon$, the big circle relation is also written as $\xi_5^3\nu\xi_2\xi_1 = 1$. Combined with (7.2), this gives $\xi_1\xi_5 = \xi_5^{-2}(\xi_2\xi_1)^{-1}$, which in turn implies (using (7.4)) that $\xi_1\xi_5 = \xi_2\xi_1$. So, again using (7.4), one deduces that $(\xi_1\xi_5)^2 = \xi_5^{-2}$. The relation (7.3) then gives $\xi_1 = \xi_5$. The equality $\xi_1\xi_5 = \xi_2\xi_1$ thus implies $\xi_2 = \xi_1$, and using the second equality in (7.1) one deduces that $\xi_4 = \xi_1$.

So, we have proved that the fundamental group $\pi_1(\boldsymbol{CP}^2 - C_5, b_0)$ is generated by a single generator. It is thus abelian.

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