

Orbit closures for representations of Dynkin quivers are regular in codimension two

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Abstract. We develop reductions for classifications of singularities of orbit closures in module varieties. Then we show that the orbit closures for representations of Dynkin quivers are regular in codimension two.

1. Introduction and the main results.

Throughout the paper, k denotes an algebraically closed field, A denotes a finitely generated associative k -algebra with identity, and by a module we mean a left A -module whose underlying k -space is finite dimensional. Let d be a positive integer and denote by $M_d(k)$ the algebra of $d \times d$ -matrices with coefficients in k . For an algebra A , the set $\text{mod}_A(d)$ of algebra homomorphisms $A \rightarrow M_d(k)$ has a natural structure of an affine variety. Indeed, if $A \simeq k\langle X_1, \dots, X_t \rangle / I$ for some two-sided ideal I , then $\text{mod}_A(d)$ can be identified with the closed subset of $(M_d(k))^t$ given by vanishing of the entries of all matrices $\rho(X_1, \dots, X_t)$, $\rho \in I$. Moreover, the general linear group $\text{GL}(d)$ acts on $\text{mod}_A(d)$ by conjugations

$$g \star (M_1, \dots, M_t) = (gM_1g^{-1}, \dots, gM_tg^{-1}),$$

and the $\text{GL}(d)$ -orbits in $\text{mod}_A(d)$ correspond bijectively to the isomorphism classes of d -dimensional modules. We shall denote by \mathcal{O}_M the $\text{GL}(d)$ -orbit in $\text{mod}_A(d)$ corresponding to a d -dimensional module M . An interesting problem is to study geometric properties of the Zariski closure $\overline{\mathcal{O}}_M$ of an orbit \mathcal{O}_M in $\text{mod}_A(d)$. We refer to [2], [3], [4], [5], [6], [12], [15], [16] and [17] for some results in this direction.

Following Hesselink [10, (1.7)] we call two pointed varieties (\mathcal{X}, x_0) and (\mathcal{Y}, y_0) smoothly equivalent if there are smooth morphisms $f : \mathcal{Z} \rightarrow \mathcal{X}$, $g : \mathcal{Z} \rightarrow \mathcal{Y}$ and a point $z_0 \in \mathcal{Z}$ with $f(z_0) = x_0$ and $g(z_0) = y_0$. This is an equivalence relation and the equivalence classes will be denoted by $\text{Sing}(\mathcal{X}, x_0)$ and called the types of singularities. If $\text{Sing}(\mathcal{X}, x_0) = \text{Sing}(\mathcal{Y}, y_0)$ then the variety \mathcal{X} is regular (respectively, normal, Cohen-Macaulay) at x_0 if and only if the same is true for the variety \mathcal{Y} at y_0 (see [9, Section 17] for more information about smooth morphisms). Obviously the regular points of the varieties give one type of singularity, which we denote by Reg . Let M and N be d -dimensional modules with $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_M$, i.e., N is a degeneration of M . We shall write $\text{Sing}(M, N)$ for $\text{Sing}(\overline{\mathcal{O}}_M, n)$, where n is an arbitrary point of \mathcal{O}_N . It was shown recently [17, Theorem 1.1] that $\text{Sing}(M, N) = \text{Reg}$ provided $\dim \mathcal{O}_M - \dim \mathcal{O}_N = 1$. In this paper

we investigate $\text{Sing}(M, N)$ when $\dim \mathcal{O}_M - \dim \mathcal{O}_N = 2$. First we prove some auxiliary result.

THEOREM 1.1. *Let M', N' and X be modules such that $\mathcal{O}_{N' \oplus X} \subset \overline{\mathcal{O}}_{M' \oplus X}$ and $\dim \mathcal{O}_{M' \oplus X} - \dim \mathcal{O}_{N' \oplus X} = 2$. Then $\mathcal{O}_{N'} \subset \overline{\mathcal{O}}_{M'}$, and one of the following cases holds:*

- (1) $\dim \mathcal{O}_{M'} - \dim \mathcal{O}_{N'} = 1$ and $\text{Sing}(M' \oplus X, N' \oplus X) = \text{Reg}$;
- (2) $\dim \mathcal{O}_{M'} - \dim \mathcal{O}_{N'} = 2$ and $\text{Sing}(M' \oplus X, N' \oplus X) = \text{Sing}(M', N')$.

This allows to restrict our attention only to the case when the modules M and N have no nonzero direct summands in common. We shall say that such modules are disjoint. We denote by $s(L)$ the number of summands in a decomposition of a module L into a direct sum of indecomposable modules. The next result give us a further reduction for the problem of description of the type $\text{Sing}(M, N)$.

THEOREM 1.2. *Let M and N be disjoint modules such that $\mathcal{O}_N \subset \overline{\mathcal{O}}_M$ and $\dim \mathcal{O}_M - \dim \mathcal{O}_N = 2$. Then $\text{Sing}(M, N) = \text{Reg}$ if $s(N) \geq 3$.*

If $A = k[\varepsilon]/(\varepsilon^2)$, $M = {}_A A$ and N is a direct sum of two simple A -modules, then $s(N) = 2$, $\mathcal{O}_N \subset \overline{\mathcal{O}}_M$, $\dim \mathcal{O}_M - \dim \mathcal{O}_N = 2$ and

$$\text{Sing}(M, N) = \text{Sing} \left(\left\{ \begin{bmatrix} x & y \\ z & -x \end{bmatrix}; x^2 + yz = 0 \right\}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

is the type of Kleinian singularity \mathbf{A}_2 . Hence orbit closures in module varieties may be singular in codimension two even for very simple algebras. However this is not true for the modules over the path algebras of Dynkin quivers. We add that Theorems 1.1 and 1.2 are used in the proof of our main result stated below.

THEOREM 1.3. *Let M be a module over the path algebra of a Dynkin quiver. Then the variety $\overline{\mathcal{O}}_M$ is regular in codimension two.*

Let $Q = (Q_0, Q_1, s, e)$ be a finite quiver. Here Q_0 is a finite set of vertices, Q_1 is a finite set of arrows, and $s, e : Q_1 \rightarrow Q_0$ are functions such that any arrow $\alpha \in Q_1$ has the starting vertex $s(\alpha)$ and the ending vertex $e(\alpha)$. Let $\mathbf{d} = (d_i)_{i \in Q_0} \in \mathbf{N}^{Q_0}$. We define the vector space

$$\text{rep}_Q(\mathbf{d}) = \prod_{\alpha \in Q_1} M_{d_{e(\alpha)} \times d_{s(\alpha)}}(k),$$

where $M_{d' \times d''}(k)$ denotes the set of $d' \times d''$ -matrices with coefficients in k for any $d', d'' \in \mathbf{N}$. The product $\text{GL}(\mathbf{d}) = \prod_{i \in Q_0} \text{GL}(d_i)$ of general linear groups acts on $\text{rep}_Q(\mathbf{d})$ via

$$g \star V = (g_{e(\alpha)} V_\alpha g_{s(\alpha)}^{-1})_{\alpha \in Q_1},$$

for any $g = (g_i)_{i \in Q_0} \in \text{GL}(\mathbf{d})$ and $V = (V_\alpha)_{\alpha \in Q_1} \in \text{rep}_Q(\mathbf{d})$. Using an equivalence described by Bongartz in [4] we can reformulate Theorem 1.3 as follows.

COROLLARY 1.4. *Let Q be a Dynkin quiver and $\mathbf{d} \in \mathbf{N}^{Q_0}$. Then the closures of the $\mathrm{GL}(\mathbf{d})$ -orbits in $\mathrm{rep}_Q(\mathbf{d})$ are regular in codimension two.*

Let $Q : 1 \xrightarrow{\alpha} 2$ be a Dynkin quiver of type A_2 and $\mathbf{d} = (2, 2) \in \mathbf{N}^{Q_0}$. Then $\mathrm{rep}_Q(\mathbf{d}) = \mathbf{M}_{2 \times 2}(k)$ and the orbit closure

$$\overline{\mathrm{GL}(\mathbf{d}) \star \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} = \{ \begin{bmatrix} x & y \\ z & t \end{bmatrix}; xt - yz = 0 \}$$

is a singular variety of dimension three. This shows that “codimension two” in Corollary 1.4 (and in Theorem 1.3) cannot be improved by “codimension three”.

We shall consider in Section 2 some properties of short exact sequences, dimensions of homomorphism spaces and degenerations of modules. Section 3 contains some sufficient conditions on regularity of $\mathrm{Sing}(M, N)$. Sections 4, 5 and 6 are devoted to the proofs of Theorems 1.1, 1.2 and 1.3, respectively.

For basic background on the representation theory of algebras and quivers we refer to [1] and [11]. The author gratefully acknowledges support from the Polish Scientific Grant KBN No. 1 P03A 018 27.

2. Degenerations of modules.

Let $\mathrm{mod} A$ denote the category of finite dimensional left A -modules and $\mathrm{rad}(\mathrm{mod} A)$ denote the Jacobson radical of the category $\mathrm{mod} A$. We can describe $\mathrm{rad}(\mathrm{mod} A)$ as the two-sided ideal of $\mathrm{mod} A$ generated by nonisomorphisms between indecomposable modules. We abbreviate by $[X, Y]$ the dimension $\dim_k \mathrm{Hom}_A(X, Y)$ for any modules X and Y . Recall that by a module we mean an object of $\mathrm{mod} A$.

LEMMA 2.1. *Let M and N be modules with $\dim_k M = \dim_k N$. Then $\dim \mathcal{O}_M - \dim \mathcal{O}_N = [N, N] - [M, M]$.*

PROOF. Let L be a d -dimensional module and choose a point l in \mathcal{O}_L . Since the isotropy group of l can be identified with the group of A -automorphisms of L and the latter is a nonempty and open subset of the vector space $\mathrm{End}_A(L)$, then we conclude the formula

$$\dim \mathcal{O}_L = \dim \mathrm{GL}(d) - [L, L].$$

We get the claim by applying the formula for $L = M$ and $L = N$. □

We shall need the following three simple facts on short exact sequences.

LEMMA 2.2. *Let X be a module and $\sigma : 0 \rightarrow U \xrightarrow{f} W \xrightarrow{g} V \rightarrow 0$ be an exact sequence in $\mathrm{mod} A$. Then:*

- (1) $\delta_\sigma(X) := [U \oplus V, X] - [W, X] \geq 0$ and the equality holds if and only if any homomorphism in $\mathrm{Hom}_A(U, X)$ factors through f ;
- (2) $\delta'_\sigma(X) := [X, U \oplus V] - [X, W] \geq 0$ and the equality holds if and only if any homomorphism in $\mathrm{Hom}_A(X, V)$ factors through g .

PROOF. The claim follow from the induced exact sequences

$$\begin{aligned}
 0 \rightarrow \text{Hom}_A(V, X) &\xrightarrow{\text{Hom}_A(g, X)} \text{Hom}_A(W, X) \xrightarrow{\text{Hom}_A(f, X)} \text{Hom}_A(U, X), \\
 0 \rightarrow \text{Hom}_A(X, U) &\xrightarrow{\text{Hom}_A(X, f)} \text{Hom}_A(X, W) \xrightarrow{\text{Hom}_A(X, g)} \text{Hom}_A(X, V). \quad \square
 \end{aligned}$$

LEMMA 2.3. *Let $\sigma : 0 \rightarrow U \xrightarrow{f} W \xrightarrow{g} V \rightarrow 0$ be an exact sequence in $\text{mod } A$. Then the following conditions are equivalent.*

- (1) *The sequence σ splits.*
- (2) *$W \simeq U \oplus V$.*
- (3) *$\delta_\sigma(U) = 0$.*
- (4) *$\delta'_\sigma(V) = 0$.*

PROOF. Clearly the condition (1) implies (2), and the condition (2) implies (3) and (4). Applying Lemma 2.2 we get that (3) implies that the endomorphism 1_U factors through f , which means that f is a section and (1) holds. Similarly, it follows from (4) that g is a retraction and (1) holds. □

LEMMA 2.4. *Let*

$$0 \rightarrow U \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} W_1 \oplus W_2 \xrightarrow{\begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix}} V_1 \oplus V_2 \rightarrow 0$$

be an exact sequence in $\text{mod } A$ such that $g_{1,1}$ is an isomorphism. Then

$$0 \rightarrow U \xrightarrow{f_2} W_2 \xrightarrow{g'} V_2 \rightarrow 0.$$

is also an exact sequence in $\text{mod } A$, where $g' = g_{2,2} - g_{2,1}g_{1,1}^{-1}g_{1,2}$.

PROOF. Straightforward. □

The next result follows from [14, Theorem 1.1] and from Lemma 2.4 and its dual.

THEOREM 2.5. *Let M and N be modules. Then the inclusion $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_M$ is equivalent to each of the following conditions:*

- (1) *There is an exact sequence $0 \rightarrow Z \xrightarrow{f} Z \oplus M \xrightarrow{g} N \rightarrow 0$ in $\text{mod } A$ for some module Z .*
- (2) *There is an exact sequence $0 \rightarrow N \xrightarrow{f'} M \oplus Z' \xrightarrow{g'} Z' \rightarrow 0$ in $\text{mod } A$ for some module Z' .*

Moreover, we may assume that f and g' belong to $\text{rad}(\text{mod } A)$.

COROLLARY 2.6. *Let*

$$\sigma : 0 \rightarrow U \rightarrow M \rightarrow V \rightarrow 0$$

be an exact sequence in $\text{mod } A$. Then $\mathcal{O}_{U \oplus V} \subseteq \overline{\mathcal{O}}_M$.

PROOF. We apply Theorem 2.5 to a direct sum of σ and the exact sequence $0 \rightarrow 0 \rightarrow U \xrightarrow{1_U} U \rightarrow 0$. □

LEMMA 2.7. Let M and N be modules such that $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_M$. Then

$$\delta_{M,N}(X) := [N, X] - [M, X] \geq 0 \quad \text{and} \quad \delta'_{M,N}(X) := [X, N] - [X, M] \geq 0$$

for any module X .

PROOF. We get an exact sequence $\sigma : 0 \rightarrow Z \rightarrow Z \oplus M \rightarrow N \rightarrow 0$ in $\text{mod } A$, by Theorem 2.5. Then the claim follows from Lemma 2.2 and the equalities $\delta_{M,N}(X) = \delta_\sigma(X)$ and $\delta'_{M,N}(X) = \delta'_\sigma(X)$ for any module X . □

Let M and N be modules with $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_M$ and σ be a short exact sequence in $\text{mod } A$. We shall use frequently without referring the following obvious properties of the nonnegative integers $\delta(L)$:

- $\delta(X) = \delta(Y)$ if $X \simeq Y$,
- $\delta(X \oplus Y) = \delta(X) + \delta(Y)$,
- $\delta(X \oplus Y) = 0$ implies $\delta(X) = 0$,

where X and Y are modules and δ is an abbreviation of δ_σ , δ'_σ , $\delta_{M,N}$ or $\delta'_{M,N}$.

3. Smooth points of orbit closures.

Throughout the section let M and N be d -dimensional modules such that $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_M$, and let $\mathcal{F}_{M,N}$ and $\mathcal{F}'_{M,N}$ denote complete sets of pairwise nonisomorphic modules X such that $\delta_{M,N}(X) = 0$ and $\delta'_{M,N}(X) = 0$, respectively.

Let $U, V \in \text{mod } A$. We denote by $\mathbf{Z}_A^1(V, U)$ the group of cocycles, i.e., the k -linear maps $Z : A \rightarrow \text{Hom}_k(V, U)$ satisfying

$$Z(aa') = Z(a)V(a') + U(a)Z(a'), \quad \text{for all } a, a' \in A.$$

The group $\mathbf{Z}_A^1(V, U)$ contains the group of coboundaries

$$\mathbf{B}_A^1(V, U) = \{hV - Uh; h \in \text{Hom}_k(V, U)\}.$$

This leads to the k -functor

$$\mathbf{Z}_A^1(-, -) : \text{mod } A \times \text{mod } A \rightarrow \text{mod } k$$

and its k -subfunctor $\mathbf{B}_A^1(-, -)$. Any cocycle Z in $\mathbf{Z}_A^1(V, U)$ induces an exact sequence

$$\sigma_Z : 0 \rightarrow U \xrightarrow{\alpha_Z} W_Z \xrightarrow{\beta_Z} V \rightarrow 0$$

in mod A . Then the cocycle Z is a coboundary if and only if the sequence σ_Z splits, which is equivalent to the fact that $W_Z \simeq U \oplus V$, by Lemma 2.3. Let

$$\mathcal{L}_{M,N}(V,U) = \{Z \in \mathbf{Z}_A^1(V,U); \delta_{\sigma_Z}(X) = 0 \text{ for any } X \in \mathcal{F}_{M,N}, \\ \delta'_{\sigma_Z}(Y) = 0 \text{ for any } Y \in \mathcal{F}'_{M,N}\}.$$

Obviously $\mathcal{L}_{M,N}(V,U)$ contains $\mathbf{B}_A^1(V,U)$ and does not depend on the choice of representatives of isomorphism classes of modules in the definition of the sets $\mathcal{F}_{M,N}$ and $\mathcal{F}'_{M,N}$.

LEMMA 3.1. *A cocycle $Z \in \mathbf{Z}_A^1(V,U)$ belongs to $\mathcal{L}_{M,N}(V,U)$ if and only if*

$$\mathbf{Z}_A^1(V,f)(Z) \in \mathbf{B}_A^1(V,X) \quad \text{and} \quad \mathbf{Z}_A^1(g,U)(Z) \in \mathbf{B}_A^1(Y,U)$$

for any modules $X \in \mathcal{F}_{M,N}$, $Y \in \mathcal{F}'_{M,N}$ and any homomorphisms $f : U \rightarrow X$, $g : Y \rightarrow V$.

PROOF. Let Z be a cocycle in $\mathbf{Z}_A^1(V,U)$. By duality, it suffices to show that $\delta_{\sigma_Z}(X) = 0$ if and only if the cocycle $\mathbf{Z}_A^1(V,f)(Z)$ is a coboundary for any homomorphism $f : U \rightarrow X$. By Lemma 2.2, the equality $\delta_{\sigma_Z}(X) = 0$ means that any homomorphism in $\text{Hom}_A(U,X)$ factors through α_Z . Let $Z' = \mathbf{Z}_A^1(V,f)(Z)$ for some homomorphism $f : U \rightarrow X$. We consider the pushout of σ_Z under f :

$$\begin{array}{ccccccccc} \sigma_Z : & 0 & \longrightarrow & U & \xrightarrow{\alpha_Z} & W_Z & \xrightarrow{\beta_Z} & V & \longrightarrow & 0 \\ & & & \downarrow f & & \downarrow & & \parallel & & \\ \sigma_{Z'} : & 0 & \longrightarrow & X & \xrightarrow{\alpha_{Z'}} & W_{Z'} & \xrightarrow{\beta_{Z'}} & V & \longrightarrow & 0. \end{array}$$

Then f factors through α_Z if and only if the sequence $\sigma_{Z'}$ splits, and the latter means that the cocycle Z' is a coboundary. □

LEMMA 3.2. *$\mathcal{L}_{M,N}(-, -)$ is a k -subfunctor of $\mathbf{Z}_A^1(-, -)$.*

PROOF. Let U and V be modules. We take $X \in \mathcal{F}_{M,N}$ and $Y \in \mathcal{F}'_{M,N}$. Then $\mathcal{L}_{M,N}(V,U)$ is a k -space, by Lemma 3.1 and since the appropriate maps $\mathbf{Z}_A^1(V,f)$ and $\mathbf{Z}_A^1(g,U)$ are k -linear. Let Z be a cocycle in $\mathcal{L}_{M,N}(V,U)$. We set $Z' = \mathbf{Z}_A^1(V,f')(Z)$, where $f' : U \rightarrow U'$ is a homomorphism for some module U' . Then

$$\mathbf{Z}_A^1(V,\tilde{f})(Z') = \mathbf{Z}_A^1(V,\tilde{f}f')(Z) \in \mathbf{B}_A^1(V,X)$$

for any homomorphism $\tilde{f} : U' \rightarrow X$ and

$$\begin{aligned} \mathbf{Z}_A^1(\tilde{g},U')(Z') &= \mathbf{Z}_A^1(\tilde{g},f')(Z) = \mathbf{Z}_A^1(Y,f')(\mathbf{Z}_A^1(\tilde{g},U)(Z)) \\ &\in \mathbf{Z}_A^1(Y,f')(\mathbf{B}_A^1(Y,U)) \subseteq \mathbf{B}_A^1(Y,U') \end{aligned}$$

for any homomorphism $\tilde{g} : Y \rightarrow V$. This shows that the cocycle Z' belongs to $\mathcal{Z}_{M,N}(V, U')$. Dually the cocycle $\mathbf{Z}_A^1(g', U)(Z)$ belongs to $\mathcal{Z}_{M,N}(V', U)$ for any module V' and any homomorphism $g' : V' \rightarrow V$. \square

The module variety $\text{mod}_A(d)$ is the underlying variety of an affine k -scheme $\underline{\text{mod}}_A^d$ of finite type, which represents the functor

$$\text{mod}_A^d : (\text{Commutative } k\text{-algebras}) \rightarrow (\text{Sets}),$$

where $\text{mod}_A^d(R)$ is the set of k -algebra homomorphisms from A to the algebra of $d \times d$ -matrices with coefficients in a commutative k -algebra R [4], [8]. We denote by $\mathcal{T}_{\mathcal{X},x}$ the tangent space of a k -scheme \mathcal{X} at a point x . Let n be a (closed) point of \mathcal{O}_N . Then the tangent space $\mathcal{T}_{\underline{\text{mod}}_A^d,n}$ corresponds to the preimage of n via the canonical map

$$\text{mod}_A^d(k[\varepsilon]/(\varepsilon^2)) \rightarrow \text{mod}_A^d(k),$$

and the latter corresponds to the group of cocycles $\mathbf{Z}_A^1(N, N)$. Hence we get a canonical k -isomorphism

$$\Phi : \mathcal{T}_{\underline{\text{mod}}_A^d,n} \xrightarrow{\cong} \mathbf{Z}_A^1(N, N).$$

Furthermore, $\Phi(\mathcal{T}_{\mathcal{O}_N,n}) = \mathbf{B}_A^1(N, N)$ which gives the isomorphism

$$\bar{\Phi} : \mathcal{T}_{\underline{\text{mod}}_A^d,n} / \mathcal{T}_{\mathcal{O}_N,n} \xrightarrow{\cong} \text{Ext}_A^1(N, N)$$

known as a Voigt result [8, Proposition 1.1]. Here and later on, the group $\text{Ext}_A^1(V, U)$ of extensions of V by U is identified with the quotient $\mathbf{Z}_A^1(V, U) / \mathbf{B}_A^1(V, U)$ for any modules U and V .

LEMMA 3.3. *Let $n \in \mathcal{O}_N$. Then $\Phi(\mathcal{T}_{\bar{\mathcal{O}}_M,n}) \subseteq \mathcal{Z}_{M,N}(N, N)$.*

PROOF. We have to recall some notation and results of Section 3 in [15] (see also the proof of [16, Proposition 2.2]). Let X be a module and

$$\text{mod}_{A,X,t}^d : (\text{Commutative } k\text{-algebras}) \rightarrow (\text{Sets})$$

be the subfunctor of mod_A^d defined in [16, (3.3)], where $t = [X, M]$. This functor is represented by an affine k -subscheme $\mathcal{X} = \underline{\text{mod}}_{A,X,t}^d$ of $\underline{\text{mod}}_A^d$ such that the underlying variety is given by

$$\mathcal{X}_{red} = \{l \in \text{mod}_A(d); [X, L] = t\}.$$

Here L denotes a module corresponding to a point l in $\text{mod}_A(d)$. Assume that $\delta'_{M,N}(X) = 0$. Then the orbits \mathcal{O}_M and \mathcal{O}_N are included in \mathcal{X}_{red} . Therefore $\mathcal{T}_{\bar{\mathcal{O}}_M,n}$ is contained in

$\mathcal{T}_{\mathcal{X},n}$. On the other hand, the tangent space $\mathcal{T}_{\mathcal{X},n}$ corresponds to the preimage of n via the canonical map

$$\text{mod}_{A,X,t}^d(k[\varepsilon]/(\varepsilon^2)) \rightarrow \text{mod}_{A,X,t}^d(k).$$

Furthermore, by [15, Lemma 3.11], the latter corresponds to the subset of $\mathbf{Z}_A^1(N, N)$ consisting of the cocycles Z such that $\delta'_{\sigma_Z}(X) = 0$. Hence $\Phi(\mathcal{T}_{\overline{\mathcal{O}}_M,n})$ is contained in

$$\{Z \in \mathbf{Z}_A^1(N, N); \delta'_{\sigma_Z}(X) = 0 \text{ for any } X \in \mathcal{F}'_{M,N}\}.$$

By duality, $\Phi(\mathcal{T}_{\overline{\mathcal{O}}_M,n})$ is also contained in

$$\{Z \in \mathbf{Z}_A^1(N, N); \delta_{\sigma_Z}(X) = 0 \text{ for any } X \in \mathcal{F}_{M,N}\},$$

and the claim follows from the definition of $\mathcal{Z}_{M,N}(N, N)$. □

We define the quotient $\mathcal{E}_{M,N}(V, U) = \mathcal{Z}_{M,N}(V, U)/\mathbf{B}_A^1(V, U)$ for any modules U and V . An immediate consequence of Lemmas 3.1 and 3.2 is the following fact.

COROLLARY 3.4. $\mathcal{E}_{M,N}(-, -)$ is a k -subfunctor of

$$\text{Ext}_A^1(-, -) : \text{mod } A \times \text{mod } A \rightarrow \text{mod } k$$

and

$$\mathcal{E}_{M,N}(V, U) = \bigcap_{\substack{X \in \mathcal{F}_{M,N} \\ f \in \text{Hom}_A(U, X)}} \text{Ker}(\text{Ext}_A^1(V, f)) \cap \bigcap_{\substack{Y \in \mathcal{F}'_{M,N} \\ g \in \text{Hom}_A(Y, V)}} \text{Ker}(\text{Ext}_A^1(g, U))$$

for any modules U and V .

Now we are ready to formulate our first sufficient conditions for regularity of points in $\overline{\mathcal{O}}_M$.

PROPOSITION 3.5. $\dim_k \mathcal{E}_{M,N}(N, N) \geq [N, N] - [M, M]$ and the equality implies that $\text{Sing}(M, N) = \text{Reg}$.

PROOF. Let $n \in \mathcal{O}_N$. Combining Lemmas 2.1 and 3.3 we get

$$\begin{aligned} \dim_k \mathcal{E}_{M,N}(N, N) &= \dim_k \mathcal{Z}_{M,N}(N, N) - \dim_k \mathbf{B}_A^1(N, N) \\ &\geq \dim_k \mathcal{T}_{\overline{\mathcal{O}}_M,n} - \dim_k \mathcal{T}_{\mathcal{O}_N,n} = \dim_k \mathcal{T}_{\overline{\mathcal{O}}_M,n} - \dim \mathcal{O}_N \\ &\geq \dim \overline{\mathcal{O}}_M - \dim \mathcal{O}_N = [N, N] - [M, M]. \end{aligned}$$

Moreover, the equality $\dim_k \mathcal{E}_{M,N}(N, N) = [N, N] - [M, M]$ implies that

$$\dim_k \mathcal{F}_{\overline{\mathcal{O}}_M, n} = \dim \overline{\mathcal{O}}_M,$$

which means that $\text{Sing}(M, N) = \text{Reg}$, as the variety $\overline{\mathcal{O}}_M$ is irreducible. □

As a consequence of the above proposition one can conclude the following useful result [16, Proposition 2.2].

PROPOSITION 3.6. *Assume that one of the following cases holds.*

- (1) *There is an exact sequence $\sigma : 0 \rightarrow Z \rightarrow Z \oplus M \rightarrow N \rightarrow 0$ in $\text{mod } A$ and $\delta'_{M,N}(Z \oplus M) = 0$ for some module Z .*
- (2) *There is an exact sequence $\sigma' : 0 \rightarrow N \rightarrow M \oplus Z' \rightarrow Z' \rightarrow 0$ in $\text{mod } A$ and $\delta_{M,N}(M \oplus Z') = 0$ for some module Z' .*

Then $\text{Sing}(M, N) = \text{Reg}$.

PROOF (1). We may assume that $Z \oplus M$ belongs to $\mathcal{F}'_{M,N}$. By Corollary 3.4, $\mathcal{E}_{M,N}(N, N)$ is contained in the kernel of the last map in the following long exact sequence induced by σ :

$$0 \rightarrow \text{Hom}_A(N, N) \rightarrow \text{Hom}_A(Z \oplus M, N) \rightarrow \text{Hom}_A(Z, N) \rightarrow \text{Ext}_A^1(N, N) \rightarrow \text{Ext}_A^1(Z \oplus M, N).$$

Consequently,

$$\dim_k \mathcal{E}_{M,N}(N, N) \leq \delta_\sigma(N) = \delta_{M,N}(N) + \delta'_{M,N}(M) = [N, N] - [M, M].$$

Hence the claim follows from Proposition 3.5.

We proceed dually in case (2). □

COROLLARY 3.7. *Let $\sigma : 0 \rightarrow U \rightarrow M \rightarrow V \rightarrow 0$ be an exact sequence in $\text{mod } A$ such that $\delta'_\sigma(U \oplus M) = 0$ or $\delta_\sigma(M \oplus V) = 0$. Then*

$$\text{Sing}(M, U \oplus V) = \text{Reg}.$$

PROOF. If $\delta'_\sigma(U \oplus M) = 0$ then it suffices to apply Proposition 3.6 for $Z = U$ and the direct sum of σ and the sequence $0 \rightarrow 0 \rightarrow U \xrightarrow{1_U} U \rightarrow 0$. We proceed in a similar way if $\delta_\sigma(M \oplus V) = 0$. □

We conclude from the proof of [17, Theorem 1.1] and its dual the following result.

THEOREM 3.8. *Assume that $\dim \mathcal{O}_M - \dim \mathcal{O}_N = 1$. Then:*

- (1) $\delta_{M,N}(M) = \delta'_{M,N}(M) = 0$ and $\delta_{M,N}(N) = \delta'_{M,N}(N) = 1$;
- (2) *there is an exact sequence $0 \rightarrow Z \rightarrow Z \oplus M \rightarrow N \rightarrow 0$ in $\text{mod } A$ for some indecomposable module Z with $\delta'_{M,N}(Z) = 0$;*
- (3) *there is an exact sequence $0 \rightarrow N \rightarrow M \oplus Z' \rightarrow Z' \rightarrow 0$ in $\text{mod } A$ for some indecomposable module Z' with $\delta_{M,N}(Z') = 0$.*

In particular $\text{Sing}(M, N) = \text{Reg}$.

4. Reduction to disjoint modules.

Combining Lemmas 2.2 and 2.4 we get the following fact.

LEMMA 4.1. *Let*

$$\sigma : 0 \rightarrow U \xrightarrow{f} W \xrightarrow{g} V_1 \oplus V_2 \rightarrow 0$$

be an exact sequence in $\text{mod } A$ such that $\delta'_\sigma(V_1) = 0$. Then $W = W_1 \oplus W_2$ for some modules $W_1 \simeq V_1$ and W_2 such that there is an exact sequence

$$\eta : 0 \rightarrow U \xrightarrow{f'} W_2 \xrightarrow{g'} V_2 \rightarrow 0$$

in $\text{mod } A$ with $f' : U \rightarrow W_2$ being a component of $f : U \rightarrow W_1 \oplus W_2$.

We denote by $\mu(L, Y)$ the multiplicity of an indecomposable module Y as a direct summand of a module L .

LEMMA 4.2. *Let M and N be modules such that $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_M$. Let Y be an indecomposable module such that $\mu(M, Y) < \mu(N, Y)$. Then $\delta_{M,N}(Y) > 0$ or $\delta'_{M,N}(Y) > 0$.*

PROOF. Applying Theorem 2.5 we get an exact sequence

$$\sigma : 0 \rightarrow Z \xrightarrow{f} Z \oplus M \rightarrow N \rightarrow 0$$

in $\text{mod } A$ such that f belongs to $\text{rad}(\text{mod } A)$. Let Y be an indecomposable A -module such that $p := \mu(N, Y) > \mu(M, Y)$. Assume that $\delta'_{M,N}(Y) = 0$. Then $\delta'_\sigma(Y^p) = \delta'_{M,N}(Y^p) = 0$ and Y^p is isomorphic to a direct summand of $Z \oplus M$, by Lemma 4.1. Therefore $\mu(Z \oplus M, Y) \geq p$ and consequently $\mu(Z, Y) > 0$. This means that there is a retraction $h : Z \rightarrow Y$. We know that h does not factor through f , as the latter belongs to $\text{rad}(\text{mod } A)$. Hence $\delta_{M,N}(Y) = \delta_\sigma(Y) > 0$, by Lemma 2.2. □

LEMMA 4.3. *Let M', N' and X be modules such that $\mathcal{O}_{N' \oplus X} \subset \overline{\mathcal{O}}_{M' \oplus X}$ and $M' \not\cong N'$. Then $[N', N'] > [M', M']$.*

PROOF. Let $M = M' \oplus X$ and $N = N' \oplus X$. Since M' and N' are not isomorphic and $\dim_k M' = \dim_k N'$, then there is an indecomposable A -module Y such that $\mu(N', Y) > \mu(M', Y)$, or equivalently, $\mu(N, Y) > \mu(M, Y)$. Consequently $\delta_{M,N}(Y) > 0$ or $\delta'_{M,N}(Y) > 0$, by Lemma 4.2. Therefore the claim follows from the inequalities

$$\begin{aligned}
 [N', N'] - [M', M'] &= \delta_{M,N}(N') + \delta'_{M,N}(M') \geq \delta_{M,N}(N') \geq \delta_{M,N}(Y), \\
 [N', N'] - [M', M'] &= \delta'_{M,N}(N') + \delta_{M,N}(M') \geq \delta'_{M,N}(N') \geq \delta'_{M,N}(Y). \quad \square
 \end{aligned}$$

We shall need the following cancellation properties proved by Bongartz [6, Corollary 2.5] [5, Theorem 2].

THEOREM 4.4. *Let M', N' and X be modules such that $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_M$ for $M = M' \oplus X$ and $N = N' \oplus X$.*

- (1) *If $\delta_{M,N}(X) = 0$ or $\delta'_{M,N}(X) = 0$ then $\mathcal{O}_{N'} \subseteq \overline{\mathcal{O}}_{M'}$.*
- (2) *If $\delta_{M,N}(X) = 0$ and $\delta'_{M,N}(X) = 0$ then $\text{Sing}(M, N) = \text{Sing}(M', N')$.*

PROOF OF THEOREM 1.1. Let M', N' and X be modules such that $\mathcal{O}_N \subset \overline{\mathcal{O}}_M$ and $\dim \mathcal{O}_M - \dim \mathcal{O}_N = 2$, where $M = M' \oplus X$ and $N = N' \oplus X$. In particular, the modules M' and N' are not isomorphic and

$$2 = [N, N] - [M, M] = ([N', N'] - [M', M']) + \delta_{M,N}(X) + \delta'_{M,N}(X).$$

On the other hand $[N', N'] - [M', M'] \geq 1$, by Lemma 4.3. Therefore

$$\dim \mathcal{O}_{M'} - \dim \mathcal{O}_{N'} = [N', N'] - [M', M'] \in \{1, 2\},$$

and at least one of the numbers $\delta_{M,N}(X)$ and $\delta'_{M,N}(X)$ is zero. Consequently $\mathcal{O}_{N'} \subseteq \overline{\mathcal{O}}_{M'}$, by Theorem 4.4.

We first consider the case $\dim \mathcal{O}_{M'} - \dim \mathcal{O}_{N'} = 1$. By duality, we may assume that $\delta'_{M,N}(X) = 0$. Using Theorem 3.8 we derive the exact sequence

$$\sigma : 0 \rightarrow Z \rightarrow Z \oplus M' \rightarrow N' \rightarrow 0$$

in mod A for some module Z such that $\delta'_{M',N'}(Z \oplus M') = 0$. Hence

$$\delta'_{M,N}(Z \oplus M) = \delta'_{M,N}(Z \oplus M') + \delta'_{M,N}(X) = \delta'_{M',N'}(Z \oplus M') = 0.$$

Let

$$0 \rightarrow Z \rightarrow Z \oplus M \rightarrow N \rightarrow 0$$

be a direct sum of σ and the short exact sequence

$$0 \rightarrow 0 \rightarrow X \xrightarrow{1_X} X \rightarrow 0.$$

Then $\text{Sing}(M, N) = \text{Reg}$, by Proposition 3.6.

It remains to consider the case $\dim \mathcal{O}_{M'} - \dim \mathcal{O}_{N'} = 2$. Then $\delta_{M,N}(X) = \delta'_{M,N}(X) = 0$. Hence $\text{Sing}(M, N) = \text{Sing}(M', N')$, by Theorem 4.4. \square

5. Reduction to at most two summands.

We shall need the following result which can be derived from the proof of [13, Theorem 2.3].

PROPOSITION 5.1. *Let $0 \rightarrow Z \xrightarrow{f} Z \oplus M \rightarrow N \rightarrow 0$ be an exact sequence in $\text{mod } A$ such that the homomorphism f belongs to $\text{rad}(\text{mod } A)$. Then there are a positive integer h and exact sequences*

$$\sigma_i: \quad 0 \rightarrow N_i \rightarrow N_{i-1} \oplus N_{i+1} \rightarrow N_i \rightarrow 0, \quad i = 1, 2, \dots, h,$$

in $\text{mod } A$ for some modules N_0, N_1, \dots, N_{h+1} such that $N_0 = 0, N_1 \simeq N, N_{h+1} \simeq N_h \oplus M$ and Z is isomorphic to a direct summand of N_h .

LEMMA 5.2. *Let $0 \rightarrow Z \xrightarrow{f} Z \oplus M \rightarrow N \rightarrow 0$ be an exact sequence in $\text{mod } A$ such that f belongs to $\text{rad}(\text{mod } A)$. Let \tilde{M} and \tilde{N} be modules such that $\mathcal{O}_{\tilde{N}} \subseteq \overline{\mathcal{O}}_{\tilde{M}}$ and $\delta_{M,N}(\tilde{M}) = \delta_{M,N}(\tilde{N}) = \delta'_{\tilde{M},\tilde{N}}(N) = 0$. Then $\delta'_{\tilde{M},\tilde{N}}(Z) = 0$.*

PROOF. We use Proposition 5.1 and the notation introduced there. Then

$$\sum_{i=1}^h \delta_{\sigma_i}(\tilde{M}) = \delta_{M,N}(\tilde{M}) = 0.$$

This implies that

$$2 \cdot [N_i, \tilde{M}] - [N_{i+1}, \tilde{M}] - [N_{i-1}, \tilde{M}] = \delta_{\sigma_i}(\tilde{M}) = 0, \quad i = 1, 2, \dots, h.$$

Proceeding by induction on i , one can show that

$$[N_i, \tilde{M}] = i \cdot [N, \tilde{M}], \quad i = 0, 1, \dots, h + 1.$$

In a similar way we get

$$[N_i, \tilde{N}] = i \cdot [N, \tilde{N}], \quad i = 0, 1, \dots, h + 1.$$

In particular

$$\delta'_{\tilde{M},\tilde{N}}(N_h) = h \cdot \delta'_{\tilde{M},\tilde{N}}(N) = 0 \quad \text{and} \quad \delta'_{\tilde{M},\tilde{N}}(Z) = 0,$$

as Z is isomorphic to a direct summand of N_h . □

PROPOSITION 5.3. *Let M', M'', N' and N'' be modules such that $M' \not\cong N', M'' \not\cong N'', \mathcal{O}_{N'} \subset \overline{\mathcal{O}}_{M'}, \mathcal{O}_{N''} \subset \overline{\mathcal{O}}_{M''}$ and*

$$\dim \mathcal{O}_{M' \oplus M''} - \dim \mathcal{O}_{N' \oplus N''} = 2.$$

Then $\text{Sing}(M' \oplus M'', N' \oplus N'') = \text{Reg}$.

PROOF. It follows from the assumptions and Lemma 2.1 that the integers

$$c' = \dim \mathcal{O}_{M'} - \dim \mathcal{O}_{N'} \quad \text{and} \quad c'' = \dim \mathcal{O}_{M''} - \dim \mathcal{O}_{N''}$$

are positive and

$$\begin{aligned} 2 &= c' + c'' + \delta_{M',N'}(N'') + \delta'_{M',N'}(N'') + \delta_{M'',N''}(M') + \delta'_{M'',N''}(M') \\ &= c' + c'' + \delta_{M',N'}(M'') + \delta'_{M',N'}(M'') + \delta_{M'',N''}(N') + \delta'_{M'',N''}(N'). \end{aligned}$$

Hence $c' = c'' = 1$ and

$$\begin{aligned} \delta_{M',N'}(N'') &= \delta'_{M',N'}(N'') = \delta_{M'',N''}(M') = \delta'_{M'',N''}(M') = 0, \\ \delta_{M',N'}(M'') &= \delta'_{M',N'}(M'') = \delta_{M'',N''}(N') = \delta'_{M'',N''}(N') = 0. \end{aligned} \tag{5.1}$$

By Theorem 3.8, there are exact sequences

$$0 \rightarrow Z' \xrightarrow{f'} Z' \oplus M' \rightarrow N' \rightarrow 0 \quad \text{and} \quad 0 \rightarrow Z'' \xrightarrow{f''} Z'' \oplus M'' \rightarrow N'' \rightarrow 0$$

in $\text{mod } A$ such that the modules Z' and Z'' are indecomposable and

$$\delta'_{M',N'}(Z' \oplus M') = \delta'_{M'',N''}(Z'' \oplus M'') = 0. \tag{5.2}$$

Observe that the homomorphisms f' and f'' belong to $\text{rad}(\text{mod } A)$, as they are not sections and Z' and Z'' are indecomposable modules. Using (5.1) and applying twice Lemma 5.2 we get

$$\delta'_{M',N'}(Z'') = \delta'_{M'',N''}(Z') = 0. \tag{5.3}$$

Let $M = M' \oplus M''$, $N = N' \oplus N''$ and $Z = Z' \oplus Z''$. Taking a direct sum of the above exact sequences we obtain an exact sequence of the form

$$0 \rightarrow Z \rightarrow Z \oplus M \rightarrow N \rightarrow 0.$$

Applying (5.1), (5.2) and (5.3) yields

$$\delta'_{M,N}(Z \oplus M) = \delta'_{M',N'}(Z' \oplus M' \oplus Z'' \oplus M'') + \delta'_{M'',N''}(Z' \oplus M' \oplus Z'' \oplus M'') = 0.$$

Hence $\text{Sing}(M, N) = \text{Reg}$, by Proposition 3.6. □

We shall need the following result proved by Bongartz in [5, Theorem 5].

PROPOSITION 5.4. *Let U, V and M be modules such that $\mathcal{O}_{U \oplus V} \subseteq \overline{\mathcal{O}}_M$ and $\delta'_{M,U \oplus V}(U) = \delta_{M,U \oplus V}(V) = 0$. Then there is an exact sequence in $\text{mod } A$ of the form $0 \rightarrow U \rightarrow M \rightarrow V \rightarrow 0$.*

PROPOSITION 5.5. *Let M and N be disjoint modules such that $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_M$. Assume*

that $N \simeq U \oplus L \oplus V$ for some modules U, L and V such that

$$\begin{aligned} \delta_{M,N}(U) = 1, \quad \delta_{M,N}(L) = 1, \quad \delta_{M,N}(V) = 0, \quad \delta_{M,N}(M) = 0, \\ \delta'_{M,N}(U) = 0, \quad \delta'_{M,N}(L) = 1, \quad \delta'_{M,N}(V) = 1, \quad \delta'_{M,N}(M) = 0. \end{aligned} \tag{5.4}$$

Then $\text{Sing}(M, N) = \text{Reg}$.

PROOF. Applying Theorem 2.5 we get an exact sequence

$$\sigma : 0 \rightarrow Z \xrightarrow{f} Z \oplus M \xrightarrow{g} N \rightarrow 0$$

in $\text{mod } A$ such that f belongs to $\text{rad}(\text{mod } A)$. Since $\delta'_{M,N}(U) = 0$ and the modules M and U are disjoint, then $Z \simeq U \oplus Y$ and there is an exact sequence

$$\tau : 0 \rightarrow Z \xrightarrow{f'} Y \oplus M \rightarrow L \oplus V \rightarrow 0$$

in $\text{mod } A$ for some module Y and some homomorphism f' in $\text{rad}(\text{mod } A)$, by Lemma 4.1. Taking a pushout of the sequence τ under a retraction $\pi : Z \rightarrow U$ leads to the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & Y & \xlongequal{\quad} & Y & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & Z & \xrightarrow{f'} & Y \oplus M & \longrightarrow & L \oplus V & \longrightarrow & 0 \\ & & \downarrow \pi & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & U & \longrightarrow & W & \longrightarrow & L \oplus V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Applying Corollary 2.6 and Theorem 2.5 to the exact sequences

$$\varepsilon : 0 \rightarrow U \xrightarrow{\alpha} W \xrightarrow{\beta} L \oplus V \rightarrow 0 \quad \text{and} \quad 0 \rightarrow Y \rightarrow Y \oplus M \rightarrow W \rightarrow 0$$

we get that $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_W$ and $\mathcal{O}_W \subseteq \overline{\mathcal{O}}_M$. We conclude from (5.4) the equality $\delta'_{M,N}(U \oplus M) = 0$. Therefore if $W \simeq M$ then $\text{Sing}(M, N) = \text{Reg}$, by Corollary 3.7 applied to the sequence ε . Thus we may assume that $W \not\simeq M$. Since f' belongs to $\text{rad}(\text{mod } A)$ then the retraction π does not factor through f' and consequently the exact sequence ε does not split. This implies that $W \not\simeq U \oplus L \oplus V \simeq N$, by Lemma 2.3. Therefore $\dim \mathcal{O}_N < \dim \mathcal{O}_W$ as well as $\dim \mathcal{O}_W < \dim \mathcal{O}_M$. Since $\dim \mathcal{O}_M - \dim \mathcal{O}_N = 2$ then

$$\dim \mathcal{O}_M - \dim \mathcal{O}_W = 1 \quad \text{and} \quad \dim \mathcal{O}_W - \dim \mathcal{O}_N = 1.$$

Applying Theorem 3.8 we get

$$\delta_{M,W}(W) = \delta'_{M,W}(W) = \delta_{W,N}(N) = \delta'_{W,N}(N) = 1, \quad \delta'_{W,N}(W) = 0. \quad (5.5)$$

Consequently

$$1 = \delta'_{W,N}(N) \geq \delta'_{W,N}(L \oplus V) = \delta'_\varepsilon(L \oplus V) > 0,$$

by Lemma 2.3. Thus

$$\delta'_{W,N}(L) + \delta'_{W,N}(V) = 1,$$

which gives two possibilities.

CASE 1. $\delta'_{W,N}(L) = 1$ and $\delta'_{W,N}(V) = 0$.

Then $\delta'_\varepsilon(V) = 0$, $W \simeq V \oplus W'$ and there is an exact sequence

$$\varepsilon' : \quad 0 \rightarrow U \rightarrow W' \rightarrow L \rightarrow 0$$

in mod A for some module W' , by Lemma 4.1.

It follows from (5.4) and (5.5) that

$$\begin{aligned} \delta'_{M,W}(W') &= \delta'_{M,W}(W) - \delta'_{M,W}(V) = 1 - (\delta'_{M,N}(V) - \delta'_{W,N}(V)) = 0, \\ \delta_{M,W}(V) &= \delta_{M,N}(V) - \delta_{W,N}(V) \leq \delta_{M,N}(V) = 0. \end{aligned} \quad (5.6)$$

Hence $\delta_{M,W}(V) = 0$ and there is an exact sequence

$$\eta : \quad 0 \rightarrow W' \rightarrow M \rightarrow V \rightarrow 0$$

in mod A , by Proposition 5.4. It follows from (5.5) that $\delta'_{W,N}(W') = 0$. Consequently, by (5.4) and (5.6),

$$\delta'_{M,N}(U \oplus W' \oplus M) = \delta'_{M,N}(W') = \delta'_{M,W}(W') + \delta'_{W,N}(W') = 0.$$

Taking a direct sum of the sequences ε' , η and $0 \rightarrow 0 \rightarrow U \xrightarrow{1_U} U \rightarrow 0$ gives an exact sequence of the form

$$0 \rightarrow U \oplus W' \rightarrow U \oplus W' \oplus M \rightarrow N \rightarrow 0.$$

Then $\text{Sing}(M, N) = \text{Reg}$, by Proposition 3.6 applied for $Z = U \oplus W'$.

CASE 2. $\delta'_{W,N}(L) = 0$ and $\delta'_{W,N}(V) = 1$.

Then $\delta'_\varepsilon(L) = 0$, $W \simeq L \oplus W''$ and there is an exact sequence

$$\varepsilon'' : 0 \rightarrow U \rightarrow W'' \rightarrow V \rightarrow 0$$

in mod A for some module W'' , by Lemma 4.1. In particular $U \oplus V \not\subseteq W''$ and $\mathcal{O}_{U \oplus V} \subseteq \overline{\mathcal{O}}_{W''}$, by Corollary 2.6. Applying Lemma 2.3 to the sequence ε yields $\delta_{W,N}(U) = \delta_\varepsilon(U) > 0$. Consequently

$$\delta_{W,N}(L) = \delta_{W,N}(N) - \delta_{W,N}(U \oplus V) \leq \delta_{W,N}(N) - \delta_{W,N}(U) \leq \delta_{W,N}(N) - 1.$$

It follows from (5.4) and (5.5) that $\delta_{W,N}(N) - 1 = 0$, $\delta_{W,N}(L) = 0$ and

$$\begin{aligned} \delta_{M,W}(W'') &= \delta_{M,W}(W) - \delta_{M,W}(L) = 1 - (\delta_{M,N}(L) - \delta_{W,N}(L)) = 0, \\ \delta'_{M,W}(W'') &= \delta'_{M,W}(W) - \delta'_{M,W}(L) = 1 - (\delta'_{M,N}(L) - \delta'_{W,N}(L)) = 0. \end{aligned} \tag{5.7}$$

Let Y be an indecomposable direct summand of W'' . Then $\delta_{M,W}(Y) = \delta'_{M,W}(Y) = 0$ and $\mu(M, Y) \geq \mu(W, Y) > 0$, by Lemma 4.2. This implies that $M \simeq W'' \oplus M'$ for some module M' not isomorphic to L . Furthermore $\mathcal{O}_L \subset \overline{\mathcal{O}}_{M'}$, by (5.7) and Theorem 4.4. Applying Proposition 5.3, we get

$$\text{Sing}(M, N) = \text{Sing}(W'' \oplus M', (U \oplus V) \oplus L) = \text{Reg}.$$

This finishes the proof of Proposition 5.5. □

PROOF OF THEOREM 1.2. We decompose $N = N_1 \oplus \dots \oplus N_s$, where N_i is an indecomposable module for $i = 1, \dots, s = s(N)$. Our assumptions and Lemma 2.1 imply that $[N, N] - [M, M] = 2$. Therefore

$$\begin{aligned} 2 &= \delta_{M,N}(M) + \sum_{i=1}^s \delta'_{M,N}(N_i) = \delta'_{M,N}(M) + \sum_{i=1}^s \delta_{M,N}(N_i), \\ 4 &= (\delta_{M,N}(M) + \delta'_{M,N}(M)) + \sum_{i=1}^s (\delta_{M,N}(N_i) + \delta'_{M,N}(N_i)). \end{aligned} \tag{5.8}$$

Since the modules M and N are disjoint then $\mu(M, N_i) = 0$ and consequently

$$\delta_{M,N}(N_i) + \delta'_{M,N}(N_i) \geq 1, \quad i = 1, \dots, s, \tag{5.9}$$

by Lemma 4.2. This implies that $s \leq 4$. Recall that $s \geq 3$, by our assumptions. Hence

$$\delta_{M,N}(M) + \delta'_{M,N}(M) \leq 1. \tag{5.10}$$

Let U and V be the direct sums of the modules N_i such that $\delta'_{M,N}(N_i) = 0$ and $\delta_{M,N}(N_i) = 0$, respectively. Then $\delta'_{M,N}(U) = 0$ and $\delta_{M,N}(V) = 0$. It follows from (5.8) and (5.9) that $N \simeq U \oplus V \oplus L$, where either $L = 0$, or $L = N_j$ for some $j \leq s$ and the equalities (5.4) hold. We get $\text{Sing}(M, N) = \text{Reg}$ in the latter case, by Proposition 5.5. Therefore we may assume that $L = 0$, or equivalently, $N \simeq U \oplus V$. Then there is an exact sequence

$$0 \rightarrow U \rightarrow M \rightarrow V \rightarrow 0$$

in $\text{mod } A$, by Proposition 5.4. Furthermore, (5.10) implies that $\delta_{M,N}(M) = 0$ or $\delta'_{M,N}(M) = 0$. Hence $\text{Sing}(M, N) = \text{Reg}$, by Corollary 3.7. This finishes the proof of Theorem 1.2. \square

6. Path algebras of Dynkin quivers.

Throughout the section, A is the path algebra of a Dynkin quiver. We shall need some special properties of modules over such algebra A described in the following three lemmas, in order to prove Theorem 1.3. The first lemma follows from [7] and the second one follows from [5, Lemma 5].

LEMMA 6.1. *There are only finitely many isomorphism classes of indecomposable modules. Moreover, for each indecomposable module Y ,*

$$\text{End}_A(Y) = \{t \cdot 1_Y; t \in k\}.$$

LEMMA 6.2. *Let M and N be disjoint modules such that $\mathcal{O}_N \subset \overline{\mathcal{O}}_M$ and $\dim \mathcal{O}_M - \dim \mathcal{O}_N = 1$. Then the inequality $\mu(M, Y) \leq 1$ holds for any indecomposable module Y .*

LEMMA 6.3. *Let M and N be disjoint modules with $\mathcal{O}_N \subset \overline{\mathcal{O}}_M$. Then there are indecomposable direct summands U and V of N such that*

$$\delta_{M,N}(U) > 0, \delta'_{M,N}(U) = 0 \quad \text{and} \quad \delta_{M,N}(V) = 0, \delta'_{M,N}(V) > 0.$$

PROOF. A complete set $\text{ind } A$ of pairwise nonisomorphic indecomposable modules is finite, by Lemma 6.1. Moreover there is a partial order \preceq on $\text{ind } A$ such that $[X, Y] > 0$ implies $X \preceq Y$ for any modules X and Y in $\text{ind } A$. Applying Theorem 2.5 we get an exact sequence

$$\eta: \quad 0 \rightarrow N \rightarrow M \oplus Z' \rightarrow Z' \rightarrow 0$$

in $\text{mod } A$. Then $\delta_{M,N}(N) = \delta_\eta(N) > 0$, by Lemma 2.3. Hence there is a \preceq -minimal $U \in \text{ind } A$ with the property $\delta_{M,N}(U) > 0$. Then $\mu(N, U) > 0$, by [6, Lemma 3.1]. Moreover, using the Auslander-Reiten formula mentioned in the proof of [6, Lemma 3.1], we get that $\delta'_{M,N}(U) = 0$. Dually we get an appropriate module V . \square

PROPOSITION 6.4. *Let $\sigma: 0 \rightarrow U \xrightarrow{f} M \xrightarrow{g} V \rightarrow 0$ be an exact sequence in $\text{mod } A$*

such that the modules M and $N = U \oplus V$ are disjoint and

$$\begin{aligned} \delta_\sigma(U) = 1, \quad \delta_\sigma(M) = 1, \quad \delta_\sigma(V) = 0, \\ \delta'_\sigma(U) = 0, \quad \delta'_\sigma(M) = 1, \quad \delta'_\sigma(V) = 1. \end{aligned} \tag{6.1}$$

Then $\text{Sing}(M, N) = \text{Reg}$.

PROOF OF PROPOSITION 6.4. The equality $\delta_\sigma(M) = 1$ implies that $M = M_1 \oplus M'$ for an indecomposable module M_1 and a module M' such that

$$\delta_\sigma(M_1) = 1 \quad \text{and} \quad \delta_\sigma(M') = 0. \tag{6.2}$$

We divide the proof into several steps.

STEP 1. *There are nonsplittable exact sequences in mod A of the form*

$$\sigma_1 : 0 \rightarrow U \xrightarrow{\begin{pmatrix} f \\ h \end{pmatrix}} M \oplus M_1 \xrightarrow{(h', -f')} X \rightarrow 0, \quad \sigma_2 : 0 \rightarrow M_1 \xrightarrow{f'} X \xrightarrow{g'} V \rightarrow 0.$$

PROOF. Since $\delta_\sigma(M_1) > 0$ then there is a homomorphism $h : U \rightarrow M_1$ which does not factor through f , by Lemma 2.2. Taking a pushout of σ under h leads to the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{f} & M & \xrightarrow{g} & V & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow h' & & \parallel & & \\ 0 & \longrightarrow & M_1 & \xrightarrow{f'} & X & \xrightarrow{g'} & V & \longrightarrow & 0, \end{array}$$

this gives the exact sequences σ_1 and σ_2 . The sequence σ_2 does not split, by our construction. Since the modules U and $M \oplus M_1$ are disjoint, the sequence σ_1 does not split as well. □

STEP 2. *The following equalities hold:*

$$\begin{aligned} \delta_{\sigma_1}(U) = 1, \quad \delta_{\sigma_1}(M) = 0, \quad \delta'_{\sigma_1}(U) = 0, \\ \delta_{\sigma_2}(V) = 0, \quad \delta'_{\sigma_2}(U) = 0, \quad \delta'_{\sigma_2}(V) = 1. \end{aligned} \tag{6.3}$$

PROOF. Since the sequences σ_1 and σ_2 do not split then the integers $\delta_{\sigma_1}(U)$, $\delta_{\sigma_2}(M_1)$ and $\delta'_{\sigma_2}(V)$ are positive, by Lemma 2.3. Hence the claim follows from (6.1), (6.2) and the equalities

$$\delta_\sigma(Y) = \delta_{\sigma_1}(Y) + \delta_{\sigma_2}(Y) \quad \text{and} \quad \delta'_\sigma(Y) = \delta'_{\sigma_1}(Y) + \delta'_{\sigma_2}(Y),$$

for any module Y . □

STEP 3. $\delta_{\sigma_1}(X) = 0$.

PROOF. Let $\tilde{M} = M \oplus M_1$. The sequence σ_1 induces the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{Hom}_A(X, U) & \longrightarrow & \text{Hom}_A(X, \tilde{M}) & \longrightarrow & \text{Hom}_A(X, X) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{Hom}_A(\tilde{M}, U) & \longrightarrow & \text{Hom}_A(\tilde{M}, \tilde{M}) & \longrightarrow & \text{Hom}_A(\tilde{M}, X) \\
 & & \downarrow & & \downarrow \alpha & & \downarrow \gamma \\
 0 & \longrightarrow & \text{Hom}_A(U, U) & \longrightarrow & \text{Hom}_A(U, \tilde{M}) & \xrightarrow{\beta} & \text{Hom}_A(U, X).
 \end{array}$$

Since $\delta_{\sigma_1}(\tilde{M}) = \delta'_{\sigma_1}(U) = 0$, then the homomorphisms α and β are surjective. Hence γ is also surjective, which implies that $\delta_{\sigma_1}(X) = 0$. \square

STEP 4. $\delta'_{\sigma_2}(M) = 0$.

PROOF. Suppose that $\delta'_{\sigma_2}(M) \geq 1$. Since $1 = \delta_{\sigma}(M) = \delta_{\sigma_1}(M) + \delta_{\sigma_2}(M)$, then

$$\delta'_{\sigma_1}(M) = 0 \quad \text{and} \quad \delta'_{\sigma_1}(M_1) = 0,$$

as M_1 is a direct summand of M . Observe that

$$\delta_{\sigma_1}(U) - \delta_{\sigma_1}(M \oplus M_1) + \delta_{\sigma_1}(X) = \delta'_{\sigma_1}(U) - \delta'_{\sigma_1}(M \oplus M_1) + \delta'_{\sigma_1}(X).$$

Applying (6.3) and Step 3 we get that $\delta'_{\sigma_1}(X) = 1$. Then $X = X_1 \oplus X'$ for an indecomposable module X_1 and a module X' such that

$$\delta'_{\sigma_1}(X_1) = 1 \quad \text{and} \quad \delta'_{\sigma_1}(X') = 0.$$

Let $\varphi : X' \rightarrow X$ be a section. Hence $\varphi = h'\tilde{h} - f'\tilde{f}$ for some homomorphisms $\tilde{h} : X' \rightarrow M$ and $\tilde{f} : X' \rightarrow M_1$, by Lemma 2.2 applied to the sequence σ_1 . Since the sequence σ_2 does not split and the module M_1 is indecomposable, then f' belongs to $\text{rad}(\text{mod } A)$. Thus $f'\tilde{f}$ belongs to $\text{rad}(\text{mod } A)$ and $h'\tilde{h}$ is a section. Consequently \tilde{h} is also a section. Applying Lemma 2.4 to σ_1 we get that $M \simeq X' \oplus M''$ and there is an exact sequence

$$\tau : \quad 0 \rightarrow U \rightarrow M'' \oplus M_1 \rightarrow X_1 \rightarrow 0$$

in $\text{mod } A$ for some module M'' . The modules U and $M'' \oplus M_1$ are disjoint, by our assumptions. The modules X_1 and $M'' \oplus M_1$ are also disjoint, since X_1 is indecomposable, $\delta'_{\sigma_1}(X_1) > 0$ and $\delta'_{\sigma_1}(M'' \oplus M_1) = 0$. Observe that

$$\dim \mathcal{O}_{M'' \oplus M_1} - \dim \mathcal{O}_{U \oplus X_1} = \delta_{\sigma_1}(U \oplus X_1) + \delta'_{\sigma_1}(M'' \oplus M_1) = 1,$$

by (6.3) and Step 2. Hence $\mu(M'', M_1) = 0$, by Lemma 6.2. Since $M_1 \oplus M'$ is isomorphic to $X' \oplus M''$ then $\mu(X, M_1) \geq \mu(X', M_1) \geq 1$ and $X \simeq M_1 \oplus X''$ for some module X'' . Hence, up to an isomorphism, the sequence σ_2 has the form

$$0 \rightarrow M_1 \xrightarrow{f' = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}} M_1 \oplus X'' \xrightarrow{g' = (\beta_1, \beta_2)} V \rightarrow 0.$$

Since the endomorphism $\alpha_1 \in \text{End}_A(M_1)$ belongs to $\text{rad}(\text{mod } A)$ and M_1 is an indecomposable module, then $\alpha_1 = 0$, by Lemma 6.1. Observe that

$$\text{Ker}(\beta_1) \subseteq \text{Ker}(g') \cap M_1 \quad \text{and} \quad \text{Ker}(g') = \text{Im}(f') \subseteq X''.$$

Therefore the homomorphism β_1 is injective and $\text{Im}(\beta_1) \cap \text{Im}(\beta_2) = \{0\}$. Thus $\text{Im}(\beta_1)$ is a direct summand of V , as g' is surjective. Consequently the homomorphism $\beta_1 : M_1 \rightarrow V$ is a section, which is impossible as M_1 and V are disjoint modules. \square

STEP 5. $\delta_\sigma(X) = 0$.

PROOF. Observe that

$$\delta_\sigma(M_1) - \delta_\sigma(X) + \delta_\sigma(V) = \delta'_{\sigma_2}(U) - \delta'_{\sigma_2}(M) + \delta'_{\sigma_2}(V).$$

Hence the claim follows from (6.1), (6.2), (6.3) and Step 4. \square

STEP 6. *There is an exact sequence $\sigma_3 : 0 \rightarrow U \rightarrow X \oplus M' \rightarrow V \oplus V \rightarrow 0$.*

PROOF. Since $M = M_1 \oplus M'$ then the sequence σ has the form

$$0 \rightarrow U \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} M_1 \oplus M' \xrightarrow{(g_1, g_2)} V \rightarrow 0.$$

We get from (6.3) the equality $\delta_{\sigma_2}(V) = 0$. Hence any homomorphism from M_1 to V factors through f' , by Lemma 2.2. Thus $g_1 = j f'$ for some homomorphism $j : X \rightarrow V$. It is easy to check that the sequence

$$0 \rightarrow U \xrightarrow{\begin{pmatrix} f' f_1 \\ f_2 \end{pmatrix}} X \oplus M' \xrightarrow{\begin{pmatrix} g' & 0 \\ j & g_2 \end{pmatrix}} V \oplus V \rightarrow 0$$

is exact. \square

We shall consider the k -functor $\mathcal{E}_{M,N}(-, -)$ defined in Section 3.

STEP 7. $\dim_k \mathcal{E}_{M,N}(V, U) \leq 2$.

PROOF. We know that $\delta_{M,N}(X \oplus M') = \delta_\sigma(X) + \delta_\sigma(M') = 0$, by (6.2) and Step 5. Applying Corollary 3.4 we get that $\mathcal{E}_{M,N}(V, U)$ is contained in the kernel of the last

map in the following long exact sequence induced by σ_3 :

$$0 \rightarrow \text{Hom}_A(V, U) \rightarrow \text{Hom}_A(V, X \oplus M') \rightarrow \text{Hom}_A(V, V \oplus V) \rightarrow \\ \rightarrow \text{Ext}_A^1(V, U) \rightarrow \text{Ext}_A^1(V, X \oplus M').$$

Consequently

$$\dim_k \mathcal{E}_{M,N}(V, U) \leq \delta'_{\sigma_3}(V) = \delta'_\sigma(V) + \delta'_{\sigma_2}(V) = 1 + 1 = 2,$$

by (6.1) and (6.3). □

STEP 8. $\dim_k \mathcal{E}_{M,N}(N, N) \leq [N, N] - [M, M].$

PROOF. Let Y be a module. We know that $\delta_{M,N}(V) = \delta_\sigma(V) = 0$ and $\delta'_{M,N}(U) = \delta'_\sigma(U) = 0$, by (6.1). Then $\mathcal{E}_{M,N}(Y, V)$ is contained in the kernel of $\text{Ext}_A^1(Y, 1_V)$ and $\mathcal{E}_{M,N}(U, Y)$ is contained in the kernel of $\text{Ext}_A^1(1_U, Y)$, by Corollary 3.4. Hence $\mathcal{E}_{M,N}(Y, V) = 0$ and $\mathcal{E}_{M,N}(U, Y) = 0$. Consequently

$$\mathcal{E}_{M,N}(N, N) \simeq \mathcal{E}_{M,N}(U \oplus V, U \oplus V) \\ \simeq \mathcal{E}_{M,N}(U, U) \oplus \mathcal{E}_{M,N}(U, V) \oplus \mathcal{E}_{M,N}(V, U) \oplus \mathcal{E}(V, V) \simeq \mathcal{E}_{M,N}(V, U).$$

Therefore the claim follows from Step 7 and the equalities

$$[N, N] - [M, M] = \delta_\sigma(U) + \delta_\sigma(V) + \delta'_\sigma(M) = 1 + 0 + 1 = 2. \quad \square$$

Step 8 together with Proposition 3.5 imply that $\text{Sing}(M, N) = \text{Reg}$, which finishes the proof of Proposition 6.4. □

PROOF OF THEOREM 1.3. Let M be a module. It follows from Lemma 6.1 that $\overline{\mathcal{O}}_M$ contains only finitely many orbits. Thus it suffices to show that $\text{Sing}(M, N) = \text{Reg}$ for any module N such that $\mathcal{O}_N \subset \overline{\mathcal{O}}_M$ and

$$c := \dim \mathcal{O}_M - \dim \mathcal{O}_N \in \{1, 2\}.$$

If $c = 1$, then the claim follows from Theorem 3.8. Therefore we may assume that $c = 2$. Applying Theorem 1.1 we reduce the problem to the case when the modules M and N are disjoint. Then $N \simeq U \oplus V \oplus L$, where U and V are indecomposable modules such that

$$\delta_{M,N}(U) > 0, \delta'_{M,N}(U) = 0 \quad \text{and} \quad \delta_{M,N}(V) = 0, \delta'_{M,N}(V) > 0,$$

by Lemma 6.3. Applying Theorem 1.2 we may assume that $L = 0$ and $N \simeq U \oplus V$. Hence there is an exact sequence

$$\sigma : 0 \rightarrow U \rightarrow M \rightarrow V \rightarrow 0,$$

by Proposition 5.4. If $\delta_\sigma(M) = 0$ or $\delta'_\sigma(M) = 0$ then $\text{Sing}(M, N) = \text{Reg}$, by Corollary 3.7. Therefore we may assume that the integers $\delta_\sigma(M)$ and $\delta'_\sigma(M)$ are positive. On the other hand, by Lemma 2.1,

$$\begin{aligned} 2 &= [N, N] - [M, M] = \delta_\sigma(U) + \delta_\sigma(V) + \delta'_\sigma(M) \\ &= \delta'_\sigma(U) + \delta'_\sigma(V) + \delta_\sigma(M), \end{aligned}$$

which implies that the equalities (6.1) hold. Thus $\text{Sing}(M, N) = \text{Reg}$, by Proposition 6.4. This finishes the proof of Theorem 1.3. \square

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