

## On continuity of minimizers for certain quadratic growth functionals

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**Abstract.** In this paper we treat the regularity problem for minimizers  $u(x) : \Omega \subset \mathbf{R}^m \rightarrow \mathbf{R}^n$  of quadratic growth functionals  $\int_{\Omega} A(x, u, Du) dx$ . About the dependence on the variable  $x$  we assume only that  $A(\cdot, u, p)$  is in the class  $VMO$  as a function of  $x$ . Namely, we do not assume the continuity of  $A(x, u, p)$  with respect to  $x$ . We will prove a partial regularity result for the case  $m \leq 4$ .

### 1. Introduction.

Let  $\Omega$  be a domain of  $\mathbf{R}^m$ . For a map  $u : \Omega \rightarrow \mathbf{R}^n$ , we consider the following type of functional

$$\mathcal{A}(u) = \int_{\Omega} A(x, u, Du) dx. \quad (1.1)$$

Here,  $A(x, u, p)$  is a nonnegative function defined on  $\Omega \times \mathbf{R}^n \times \mathbf{R}^{mn}$  which is of class  $VMO$  as a function of  $x$ , continuous in  $u$  and of class  $C^2$  with respect to  $p$ . We also assume that for some positive constants  $\mu_0 \leq \mu_1$ ,

$$\mu_0 |p|^2 \leq A(x, u, p) \leq \mu_1 |p|^2 \quad \text{for all } (x, u, p) \in \Omega \times \mathbf{R}^n \times \mathbf{R}^{mn}.$$

A *local minimizer* of the functional  $\mathcal{A}$  is a function  $u \in H_{\text{loc}}^{1,2}(\Omega, \mathbf{R}^n)$  which satisfies

$$\mathcal{A}(u; \text{supp}\varphi) \leq \mathcal{A}(u + \varphi; \text{supp}\varphi)$$

for every  $\varphi \in H^{1,2}(\Omega, \mathbf{R}^n)$  with  $\text{supp}\varphi \subset\subset \Omega$ .

Except in the two-dimensional case, the regularity theory for vector valued minimizers or solutions of elliptic systems is far different from the one for scalar valued case. We wish to recall the fundamental paper [5] by De Giorgi where it is proved that the famous result so-called De Giorgi-Nash's theorem for second order elliptic equations with  $L^\infty$ -coefficients cannot be extended to linear elliptic systems. He considered the functional

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$$\int_{\Omega} \left\{ \left( (m-2) \sum_{h=1}^m \frac{\partial u^h}{\partial x^h} + m \sum_{h,k=1}^m \frac{x^h x^k}{|x|^2} \frac{\partial u^h}{\partial x^k} \right)^2 + \sum_{h,k=1}^n \left( \frac{\partial u^h}{\partial x^k} \right)^2 \right\} dx \tag{1.2}$$

for  $u \in H^{1,2}(\Omega, \mathbf{R}^m)$  with  $\Omega$  being the unit ball around the origin in  $\mathbf{R}^m, m \geq 3$ . It is easy to see that the integrand is coercive and their coefficients are bounded. The vector valued function

$$u(x) = x \cdot |x|^{-\theta}, \quad \theta = \frac{m}{2} [1 - (1 + 4(n-1)^2)^{-\frac{1}{2}}],$$

which belongs to  $H^{1,2}(\Omega, \mathbf{R}^n)$ , is not bounded and it is a minimizer of (1.2). Then, De Giorgi’s result shows that it is not possible to have regularity of the extremals of variational integrals in the same way as in the scalar case.

A modification of De Giorgi’s example due to Giusti and Miranda in [12] gives an example of a functional of the type

$$\int_{\Omega} A_{ij}^{\alpha\beta}(u) D_{\alpha} u^i D_{\beta} u^j \tag{1.3}$$

which has a minimizer with singularity.

Also we point out that independently by Giusti and Miranda, analogous examples of non-continuous minimizers of functionals with analytic coefficients were proved by Maz’ya in [18].

The works by Giusti and Miranda [13] and Morrey [19] start the study of partial regularity in the vector value case, which means regularity except on a “small” set.

For a general quadratic growth functional  $\mathcal{A}(u)$  as (1.1), Giaquinta and Giusti [10] proved that a minimizer  $u$  of  $\mathcal{A}(u)$  is of class  $C^{1,\alpha}(\Omega_0)$  for an open set  $\Omega_0 \subset \Omega$  with  $\mathcal{L}^m(\Omega \setminus \Omega_0) = 0$ . Namely, the singular set of a local minimizer is at most null set.

For functionals with specific structure, so-called *quadratic functionals*, we can see that the singular sets of minimizers are smaller than null sets. In [11], Giaquinta and Giusti treated *quadratic functionals* of the type

$$\int_{\Omega} g^{\alpha\beta}(x) h^{ij}(u) D_{\alpha} u^i D_{\beta} u^j dx \tag{1.4}$$

with  $(g^{\alpha\beta})$  and  $(h^{ij})$  symmetric positive definite matrices with smooth coefficients. For a local minimizer  $u$  of (1.4), they showed that the singular set of  $u$  has Hausdorff dimension  $d \leq m - 3$ .

We stress that in general we can not have everywhere regularity for local minimizers of quadratic growth functionals even if the coefficients are regular, we can obtain global regularity only in some particular cases (see e.g. [7], [23]).

We also mention that in the above results the integrands of the considered functionals are always assumed to be continuous with respect to the variable  $x$ . In the present paper we are interested in the study of partial regularity of local minimizers of functionals (1.1) whose integrands  $A(x, u, p)$  are discontinuous in  $x$ .

If we consider the *linear* elliptic system

$$\operatorname{div}(A(x)Du(x)) = F(x) \tag{1.5}$$

where  $A(x)$  are Hölder continuous in  $\overline{\Omega}$  ( $\Omega$  bounded open set), regularity results have been obtained by Campanato in [2]. As for the case where the coefficients belong to a class which neither contains nor is contained in  $C^0(\overline{\Omega})$  (a class of small multipliers of the *BMO* class defined by John and Nirenberg in [14]), Acquistapace in [1] proves *BMO* regularity results for the gradient of the solutions of (1.5) dropping the assumption  $A \in C^0(\overline{\Omega})$ . Later Huang in [17] investigates regularity results for elliptic systems assuming that  $A(x)$  belong to the subclass of *BMO* of vanishing mean oscillation functions, then he generalizes both Acquistapace and Campanato results.

In the *nonlinear* case,  $L^{2,\lambda}$  regularity results of derivatives of functions minimizing variational integrals have been considered by Daněček and Viszus, in [4]. (Here,  $L^{2,\lambda}$  are the Morrey spaces. For precise definition see Definition 2.1.) They consider the functional

$$\int_{\Omega} \{A_{ij}^{\alpha\beta}(x)D_{\alpha}u^iD_{\beta}u^j + g(x, u, Du)\}dx,$$

where  $A_{ij}^{\alpha\beta}$  are in the *VMO* class and satisfy strong ellipticity condition while the lower order term  $g$  is a Charathéodory function and satisfy the following inequality

$$|g(x, u, z)| \leq f(x) + H|z|^{\kappa} \tag{1.6}$$

where  $f \geq 0$ , a.e. in  $\Omega$ ,  $f \in L^p(\Omega)$ ,  $2 < p \leq \infty$ ,  $H \geq 0$ ,  $0 \leq \kappa < 2$ . Later in [6], Di Gironimo, Esposito and Sgambati considered the quadratic functionals

$$\int_{\Omega} A_{ij}^{\alpha\beta}(x, u)D_{\alpha}u^iD_{\beta}u^jdx,$$

where  $(A_{ij}^{\alpha\beta}(x, u))$  is elliptic, of the class *VMO* in the variable  $x$  and satisfies

$$|A_{ij}^{\alpha\beta}(x, u) - A_{ij}^{\alpha\beta}(x, v)| \leq \omega(|u - v|^2), \quad \forall x \in \Omega, \quad \forall u, v \in \mathbf{R}^n$$

for some concave function  $\omega$  with

$$\omega : \mathbf{R}^+ \rightarrow \mathbf{R}^+, \quad \omega(0) = 0, \quad 0 \leq \omega \leq 1.$$

They proved  $L^{2,\lambda}$  regularity for minimizers of such functionals.

Let us point out that continuity assumption with respect to  $u$  cannot be removed because *VMO* property is not preserved under composition with minimizer  $u$ .

In the paper [21] the authors extend the results contained in [6] obtaining  $L^{2,\lambda}$  regularity in the case that  $g(x, u, z)$  is not equal to zero but it is a Charathéodory function

and also that  $g(x, u, z)$  verify condition (1.6), for a.a.  $x \in \Omega$ .

In the present paper we generalize our previous paper [21] in the case where  $m \leq 4$ . We prove Morrey regularity of minimizers for quadratic functionals whose integrand function has a more generalized form depending on  $x, u$  and also on  $Du$ .

**2. Definitions and main theorem.**

We set, throughout the paper, by  $|\cdot|$  the norm in  $\mathbf{R}^m$  as well as in  $\mathbf{R}^n$  and in  $\mathbf{R}^{mn}$ , and by

$$B(x, r) = \{y \in \mathbf{R}^m : |y - x| < r\}$$

a generic ball in  $\mathbf{R}^m$  centered at  $x$  with radius  $r$ .

Let us now give the definition of the Morrey space  $L^{p,\lambda}$ . In the sequel we are interested in the Morrey regularity of the gradient of  $u$  for  $p = 2$ .

DEFINITION 2.1 (see [16], [20]). Let  $1 < p < \infty$ ,  $0 \leq \lambda < m$ . A measurable function  $f \in L^1_{loc}(\Omega, \mathbf{R}^n)$  is in the Morrey class  $L^{p,\lambda}(\Omega, \mathbf{R}^n)$  if the following norm is finite

$$\|f\|_{L^{p,\lambda}(\Omega)} = \sup_{\substack{0 < \rho < \text{diam } \Omega \\ x \in \Omega}} \frac{1}{\rho^\lambda} \int_{\Omega \cap B(x, \rho)} |f(y)|^p dy,$$

where  $B(x, \rho)$  ranges in the class of the balls.

DEFINITION 2.2. Let  $f \in L^1(\Omega, \mathbf{R}^n)$  we set the integral mean  $f_{x,R}$  by

$$f_{x,R} = \int_{\Omega \cap B(x,R)} f(y) dy = \frac{1}{|\Omega \cap B(x,R)|} \int_{\Omega \cap B(x,R)} f(y) dy$$

where  $|\Omega \cap B(x, R)|$  is the Lebesgue measure of  $\Omega \cap B(x, R)$ .

If we are not interested in specifying which the center is, we only set  $f_R$ .

Let us define the John Nirenberg class of Bounded Mean Oscillation functions (see [14]).

DEFINITION 2.3. Let  $f \in L^1_{loc}(\mathbf{R}^m)$ . We say that  $f$  belongs to  $BMO(\mathbf{R}^m)$  if the seminorm

$$\|f\|_* \equiv \sup_{B(x,R)} \frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y) - f_{x,R}| dy < \infty.$$

Let us now introduce the space of vanishing mean oscillation functions ([22]).

DEFINITION 2.4. Let  $f \in BMO(\mathbf{R}^m)$  and

$$\eta(f, R) = \sup_{\rho \leq R} \frac{1}{|B(x, \rho)|} \int_{B(x, \rho)} |f(y) - f_\rho| dy$$

where  $B(x, \rho)$  ranges over the class of the balls of  $\mathbf{R}^m$  of radius  $\rho$ . We say that  $f \in VMO(\Omega)$  if

$$\lim_{R \rightarrow 0} \eta(f, R) = 0.$$

Let us observe that substituting  $\mathbf{R}^m$  for  $\Omega$  we obtain the definitions of  $BMO(\Omega)$  and  $VMO(\Omega)$  preserving its character.

Let  $\Omega \subset \mathbf{R}^m$  be a domain. Let  $A(x, u, p)$  be a nonnegative function defined on  $\Omega \times \mathbf{R}^n \times \mathbf{R}^{mn}$  which satisfies the following conditions.

(A-1) For every  $(u, p) \in \mathbf{R}^n \times \mathbf{R}^{mn}$ ,  $A(\cdot, u, p) \in VMO(\Omega)$  and the mean oscillation of  $A(\cdot, u, p)/|p|^2$  vanishes uniformly with respect to  $u, p$  in the following sense: For some nonnegative function  $\sigma(y, \rho)$  with

$$\lim_{R \rightarrow 0} \sup_{\rho < R} \int_{B(x, \rho)} \sigma(y, \rho) dy = 0, \tag{2.7}$$

$A(\cdot, u, p)$  satisfies

$$|A(y, u, p) - A_{x, \rho}(u, p)| \leq \sigma(y, \rho) |p|^2 \quad \forall (u, p) \in \mathbf{R}^n \times \mathbf{R}^{mn}, \tag{2.8}$$

where

$$A_{x, \rho}(u, p) = \int_{B(x, \rho)} A(y, u, p) dy.$$

(A-2) For every  $x \in \Omega$ ,  $p \in \mathbf{R}^{mn}$  and  $u, v \in \mathbf{R}^n$ ,

$$|A(x, u, p) - A(x, v, p)| \leq \omega(|u - v|^2) |p|^2$$

for some monotone increasing concave function  $\omega$  with  $\omega(0) = 0$ .

(A-3) For almost all  $x \in \Omega$  and all  $u \in \mathbf{R}^n$ ,  $A(x, u, \cdot) \in C^2(\mathbf{R}^{mn})$ .

(A-4) There exist positive constants  $\mu_0 \leq \mu_1$  such that

$$\mu_0 |p|^2 \leq A(x, u, p) \leq \mu_1 |p|^2 \quad \text{for all } (x, u, p) \in \Omega \times \mathbf{R}^n \times \mathbf{R}^{mn}.$$

In this paper we show the following theorem.

**THEOREM 2.5.** *Let  $u \in W^{1,2}(\Omega, \mathbf{R}^n)$  be a local minimizer of the functional  $\mathcal{A}(u, \Omega)$  defined by (1.1). Suppose that assumptions (A-1), (A-2), (A-3) and (A-4) are satisfied. Then for every  $0 < \lambda < \min\{2 + \varepsilon, m\}$  for some  $\varepsilon > 0$  we have*

$$Du \in L_{\text{loc}}^{2,\lambda}(\Omega_0, \mathbf{R}^{mn}) \tag{2.9}$$

where

$$\Omega_0 = \left\{ x \in \Omega : \liminf_{R \rightarrow 0} \frac{1}{R^{m-2}} \int_{B(x,R)} |Du(y)|^2 dy = 0 \right\}.$$

Moreover, we have

$$\mathcal{H}^{m-2-\delta}(\Omega \setminus \Omega_0) = 0$$

for some  $\delta > 0$ , where  $\mathcal{H}^r$  denotes the  $r$ -dimensional Hausdorff measure.

As a corollary of the above theorem we have the following partial Hölder regularity result.

**COROLLARY 2.6.** *Let  $u$  and  $\Omega_0$  be as in Theorem 2.5, and assume that  $m \leq 4$ . Then, for some  $\alpha \in (0, 1)$ , we have*

$$u \in C^{0,\alpha}(\Omega_0, \mathbf{R}^N). \tag{2.10}$$

### 3. Proof of main theorem.

For some fixed point  $x_0 \in \Omega$  and  $R > 0$  with  $B(2R) := B(x_0, 2R) \subset\subset \Omega$ , let us define  $A^0(p)$  and  $\mathcal{A}^0(u)$  by

$$A^0(p) = A_R(u_R, p) := \int_{B(x_0,R)} A(y, u_R, p) dy, \tag{3.11}$$

$$\mathcal{A}^0(v) := \int_{B(x_0,R)} A^0(Dv) dx, \tag{3.12}$$

where

$$u_R = u_{x_0,R} = \int_{B(x_0,R)} u(y) dy.$$

Let  $v \in H^{1,2}(B_R)$  be a minimizer of  $\mathcal{A}^0$  in the class

$$\{v \in H^{1,2}(B(R)) ; u - v \in H_0^{1,2}(B(R))\}.$$

Then, by [3, Theorem 3.1], we have the following.

**LEMMA 3.1.** *Let  $v$  be as above, then  $v$  satisfies*

$$\int_{B(r)} |Dv|^2 dx \leq c \left(\frac{r}{R}\right)^\lambda \int_{B(R)} |Dv|^2 dx, \tag{3.13}$$

where

$$\lambda = \min\{2 + \varepsilon, m\} \tag{3.14}$$

for some positive constant  $\varepsilon$ . Here,  $\varepsilon$  and  $c$  do not depend on  $r, R, x_0$ .

Moreover, we have the following  $L^p$ -estimate for  $v$  as a direct consequence of [15, Lemma 1].

LEMMA 3.2. *Let  $v$  be as above. Suppose that  $Du \in L^{2+\delta}$  for some  $\delta \in (0, 1)$ . Then for some  $\delta_0 \in (0, \delta)$  and  $c > 0$ , we have*

$$\int_{B(R)} |Dv|^p dx \leq c \int_{B(R)} |Du|^p dx \tag{3.15}$$

for every  $p \in (2, 2 + \delta_0)$ .

We show the partial regularity of  $u$  by comparing  $u$  with  $v$ . For this purpose, we need the following lemma.

LEMMA 3.3 ([10, Lemma 2.1]). *Let  $v$  be as above. Then we have*

$$\int_{B(R)} |Du - Dv|^2 dx \leq c \{ \mathcal{A}^0(u) - \mathcal{A}^0(v) \}. \tag{3.16}$$

We have the following reverse Hölder type inequality for  $u$ .

LEMMA 3.4 ([9, Theorem 4.1]). *Let  $u \in H^{1,2}(\Omega, \mathbf{R}^n)$  be a minimizer of the functional  $\mathcal{A}(u, \Omega)$ . Then*

$$Du \in L_{loc}^{s_0}(\Omega, \mathbf{R}^n) \text{ for some } s_0 > 2.$$

Moreover, there exist positive constants  $C = C(\nu_0, \|A_{ij}^{\alpha\beta}\|_\infty)$  and  $\bar{R}$  such that for every ball  $B(x, R) \subset B(x, \bar{R})$  with  $B(x, 2R) \subset \Omega$  and every  $p \in (2, s_0)$  the following estimate holds.

$$\left( \int_{B(x,R)} |Du|^p dx \right)^{\frac{1}{p}} \leq C \left( \int_{B(x,2R)} |Du|^2 dx \right)^{\frac{1}{2}}.$$

Now, we can prove our main theorem.

PROOF OF THEOREM 2.5. Let  $w = u - v$ . First we will estimate  $\int_{B(R)} |Dw|^2 dx$ . By Lemma 3.3 we can see that

$$\begin{aligned}
 \int_{B(R)} |Dw|^2 dx &\leq c\{\mathcal{A}^0(u) - \mathcal{A}^0(v)\} \\
 &\leq c \int_{B(R)} |A_R(u_R, Du) - A(x, u_R, Du)| dx \\
 &\quad + c \int_{B(R)} |A(x, u_R, Du) - A(x, u, Du)| dx \\
 &\quad + c \int_{B(R)} |A(x, v, Dv) - A(x, u_R, Dv)| dx \\
 &\quad + c \int_{B(R)} |A(x, u_R, Dv) - A_R(u_R, Dv)| dx. \tag{3.17}
 \end{aligned}$$

Here we used the minimality of  $u$ . So, using the assumptions on  $A$ , we get

$$\begin{aligned}
 \int_{B(R)} |Dw|^2 dx &\leq \int_{B(R)} \{\sigma(x, R) + \omega(|u - u_R|^2)\} |Du(x)|^2 dx \\
 &\quad + \int_{B(R)} \{\sigma(x, R) + \omega(|v - u_R|^2)\} |Dv(x)|^2 dx. \tag{3.18}
 \end{aligned}$$

Using Lemma 3.4 and the boundedness of  $\omega$  and  $\sigma$ , we have

$$\begin{aligned}
 &\int_{B(R)} \{\sigma(x, R) + \omega(|u - u_R|^2)\} |Du(x)|^2 dx \\
 &\leq \left\{ \left( \int \sigma(x, R)^{\frac{p-2}{p-2}} dx \right)^{\frac{p-2}{p}} + \left( \int \omega(|u - u_R|^2)^{\frac{p-2}{p-2}} dx \right)^{\frac{p-2}{p}} \right\} \left( \int |Du|^p dx \right)^{\frac{2}{p}} \\
 &\leq C \left\{ \left( \int_{B(R)} \sigma(x, R) dx \right)^{\frac{p-2}{p}} + \left( \int_{B(R)} \omega(|u - u_R|^2) dx \right)^{\frac{p-2}{p}} \right\} \cdot \int_{B(2R)} |Du|^2 dx. \tag{3.19}
 \end{aligned}$$

Using Lemma 3.2, we get similarly

$$\begin{aligned}
 &\int_{B(R)} \{\sigma(x, R) + \omega(|v - u_R|^2)\} |Dv(x)|^2 dx \\
 &\leq C \left\{ \left( \int_{B(R)} \sigma(x, R) dx \right)^{\frac{p-2}{p}} + \left( \int_{B(R)} \omega(|v - u_R|^2) dx \right)^{\frac{p-2}{p}} \right\} \cdot \int_{B(2R)} |Dv|^2 dx. \tag{3.20}
 \end{aligned}$$

By virtue of concavity of  $\omega$ , using Jensen’s inequality and Poincaré inequality, we have



$$\begin{aligned} & \int_{B(R)} \omega(|u - u_R|^2) dx, \quad \int_{B(R)} \omega(|v - u_R|^2) dx \\ & \leq C\omega \left( R^{2-m} \int_{B(R)} |Du|^2 dx \right). \end{aligned} \tag{3.21}$$

Combining (3.18)–(3.21), we obtain

$$\begin{aligned} & \int_{B(R)} |Dw|^2 dx \\ & \leq C \left\{ \left( \int_{B(R)} \sigma(x, R) dx \right)^{\frac{p-2}{p}} + \omega \left( R^{2-m} \int_{B(R)} |Du|^2 dx \right)^{\frac{p-2}{p}} \right\} \cdot \int_{B(2R)} |Du|^2 dx. \end{aligned} \tag{3.22}$$

Now, from Lemma 3.1 and (3.22), we get

$$\begin{aligned} \int_{B(r)} |Du|^2 dx & \leq \int_{B(r)} (|Dv|^2 + |Dw|^2) dx \\ & \leq C \left\{ \left( \frac{r}{R} \right)^\lambda + \left( \int_{B(R)} \sigma(x, R) dx \right)^{\frac{p-2}{p}} + \omega \left( R^{2-m} \int_{B(R)} |Du|^2 dx \right)^{\frac{p-2}{p}} \right\} \\ & \quad \cdot \int_{B(2R)} |Du|^2 dx. \end{aligned} \tag{3.23}$$

By the assumption (A-1), we have

$$\int_{B(R)} \sigma(x, R) dx \rightarrow 0 \quad \text{as } R \rightarrow 0.$$

So, using “A useful lemma” on p. 44 of [8] and [8, Theorem 6.1], we get the assertion.  $\square$

Now, Corollary 2.6 is a direct consequence of Theorem 2.5 and Morrey’s theorem on the growth of the Dirichlet integral (see, for example, p. 43 of [8]).

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