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3-transposition groups of symplectic type and vertex operator algebras

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Abstract. The 3-transposition groups that act on vertex operator algebras in the way described by Miyamoto in [**Mi**] are classified under the assumption that the group is centerfree and the VOA carries a positive-definite invariant Hermitian form.

1. Introduction.

Let $V = \bigoplus_{n\in\boldsymbol{Z}} V^n$ be a vertex operator algebra (VOA) over the field $\boldsymbol{C}$ of complex numbers with $V^n = 0$ for $n < 0$, $V^0 = \boldsymbol{C}\mathbf{1}$ where $\mathbf{1}$ is the vacuum vector and $V^1 = 0$ having a positive-definite Hermitian form. In this paper we will denote the operation of VOA by $Y(a,z) = \sum_{n\in\boldsymbol{Z}} a_{(n)} z^{-n-1}$.

Suppose that V^2 is spanned by a set E such that each $e \in E$ generates an action of the Virasoro algebra of central charge $1/2$ by which the space V decomposes to $V_e(0) \oplus V_e(1/2)$ where $V(h)$ is the sum of irreducible components of lowest conformal weight h.

In [**Mi**], Miyamoto showed that the set D of the automorphisms of V of the form

$$\sigma_e = \begin{cases} \mathrm{id}, & \text{on } V_e(0), \\ -\mathrm{id}, & \text{on } V_e(1/2), \end{cases} \tag{1}$$

generates a 3-transposition group G with D being the class of 3-transpositions if the set E is stable under the action of D.

Let us say in this paper that a 3-transposition group (G, D) is *realizable by a VOA* (or *realized by* V) if it is obtained in this way. 3-transposition groups possibly realizable by code VOAs were considered by Kitazume and Miyamoto in [**KM**].

The purpose of this paper is to give a complete classification of the realizable 3-transposition groups without assuming that the VOA is a code VOA. Namely, we will show the following result:

THEOREM 1. *A centerfree 3-transposition group is realizable by a VOA having a positive-definite Hermitian form if and only if it is the direct product of a finite number of groups of the following type*:

$$S_n,\ (n \geq 3);\ F : S_n,\ (n \geq 4);\ F^2 : S_n,\ (n \geq 4);$$

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$O_6^-(2),\ 2^6 : O_6^-(2),\ O_8^-(2);\ Sp_6(2),\ 2^6 : Sp_6(2),\ Sp_8(2);\ O_8^+(2),\ 2^8 : O_8^+(2),\ O_{10}^+(2);$

where $F = 2^{2m}$ *if* $n = 2m+1$ *or* $2m+2$.

The groups $F^2 : S_n$ and $O_{10}^+(2)$ are in fact realized by the VOA $V_{\sqrt{2}D_n}^+$ and by $V_{\sqrt{2}E_8}^+$ respectively, as already observed in [**DLMN**] and [**Gr**], their 3-transposition subgroups are realized by taking subVOAs, and the direct product is realized by taking the tensor product of VOAs.

In deriving our result, the key is the following arguments, both of which use the existence of a positive-definite invariant Hermitian form on our VOA. One is to consider the decomposition with respect not only to the Virasoro action of central charge 1/2 but also to that of central charge 7/10. Then, using the classification of unitary highest weight representations of Virasoro minimal models ([**FQS**]), we will see that a subgroup of shape $3^2 : 2$ does not occur if the group is realizable by a VOA with a positive-definite invariant Hermitian form, so that our (G, D) is a 3-transposition group of symplectic type (Section 4, Proposition 1). We can now use the classification of such groups by J. I. Hall [**Ha1**], [**Ha2**] (cf. [**Fi**]).

The other is to consider the adjacency matrix of a certain graph associated to (G, D). We will see that if (G, D) is realizable by a VOA with a positive-definite invariant Hermitian form then the least eigenvalue of the matrix must be greater than or equal to -8 (Section 5, Proposition 2). We will compute the least eigenvalues for the 3-transposition groups of symplectic type (Table 2); we can eliminate most groups from our list.

We then check, using some general results on subVOAs (Section 6, Proposition 4), that the groups that passed these tests (Table 3) are all realized by taking subVOAs of $V_{\sqrt{2}R}^+$ associated with a simply-laced root system R (Sections 7 and 8).

We refer the reader to [**Bor**], [**FHL**] and [**MN**] for generalities on VOAs, to [**Gi**] and [**DMZ**] for Virasoro actions, and to [**Fi**], [**CH**] and [**As**] for 3-transposition groups.

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2. 3-transposition automorphism groups of VOAs.

Let $V = \bigoplus_{n \in \boldsymbol{Z}} V^n$ be a vertex operator algebra (VOA) with the vacuum vector $\mathbf{1} \in V^0$ and the conformal vector $\omega \in V^2$ (called the Virasoro element in [**FLM**]).

Throughout the paper, we assume the following.

(I) $V = \bigoplus_{n=0}^{\infty} V^n$, $V^0 = \boldsymbol{C}\mathbf{1}$ and $V^1 = 0$.

We set $B = V^2$. Then the space B has a structure of a nonassociative algebra equipped with an invariant symmetric bilinear form by setting

$$a \cdot b = a_{(1)} b, \quad (a|b)\mathbf{1} = a_{(3)} b. \tag{2}$$

Then the conformal vector $\omega \in V^2$ is twice the unit of B. The subspace B with this structure is called the Griess algebra of the VOA V.

We next assume the following condition, which is the most crucial in our argument:

(II) The VOA V has a positive-definite invariant Hermitian form.

In particular, the form $(\ |\ )$ on B restricted onto the real part agrees with the restriction of the Hermitian form on V normalized so that the norm of the vacuum vector is 1.

Now consider a real vector e satisfying $e \cdot e = 2e$ and $(e|e) = 1/4$. Then the operators $L_n = e_{(n+1)}$ give rise to an action of Virasoro algebra of central charge 1/2. By the assumption (II), the space V decomposes as $V = V_e(0) \oplus V_e(1/2) \oplus V_e(1/16)$, where $V_e(h)$ is the sum of the components isomorphic to the irreducible Virasoro module of lowest conformal weight h.

Let E be a set of such vectors e as above which further satisfies that the component $V_e(1/16)$ is zero. For each $e \in E$, consider the involution σ_e on V given by (1) and suppose that the set E is invariant under the involutions. Then, by Miyamoto [**Mi**], the set $D_V = \{\sigma_e \,|\, e \in E\}$ generates a 3-transposition subgroup G_V of the full automorphism group $\mathrm{Aut}(V)$. Namely, for any $e, f \in E$, the order of $\sigma_e \sigma_f$ is either 1, 2 or 3. More precisely, we have one of the following (cf. [**Co**]).

(i) $(e|f) = 1/4$ and $e = f$.
(ii) $(e|f) = 0$ and $e \cdot f = 0$.
(iii) $(e|f) = 1/32$ and $e \cdot f = (1/4)(e + f - g)$ where $g = \sigma_e(f) = \sigma_f(e)$.

We will write $e = f$, $e \perp f$ and $e \sim f$ when the pair e, f falls into (i), (ii) and (iii) respectively. In the case (iii), we will denote the element g by $e \circ f$, which belongs to the set E.

We now assume the following:

(III) The VOA V is generated by B and the space B is spanned by the set E.

Let G_E be the image of the group G_V in the symmetric group on the set E. Under the assumption (III), the natural map $G_V \to G_E$ is an isomorphism.

We say that a 3-transposition group (G, D) is *realizable* by a VOA if it is isomorphic to (G_V, D_V) as 3-transposition groups for a VOA V satisfying (I)–(III) with a set E. We often say that G is *realized* by V as far as the choice of D and E is clear from the context or it is not relevant in the context.

3. Indecomposability and centerfreeness.

Let V be a VOA satisfying (I)–(III) and suppose given a decomposition $E = E' \sqcup E''$ such that $E' \perp E''$. Let D' and D'' be the corresponding sets of involutions and let G' and G'' be the subgroups of $G = G_V$ generated by D' and D''. Let V' and V'' be the subVOA generated by E' and E'' respectively. Then 3-transposition groups (G', D') and (G'', D'') are realized by V' and V'' respectively. (See Proposition 4 for a related statement.)

Conversely, if 3-transposition groups (G', D') and (G'', D'') are realized by VOAs V' and V'' respectively, then the direct product $(G' \times G'', (D' \times 1) \sqcup (1 \times D''))$ is realized by the tensor product $V' \otimes V''$ of VOAs.

Hence the classification of realizable 3-transposition groups reduces to that of indecomposable ones.

Now let E be indecomposable: E does not have nontrivial orthogonal decompositions. If E consists of one element e then the involution σ_e generate a group of order 2 which is abelian. In order to avoid exceptions caused by this trivial case, we will assume the following.

(IV) The set E is indecomposable and $|E| \geq 3$.

Under the assumptions (I)–(IV), the map $\sigma : E \to D$ is bijective and the group G is centerfree. We will sometimes identify the sets D and E via the map σ.

4. 3-transposition groups of symplectic type.

Let (G, D) be an indecomposable centerfree 3-transposition group. Consider subgroups of G isomorphic to S_3 generated by some elements of D. Two of such subgroups generate either $S_3 \times S_3$, S_4 or a group of shape $3^2 : 2$. If the latter group does not occur then (G, D) is called a 3-transposition group of symplectic type ([**Ha1**], [**Ha2**]).

The following is the key observation of this paper, where the assumption (II) is crucial.

Proposition 1. *A 3-transposition group realizable by a VOA is of symplectic type.*

Proof. Suppose that a 3-transposition group is realized by a VOA V and consider the set E. Suppose that E contains a subset X with which the associated involutions generate $3^2 : 2$. Then the configuration of X is the affine plane of order 3: we may index the set as $X = \{x_{i,j} \,|\, i, j \in \boldsymbol{Z}/3\boldsymbol{Z}\}$ so that if $(i, j) \neq (k, \ell)$ then $x_{i,j} \sim x_{k,\ell}$ and $x_{i,j} \circ x_{k,\ell} = x_{i+k,j+\ell}$. Now consider the following vectors:

$$\eta = \frac{4}{5}(x_{00} + x_{01} + x_{02}) - x_{00}, \quad w = x_{10} + x_{11} + x_{12} - x_{20} - x_{21} - x_{22}. \tag{3}$$

Then since $(w|w) = 3/2$ the vector w is nonzero, and the vector η generates an action of Virasoro algebra of central charge $c = 7/10$, for which the vector w is of conformal weight $h = 7/10$. However, this is impossible because the lowest conformal weight of unitary irreducible highest weight representations of central charge $c = 7/10$ are only 0, 1/10, 3/5, 3/2, 3/80 and 7/16 ([**FQS**]). □

Centerfree 3-transposition groups of symplectic type are classified by J. I. Hall in [**Ha1**] and [**Ha2**].

Theorem (J. I. Hall). *An indecomposable centerfree 3-transposition group of symplectic type is isomorphic to the extension of one of the groups $S_3; S_n$, $(n \geq 5); Sp_{2n}(2)$, $(n \geq 3); O^+_{2n}(2)$, $(n \geq 4)$; and $O^-_{2n}(2)$, $(n \geq 3)$, by the direct sum of copies of the natural module.*

Here the natural module, which we will denote by F in the sequel, is isomorphic to 2^{2n} for $O^{\pm}_{2n}(2)$, or $Sp_{2n}(2)$. Since S_{2n+1} and S_{2n+2} are embedded in the symplectic transformation on the space 2^{2n}, we understand that the natural module for these groups is 2^{2n}. Note that $S_4 \simeq 2^2 : S_3$.

Consider the *reduced* case of being without an extension by the natural modules.

G	$\|D\|$	$G^{(1)}$	$G^{(2)}$
S_3	3	—	—
S_4	6	S_2	—
S_n $n \geq 5$	$\frac{n(n-1)}{2}$	S_{n-2}	S_{n-3}
$O^+_{2n}(2)$ $n \geq 4$	$2^{2n-1} - 2^{n-1}$	$Sp_{2n-2}(2)$	$O^-_{2n-2}(2)$
$O^-_{2n}(2)$ $n \geq 3$	$2^{2n-1} + 2^{n-1}$	$Sp_{2n-2}(2)$	$O^+_{2n-2}(2)$
$Sp_{2n}(2)$ $n \geq 3$	$2^{2n} - 1$	$2^{2n-2} : Sp_{2n-2}(2)$	$Sp_{2n-2}(2)$

Table 1. Inductive structure of 3-transposition groups of symplectic type.

Then the group G is a rank 3 permutation group on the set D. Let $D^{(1)}$ be the elements of D which commutes with a fixed element of D and $D^{(2)}$ be that with two fixed noncommuting elements of D. (They are usually denoted by D_d and $D_{d,e}$ respectively.) The elements of $D^{(2)}$ actually commute with 3 elements that correspond to a subgroup isomorphic to S_3. We set $G^{(1)} = \langle D^{(1)} \rangle$ and $G^{(2)} = \langle D^{(2)} \rangle$.

Table 1 summarizes the inductive structure of those 3-transposition groups of symplectic type (cf. [**We**]).

5. Least eigenvalue of the graph.

Let (G, D) be an indecomposable centerfree 3-transposition group of symplectic type. For distinct elements $x, y \in D$, we write $x \perp y$ if x and y commute and $x \sim y$ otherwise. In the latter case, we denote by $x \circ y$ the other involution in the S_3 generated by x and y.

Assign the symbol $\tilde{x}$ to each $x \in D$ and regard the set $\{\tilde{x} \mid x \in D\}$ as a basis of the vector space

$$\tilde{B} = \bigoplus_{x \in D} \boldsymbol{C}\tilde{x}. \tag{4}$$

We make this space into an algebra equipped with a symmetric bilinear form by the same rule as in the algebra B. Namely,

(i)′ $(\tilde{x}|\tilde{y}) = 1/4$ and $\tilde{x} \cdot \tilde{y} = 2\tilde{y}$ when $x = y$,
(ii)′ $(\tilde{x}|\tilde{y}) = 0$ and $\tilde{x} \cdot \tilde{y} = 0$ when $x \perp y$,
(iii)′ $(\tilde{x}|\tilde{y}) = 1/32$ and $\tilde{x} \cdot \tilde{y} = (1/4)(\tilde{x} + \tilde{y} - \widetilde{x \circ y})$ when $x \sim y$.

Now suppose that (G, D) is realized by a VOA V. Then we have a canonical surjective homomorphism of algebras $\tilde{B} \to B$ which preserves the symmetric bilineary forms.

Let A be the adjacency matrix of the graph on the vertex set D given by the relation $x \sim y$.

Proposition 2. *The least eigenvalue of the adjacency matrix A of a 3-transposition group realizable by a VOA is greater than or equal to -8.*

G	ν	k	λ	s	g
S_3	3	2	1	-1	2
S_n $n \geq 4$	$\frac{n(n-1)}{2}$	$2n-4$	$n-2$	-2	$\frac{n(n-3)}{2}$
$O^+_{2n}(2)$ $n \geq 4$	$2^{2n-1}-2^{n-1}$	$2^{2n-2}-2^{n-1}$	$2^{2n-3}-2^{n-2}$	-2^{n-2}	$\frac{2^{2n}-4}{3}$
$O^-_{2n}(2)$ $n \geq 3$	$2^{2n-1}+2^{n-1}$	$2^{2n-2}+2^{n-1}$	$2^{2n-3}+2^{n-1}$	-2^{n-1}	$\frac{2^{2n}+3\cdot 2^n+2}{6}$
$Sp_{2n}(2)$ $n \geq 3$	$2^{2n}-1$	2^{2n-1}	2^{2n-2}	-2^{n-1}	$2^{2n-1}+2^{n-1}-1$

Table 2. Parameters of the graph.

PROOF. The Gram matrix of the form on $\tilde{B}$ is given by

$$\frac{1}{4}\left(I+\frac{1}{8}A\right). \tag{5}$$

Then since the form $(\ |\ )$ on B is positive-definite by the assumption (II), the form on $\tilde{B}$ is positive-semidefinite. In other words, the least eigenvalue of the matrix A is greater than or equal to -8. $\square$

Let us compute the least eigenvalue of the graph for each indecomposable centerfree 3-transposition group of symplectic type. We first consider the reduced case. From Table 1, we can compute the standard parameters (ν, k, λ, μ) of the graph by standard techniques (cf. [**Bos**], [**Hi**]) using the inductive structure summarized in Table 1. For our later convenience, we include the multiplicity g of the least eigenvalue s. The results are summarized in Table 2.

Let us next consider the case with an extension. By the theorem of Hall mentioned above, an indecomposable centerfree 3-transposition group is of the form $F^m : G$ for some m where F is the natural module over a reduced group G. Then the graph associated with $F^m : G$ is given by the set $2^m \times X$, where X is the graph for G, with (p, x) and (q, y) being adjacent if and only if $x \sim y$. Hence the Gram matrix for $F^m : G$ is given by

$$\underbrace{\begin{pmatrix}1&1\\1&1\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}1&1\\1&1\end{pmatrix}}_{m}\otimes A. \tag{6}$$

Therefore the eigenvalues are simply 0 and 2^m times the eigenvalues of A. In particular, the least eigenvalue is $2^m s$ as the least eigenvalue s for the reduced G is negative.

Among the groups in Table 2, the ones with $s \geq -8$ are only:

$$S_3;\ S_n,\ (n \geq 5);\ O_8^+(2);\ O_{10}^+(2);\ O_6^-(2);\ O_8^-(2);\ Sp_6(2);\ Sp_8(2)$$

G	ν	k	s	g	c	d
S_3	3	2	-1	2	$\frac{6}{5}$	3
$S_n\ n \geq 4$	$\frac{n(n-1)}{2}$	$2n-4$	-2	$\frac{n(n-3)}{2}$	$\frac{n(n-1)}{n+2}$	$\frac{n(n-1)}{2}$
$F : S_n\ n \geq 4$	$n(n-1)$	$4n-8$	-4	$\frac{n(n-3)}{2}$	$n-1$	$n(n-1)$
$F^2 : S_n\ n \geq 4$	$2n(n-1)$	$8n-16$	-8	$\frac{n(n-3)}{2}$	n	$\frac{(3n-1)n}{2}$
$O_6^-(2)$	36	20	-4	15	$\frac{36}{7}$	36
$Sp_6(2)$	63	32	-4	35	$\frac{63}{10}$	63
$O_8^+(2)$	120	56	-4	84	$\frac{15}{2}$	120
$2^6 : O_6^-(2)$	72	40	-8	15	6	57
$2^6 : Sp_6(2)$	126	64	-8	35	7	91
$2^8 : O_8^+(2)$	240	112	-8	84	8	156
$O_8^-(2)$	136	72	-8	51	$\frac{34}{5}$	85
$Sp_8(2)$	255	128	-8	135	$\frac{15}{2}$	120
$O_{10}^+(2)$	496	240	-8	340	8	156

Table 3. List of realizable groups.

and the allowed extensions are only:

$$S_4 = 2^2 : S_3;\ F : S_n;\ F^2 : S_n,\ (n \geq 4);\ 2^8 : O_8^+(2);\ 2^6 : O_6^-(2);\ 2^6 : Sp_6(2).$$

The parameters are listed in Table 3, where c denotes the possible central charge and d denotes the possible dimension of B: they are determined by the formulas

$$c = \frac{4\nu}{k+8} \quad \text{and} \quad d = \begin{cases} \nu, & \text{if } s > -8, \\ \nu - g, & \text{if } s = -8. \end{cases} \tag{7}$$

6. Vertex operator subalgebras.

In order to show that the groups in Table 3 are realizable by VOAs, we need some general results on vertex operator subalgebras.

We mean by a vertex operator subalgebra of a VOA V, or rather by a *subVOA*, a graded subspace U which has a structure of a VOA such that the operations and the grading of U agrees with the restriction of those of V and that U and V share the same vacuum vector. However, we do *not* assume that they share the same conformal vector. When they do, we will call U a *full* subVOA.

Let S be any subset of V. Consider the subspace $\langle S\rangle_{\mathrm{VOA}}$ generated by S, i.e., the span of the elements of V of the form $a^1_{(n_1)}a^2_{(n_2)}\cdots a^k_{(n_k)}\mathbf{1}$ where k is a nonnegative integer, $a^1,\ldots,a^k \in S$ and $n_1,\ldots,n_k \in \boldsymbol{Z}$. Then the subspace $\langle S\rangle_{\mathrm{VOA}}$ is in fact closed under the operations of the VOA V. If it owns a conformal vector with appropriate properties then it is a subVOA in our sense.

Lemma 3. *Let V be a VOA and let U be a graded subspace of V containing the vacuum vector $\mathbf{1}$ of V such that $U \cap V^2$ is nonzero. Suppose that U is closed under the operations of the VOA V and that there exists a subspace W such that $V = U \oplus W$ and $U_{(n)}W \subseteq W$. Then U has a unique conformal vector that gives U a structure of a subVOA of V.*

Proof. Let $\omega = \xi + \eta$ be the decomposition of the conformal vector ω of V with respect to $V = U \oplus W$. We will show that the vector ξ has the desired properties. First note that $\xi, \eta \in V^2$, so $2\xi = \omega\cdot\xi = (\xi+\eta)\cdot\xi = \xi\cdot\xi + \xi\cdot\eta$. Since $\xi \in U$, $\xi\cdot\xi \in U$ and $\xi\cdot\eta \in W$, we have $\xi\cdot\xi = 2\xi$ and $\xi\cdot\eta = 0$. Hence the vector ξ gives rise to an action of the Virasoro algebra on U. Furthermore, for any $u \in U \cap V^n$, we have $nu = L_0 u = \omega_{(1)}u = (\xi+\eta)_{(1)}u = \xi_{(1)}u + \eta_{(1)}u$. Since nu and $\xi_{(1)}u$ belong to U and $\eta_{(1)}u$ belongs to W, we see that $L^{\xi}_0 u = \xi_{(1)}u = nu$. Hence the grading of U with respect to L^{ξ}_0 agrees with the restriction of the grading of V. Finally, for any $u \in U$ we have $u_{(-2)}\mathbf{1} = \omega_{(0)}u = \xi_{(0)}u + \eta_{(0)}u$. However, since $u_{(-2)}\mathbf{1} \in U$, $\xi_{(0)}u \in U$ and $\eta_{(0)}u \in W$, we see that $u_{(-2)}\mathbf{1} = \xi_{(0)}u$. The uniqueness is obvious. □

The following proposition will be used in Section 8 to show that certain subgroups of realizable 3-transposition groups are again realizable.

Proposition 4. *Let V be a VOA satisfying the conditions* (I) *and* (II). *Let A be a nontrivial subalgebra of B such that $A = \{a \in B \mid a\cdot T = 0\}$ for some real subset T of B. Then the VOA $\langle A\rangle_{\mathrm{VOA}}$ generated by A has a unique structure of a subVOA of V such that $\langle A\rangle_{\mathrm{VOA}} \cap V^2 = A$.*

Proof. Set $U = \langle A\rangle_{\mathrm{VOA}}$ and let W be the orthogonal complement of U in V. Then U is a graded subspace of V which is closed under the operations of VOA. So we have $(U|u_{(n)}w) \subseteq (U|w) = 0$ for any real $u \in U$ and $w \in W$. Hence by the lemma above, U has a unique structure of a subVOA of V. Now let $a \in A$ be a real vector and $t \in T$. Note that $(a|t) = (a\cdot u|t) = (u|a\cdot t) = 0$ where $u = \omega/2$ is the unity of B. Hence $a_{(3)}t = (a|t)\mathbf{1} = 0$. Since $(a_{(1)}a)_{(1)}t + (a_{(2)}a)_{(0)}t = a_{(2)}(a_{(0)}t) - a_{(1)}(a_{(1)}t) + a_{(1)}(a_{(1)}t) - a_{(0)}(a_{(2)}t)$, we have $a_{(2)}(a_{(0)}t) = 0$. Hence $(a_{(0)}t|a_{(0)}t) = (t|a_{(2)}a_{(0)}t) = 0$ and we have $a_{(0)}t = 0$ by the condition (II). Thus $A_{(n)}T = 0$ for all $n \geq 0$, which yields $U_{(n)}T$ for all $n \geq 0$. In particular, we have $(U \cap V^2)\cdot T = 0$, which shows that $U \cap V^2 = A$. □

7. Groups related to root systems.

Let us recall the structures of certain VOAs associated with $\sqrt{2}$ times root lattices which realizes 3-transposition groups.

Let R be a root system of simply-laced type of rank greater than 1. We denote the root lattice by the same symbol R. Let $\sqrt{2}R$ be the root lattice with the norm

R	G	G'	c_η	eigenvalues of η on B
A_{n-1}	$F : S_n$	S_n	$\frac{2(n-1)}{n+2}$	$0,\ \frac{6}{n+2}, \frac{n+4}{n+2}, 2$
D_n	$F^2 : S_n$	$F : S_n$	1	$0,\ \frac{4}{n},\ 1,\ 2$
E_6	$2^6 : O_6^-(2)$	$O_6^-(2)$	$\frac{6}{7}$	$0,\ \frac{7}{5},\ 2$
E_7	$2^6 : Sp_6(2)$	$Sp_6(2)$	$\frac{7}{10}$	$0,\ \frac{3}{5},\ 2$
E_8	$2^8 : O_8^+(2)$	$O_8^+(2)$	$\frac{1}{2}$	$0,\ \frac{1}{2},\ 2$

Table 4. Groups related to root systems.

being multiplied by 2, which is denoted by $R(2)$ in other areas of mathematics. Let $V_{\sqrt{2}R}$ denote the VOA associated with this lattice and let $V = V_{\sqrt{2}R}^+$ be the fixed-point subspace with respect to the involution θ which is a lift of the -1 isometry of the lattice ([**FLM**]).

It is well known that $V_{\sqrt{2}R}^+$ is a full subVOA of $V_{\sqrt{2}R}$ satisfying the properties (I)–(IV). The structure of the algebra B for the VOA $V = V_{\sqrt{2}R}^+$ is described by Dong et al. [**DLMN**].

Let R^+ be the set of positive roots and consider the set $X = 2 \times R^+$. We call a 3-set $\{(p,\alpha),(q,\beta),(r,\gamma)\}$ a line if and only if $\{\alpha,\beta,\gamma\}$ is the set of the three positive roots in a subsystem of type A_2 and $p+q+r \equiv 0$ modulo 2. For distinct elements $x, y \in X$, we write $x \perp y$ if x and y are not on a line and $x \sim y$ otherwise. In the latter case, we denote by $x \circ y$ the other element on the line.

Consider the space $\tilde{B}$ spanned by X as a basis, and give it a structure of an algebra equipped with a symmetric bilinear form by the same rule as in Section 5. Then we have a surjective homomorphism of algebras $\tilde{B} \to B$ which preserves the bilinear forms, which is an isomorphism if and only if the least eigenvalue of the associated graph is equal to -8. The associated group is listed as G in Table 4.

Note 5. The full automorphism group of $V_{\sqrt{2}R}^+$ is described in [**Sh**]. See [**Gr**] for type E_8 and [**MM**] for type D_4.

8. Realizability of certain subgroups.

Let us now show that the rest of the groups in Table 3 are all realizable by VOAs.

As we already know that each of the groups G in Table 4 are realized by the VOA $V_{\sqrt{2}R}^+$, we will show that the subgroup G' is also realizable by a VOA by taking a suitable subVOA of $V_{\sqrt{2}R}^+$.

Let $\tilde{B}$ be as in the preceding section and let $\tilde{B}'$ be the subalgebra spanned by the subset $X' = \{0\} \times R^+$. Let B be the degree 2 subspace of $V_{\sqrt{2}R}^+$ and let B' be the image of $\tilde{B}'$. Let (G', D') be the corresponding subgroup of (G, D) listed in Table 4 and consider the subVOA V' generated by the subalgebra B'. Let us show that the degree 2 subspace of V' agrees with B' to ensure the property (III).

Let $\tilde{\omega}$ be twice the unit of the larger algebra $\tilde{B}$ and let $\tilde{\omega}'$ be that of $\tilde{B}'$. Set $\tilde{\eta} = \tilde{\omega} - \tilde{\omega}'$. Let η denote the image of $\tilde{\eta}$ in B. Then the vector η generates an action of Virasoro algebra, whose central charge is listed as c_η in Table 4.

Lemma 6. *The subspace B' agrees with the eigenspace of eigenvalue 0 with respect to the adjoint action by η on B.*

Proof. Consider the eigenspace decomposition of the large algebra $\tilde{B}$ with respect to the adjoint action by $\tilde{\eta}$. Obviously, the small algebra $\tilde{B}'$ is contained in the eigenspace with the eigenvalue 0. Then the induced action of $\tilde{\eta}$ on the quotient space $\tilde{B}/\tilde{B}'$ is given by the following matrix:

$$\frac{2}{8+k}(8I + A'). \tag{8}$$

Here I is the identity matrix of size $|X'|$ and A' is the adjacency matrix of the graph associated with X'. Now this matrix is positive since the least eigenvalue of the matrix A' is greater than or equal to -4. □

The realizability of the groups (G', D') in Table 4 now follows from Proposition 4.

There are some more groups in Table 3. The group $O_{10}^+(2)$ is also realized by $V_{\sqrt{2}E_8}^+$ as shown in [**Gr**] and the VOA realizing $Sp_8(2)$ is constructed in [**KM**]. Since $O_8^-(2) = O_{10}^+(2)^{(2)}$, we can show by the same argument as above that this group is also realized by an appropriate subVOA of $V_{\sqrt{2}E_8}^+$. Thus the groups in Table 3 are all shown to be realizable by appropriate VOAs, and the proof of Theorem 1 is completed.

Note 7. For E_8 type root system, the central charge of the Virasoro action generated by the vector η is equal to $1/2$. As is well known, this is the origin of the presence of the bigger 3-transposition group $\operatorname{Aut} V_{\sqrt{2}E_8}^+ \simeq O_{10}^+(2)$ as described in [**Gr**].

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