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Classification of singular fibres of stable maps of 4-manifolds into 3-manifolds and its applications

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Abstract. In this paper we classify the singular fibres of stable maps of closed (possibly non-orientable) 4-manifolds into 3-manifolds up to the C^∞ equivalence. Furthermore, we obtain several results on the co-existence of the singular fibres of such maps. As a consequence, we show that under certain conditions, the Euler number of the source 4-manifold has the same parity as the total number of certain singular fibres. This generalises Saeki's result in the orientable case.

1. Introduction.

As pioneers, Kushner, Levine and Porto studied the singular fibres of stable maps of closed 3-manifolds into the plane in [**7**] and [**8**]. However, they did not state clearly the definitions of singular fibres and the equivalence relation among them. Recently, in the book [**15**], Saeki stated the precise definition of singular fibres, introduced an equivalence relation among them, and classified the singular fibres of stable maps of closed orientable 4-manifolds into 3-manifolds. Moreover, he proved the following: For any stable map of an orientable closed 4-manifold into a connected 3-manifold, the number of singular fibres of III^{12} type as depicted in Figure 1 and the Euler number of the source 4-manifold are of the same parity. (In the book [**15**], the symbol "III^{8}" is used instead of "III^{12}".)

Then it is natural to ask:

Is there any formula of the same type if the source manifold is non-orientable?

In this paper we generalise Saeki's Euler number formula, giving an answer to the above question. We first classify the singular fibres in the general case where the source 4-manifold may possibly be non-orientable (see Theorem 2.4). Then we prove Theorem 4.7: Under certain homological conditions, for a stable map $f : M \to N$ of a closed 4-manifold M into a connected 3-manifold N, the total number of certain singular fibres and the Euler number $\chi(M)$ of the source 4-manifold M are of the same parity: namely,

$$\begin{aligned}\chi(M) \equiv |\mathrm{III}^{2,2,2}(f)| + |\mathrm{III}^{2,7}(f)| + |\mathrm{III}^{12}(f)| + |\mathrm{III}^{13}_{e}(f)| \\ + |\mathrm{III}^{13}_{B}(f)| + |\mathrm{III}^{25}(f)| + |\mathrm{III}^{26}(f)| \pmod 2,\end{aligned}$$

where $|\mathscr{F}(f)|$ denotes the number of singular fibres of f of type $\mathscr{F}$. For the notation of

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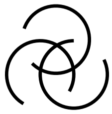

Figure 1. The singular fibre of III^{12} type.

singular fibres, refer to Theorem 2.4 and the succeeding paragraphs.

We note that the C^∞ equivalence classes of fibres over points of N give a natural stratification of N. Then the set of regular values $N \setminus f(S(f))$ consists of 3-dimensional strata, while the set of singular values $f(S(f))$ consists of 2-, 1- and 0-dimensional strata, where $S(f)$ ($\subset M$) denotes the set of singular points of f. We assign to each 3-dimensional stratum the number of connected components of the fibre over a point in the stratum. Note that the number is constant on each stratum.

Under certain homological conditions, which will be called the *two colourable condition*, for $f|_{S(f)} : S(f) \to N$ there exist disjoint open subsets R and B of N such that $R \cup B = N \setminus f(S(f))$ and $\overline{R} \cap \overline{B} = \partial R = \partial B = f(S(f))$ (for details, see §5). For points $q \in f(S(f))$, by combining the colourings (i.e. R or B) of the 3-dimensional strata adjacent to q and the numbers of connected components of the fibres corresponding to these 3-dimensional strata, we can divide several C^∞ equivalence classes of singular fibres into two types A and B. In the formula of Theorem 4.7, III^{13}_B denotes such a subclass of III^{13}. We note that III^{13}_e is also a subclass of III^{13}, which consists of those singular fibres of type III^{13} with an even number of connected components.

If M is orientable, then $f|_{S(f)} : S(f) \to N$ always satisfies the two colourable condition (for details, see §5). In other words, the assumption of Theorem 4.7 is automatically satisfied. Furthermore, the singular fibres of types other than III^{12} in the formula never appear. Thus Theorem 4.7 gives the Euler number formula obtained in [**15**] when the source 4-manifold is orientable.

In [**6**] Kobayashi constructed a stable map $g : \boldsymbol{C}P^2 \to \boldsymbol{R}^3$ such that $g(S(g))$ has exactly two triple points, which correspond to the singular fibres $\mathrm{III}^{0,0,0}$ and III^{12}. Theorem 4.7 implies that $\chi(\boldsymbol{C}P^2)$ is odd. In fact, we have $\chi(\boldsymbol{C}P^2) = 3$. In [**12**, Example 3.7] Saeki constructed a non-orientable closed 4-manifold E as the total space of an $\boldsymbol{R}P^2$-bundle over $\boldsymbol{R}P^2$ together with a stable map $h : E \to \boldsymbol{R}^3$ such that $h(S(h))$ has 27 triple points. They consist of eight $\mathrm{III}^{0,0,0}$ points, twelve $\mathrm{III}^{0,0,2}$ points, six $\mathrm{III}^{0,2,2}$ points and one $\mathrm{III}^{2,2,2}$ point. Theorem 4.7 shows that $\chi(E)$ must be an odd number. Actually, we have $\chi(E) = 1$.

We have some direct consequences of Theorem 4.7 as follows.

Corollary 1.1. *Let M be a closed 4-manifold with odd Euler number and $f : M \to N$ a stable map of M into a connected 3-manifold N with $H_1(N; \boldsymbol{Z}_2) = 0$. Then $f|_{S(f)}$ has at least one triple point.*

Corollary 1.2. *Let f be a stable map as in Corollary 1.1. Then f has at least one singular fibre of III^{12} type or I^2 type.*

Corollary 1.2 is proved as follows. The stable map f has at least one singular fibre as appearing in the formula of Theorem 4.7, since $\chi(M)$ is odd. If the source manifold of f is orientable, then f must have a singular fibre of type III^{12}. If the source manifold is non-orientable and f has no singular fibre of type III^{12}, then by using the description of local nearby fibres of singular fibres of types other than III^{12} appearing in Theorem 4.7 (e.g. see Figure 6), we see that f has a singular fibre of type I^2.

We say that a stable map is *simple* if each connected component of every fibre has at most one singular point.

COROLLARY 1.3. *Let f be a stable map as in Corollary* 1.1. *If f is simple, then f has at least one singular fibre of type* $\mathrm{III}^{2,2,2}$.

The hypothesis that the Euler number should be odd in Corollaries 1.1–1.3 is essential. In fact there exists a special generic map $f : S^3 \widetilde{\times} S^1 \to \boldsymbol{R}^3$ such that $f|_{S(f)}$ is an embedding [**13**], where $S^3 \widetilde{\times} S^1$ is the total space of a non-orientable S^3 bundle over S^1. A *special generic map* is a stable map which has only definite fold points as its singular points. In fact, all the singular fibres of f are of type I^0. We note that $\chi(S^3 \widetilde{\times} S^1) = 0$.

The paper is organised as follows. In §2, we give the definitions of equivalence relations among the fibres of stable maps. Furthermore, we classify the singular fibres up to C^∞ equivalence, and give a table of singular fibres for stable maps of closed possibly non-orientable 4-manifolds into 3-manifolds. In §3, we give some co-existence relations of singular fibres for stable maps of closed 4-manifolds into 3-manifolds. In §4, we show that the Euler number of the source manifold has the same parity as the total number of certain singular fibres (Theorem 4.7 and Proposition 4.5).

Throughout the paper, all manifolds and maps are of class C^∞ unless otherwise stated. For a finite set P, we denote by $|P|$ the number of its elements. For a topological space Y and a subset $X \subset Y$, $\overline{X}$ is the topological closure of X in Y, ∂X is the boundary of X which is defined to be $\overline{X} \setminus X$, and $\mathrm{Int}(X)$ is the interior of X which is defined to be $X \setminus \partial X$. We denote by $\chi(X)$ the Euler number of X. A part of the results of this paper has been obtained in the author's master's thesis [**17**].

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2. Singular fibres of stable maps of 4-manifolds into 3-manifolds.

In this section, we classify the singular fibres of stable maps of closed 4-manifolds into 3- manifolds.

Let us begin with some fundamental definitions.

DEFINITION 2.1. Let M_i be smooth manifolds and A_i subsets of M_i, $i = 0, 1$. A continuous map $g : A_0 \to A_1$ is said to be *smooth* if for every point $q \in A_0$, there exists a smooth map $\tilde{g} : V \to M_1$ defined on a neighbourhood V of q in M_0 such that $\tilde{g}|_{V \cap A_0} = g|_{V \cap A_0}$. A smooth map $g : A_0 \to A_1$ is a *diffeomorphism* if it is a homeomorphism and its inverse is also smooth. When there exists a diffeomorphism between A_0 and A_1, we say that they are *diffeomorphic.*

Let $f_i : M_i \to N_i$ be smooth maps, $i = 0, 1$. For $q_i \in N_i$, $i = 0, 1$, we say that the fibres over q_0 and q_1 are *diffeomorphic* if $f_0^{-1}(q_0) \subset M_0$ and $f_1^{-1}(q_1) \subset M_1$ are diffeomorphic in the above sense. Furthermore, we say that the fibres over q_0 and q_1 are C^∞ *equivalent* or *right-left equivalent* if for some open neighbourhood U_i of q_i, there exist diffeomorphisms $\Phi : f_0^{-1}(U_0) \to f_1^{-1}(U_1)$ and $\varphi : U_0 \to U_1$ with $\varphi(q_0) = q_1$ which make the following diagram commutative:

$$\begin{array}{ccc} \big(f_0^{-1}(U_0), f_0^{-1}(q_0)\big) & \xrightarrow{\Phi} & \big(f_1^{-1}(U_1), f_1^{-1}(q_1)\big) \\ {\scriptstyle f_0}\big\downarrow & & \big\downarrow{\scriptstyle f_1} \\ (U_0, q_0) & \xrightarrow{\varphi} & (U_1, q_1). \end{array}$$

If $q \in N$ is a regular value of a smooth map $f : M \to N$ between manifolds, then we call the map germ $f : (M, f^{-1}(q)) \to (N, q)$ along the set $f^{-1}(q)$ a *regular fibre*; otherwise, we call it a *singular fibre*.

Let us recall a characterization of stable maps of closed 4-manifolds into 3-manifolds. For smooth manifolds M and N, let us denote by $C^\infty(M, N)$ the space of all C^∞-maps $M \to N$, equipped with the Whitney C^∞-topology. In general, we say that $f \in C^\infty(M, N)$ is C^∞ *stable* (or *stable* for short) if the $\mathscr{A}$-orbit of f is open in $C^\infty(M, N)$. Here the $\mathscr{A}$-orbit of $f \in C^\infty(M, N)$ is defined as follows. Let $\mathrm{Diff}(N)$ denote the group of self-diffeomorphisms of N. Then the group $\mathrm{Diff}(M) \times \mathrm{Diff}(N)$ acts on $C^\infty(M, N)$ by $(\Phi, \Psi)f = \Psi \circ f \circ \Phi^{-1}$, where $(\Phi, \Psi) \in \mathrm{Diff}(M) \times \mathrm{Diff}(N)$ and $f \in C^\infty(M, N)$. Then the $\mathscr{A}$-orbit of $f \in C^\infty(M, N)$ is the orbit through f with respect to this action.

Proposition 2.2. *A smooth map $f : M \to N$ of a closed 4-manifold M into a 3-manifold N is C^∞ stable if and only if the following conditions are satisfied.*

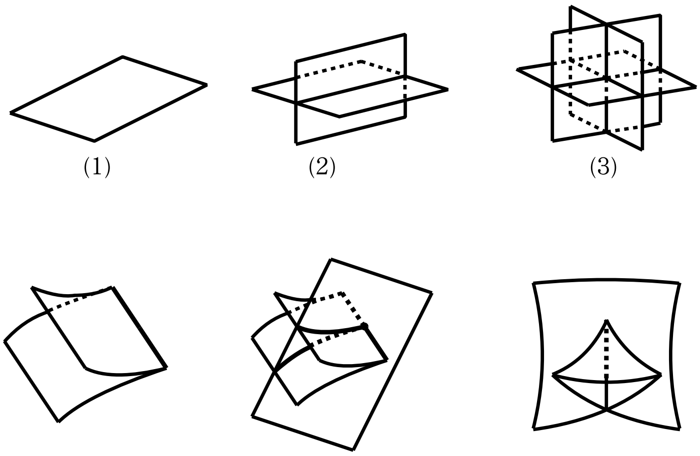
(i) (*Local condition*) *For every $p \in M$, there exist local coordinates (x, y, z, w) and (X, Y, Z) around $p \in M$ and $f(p) \in N$, respectively, such that one of the following holds*:

$$(X \circ f, Y \circ f, Z \circ f)$$
$$= \begin{cases} (x, y, z), & p : \textit{regular point,} \\ (x, y, z^2 + w^2), & p : \textit{definite fold point,} \\ (x, y, z^2 - w^2), & p : \textit{indefinite fold point,} \\ (x, y, z^3 + xz - w^2), & p : \textit{cusp point,} \\ (x, y, z^4 + xz^2 + yz + w^2), & p : \textit{definite swallow-tail point,} \\ (x, y, z^4 + xz^2 + yz - w^2), & p : \textit{indefinite swallow-tail point.} \end{cases}$$

(ii) (*Global condition*) *Set $S(f) = \{p \in M | \mathrm{rank}\, df_p < 3\}$, which is a closed 2-dimensional submanifold of M under the above local condition. Then, for every $q \in f(S(f))$, $f^{-1}(q) \cap S(f)$ consists of at most three points and the multi-germ*

$$\big(f|_{S(f)}, f^{-1}(q) \cap S(f)\big)$$

is right-left equivalent to one of the six multi-germs as described in Figure 2 : (1) *corresponds to a single fold point,* (2) *and* (3) *represent normal crossings of two and three immersion germs, respectively, each of which corresponds to a fold point,*



(1) (2) (3)

(4) (5) (6)

Figure 2. Multi-germs of $f|_{S(f)}$.

(4) *corresponds to a cusp point,* (5) *represents a transverse crossing of a cuspidal edge as in* (4) *and an immersion germ corresponding to a fold point as in* (1)*, and* (6) *corresponds to a swallow-tail point.*

Proposition 2.2 can be proved by using the transversality theorem and the multi-transversality theorem, since the dimensions pair $(4, 3)$ is in the nice range in the sense of Mather [**10**] (for details, see [**4**], [**9**] or [**5**]).

Let $f : M \to N$ be a stable map of a closed 4-manifold M into a 3-manifold N. For each regular point $x \in M$ of f, the fibre through x is a 1-dimensional submanifold near the point. For each singular point $p \in M$ of f, based on the local condition of Proposition 2.2, it is easy to determine the diffeomorphism type of a neighbourhood of p in $f^{-1}(f(p))$ as follows.

Lemma 2.3. *Every singular point p of a stable map $f : M \to N$ of a closed 4-manifold M into a 3-manifold N has one of the following neighbourhoods in its corresponding singular fibre $f^{-1}(f(p))$ (see Figure 3) :*

(i) *isolated point diffeomorphic to $\{(x, y) \in \boldsymbol{R}^2 \mid x^2 + y^2 = 0\}$, if p is a definite fold point,*
(ii) *union of two transverse arcs diffeomorphic to $\{(x, y) \in \boldsymbol{R}^2 \mid x^2 - y^2 = 0\}$, if p is an indefinite fold point,*
(iii) *cuspidal arc diffeomorphic to $\{(x, y) \in \boldsymbol{R}^2 \mid x^3 - y^2 = 0\}$, if p is a cusp point,*
(iv) *isolated point diffeomorphic to $\{(x, y) \in \boldsymbol{R}^2 \mid x^4 + y^2 = 0\}$, if p is a definite swallowtail point,*
(v) *union of two tangent arcs diffeomorphic to $\{(x, y) \in \boldsymbol{R}^2 \mid x^4 - y^2 = 0\}$, if p is an indefinite swallowtail point.*

We note that in Figure 3, both the black dot (1) and the black square (4) represent an isolated point, although the corresponding map germs are not C^∞ equivalent to each

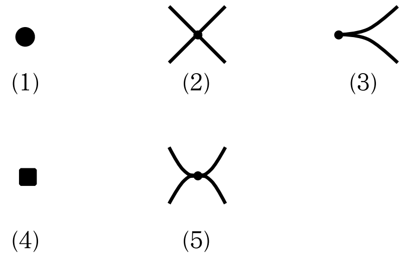


Figure 3. Neighbourhoods of singular points in singular fibres.

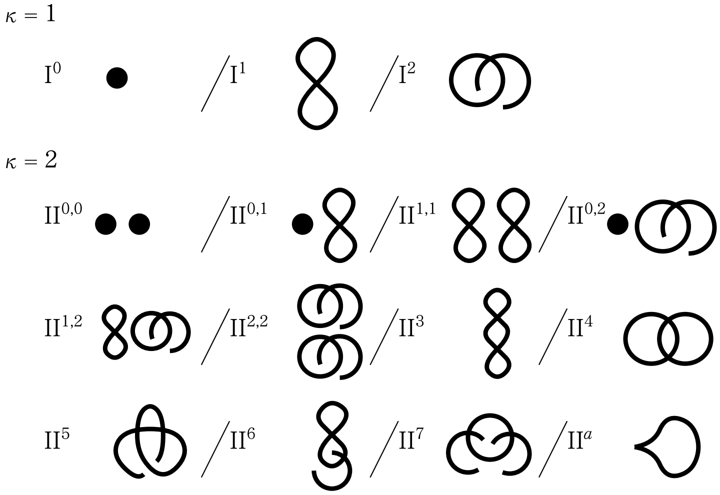


Figure 4. List of singular fibres; 1.

other; we use distinct symbols in order to distinguish them. We note that each singular point $p \in M$, except for a definite fold point and a definite swallow-tail point, is incident to some edges in its neighbourhood in $f^{-1}(f(p))$.

We note that a regular fibre of f is a closed 1-dimensional submanifold of M, namely a disjoint union of a finite number of circles. Thus, for a regular value q of f, the fibre of f over q is C^∞ equivalent to the disjoint union of a finite number of copies of a fibre of a trivial circle bundle. For the singular fibres of f, we have the following.

THEOREM 2.4. *Let $f : M \to N$ be a stable map of a closed 4-manifold M into a 3-manifold N. Then, every singular fibre of f is C^∞ equivalent to the disjoint union of one of the fibres in the following list and a finite number of copies of a fibre of a trivial circle bundle*:

(i) *one of the fibres as depicted in Figure* 4*,*
(ii) *a disconnected fibre* $\mathrm{III}^{0,0,0}$, $\mathrm{III}^{0,0,1}$, $\mathrm{III}^{0,1,1}$, $\mathrm{III}^{1,1,1}$, $\mathrm{III}^{0,0,2}$, $\mathrm{III}^{0,2,2}$, $\mathrm{III}^{1,1,2}$, $\mathrm{III}^{1,2,2}$, $\mathrm{III}^{0,1,2}$, $\mathrm{III}^{2,2,2}$, $\mathrm{III}^{0,3}$, $\mathrm{III}^{0,4}$, $\mathrm{III}^{0,5}$, $\mathrm{III}^{0,6}$, $\mathrm{III}^{0,7}$, $\mathrm{III}^{1,3}$, $\mathrm{III}^{1,4}$, $\mathrm{III}^{1,5}$, $\mathrm{III}^{1,6}$, $\mathrm{III}^{1,7}$, $\mathrm{III}^{2,3}$, $\mathrm{III}^{2,4}$, $\mathrm{III}^{2,5}$, $\mathrm{III}^{2,6}$, $\mathrm{III}^{2,7}$, $\mathrm{III}^{0,a}$, $\mathrm{III}^{1,a}$ *or* $\mathrm{III}^{2,a}$*,*
(iii) *one of the connected fibres as depicted in Figure* 5*.*

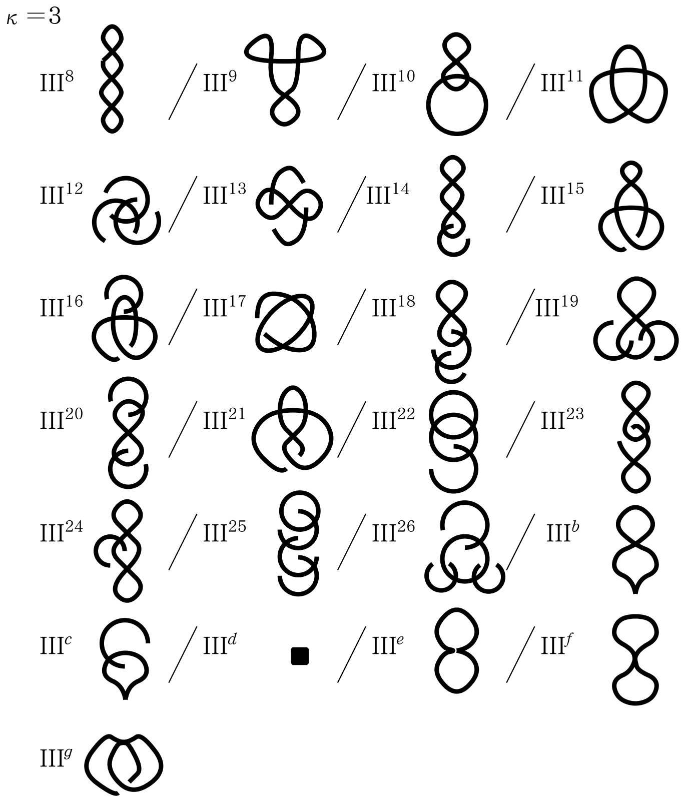


Figure 5. List of singular fibres; 2.

The figure corresponding to each fibre listed in Theorem 2.4 (2) can be obtained by taking the disjoint union of the fibres in Figure 4 corresponding to the numbers or letters appearing in the superscript. For example, the figure of the fibre $\mathrm{III}^{0,0,2}$ consists of two dots and a figure of I^2 type.

In Figures 4 and 5, κ denotes the codimension of the set of points in N whose corresponding fibres are C^∞ equivalent to the relevant one (for details, see [**15**]). Furthermore, I^*, II^*, III^* mean the names of the corresponding singular fibres, and "/" is used only for separating the figures.

We note that the list of singular fibres as in Figure 4 coincides with that appearing in the introduction of [**8**].

We note that the conclusion of Theorem 2.4 holds if f is proper even if M is not closed, where a continuous map is said to be *proper* if the inverse image of a compact set is always compact.

Theorem 2.4 can be proved in two steps. First we show that for a singular value q of f, the union of the components of $f^{-1}(q)$ containing singular points is diffeomorphic to one of the fibres listed in Theorem 2.4 in the sense of Definition 2.1. Second we show that if two singular fibres are diffeomorphic to each other, then they are C^∞ equivalent in the sense of Definition 2.1, except for the two types of fibres I^0 and III^d. The proof is very similar to that of [**15**, Theorem 3.5], as we omit the proof here.

Remark 2.5. Each singular fibre described in Theorem 2.4 can be realized as a component (or as a union of some components) of a singular fibre of a stable map of a closed 4-manifold into $\boldsymbol{R}^3$. This can be seen as follows. Given a singular fibre, we can realize it as a singular fibre of a Morse function parameterized on D^2, $f_t : S \to [-1,1], t \in D^2$, of a compact surface with boundary S into $[-1,1]$, where D^2 denotes the unit disk in $\boldsymbol{R}^2$. We note that $F : S \times D^2 \to [-1,1] \times D^2$, defined by $F(x,t) = (f_t(x), t)$, is a smooth map and that F has the given singular fibre over $(0,0)$. We call S *a transverse surface* corresponding to the singular fibre (for details, see [**8**]). In this way we obtain a proper smooth map $F|_{\mathrm{Int}(S\times D^2)} : \mathrm{Int}(S \times D^2) \to \mathrm{Int}([-1,1] \times D^2)$. Then we can extend the map to a smooth map of a closed 4-manifold containing $\mathrm{Int}(S \times D^2)$ into $\boldsymbol{R}^3$. Perturbing the extended map slightly, we obtain a desired stable map.

If the source 4-manifold is orientable, then any transverse surface for any singular fibre is orientable. If the source 4-manifold is non-orientable, then there may exist a non-orientable transverse surface. The transverse surface which corresponds to the singular fibre of I^2 type is a punctured Möbius band. We note that there exists a stable map of a non-orientable 4-manifold into a 3-manifold such that the transverse surface is orientable for any fibre. (For instance, see the example just after Corollary 1.3 in §1.)

We note that for a stable map $f : M \to N$ of an orientable closed 4-manifold M into a 3-manifold N, the singular fibres of the following types never appear, since they have non-orientable transverse surfaces: I^2, $\mathrm{II}^{0,2}$, $\mathrm{II}^{1,2}$, $\mathrm{II}^{2,2}$, II^5, II^6, II^7, $\mathrm{III}^{0,0,2}$, $\mathrm{III}^{0,2,2}$, $\mathrm{III}^{1,1,2}$, $\mathrm{III}^{1,2,2}$, $\mathrm{III}^{0,1,2}$, $\mathrm{III}^{2,2,2}$, $\mathrm{III}^{0,5}$, $\mathrm{III}^{0,6}$, $\mathrm{III}^{0,7}$, $\mathrm{III}^{1,5}$, $\mathrm{III}^{1,6}$, $\mathrm{III}^{1,7}$, $\mathrm{III}^{2,3}$, $\mathrm{III}^{2,4}$, $\mathrm{III}^{2,5}$, $\mathrm{III}^{2,6}$, $\mathrm{III}^{2,7}$, $\mathrm{III}^{2,a}$, III^{13}, III^{14}, III^{15}, III^{16}, III^{17}, III^{18}, III^{19}, III^{20}, III^{21}, III^{22}, III^{23}, III^{24}, III^{25} III^{26}, III^c and III^g.

Remark 2.6. For stable maps f of 4-manifolds into 3-manifolds, the triple points of $f|_{S(f)}$ correspond to the singular fibre of types $\mathrm{III}^{0,0,0}$, $\mathrm{III}^{0,0,1}$, $\mathrm{III}^{0,1,1}$, $\mathrm{III}^{1,1,1}$, $\mathrm{III}^{0,0,2}$, $\mathrm{III}^{0,2,2}$, $\mathrm{III}^{1,1,2}$, $\mathrm{III}^{1,2,2}$, $\mathrm{III}^{0,1,2}$, $\mathrm{III}^{2,2,2}$, $\mathrm{III}^{0,3}$, $\mathrm{III}^{0,4}$, $\mathrm{III}^{0,5}$, $\mathrm{III}^{0,6}$, $\mathrm{III}^{0,7}$, $\mathrm{III}^{1,3}$, $\mathrm{III}^{1,4}$, $\mathrm{III}^{1,5}$, $\mathrm{III}^{1,6}$, $\mathrm{III}^{1,7}$, $\mathrm{III}^{2,3}$, $\mathrm{III}^{2,4}$, $\mathrm{III}^{2,5}$, $\mathrm{III}^{2,6}$, $\mathrm{III}^{2,7}$, III^8, III^9, III^{10}, III^{11}, III^{12}, III^{13}, III^{14}, III^{15}, III^{16}, III^{17}, III^{18}, III^{19}, III^{20}, III^{21}, III^{22}, III^{21}, III^{22}, III^{23}, III^{24}, III^{25} and III^{26} of f. Thus the number of triple points of $f|_{S(f)}$ coincides with the total number of singular fibre of types as above.

For stable maps of a closed 4-manifolds into 3-manifolds, if the source 4-manifolds are orientable, then the classification of the singular fibres with respect to the C^∞ equivalence coincides with respect to the C^0 equivalence (for details, see [**15**, Corollary 3.9]). Thus we pose the following problem.

Problem 2.7. For two singular fibres of proper stable maps of (possibly non-orientable) 4-manifolds into 3-manifolds, are the following two equivalent?

(1) They are C^∞ equivalent.
(2) They are C^0 equivalent.

3. Relations among the numbers of singular fibres.

Let $f : M \to N$ be a stable map of a closed 4-manifold M into a 3-manifold N. In this section, we consider a natural stratification of N induced by the C^∞ equivalence classes of the fibres of f, and obtain some relations among the numbers of singular fibres of codimension three.

Let $f : M \to N$ be as above, and $\mathscr{F}$ be a C^∞ equivalence class of one of the singular fibres appearing in Theorem 2.4. We define $\mathscr{F}(f)$ to be the set of points $q \in N$ such that the fibre $f^{-1}(q)$ over q is C^∞ equivalent to the union of $\mathscr{F}$ and some regular fibres. Then we obtain a "stratification" of N which consists of the components of $\mathscr{F}(f)$ together with $N \setminus f(S(f))$, where $\mathscr{F}$ runs over all C^∞ equivalence classes of singular fibres[1].

We define $\mathscr{F}_o(f)$ (resp. $\mathscr{F}_e(f)$) to be the subset of $\mathscr{F}(f)$ consisting of the points $q \in \mathscr{F}(f)$ such that the number of connected components of $f^{-1}(q)$ is odd (resp. even). It is easy to see that the closures $\overline{\mathscr{F}(f)}$, $\overline{\mathscr{F}_o(f)}$ and $\overline{\mathscr{F}_e(f)}$ of $\mathscr{F}(f)$, $\mathscr{F}_o(f)$ and $\mathscr{F}_e(f)$, respectively, in N are $(3-\kappa)$-dimensional complexes in N, where κ is the codimension of $\mathscr{F}$. In particular, if the codimension κ is equal to two, then $\overline{\mathscr{F}_o(f)}$ and $\overline{\mathscr{F}_e(f)}$ are finite graphs embedded in N. Their vertices correspond to points over which f has a singular fibre of codimension three. For a C^∞ equivalence class $\mathscr{G}$ of singular fibres of codimension three, the degree of the vertex corresponding to $\mathscr{G}_o(f)$ (or $\mathscr{G}_e(f)$) in the graph $\overline{\mathscr{F}_o(f)}$ is given in Tables 1, 2 and 3. In the tables, only non-zero degrees are given: an empty column means that the corresponding degree is equal to zero. We note that the graphs $\overline{\mathrm{II}^*_o(f)}$ or $\overline{\mathrm{II}^*_e(f)}$, or the vertices $\mathrm{III}^*_o(f)$ or $\mathrm{III}^*_e(f)$ may possibly be empty depending on the stable map f.

These tables can be obtained by using the description of local nearby fibres as shown in Figure 6. We note that the degrees in the graph $\overline{\mathscr{F}_e(f)}$ can be obtained by interchanging $\mathscr{G}_o(f)$ with $\mathscr{G}_e(f)$ in the table corresponding to the graph $\overline{\mathscr{F}_o(f)}$.

The handshake lemma of the classical graph theory claims that for a finite graph, the sum of the degrees over all vertices is equal to the double of the number of edges and hence is always even. We apply this lemma to the graphs $\overline{\mathrm{II}^{0,0}_o(f)}$, $\overline{\mathrm{II}^{0,0}_e(f)}$, $\overline{\mathrm{II}^{0,1}_o(f)}$, $\overline{\mathrm{II}^{0,1}_e(f)}$, $\overline{\mathrm{II}^{1,1}_o(f)}$, $\overline{\mathrm{II}^{1,1}_e(f)}$, $\overline{\mathrm{II}^{0,2}_o(f)}$, $\overline{\mathrm{II}^{0,2}_e(f)}$, $\overline{\mathrm{II}^{1,2}_o(f)}$, $\overline{\mathrm{II}^{1,2}_e(f)}$, $\overline{\mathrm{II}^{2,2}_o(f)}$, $\overline{\mathrm{II}^{2,2}_e(f)}$, $\overline{\mathrm{II}^{3}_o(f)}$, $\overline{\mathrm{II}^{3}_e(f)}$, $\overline{\mathrm{II}^{4}_o(f)}$, $\overline{\mathrm{II}^{4}_e(f)}$, $\overline{\mathrm{II}^{5}_o(f)}$, $\overline{\mathrm{II}^{5}_e(f)}$, $\overline{\mathrm{II}^{6}_o(f)}$, $\overline{\mathrm{II}^{6}_e(f)}$, $\overline{\mathrm{II}^{7}_o(f)}$, $\overline{\mathrm{II}^{7}_e(f)}$, $\overline{\mathrm{II}^{a}_o(f)}$ and $\overline{\mathrm{II}^{a}_e(f)}$. Then we obtain 24 relations among the numbers of elements of $\mathscr{G}_o(f)$ and $\mathscr{G}_e(f)$ for C^∞ equivalence classes $\mathscr{G}$ of singular fibres of codimension three. We combine the relation obtained from $\overline{\mathrm{II}^*_o(f)}$ and that obtained from $\overline{\mathrm{II}^*_e(f)}$. Then we obtain the following.

Proposition 3.1. *Let $f : M \to N$ be a stable map of a closed 4-manifold M into a 3-manifold N. Then the following numbers are always even:*

(1) $|\mathrm{III}^{0,a}(f)| + |\mathrm{III}^{d}(f)|$,
(2) $|\mathrm{III}^{1,a}(f)| + |\mathrm{III}^{8}(f)|$,
(3) $|\mathrm{III}^{2,a}(f)| + |\mathrm{III}^{c}(f)|$,
(4) $|\mathrm{III}^{2,a}(f)| + |\mathrm{III}^{14}(f)|$,
(5) $|\mathrm{III}^{20}(f)|$,
(6) $|\mathrm{III}^{8}(f)| + |\mathrm{III}^{11}(f)| + |\mathrm{III}^{17}(f)| + |\mathrm{III}^{b}(f)| + |\mathrm{III}^{f}(f)|$,
(7) $|\mathrm{III}^{11}(f)| + |\mathrm{III}^{13}(f)| + |\mathrm{III}^{21}(f)| + |\mathrm{III}^{e}(f)|$,
(8) $|\mathrm{III}^{17}(f)| + |\mathrm{III}^{21}(f)| + |\mathrm{III}^{g}(f)|$,
(9) $|\mathrm{III}^{14}(f)| + |\mathrm{III}^{c}(f)|$,
(10) $|\mathrm{III}^{13}(f)| + |\mathrm{III}^{20}(f)|$,
(11) $|\mathrm{III}^{0,a}(f)| + |\mathrm{III}^{1,a}(f)| + |\mathrm{III}^{b}(f)|$.

[1]In this paper, each stratum of a stratification may not necessarily be connected.

	$* = 0,0$	$0,1$	$1,1$	$0,2$	$1,2$	$2,2$	3	4	5	6	7	a
$\mathrm{III}_o^{0,0,0}(f)$	3											
$\mathrm{III}_e^{0,0,0}(f)$	3											
$\mathrm{III}_o^{0,0,1}(f)$	1	2										
$\mathrm{III}_e^{0,0,1}(f)$	1	2										
$\mathrm{III}_o^{0,1,1}(f)$		2	1									
$\mathrm{III}_e^{0,1,1}(f)$		2	1									
$\mathrm{III}_o^{1,1,1}(f)$			3									
$\mathrm{III}_e^{1,1,1}(f)$			3									
$\mathrm{III}_o^{0,0,2}(f)$	2			2								
$\mathrm{III}_e^{0,0,2}(f)$				2								
$\mathrm{III}_o^{0,2,2}(f)$				4		1						
$\mathrm{III}_e^{0,2,2}(f)$						1						
$\mathrm{III}_o^{1,1,2}(f)$			2		2							
$\mathrm{III}_e^{1,1,2}(f)$					2							
$\mathrm{III}_o^{1,2,2}(f)$					4	1						
$\mathrm{III}_e^{1,2,2}(f)$						1						
$\mathrm{III}_o^{0,1,2}(f)$		2		1	1							
$\mathrm{III}_e^{0,1,2}(f)$				1	1							
$\mathrm{III}_o^{2,2,2}(f)$						6						
$\mathrm{III}_e^{2,2,2}(f)$												
$\mathrm{III}_o^{0,3}(f)$		2					1					
$\mathrm{III}_e^{0,3}(f)$		2					1					
$\mathrm{III}_o^{0,4}(f)$		4						1				
$\mathrm{III}_e^{0,4}(f)$								1				
$\mathrm{III}_o^{0,5}(f)$		2		2					1			
$\mathrm{III}_e^{0,5}(f)$									1			
$\mathrm{III}_o^{0,6}(f)$		2		1						1		
$\mathrm{III}_e^{0,6}(f)$				1						1		
$\mathrm{III}_o^{0,7}(f)$				4							1	
$\mathrm{III}_e^{0,7}(f)$											1	
$\mathrm{III}_o^{1,3}(f)$			2				1					
$\mathrm{III}_e^{1,3}(f)$			2				1					
$\mathrm{III}_o^{1,4}(f)$			4					1				
$\mathrm{III}_e^{1,4}(f)$								1				
$\mathrm{III}_o^{1,5}(f)$			2		2				1			
$\mathrm{III}_e^{1,5}(f)$									1			
$\mathrm{III}_o^{1,6}(f)$			2		1					1		
$\mathrm{III}_e^{1,6}(f)$					1					1		
$\mathrm{III}_o^{1,7}(f)$					4						1	
$\mathrm{III}_e^{1,7}(f)$											1	
$\mathrm{III}_o^{2,3}(f)$					2		2					
$\mathrm{III}_e^{2,3}(f)$					2							
$\mathrm{III}_o^{2,4}(f)$					4			2				
$\mathrm{III}_e^{2,4}(f)$												

Table 1. The degree of each vertex in the graphs $\overline{\mathrm{II}_o^*(f)}$.

	$*=0,0$	$0,1$	$1,1$	$0,2$	$1,2$	$2,2$	3	4	5	6	7	a
$\mathrm{III}^{2,5}_o(f)$					2	2			2			
$\mathrm{III}^{2,5}_e(f)$												
$\mathrm{III}^{2,6}_o(f)$					2	1				2		
$\mathrm{III}^{2,6}_e(f)$						1						
$\mathrm{III}^{2,7}_o(f)$						4					2	
$\mathrm{III}^{2,7}_e(f)$												
$\mathrm{III}^{0,a}_o(f)$		1										1
$\mathrm{III}^{0,a}_e(f)$	1											1
$\mathrm{III}^{1,a}_o(f)$			1									1
$\mathrm{III}^{1,a}_e(f)$		1										1
$\mathrm{III}^{2,a}_o(f)$					1							2
$\mathrm{III}^{2,a}_e(f)$				1								
$\mathrm{III}^{8}_o(f)$							3					
$\mathrm{III}^{8}_e(f)$			1				2					
$\mathrm{III}^{9}_o(f)$							3					
$\mathrm{III}^{9}_e(f)$							3					
$\mathrm{III}^{10}_o(f)$							4	1				
$\mathrm{III}^{10}_e(f)$								1				
$\mathrm{III}^{11}_o(f)$							3	3				
$\mathrm{III}^{11}_e(f)$												
$\mathrm{III}^{12}_o(f)$								6				
$\mathrm{III}^{12}_e(f)$												
$\mathrm{III}^{13}_o(f)$								1	4		1	
$\mathrm{III}^{13}_e(f)$												
$\mathrm{III}^{14}_o(f)$							2			2		
$\mathrm{III}^{14}_e(f)$					1					1		
$\mathrm{III}^{15}_o(f)$							2		1	2		
$\mathrm{III}^{15}_e(f)$									1			
$\mathrm{III}^{16}_o(f)$									2	2	2	
$\mathrm{III}^{16}_e(f)$												
$\mathrm{III}^{17}_o(f)$							3		3			
$\mathrm{III}^{17}_e(f)$												
$\mathrm{III}^{18}_o(f)$										4	1	
$\mathrm{III}^{18}_e(f)$											1	
$\mathrm{III}^{19}_o(f)$										4	1	
$\mathrm{III}^{19}_e(f)$											1	
$\mathrm{III}^{20}_o(f)$										4	1	
$\mathrm{III}^{20}_e(f)$						1						
$\mathrm{III}^{21}_o(f)$								1	3	2		
$\mathrm{III}^{21}_e(f)$												
$\mathrm{III}^{22}_o(f)$								2		4		
$\mathrm{III}^{22}_e(f)$												
$\mathrm{III}^{23}_o(f)$							2			2		
$\mathrm{III}^{23}_e(f)$										2		
$\mathrm{III}^{24}_o(f)$							2			2		
$\mathrm{III}^{24}_e(f)$										2		

Table 2. The degree of each vertex in the graphs $\overline{\mathrm{II}^{*}_o(f)}$.

	$*=0,0$	$0,1$	$1,1$	$0,2$	$1,2$	$2,2$	3	4	5	6	7	a
$\mathrm{III}_o^{25}(f)$											6	
$\mathrm{III}_e^{25}(f)$												
$\mathrm{III}_o^{26}(f)$											6	
$\mathrm{III}_e^{26}(f)$												
$\mathrm{III}_o^{b}(f)$							1					1
$\mathrm{III}_e^{b}(f)$		1										1
$\mathrm{III}_o^{c}(f)$										1		2
$\mathrm{III}_e^{c}(f)$				1								
$\mathrm{III}_o^{d}(f)$												2
$\mathrm{III}_e^{d}(f)$	1											
$\mathrm{III}_o^{e}(f)$								1				2
$\mathrm{III}_e^{e}(f)$												
$\mathrm{III}_o^{f}(f)$							1					
$\mathrm{III}_e^{f}(f)$												2
$\mathrm{III}_o^{g}(f)$									1			2
$\mathrm{III}_e^{g}(f)$												

Table 3. The degree of each vertex in the graphs $\overline{\mathrm{II}_o^*(f)}$.

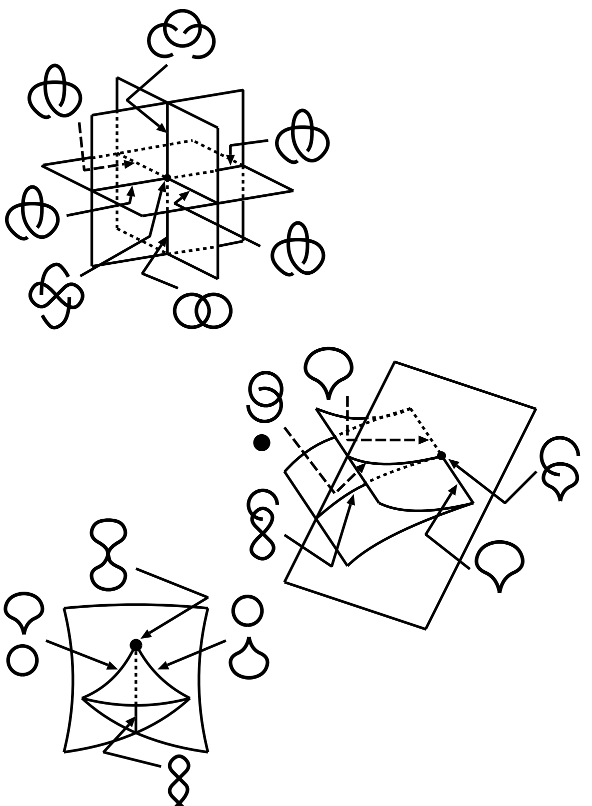


Figure 6. Descriptions of local nearby fibres of types III^{13}, III^c and III^f.

In fact, items (1)–(11) of the above proposition correspond to the relations obtained by combining the relations obtained from $\overline{\mathrm{II}_o^{0,0}(f)}$ and $\overline{\mathrm{II}_e^{0,0}(f)}$, $\overline{\mathrm{II}_o^{1,1}(f)}$ and $\overline{\mathrm{II}_e^{1,1}(f)}$, $\overline{\mathrm{II}_o^{0,2}(f)}$ and $\overline{\mathrm{II}_e^{0,2}(f)}$, $\overline{\mathrm{II}_o^{1,2}(f)}$ and $\overline{\mathrm{II}_e^{1,2}(f)}$, $\overline{\mathrm{II}_o^{2,2}(f)}$ and $\overline{\mathrm{II}_e^{2,2}(f)}$, $\overline{\mathrm{II}_o^{3}(f)}$ and $\overline{\mathrm{II}_e^{3}(f)}$, $\overline{\mathrm{II}_o^{4}(f)}$ and $\overline{\mathrm{II}_e^{4}(f)}$, $\overline{\mathrm{II}_o^{5}(f)}$ and $\overline{\mathrm{II}_e^{5}(f)}$, $\overline{\mathrm{II}_o^{6}(f)}$ and $\overline{\mathrm{II}_e^{6}(f)}$, $\overline{\mathrm{II}_o^{7}(f)}$ and $\overline{\mathrm{II}_e^{7}(f)}$, $\overline{\mathrm{II}_o^{0,1}(f)}$ and $\overline{\mathrm{II}_e^{0,1}(f)}$, respectively. We note that item (11) is also obtained from the two graphs $\overline{\mathrm{II}_o^{a}(f)}$ and $\overline{\mathrm{II}_e^{a}(f)}$.

Remark 3.2. It is easy to see that the eleven numbers appearing in Proposition 3.1 are all even if and only if the following formulae hold:

(1) $|\mathrm{III}^{0,a}(f)| \equiv |\mathrm{III}^{d}(f)| \pmod 2$,
(2) $|\mathrm{III}^{1,a}(f)| \equiv |\mathrm{III}^{8}(f)| \pmod 2$,
(3) $|\mathrm{III}^{2,a}(f)| \equiv |\mathrm{III}^{14}(f)| \equiv |\mathrm{III}^{c}(f)| \pmod 2$,
(4) $|\mathrm{III}^{13}(f)| \equiv |\mathrm{III}^{20}(f)| \equiv 0 \pmod 2$,
(5) $|\mathrm{III}^{d}(f)| + |\mathrm{III}^{e}(f)| + |\mathrm{III}^{f}(f)| + |\mathrm{III}^{g}(f)| \equiv 0 \pmod 2$,
(6) $|\mathrm{III}^{0,a}(f)| + |\mathrm{III}^{1,a}(f)| + |\mathrm{III}^{b}(f)| \equiv 0 \pmod 2$.

We note that the left hand side of the congruence (5) of Remark 3.2 is nothing but the total number of definite and indefinite swallow-tail points of a stable map f. The congruences (3) and (6) of Remark 3.2 imply that the total number of multi-germs which correspond to the transverse intersection of a fold sheet and a cuspidal edge in the target is always even for a stable map f.

By combining the above 24 co-existence relations in a certain manner, we obtain the following co-existence formula:

$$\begin{aligned}
&|\mathrm{III}^{0,0,0}(f)| + |\mathrm{III}^{0,0,1}(f)| + |\mathrm{III}^{0,1,1}(f)| + |\mathrm{III}^{1,1,1}(f)| + |\mathrm{III}^{0,2,2}(f)| + |\mathrm{III}^{1,2,2}(f)| \\
&\quad + |\mathrm{III}^{0,3}(f)| + |\mathrm{III}^{0,4}(f)| + |\mathrm{III}^{0,5}(f)| + |\mathrm{III}^{0,7}(f)| + |\mathrm{III}^{1,3}(f)| + |\mathrm{III}^{1,4}(f)| \\
&\quad + |\mathrm{III}^{1,5}(f)| + |\mathrm{III}^{1,7}(f)| + |\mathrm{III}^{2,6}(f)| + |\mathrm{III}^{8}(f)| + |\mathrm{III}^{9}(f)| + |\mathrm{III}^{10}(f)| \\
&\quad + |\mathrm{III}^{11}(f)| + |\mathrm{III}^{13}(f)| + |\mathrm{III}^{15}(f)| + |\mathrm{III}^{18}(f)| + |\mathrm{III}^{19}(f)| + |\mathrm{III}^{20}(f)| \\
&\quad + |\mathrm{III}^{21}(f)| + |\mathrm{III}_e^{d}(f)| + |\mathrm{III}_e^{e}(f)| + |\mathrm{III}_o^{f}(f)| + |\mathrm{III}_o^{g}(f)| \equiv 0 \pmod 2. \qquad (*)
\end{aligned}$$

This formula $(*)$ will be used in the following section.

4. Parity of the Euler characteristic.

In this section we study the relationship between the total number of certain singular fibres and the Euler number of the source 4-manifold, based on the co-existence results among singular fibres obtained in the previous section.

Let $f : M \to N$ be a stable map of a closed 4-manifold M into a 3-manifold N. Recall the stratification of N by the C^∞ equivalence classes of singular fibres obtained in the previous section. We further subdivide it in the following way. For $n = 0, 1, 2, \ldots$, we define $\mathscr{O}_n(f)$ to be the set of points in $N \setminus f(S(f))$ such that the number of connected components of the associated fibre is equal to n. Then we obtain another stratification of N, which consists of the components of $\mathscr{F}(f)$ together with $\mathscr{O}_n(f)$, where $\mathscr{F}$ runs over

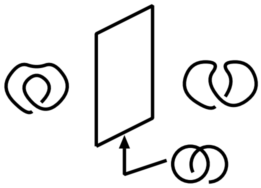


Figure 7. Deformation of the I^2 fibre.

all C^∞ equivalence classes of singular fibres appearing in Theorem 2.4 and $n = 0, 1, 2, \ldots$. Thus the set of regular values $N \setminus f(S(f))$ consists of 3-dimensional strata $\mathscr{O}_n(f)$ ($n = 0, 1, 2, \ldots$), while the set of singular values $f(S(f))$ consists of 2-, 1- and 0-dimensional strata. Throughout this section, we consider this subdivided stratification and not that obtained in § 3. Then we assign to each 3-dimensional stratum $\mathscr{O}_n(f)$ the number n.

Let X be a closed subset of a manifold Y. We say that $Y \setminus X$ has a *two colour decomposition* if there exist non-empty disjoint open subsets R and B of Y such that $Y \setminus X = R \cup B$ and $\overline{R} \cap \overline{B} = \partial R = \partial B = X$. We call the pair (R, B) a *two colour decomposition* (or a *colouring*) for $Y \setminus X$. In the following, we study the condition under which $N \setminus f(S(f))$ has a colouring (R, B) for a stable map $f : M \to N$ as above.

Let $f : M \to N$ be as above, and we define $\triangle_o(f)$ (resp. $\triangle_e(f)$) as the set of points in $N \setminus f(S(f))$ such that the number of connected components of the associated fibre is odd (resp. even). Then $\triangle_o(f)$ and $\triangle_e(f)$ are unions of 3-dimensional strata of the above stratification. It is easy to see that they are disjoint open subsets of N. If M is orientable, then we have $\overline{\triangle_o(f)} \cap \overline{\triangle_e(f)} = \partial\triangle_o(f) = \partial\triangle_e(f) = f(S(f))$, since the difference between the numbers of connected components of the fibres associated with the two 3-dimensional strata adjacent to each 2-dimensional stratum is always equal to one. Therefore, if M is orientable, then the pair $(\triangle_o(f), \triangle_e(f))$ is a colouring for $N \setminus f(S(f))$. If $S \subset f(S(f))$ is a 2-dimensional stratum whose corresponding fibre is of I^2 type, then the numbers of connected components of the fibres associated with the two 3-dimensional strata adjacent to S are the same (see Figure 7). Hence, if f has a fibre of I^2 type, then we have $\overline{\triangle_o(f)} \cap \overline{\triangle_e(f)} \neq f(S(f))$. Thus if M is non-orientable, then the pair ($\triangle_o(f)$, $\triangle_e(f)$) may not be a two colour decomposition for $N \setminus f(S(f))$.

The following lemma which claims that $N \setminus f(S(f))$ has a two colour decomposition under certain homological conditions is well known (see [**11**]).

Lemma 4.1. *Let $g : X \to Y$ be a stable map of a closed surface X into a connected 3-manifold Y such that either $H_1(Y; \boldsymbol{Z}_2) = 0$ or $g_*[X] = 0 \in H_2(Y; \boldsymbol{Z}_2)$, where $[X] \in H_2(X; \boldsymbol{Z}_2)$ is the fundamental class of X. Then $Y \setminus g(X)$ has a two colour decomposition (R, B).*

We call the assumption that $g : X \to Y$ satisfies either $H_1(Y; \boldsymbol{Z}_2) = 0$ or $g_*[X] = 0 \in H_2(Y; \boldsymbol{Z}_2)$, the *two colourable condition.* We note that this is a sufficient condition for the existence of a two colour decomposition and that this is not a necessary condition in general.

Remark 4.2. There is a characterization of stable maps of closed surfaces into 3-manifolds similar to Proposition 2.2. We note that a stable map g of a closed surface

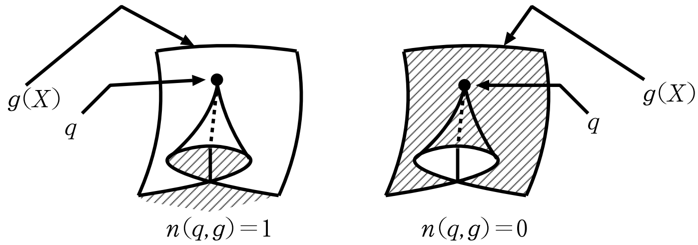


Figure 8. Index of a Whitney umbrella point q.

into a 3-manifold has only Whitney umbrella points as its singular points and g is an immersion with normal crossings outside of the Whitney umbrella points.

We note that if $f : M \to N$ is a stable map of a closed 4-manifold into a 3-manifold, then the singular point set $S(f)$ is a smooth submanifold of M of codimension two and the map $f|_{S(f)} : S(f) \to N$ is a topologically stable singular surface. Here, a smooth map $g : X \to Y$ of a closed surface into a 3-manifold is a *topologically stable singular surface* if there exist a C^∞ stable map $\tilde{g} : \tilde{X} \to \tilde{Y}$ of a closed surface into a 3-manifold, and homeomorphisms $\phi : X \to \tilde{X}$ and $\psi : Y \to \tilde{Y}$ such that $\psi \circ g = \tilde{g} \circ \phi$. We note that Lemma 4.1 can be applied to topologically stable singular surfaces as well. Thus we can apply this lemma to $f|_{S(f)}$. For a stable map $f : M \to N$ as above, if we assume that the map $f|_{S(f)} : S(f) \to N$ satisfies the two colourable condition, then there exists a two colour decomposition (R, B) for $N \setminus f(S(f))$.

The following theorem relates the number of triple points of g in the target manifold and the topology of the source manifold.

Theorem 4.3 (A. Szűcs [**16**], J. J. Nuño Ballesteros-O. Saeki [**11**]). *Let $g : X \to Y$ be a stable map of a closed surface X into a connected 3-manifold Y. Suppose g satisfies the two colourable condition. Then we have*

$$T(g) + \sum_{q:\text{Whitney umbrella}} n(q,g) \equiv \chi(X) \qquad (\text{mod } 2),$$

where $T(g)$ is the total number of triple points of g in the target, and $n(q,g) \in \{0,1\}$ is the index of a Whitney umbrella point q defined by using the two colour decomposition (R, B) for $Y \setminus g(X)$ as in Figure 8 in which the shadowed regions indicate R.

Each swallow-tail point of a stable map f of a 4-manifold into a 3-manifold corresponds to a Whitney umbrella point of the topologically stable singular surface $f|_{S(f)}$. We note that Theorem 4.3 can also be applied to topologically stable singular surfaces.

Let us combine the co-existence relation of singular fibres $(*)$ obtained in § 3 and the relation between the number of singular fibres of f and the number of triple points of $f|_{S(f)}$ obtained in Remark 2.6. Then we obtain

$$\begin{aligned} T(f|_{S(f)}) \equiv & |\mathrm{III}^{0,0,2}(f)| + |\mathrm{III}^{1,1,2}(f)| + |\mathrm{III}^{0,1,2}(f)| + |\mathrm{III}^{2,2,2}(f)| + |\mathrm{III}^{0,6}(f)| \\ & + |\mathrm{III}^{1,6}(f)| + |\mathrm{III}^{2,3}(f)| + |\mathrm{III}^{2,4}(f)| + |\mathrm{III}^{2,5}(f)| + |\mathrm{III}^{2,7}(f)| \\ & + |\mathrm{III}^{12}(f)| + |\mathrm{III}^{14}(f)| + |\mathrm{III}^{16}(f)| + |\mathrm{III}^{17}(f)| + |\mathrm{III}^{22}(f)| \\ & + |\mathrm{III}^{23}(f)| + |\mathrm{III}^{24}(f)| + |\mathrm{III}^{25}(f)| + |\mathrm{III}^{26}(f)| \pmod 2. \end{aligned}$$

If $f : M \to \boldsymbol{R}^3$ is a stable map of a closed 4-manifold M into $\boldsymbol{R}^3$ which has no swallow-tail points, then we have

$$T\bigl(f|_{S(f)}\bigr) \equiv \chi(S(f)) \pmod 2$$

by Theorem 4.3 or by the following result of Banchoff [**1**]: *For any self-transverse immersion $f : X \to \boldsymbol{R}^3$ of a closed surface X into $\boldsymbol{R}^3$, the number of triple points of f in $\boldsymbol{R}^3$ and the Euler number of the surface X have the same parity.* We note that if f has no swallow-tail points, then singular fibres of types III^d, III^e, III^f and III^g never appear.

Furthermore, we have the following.

Theorem 4.4 (T. Fukuda [**2**], O. Saeki [**14**]). *Let $h : V \to N$ be a stable map of a closed n-dimensional manifold V $(n \geq 3)$ into a 3-manifold N. Then we have*

$$\chi(V) \equiv \chi(S(h)) \pmod 2.$$

Thus we obtain the following proposition.

Proposition 4.5. *Let $f : M \to \boldsymbol{R}^3$ be a stable map of a closed 4-manifold into $\boldsymbol{R}^3$ which has no swallow-tail points. Then we have*

$$\begin{aligned} \chi(M) \equiv & |\mathrm{III}^{0,0,2}(f)| + |\mathrm{III}^{1,1,2}(f)| + |\mathrm{III}^{0,1,2}(f)| + |\mathrm{III}^{2,2,2}(f)| + |\mathrm{III}^{0,6}(f)| \\ & + |\mathrm{III}^{1,6}(f)| + |\mathrm{III}^{2,3}(f)| + |\mathrm{III}^{2,4}(f)| + |\mathrm{III}^{2,5}(f)| + |\mathrm{III}^{2,7}(f)| \\ & + |\mathrm{III}^{12}(f)| + |\mathrm{III}^{14}(f)| + |\mathrm{III}^{16}(f)| + |\mathrm{III}^{17}(f)| + |\mathrm{III}^{22}(f)| \\ & + |\mathrm{III}^{23}(f)| + |\mathrm{III}^{24}(f)| + |\mathrm{III}^{25}(f)| + |\mathrm{III}^{26}(f)| \pmod 2. \end{aligned}$$

In the following, we consider a stable map $f : M \to N$ of a closed 4-manifold M into a connected 3-manifold N such that $f|_{S(f)}$ satisfies the two colourable condition. Then, based on a fixed colouring (R, B) for $N \setminus f(S(f))$, we divide several C^∞ equivalence classes of singular fibres into two types, A and B, as follows.

First, for any C^∞ equivalence class $\mathscr{E}$ of singular fibres of codimension one, the 2-dimensional stratum $\mathscr{E}(f)$ is locally adjacent to two 3-dimensional strata. If $\mathscr{E} = \mathrm{I}^0$ or I^1, then the difference between the numbers of connected components of the fibres associated with the two 3-dimensional strata adjacent to $\mathscr{E}(f)$ is always equal to one. Let us take a point $y \in \mathscr{E}(f)$ with $\mathscr{E} = \mathrm{I}^0$ or I^1. If the 3-dimensional stratum adjacent to y which has a larger associated number is contained in R, then we say that the fibre over y is of type $\mathscr{E}_A$, otherwise $\mathscr{E}_B$. In this way, we can divide the stratum $\mathscr{E}(f)$ into $\mathscr{E}_A(f)$

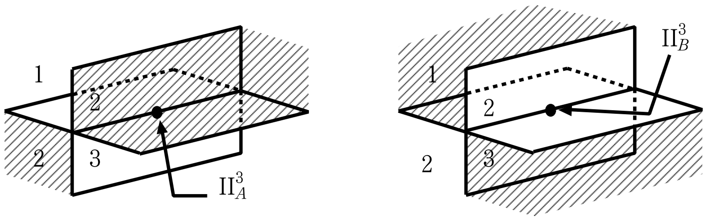


Figure 9. Types A and B for II^3.

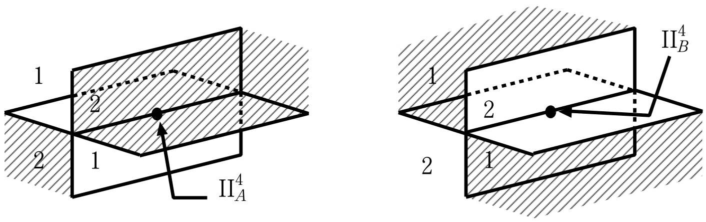


Figure 10. Types A and B for II^4.

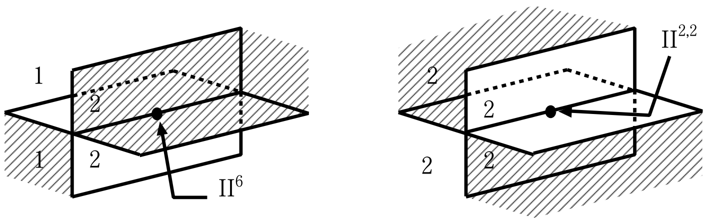


Figure 11. Codimension 2 strata which cannot be divided into two types.

and $\mathscr{E}_B(f)$. If $\mathscr{E} = \mathrm{I}^2$, then the difference between the numbers of connected components of the fibres associated with the two 3-dimensional strata adjacent to $\mathrm{I}^2(f)$ is always equal to zero, as shown in Figure 7. Thus we cannot divide singular fibres of I^2 type into two types by this method.

For any C^∞ equivalence class $\mathscr{F}$ of singular fibres of codimension two, except for II^a, $\mathscr{F}(f)$ is locally adjacent to four 3-dimensional strata. Now we combine the numbers of connected components of the fibres and the "colours" of these 3-dimensional strata. We divide $\mathscr{F}(f)$ into two types A and B in the following way. Let us take a point $y \in \mathscr{F}(f)$ with $\mathscr{F} = \mathrm{II}^{0,0}$, $\mathrm{II}^{0,1}$, $\mathrm{II}^{1,1}$, II^3 or II^5. If the two 3-dimensional strata adjacent to y which are contained in R have the same associated number then we say that the fibre over y is of type $\mathscr{F}_A$, otherwise $\mathscr{F}_B$ (see Figure 9). Let us take a point $y \in \mathscr{F}(f)$ with $\mathscr{F} = \mathrm{II}^4$. If the two 3-dimensional strata adjacent to y which have a larger associated number are contained in R, then we say that the fibre over y is of type $\mathscr{F}_A$, otherwise $\mathscr{F}_B$ (see Figure 10). In a way similar to that for I^0 and I^1, singular fibres of II^a type are divided into two types A and B. In this way, we can divide the stratum $\mathscr{F}(f)$ into $\mathscr{F}_A(f)$ and $\mathscr{F}_B(f)$. However, if $\mathscr{F} = \mathrm{II}^{0,2}$, $\mathrm{II}^{1,2}$, $\mathrm{II}^{2,2}$, II^6 and II^7, then we cannot divide into two types by these method (see Figure 11).

In Figures 9 and 11 the numbers attached to 3-dimensional strata are the numbers

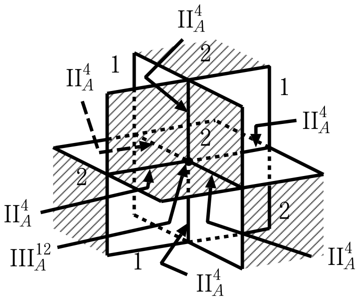


Figure 12. Types A and B for III^{12}.

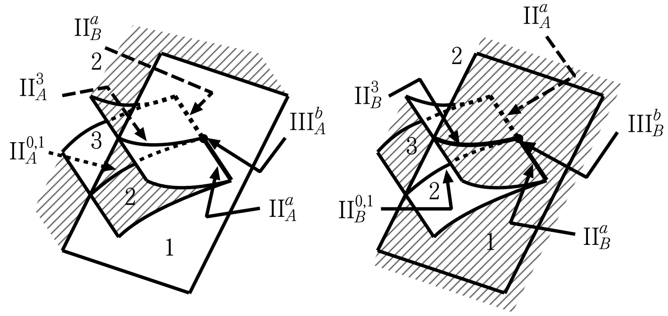


Figure 13. Types A and B for III^b.

of connected components of the associated fibres when the singular value at the centre has no regular component in its fibre. Furthermore, the shadowed regions indicate R.

For the equivalence classes of codimension three, we see that $\mathrm{III}^{0,0,0}(f)$, $\mathrm{III}^{0,0,1}(f)$, $\mathrm{III}^{0,1,1}(f)$, $\mathrm{III}^{1,1,1}(f)$, $\mathrm{III}^{0,3}(f)$, $\mathrm{III}^{0,4}(f)$, $\mathrm{III}^{0,5}(f)$, $\mathrm{III}^{1,3}(f)$, $\mathrm{III}^{1,4}(f)$, $\mathrm{III}^{1,5}(f)$, $\mathrm{III}^{0,a}(f)$, $\mathrm{III}^{1,a}(f)$, $\mathrm{III}^{8}(f)$, $\mathrm{III}^{9}(f)$, $\mathrm{III}^{10}(f)$, $\mathrm{III}^{11}(f)$, $\mathrm{III}^{12}(f)$, $\mathrm{III}^{13}(f)$, $\mathrm{III}^{15}(f)$, $\mathrm{III}^{17}(f)$, $\mathrm{III}^{21}(f)$, $\mathrm{III}^{b}(f)$, $\mathrm{III}^{d}(f)$, $\mathrm{III}^{e}(f)$, $\mathrm{III}^{f}(f)$ and $\mathrm{III}^{g}(f)$ can be divided into two types A and B, in a way similar to that for the codimension 2 case (see Figures 12, 13 and 14). In Figures 12, 13 and 14 the numbers attached to 3-dimensional strata are chosen in the same way as in Figures 9 and 11 and the shadowed regions indicate R. Singular fibres of codimension three of the other types cannot be divided into two types A and B.

Let $f : M \to N$ be as above and suppose that $f|_{S(f)}$ satisfies the two colourable condition. Let $\mathscr{F}$ be the C^∞ equivalence class of one of the singular fibres of codimension two appearing in Figure 4. If $\mathscr{F}$ can be divided into two types A and B, then we define $\mathscr{F}_A(f)$ (or $\mathscr{F}_B(f)$) to be the set of points $y \in \mathscr{F}(f)$ such that the fibre over y is of type A (resp. type B). If $\mathscr{F}$ cannot be divided into two types, then $\mathscr{F}_A(f)$ or $\mathscr{F}_B(f)$ is not defined and we just consider $\mathscr{F}(f)$. Note that then $\overline{\mathscr{F}_A(f)}$, $\overline{\mathscr{F}_B(f)}$ and $\overline{\mathscr{F}(f)}$ are finite graphs embedded in N. Their vertices correspond to points over which f has a singular fibre of codimension three.

We again apply the handshake lemma to the graphs $\overline{\mathrm{II}_A^{0,0}(f)}$, $\overline{\mathrm{II}_B^{0,0}(f)}$, $\overline{\mathrm{II}_A^{0,1}(f)}$, $\overline{\mathrm{II}_B^{0,1}(f)}$, $\overline{\mathrm{II}_A^{1,1}(f)}$, $\overline{\mathrm{II}_B^{1,1}(f)}$, $\overline{\mathrm{II}^{0,2}(f)}$, $\overline{\mathrm{II}^{1,2}(f)}$, $\overline{\mathrm{II}^{2,2}(f)}$, $\overline{\mathrm{II}_A^{3}(f)}$, $\overline{\mathrm{II}_B^{3}(f)}$, $\overline{\mathrm{II}_A^{4}(f)}$, $\overline{\mathrm{II}_B^{4}(f)}$, $\overline{\mathrm{II}_A^{5}(f)}$, $\overline{\mathrm{II}_B^{5}(f)}$, $\overline{\mathrm{II}^{6}(f)}$, $\overline{\mathrm{II}^{7}(f)}$, $\overline{\mathrm{II}_A^{a}(f)}$ and $\overline{\mathrm{II}_B^{a}(f)}$. Then we obtain 19 co-existence re-

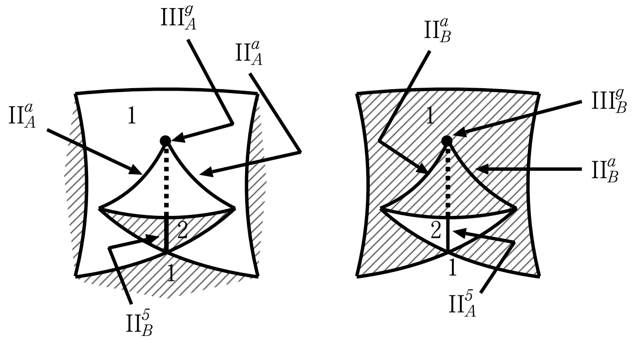


Figure 14. Types A and B for III^g.

lations among the numbers of singular fibres of codimension three. We note that the relation obtained from the graph $\overline{\mathrm{II}^*_B(f)}$ can also be obtained from the graph $\overline{\mathrm{II}^*_A(f)}$, by interchanging $\mathrm{III}^*_A(f)$ and $\mathrm{III}^*_B(f)$. In the following Proposition 4.6, we give the relations obtained from the graph $\overline{\mathrm{II}^*_A(f)}$ and $\overline{\mathrm{II}^*(f)}$.

PROPOSITION 4.6. *Let $f: M \to N$ be a stable map of a closed 4-manifold M into a connected 3-manifold N. Suppose that $f|_{S(f)}$ satisfies the two colourable condition. Then the following numbers are always even*:

(1) $|\mathrm{III}^{0,0,0}(f)| + |\mathrm{III}^{0,0,1}(f)| + |\mathrm{III}^{0,0,2}(f)| + |\mathrm{III}^{0,a}_A(f)| + |\mathrm{III}^d_A(f)|$,
(2) $|\mathrm{III}^{0,1,2}(f)| + |\mathrm{III}^{0,6}(f)| + |\mathrm{III}^{0,a}_A(f)| + |\mathrm{III}^{1,a}_A(f)| + |\mathrm{III}^b_A(f)|$,
(3) $|\mathrm{III}^{0,1,1}(f)| + |\mathrm{III}^{1,1,1}(f)| + |\mathrm{III}^{1,1,2}(f)| + |\mathrm{III}^{1,6}(f)| + |\mathrm{III}^{1,a}_A(f)| + |\mathrm{III}^8_B(f)|$,
(4) $|\mathrm{III}^{2,a}(f)| + |\mathrm{III}^c(f)|$,
(5) $|\mathrm{III}^{2,a}(f)| + |\mathrm{III}^{14}(f)|$,
(6) $|\mathrm{III}^{20}(f)|$,
(7) $|\mathrm{III}^{0,3}(f)| + |\mathrm{III}^{1,3}(f)| + |\mathrm{III}^{2,3}(f)| + |\mathrm{III}^8_A(f)| + |\mathrm{III}^9(f)| + |\mathrm{III}^{11}_A(f)| + |\mathrm{III}^{14}(f)| + |\mathrm{III}^{17}_A(f)| + |\mathrm{III}^{23}(f)| + |\mathrm{III}^{24}(f)| + |\mathrm{III}^b_A(f)| + |\mathrm{III}^f_A(f)|$,
(8) $|\mathrm{III}^{0,4}(f)| + |\mathrm{III}^{1,4}(f)| + |\mathrm{III}^{2,4}(f)| + |\mathrm{III}^{10}(f)| + |\mathrm{III}^{11}_A(f)| + |\mathrm{III}^{13}_B(f)| + |\mathrm{III}^{21}_A(f)| + |\mathrm{III}^{22}(f)| + |\mathrm{III}^e_B(f)|$,
(9) $|\mathrm{III}^{0,5}(f)| + |\mathrm{III}^{1,5}(f)| + |\mathrm{III}^{2,5}(f)| + |\mathrm{III}^{15}(f)| + |\mathrm{III}^{16}(f)| + |\mathrm{III}^{17}_A(f)| + |\mathrm{III}^{21}_B(f)| + |\mathrm{III}^g_B(f)|$,
(10) $|\mathrm{III}^{14}(f)| + |\mathrm{III}^c(f)|$,
(11) $|\mathrm{III}^{13}(f)| + |\mathrm{III}^{20}(f)|$,
(12) $|\mathrm{III}^{0,a}(f)| + |\mathrm{III}^{1,a}(f)| + |\mathrm{III}^{2,a}(f)| + |\mathrm{III}^b(f)| + |\mathrm{III}^c(f)|$.

Items (1)–(11) of the above proposition correspond to the graphs $\overline{\mathrm{II}^{0,0}_A(f)}$, $\overline{\mathrm{II}^{0,1}_A(f)}$, $\overline{\mathrm{II}^{1,1}_A(f)}$, $\overline{\mathrm{II}^{0,2}(f)}$, $\overline{\mathrm{II}^{1,2}(f)}$, $\overline{\mathrm{II}^{2,2}(f)}$, $\overline{\mathrm{II}^3_A(f)}$, $\overline{\mathrm{II}^4_A(f)}$, $\overline{\mathrm{II}^5_A(f)}$, $\overline{\mathrm{II}^6(f)}$ and $\overline{\mathrm{II}^7(f)}$ respectively. Item (12) corresponds to both $\overline{\mathrm{II}^a_A(f)}$ and $\overline{\mathrm{II}^a_B(f)}$. If we combine the relations obtained from $\overline{\mathrm{II}^*_A(f)}$ and that obtained from $\overline{\mathrm{II}^*_B(f)}$, then we obtain exactly the same co-existence relations as those in Proposition 3.1. We omit the table of the degrees of the vertices in these graphs.

For a stable map $f: M \to N$ of a closed 4-manifold into a 3-manifold, the symbol $\mathrm{III}^*_o(f)$ (resp. $\mathrm{III}^*_e(f)$) denotes the set of points $y \in N$ such that the fibre $f^{-1}(y)$ over

y is C^∞ equivalent to the union of the III* type fibre and some copies of a fibre of the trivial circle bundle and that the total number of connected components of $f^{-1}(y)$ is odd (resp. even). Furthermore, the symbol $\mathrm{III}^*_A(f)$ (resp. $\mathrm{III}^*_B(f)$) denotes the set of points $y \in N$ such that the fibre $f^{-1}(y)$ over y is C^∞ equivalent to the union of the III* type fibre and some copies of a fibre of the trivial circle bundle and that the colouring of a neighbourhood of $y \in N$ is of A type (resp. B type).

Let us now state and prove the main theorem of this paper.

Theorem 4.7. *Let $f : M \to N$ be a stable map of a closed 4-manifold M into a connected 3-manifold N such that $H_1(N; \boldsymbol{Z}_2) = 0$ or $f_*[S(f)] = 0 \in H_2(N; \boldsymbol{Z}_2)$, where $S(f) \subset M$ denotes the singular point set of f and $[S(f)] \in H_2(M; \boldsymbol{Z}_2)$ is the homology class represented by $S(f)$. Then we have*

$$\begin{aligned}\chi(M) \equiv |\mathrm{III}^{2,2,2}(f)| + |\mathrm{III}^{2,7}(f)| + |\mathrm{III}^{12}(f)| + |\mathrm{III}^{13}_e(f)| \\ + |\mathrm{III}^{13}_B(f)| + |\mathrm{III}^{25}(f)| + |\mathrm{III}^{26}(f)| \pmod 2,\end{aligned}$$

where $\chi(M)$ denotes the Euler number of M.

By Remark 3.2 or Proposition 4.6, the number of singular fibres of type III^{13} is even for any stable map $f : M \to N$ as in Theorem 4.7. Thus, $|\mathrm{III}^{13}_A(f)|$ and $|\mathrm{III}^{13}_B(f)|$ have the same parity. Therefore, we may replace $\mathrm{III}^{13}_A(f)$ with $\mathrm{III}^{13}_B(f)$ in Theorem 4.7. Similarly we may replace $\mathrm{III}^{13}_o(f)$ with $\mathrm{III}^{13}_e(f)$.

Proof of Theorem 4.7. We add up those items which correspond to the graphs $\overline{\mathrm{II}^{0,0}_A(f)}$, $\overline{\mathrm{II}^{0,1}_A(f)}$, $\overline{\mathrm{II}^{1,1}_A(f)}$, $\overline{\mathrm{II}^{3}_A(f)}$, $\overline{\mathrm{II}^{4}_B(f)}$ and $\overline{\mathrm{II}^{5}_B(f)}$ of Proposition 4.6, and we further add to it the co-existence relations obtained by the graphs $\overline{\mathrm{II}^{2,2}_o(f)}$ and $\overline{\mathrm{II}^{7}_o(f)}$ in § 4:

$$\begin{aligned}&\overline{\mathrm{II}^{2,2}_o(f)} : |\mathrm{III}^{0,2,2}(f)| + |\mathrm{III}^{1,2,2}(f)| + |\mathrm{III}^{2,6}(f)| + |\mathrm{III}^{20}_e(f)| \equiv 0 \pmod 2, \\ &\overline{\mathrm{II}^{7}_o(f)} : |\mathrm{III}^{0,7}(f)| + |\mathrm{III}^{1,7}(f)| + |\mathrm{III}^{13}_o(f)| + |\mathrm{III}^{18}(f)| + |\mathrm{III}^{19}(f)| + |\mathrm{III}^{20}_o(f)| \\ &\qquad \equiv 0 \pmod 2.\end{aligned}$$

In this way we obtain

$$\begin{aligned}&|\mathrm{III}^{0,0,0}(f)| + |\mathrm{III}^{0,0,1}(f)| + |\mathrm{III}^{0,1,1}(f)| + |\mathrm{III}^{1,1,1}(f)| + |\mathrm{III}^{0,0,2}(f)| + |\mathrm{III}^{0,2,2}(f)| \\ &\quad + |\mathrm{III}^{1,1,2}(f)| + |\mathrm{III}^{1,2,2}(f)| + |\mathrm{III}^{0,1,2}(f)| + |\mathrm{III}^{0,3}(f)| + |\mathrm{III}^{0,4}(f)| + |\mathrm{III}^{0,5}(f)| \\ &\quad + |\mathrm{III}^{0,6}(f)| + |\mathrm{III}^{0,7}(f)| + |\mathrm{III}^{1,3}(f)| + |\mathrm{III}^{1,4}(f)| + |\mathrm{III}^{1,5}(f)| + |\mathrm{III}^{1,6}(f)| \\ &\quad + |\mathrm{III}^{1,7}(f)| + |\mathrm{III}^{2,3}(f)| + |\mathrm{III}^{2,4}(f)| + |\mathrm{III}^{2,5}(f)| + |\mathrm{III}^{2,6}(f)| + |\mathrm{III}^{8}(f)| + |\mathrm{III}^{9}(f)| \\ &\quad + |\mathrm{III}^{10}(f)| + |\mathrm{III}^{11}(f)| + |\mathrm{III}^{13}_o(f)| + |\mathrm{III}^{13}_A(f)| + |\mathrm{III}^{14}(f)| + |\mathrm{III}^{15}(f)| + |\mathrm{III}^{16}(f)| \\ &\quad + |\mathrm{III}^{17}(f)| + |\mathrm{III}^{18}(f)| + |\mathrm{III}^{19}(f)| + |\mathrm{III}^{20}(f)| + |\mathrm{III}^{21}(f)| + |\mathrm{III}^{22}(f)| + |\mathrm{III}^{23}(f)| \\ &\quad + |\mathrm{III}^{24}(f)| + |\mathrm{III}^{d}_A(f)| + |\mathrm{III}^{e}_A(f)| + |\mathrm{III}^{f}_A(f)| + |\mathrm{III}^{g}_A(f)| \equiv 0 \pmod 2.\end{aligned}$$

Combining this formula with the relation between the number of singular fibres of f and the number of triple points of $f|_{S(f)}$ obtained in Remark 2.6, we obtain

$$\begin{aligned} T(f|_{S(f)}) = &|\mathrm{III}^{2,2,2}(f)| + |\mathrm{III}^{2,7}(f)| + |\mathrm{III}^{12}(f)| + |\mathrm{III}^{13}_e(f)| + |\mathrm{III}^{13}_B(f)| \\ &+ |\mathrm{III}^{25}(f)| + |\mathrm{III}^{26}(f)| + |\mathrm{III}^{d}_A(f)| + |\mathrm{III}^{e}_A(f)| + |\mathrm{III}^{f}_A(f)| \\ &+ |\mathrm{III}^{g}_A(f)| \pmod 2. \end{aligned}$$

On the other hand, by Theorem 4.3, we have

$$\chi(S(f)) \equiv T(f|_{S(f)}) + \sum_{q\text{: swallow-tail point of } f} n(q, f|_{S(f)}) \pmod 2.$$

Then we obtain

$$\begin{aligned} \chi(S(f)) \equiv &|\mathrm{III}^{2,2,2}(f)| + |\mathrm{III}^{2,7}(f)| + |\mathrm{III}^{12}(f)| + |\mathrm{III}^{13}_e(f)| + |\mathrm{III}^{13}_B(f)| \\ &+ |\mathrm{III}^{25}(f)| + |\mathrm{III}^{26}(f)| + |\mathrm{III}^{d}_A(f)| + |\mathrm{III}^{e}_A(f)| + |\mathrm{III}^{f}_A(f)| \\ &+ |\mathrm{III}^{g}_A(f)| + \sum_{q\text{: swallow-tail point of } f} n(q, f|_{S(f)}) \pmod 2. \end{aligned}$$

Furthermore, by the definitions of $n(q, f|_{S(f)})$ and type A (see Figures 8 and 14), we have

$$\sum_{q\text{: swallow-tail point of } f} n(q, f|_{S(f)}) = |\mathrm{III}^{d}_A(f)| + |\mathrm{III}^{e}_A(f)| + |\mathrm{III}^{f}_A(f)| + |\mathrm{III}^{g}_A(f)|.$$

Combining these equations with Theorem 4.4, we obtain Theorem 4.7. □

The homological hypothesis in Theorem 4.7 and in Corollaries 1.1 and 1.3 is essential. In fact, let $f : \boldsymbol{R}P^2 \times \boldsymbol{R}P^2 \to \boldsymbol{R}P^2 \times \boldsymbol{R}$ be defined by $f(x, y) = (x, \varphi(y))$, where φ is any stable Morse function on $\boldsymbol{R}P^2$. Then $f|_{S(f)}$ has no triple points, although $\chi(\boldsymbol{R}P^2 \times \boldsymbol{R}P^2) = 1$. We note that $H_1(\boldsymbol{R}P^2 \times \boldsymbol{R}; \boldsymbol{Z}_2) \neq 0$ and $f_*[S(f)] \neq 0$ in $H_2(\boldsymbol{R}P^2 \times \boldsymbol{R}; \boldsymbol{Z}_2)$.

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