Maximal regularity for the Stokes system
on noncylindrical space-time domains

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Abstract. We prove $L^p - L^q$ maximal regularity estimates for the Stokes equations in spatial regions with moving boundary. Our result includes bounded and unbounded regions. The method relies on a reduction of the problem to an equivalent nonautonomous system on a cylindrical space-time domain. By applying suitable abstract results for nonautonomous Cauchy problems we show maximal regularity of the associated propagator which yields the result. The abstract results, also proved in this note, are a modified version of a nonautonomous maximal regularity result of Y. Giga, M. Giga, and H. Sohr and a suitable perturbation result. Finally we describe briefly the application to the special case of rotating regions.

1. Introduction and main results.

For $T > 0$ let $Q_T := \bigcup_{t \in (0,T)} \Omega(t) \times \{t\} \subseteq \mathbf{R}^{n+1}$ be a noncylindrical space-time domain. In this note we consider the following Stokes equations:

$$(SE)^{\Omega(t)}_{f,v_0} \begin{cases} v_t - \Delta v + \nabla p = f & \text{in } Q_T, \\ \operatorname{div} v = 0 & \text{in } Q_T, \\ v = 0 & \text{on } \bigcup_{t \in (0,T)} \partial\Omega(t) \times \{t\}, \\ v|_{t=0} = v_0 & \text{in } \Omega(0) =: \Omega_0, \end{cases}$$

with velocity field v and pressure p. Here we assume the moving boundary, i.e. the evolution of the domain $\Omega(t)$ to be determined by a level-preserving diffeomorphism

$$\psi : \overline{\Omega_0 \times (0,T)} \to \overline{Q_T}, \quad (\xi, t) \mapsto (x,t) = \psi(\xi, t) := (\phi(\xi; t), t)$$

such that for each $t \in [0,T]$, $\phi(\cdot; t)$ maps Ω_0 onto $\Omega(t)$. More precisely we assume the following conditions on ϕ respectively ψ.

ASSUMPTION 1.1. Let $T \in (0, \infty]$, $\Omega_0 \subseteq \mathbf{R}^n$ be a domain of class C^3 either bounded, exterior, or a perturbed half-space. The domains $\Omega(t)$, $t \in [0,T]$, shall all be of the same type as Ω_0, i.e. $\{\Omega(t)\}_{t \in [0,T]}$ is either a family of bounded domains, exterior domains, or perturbed half-spaces. Furthermore:

1. For each $t \in [0,T]$, $\phi(\cdot; t) : \overline{\Omega_0} \to \overline{\Omega(t)}$ is a C^3-diffeomorphism. Its inverse we denote by $\phi^{-1}(\cdot; t)$ (to emphasize that ϕ^{-1} is merely the inverse w.r.t. the space

variables we use the semicolon notation $(\xi; t)$ for the argument of ϕ and ϕ^{-1}).

2. If we regard ϕ as a function from $Q_0 := \Omega_0 \times (0, T)$ into $\mathbf{R}^n$ it shall satisfy $\phi \in C_b^{3,1}(Q_0) := \{f \in C(Q_0) : \partial_t^k D_x^\alpha f \in C_b(Q_0),\ 1 \leq 2k + |\alpha| \leq 3,\ k \in \mathbf{N}_0, \alpha \in \mathbf{N}_0^n\}$, where $C_b(Q_0)$ denotes the space of all bounded and continuous functions on Q_0.

3. We have $\det \nabla_\xi \phi(\xi, t) \equiv 1$, $(\xi, t) \in \overline{Q_0}$, (volume preserving).

4. If $T = \infty$ we demand $\partial_t^k \phi(\cdot; t) \to \partial_t^k \phi(\cdot; \infty)$ in $C_b^{3-2k}(\Omega_0)$, $k = 0, 1$, for $t \to \infty$. (Note that $\phi(\cdot; \infty)$ does no more depend on t, i.e. the time derivative of ϕ tends to 0.)

Observe that for the case $T = \infty$ Assumption 1.1.4 says that the moving domain $\Omega(t)$ tends to some fixed domain $\Omega(\infty)$ for large values of t.

We remark that in view of realistic physical situations problem $(SE)_{f,v_0}^{\Omega(t)}$ should be considered with a certain boundary condition $v = b \neq 0$ at $\bigcup_{t \in (0,T)} \partial\Omega(t) \times \{t\}$. On the other hand, by assuming the existence of a solenoidal field $\beta : \overline{Q_T} \to \mathbf{R}^n$ such that $\beta = b$ on the boundary, the problem with $b \neq 0$ can be reduced to the case $b = 0$ as described in [19] and [18]. Therefore we restrict our considerations to the system $(SE)_{f,v_0}^{\Omega(t)}$ with zero boundary conditions.

Also, note that in certain concrete situations the existence of the diffeormorpism ψ is established. For instance in [18] the authors give as a nice example of a moving domain $\Omega(t)$ a bowl with swimming goldfishes (note that kisses are not allowed!). As a reference for the existence of ψ in such a situation they gave [26] and [7].

Now define $\mathscr{I}^p(A) := (X, D(A))_{1-\frac{1}{p}, p}$, where the latter space denotes the real interpolation space of a Banach space X and the domain $D(A)$ of a closed operator A in X. For $t \in [0, T)$ we denote by

$$A_{\Omega(t)} = -P_{\Omega(t)}\Delta \quad \text{defined on}$$

$$D(A_{\Omega(t)}) = W^{2,q}(\Omega(t)) \cap W_0^{1,q}(\Omega(t)) \cap L_\sigma^q(\Omega(t))$$

the Stokes operator in the space of solenoidal fields $L_\sigma^q(\Omega(t)) = \overline{C_{c,\sigma}^\infty(\Omega(t))}^{L^q(\Omega(t))}$, where $C_{c,\sigma}^\infty(\Omega(t)) := \{u \in C_c^\infty(\Omega(t)) : \operatorname{div} u = 0\}$. Here $P_{\Omega(t)} : L^q(\Omega(t)) \to L_\sigma^q(\Omega(t))$ denotes the Helmholtz projection associated to the Helmholtz decomposition $L^q(\Omega(t)) = L_\sigma^q(\Omega(t)) \oplus G_q(\Omega(t))$, where $G_q(\Omega(t)) = \{\nabla p \,;\ p \in \widehat{W}^{1,q}(\Omega(t))\}$. Note that it is well known that there exists a compatible family $\{P_{\Omega,q}\}_{q \in (1,\infty)}$ of bounded projections $P_\Omega = P_{\Omega,q} : L^q(\Omega) \to L_\sigma^q(\Omega)$ for all types of domains Ω considered in this note, see e.g. [10], [34], [32]. For the above types of moving domains our main result is

THEOREM 1.2. *Let $n \geq 2$, $1 < p, q < \infty$, and $T \in (0, \infty]$. Let the evolution of $\Omega(t)$, $t \in [0, T]$, be determined by a function ψ satisfying Assumptions 1.1. Then problem $(SE)_{f,0}^{\Omega(t)}$ has a unique solution $t \mapsto (v(t), p(t)) \in D(A_{\Omega(t)}) \times \widehat{W}^{1,q}(\Omega(t))$, $t \in [0, T]$. Furthermore, for $T < \infty$, this solution satisfies the estimate*

$$\int_0^T \left[\|v_t(t)\|_{L^q(\Omega(t))}^p + \|v(t)\|_{W^{2,q}(\Omega(t))}^p + \|\nabla p(t)\|_{L^q(\Omega(t))}^p \right] \mathrm{d}t$$

$$\leq C(T)\left(\|v_0\|_{\mathscr{I}^p(A_{\Omega_0})}^p + \int_0^T \|f(t)\|_{L^q(\Omega(t))}^p \mathrm{d}t \right) \tag{1}$$

for all $v_0 \in \mathscr{I}^p(A_{\Omega_0})$ and $f \in L^p((0,T); L^q(\Omega(t)))$. If Ω_0 is bounded then the above inequality is also valid for $T = \infty$ with a finite constant $C(\infty) > 0$.

REMARK 1.3.

(1) The proof of Theorem 1.2 shows that the norms $\|\cdot\|_{D(A(t))}$ (graph norm) and $\|\cdot\|_{W^{2,q}(\Omega(t))}$ are equivalent with equivalence constants independent of t. In particular we may replace $\|v(t)\|_{W^{2,q}(\Omega(t))}$ by the term $\|v(t)\|_{D(A(t))}$ in inequality (1).

(2) By checking the details of the proof in Section 3 below one will realize that the assumption on the family $\{\Omega(t)\}_{t \in [0,T]}$ which we intrinsically use is not the particular geometric type of the domains, but the property of having maximal regularity of the corresponding Stokes operator $A_{\Omega(t)}$ for each fixed $t \in [0,T]$. Therefore, Theorem 1.2 stays true for each family $\{\Omega(t)\}_{t \in [0,T]}$ with $\Omega(t)$ of the "same type" for all $t \in [0,T]$ such that $A_{\Omega(t)}$ has maximal regularity. This property, for instance, is also known to be valid for families of asymptotically flat layers as examined in [2], [1].

(3) Moreover, to prove the assertion for $T = \infty$, in our proof we only need the additional assumption that $A_{\Omega(t)}$ is boundedly invertible for t close to infinity. This is also well known in the case of asymptotically flat layers, see again [2], [1]. Hence the full assertion of Theorem 1.2 is valid for families of domains of that type.

Some special cases of the situation in Theorem 1.2 are considered in several former works. First investigations of the solvability of $(SE)_{f,v_0}^{\Omega(t)}$ and the corresponding Navier-Stokes equations can be found in [31]. However, until now there are only existence results in $L^2(\Omega(t))$ available under the restrictive assumptions that $T < \infty$ and $\{\Omega(t)\}_{t \in [0,T]}$ is a family of bounded domains. In that situation for instance in [18] the existence of a unique solution of $(SE)_{f,v_0}^{\Omega(t)}$ is proved. By regarding ψ as a variable transform the authors reduced $(SE)_{f,v_0}^{\Omega(t)}$ to a transformed nonautonomous problem and then applied an abstract result in [36] to the associated Cauchy problem. As an application of their result for the linear part, they also proved a local existence result for the corresponding nonlinear Navier-Stokes equations in Q_T. The existence of global weak solutions in $L^2(\Omega(t))$ for the Navier-Stokes equations was already shown in [9] (see also [5]). The periodic case was obtained in [25], i.e. if $t \mapsto \Omega(t)$ is periodic then there exists a periodic weak solution of the Navier-Stokes equations. Another result for more regular periodic solutions can be found in [23]. For periodic solutions of the Stokes equations we refer to [37]. Another existence result of local strong solutions of the Navier-Stokes equations on Q_T is obtained in [28]. There the authors could relax a restrictive decay condition on the right hand side f assumed in [18], but nevertheless they obtained a more regular solution. We want to remark that the assumptions on the evolution and regularity of $\Omega(t)$ differ in the above cited papers. This depends mainly on the method that the authors use in their works. The approach presented here is closely related to the method used in [18]. Therefore we have similar assumptions on ψ (hence also on $\Omega(t)$) as in that paper.

Theorem 1.2 generalizes the known results for $(SE)_{f,v_0}^{\Omega(t)}$ in several directions. Firstly, we handle L^q-spaces for the full scale $1 < q < \infty$ and not only the Hilbert space case. Secondly, we also treat various families $\{\Omega(t)\}_{t \in [0,T)}$ of unbounded domains. In the case of bounded domains (and perturbed layers) we also obtain a result for the case $T = \infty$ under the assumption that $\Omega(t)$ tends to some fixed domain $\Omega(\infty)$ in the sense of Assumption 1.1.4. Moreover, we do not only prove existence of solutions, but even

maximal regularity for all types of treated families of domains and L^q-spaces.

Although the results presented here are of independent interest, with our work we want to establish the basis for further regularity investigations of the Navier-Stokes equations on noncylindrical space-time domains. For example in the cylindrical case it is known, that the maximal regularity of the linear equations can be used to obtain regularity of weak solutions of the corresponding Navier-Stokes equations, see [**14**], or to prove uniqueness of mild solutions, see [**24**], [**20**]. The results in [**33**] show that we even can get a global strong solution of the two-dimensional Navier-Stokes equations by using maximal regularity. Besides these improvements we further hope that our methods can be used to obtain also higher regularity for the solutions of free boundary value problems related to the Navier-Stokes equations. The idea is that, once existence is proved, the movement of the free boundary is known. Then we are in the situation to apply a regularity result for a moving boundary problem.

As we already mentioned, our approach is similar to the one in [**18**], i.e. we transform the problem $(SE)_{f,0}^{\Omega(t)}$ via ψ back to a problem on the cylindrical domain $\Omega_0 \times (0,T)$. To the propagator $A_{\Omega_0}(\cdot)$ of the associated Cauchy problem we then will apply an abstract result. In order to formulate this result let $p \in (1,\infty)$ and denote by $\mathrm{MR}_p(X,K)$ the class of all operators (and propagators) $A(\cdot)$ having maximal (L^p-) regularity on X with a maximal regularity constant not exceeding K, i.e. there exists a unique solution $t \mapsto u(t) \in D(A(t))$ of the (eventually non-autonomous) Cauchy problem

$$\begin{cases} u' + A(\cdot)u = f & \text{in } (0,T), \\ \quad\quad u(0) = u_0, \end{cases} \tag{2}$$

satisfying the estimate

$$\|u'\|_{L^p((0,T);X)} + \|A(\cdot)u\|_{L^p((0,T);X)} \le K\big(\|f\|_{L^p((0,T);X)} + \|u_0\|_{\mathscr{I}^p(A(0))}\big)$$

for $f \in L^p((0,T);X)$ and $u_0 \in \mathscr{I}^p(A(0))$. We write $A(\cdot) \in \mathrm{MR}_p(X) := \bigcup_{K \in (0,\infty)} \mathrm{MR}_p(X,K)$ if the dependence of the maximal regularity constant K can be neglected. The abstract result which we will use reads as

THEOREM 1.4. *Let X be a Banach space, $1 < p < \infty$, $T \in (0,\infty]$, and $\{A(t)\}_{t \in [0,T]}$ a family of boundedly invertible sectorial operators in X satisfying*

1. *$D(A(t)) = D(A(0))$, $t \in [0,T]$.*
2. *The mapping $A(\cdot) : [0,T] \to \mathscr{L}(D(A(0),X))$ is continuous, where $D(A(0))$ is endowed with the graph norm.*
3. *$A(t) \to A(T)$ in $\mathscr{L}(D(A(0),X))$, as $t \to T$.*
4. *For each $t \in [0,T]$ we have $A(t) \in \mathrm{MR}_p(X,C(t))$, for some constant $C(t) > 0$.*

Then $A(\cdot) \in \mathrm{MR}_p(X)$. More precisely, for each $f \in L^p((0,T);X)$ and $u_0 \in \mathscr{I}^p(A(0))$ there is a unique solution $u \in W^{1,p}((0,T);X) \cap L^p((0,T);D(A(0)))$ of problem (2) such that

$$\|u\|_{W^{1,p}((0,T);X)} + \|A(\cdot)u\|_{L^p((0,T);X)} \le C\big(\|f\|_{L^p((0,T);X)} + \|u_0\|_{\mathscr{I}^p(A(0))}\big) \tag{3}$$

with a constant $C > 0$ independent of f and u_0.

Observe that assumption 4 is a pointwise condition, i.e. we assume that for each (fixed) $t \in [0, T]$ the (autonomous) operator $A(t)$ has maximal regularity. In Section 2 we will prove that the assumptions of Theorem 1.4 imply the assumptions of the Theorem in [**12**]. This will yield the assertion of Theorem 1.4. We want to remark that for the case of finite $T > 0$ Theorem 1.4 can be found in [**29**]. There the authors give a comprehensive discussion of the corresponding evolution operator. Another result of that type for UMD spaces X is given in [**3**]. There, besides the continuity of $t \mapsto A(t)$, the essential assumption, instead of the pointwise maximal regularity in assumption 4, is the stronger property of bounded imaginary powers of $A(t)$. For our concrete application to the Stokes equations this might be no problem. However, the additional assumption for the case $T = \infty$ in [**3**] is different from the assumption in [**12**] and to the author of the present note the condition in [**12**] seems to be more suitable for our purposes.

For further results concerning the maximal regularity of nonautonomous abstract Cauchy problems we refer to [**35**], and the references given there. However, there the authors also deal with the case $T < \infty$ only, and their results are based on commutator conditions on the operators $A(t)$, $t \in [0, T]$, which in general are not easy to verify. For this reason we decided to base Theorem 1.4 on the result in [**12**].

After stating the abstract result, in Section 3 we will turn to the proof of Theorem 1.2. By regarding ψ as a variable transform we will reduce $(SE)_{f,v_0}^{\Omega(t)}$ to an equivalent system of transformed equations on the cylindrical domain Q_0. The price we have to pay is that we are now left with a nonautonomous system of partial differential equations, i.e. the coefficients of these transformed equations depend on space and time in general. Here Assumption 1.1.2 assures that they are at least continuous. Another important point is that the transformed functions belong to the solenoidal space $L_\sigma^q(\Omega_0)$, which relies essentially on Assumption 1.1.3. More precisely this condition assures that the divergence operator is invariant under the chosen transform.

Similar to the autonomous Stokes equations this will give us the possibility to formulate an associated abstract Cauchy problem with operators acting in $L_\sigma^q(\Omega_0)$. The idea here is to use the family of projections $P_{\Omega_0}(t) : L^q(\Omega_0) \to L_\sigma^q(\Omega_0)$, which are exactly the transformed Helmholtz projections $P_{\Omega(t)}$. In order to verify condition 2 of Theorem 1.4 for the operator of the associated Cauchy problem, one difficulty is to show the continuity of $t \mapsto P_{\Omega_0}(t)$. This will be handled by considering the associated transformed Neumann problems to $P_{\Omega_0}(t)$.

Another problem is that in our case $A(t)$ is not only given by the transformed Stokes operator $A_{\Omega_0}(t)$, but we also have a perturbation $B(t)$ which arises from the fact that the transform ψ depends on t, and therefore does not commute with ∂_t. Here we will apply an abstract perturbation result for propagators established also in Section 2. This will lead to the maximal regularity of the shifted propagator $\mu + A_{\Omega_0}(\cdot) + B(\cdot)$ which implies the assertion of Theorem 1.2 for the case $T < \infty$.

In order to prove the statement for bounded Ω_0 and $T = \infty$, Assumption 1.1.4 will be the essential ingredient. It allows us to apply again the abstract perturbation result to the full operator $A_{\Omega_0}(t) + B(t)$ for t close to infinity, since then the coefficients of $B(t)$ are small. In combination with the result for the case $T < \infty$ this completes the proof.

In the last section we will give a brief application of Theorem 1.2 to the special case

of rotations, as they are of greatest interest in the present research.

We introduce some notation used in the sequel. By $C^k(\Omega)$ we denote the space of all k-times continuously differentiable functions in an open subset Ω of $\mathbf{R}^n$, and by $C_b^k(\Omega)$ its subspace of k-times bounded continuously differentiable functions. As usual $W^{k,q}(\Omega)$ is the Sobolev space with norm $\|\cdot\|_{k,q} = (\sum_{j=0}^k \|\nabla^j \cdot\|_q^q)^{1/q}$ and $L^q(\Omega) = W^{0,q}(\Omega)$ the Lebesgue space of q-integrable functions. We also make use of the homogeneous Sobolev space $\widehat{W}^{1,q}(\Omega)$, consisting of all locally integrable functions f in Ω with $\|\nabla f\|_q < \infty$, modulo constants. Furthermore, $\mathscr{L}(X,Y)$ denotes the class of all bounded operators from X to Y and $\mathscr{I}som(X,Y)$ its subclass of isomorphisms. If $X = Y$ we set $\mathscr{L}(X) := \mathscr{L}(X,X)$ and $\mathscr{I}som(X) := \mathscr{I}som(X,X)$. The domain of an operator A in a Banach space X we denote by $D(A)$, its range by $R(A)$, and its resolvent set by $\rho(A)$. Finally, by a sectorial operator in X we understand a closed injective operator A, such that $\overline{D(A)} = \overline{R(A)} = X$, $(0,\infty) \subseteq \rho(-A)$, and $\|\lambda(\lambda + A)^{-1}\|_{\mathscr{L}(X)} \leq C$, $\lambda > 0$, for some $C > 0$. Note that $\Sigma_{\pi-\varphi_0} := \{z \in \mathbf{C} \setminus \{0\} : |\arg z| < \pi - \varphi_0\} \subseteq \rho(A)$ for some $\varphi_0 \in (0,\pi)$, if A is sectorial, and the resolvent estimates are still valid for all $\lambda \in \Sigma_{\pi-\varphi_0}$ with a certain constant $C_{\varphi_0} > 0$. See [**6**] for more information about sectorial operators.

2.   The abstract nonautonomous maximal regularity result.

We start with a simple but in the sequel useful lemma about uniform boundedness of families of operators.

LEMMA 2.1.   *Let X, Y be Banach spaces and $T \in (0,\infty]$. Let $\{\Psi(t)\}_{t \in [0,T]}$ be a family of operators $\Psi(t) \in \mathscr{L}(Y,X)$, $t \in [0,T]$, such that $\Psi : [0,T) \to \mathscr{L}(Y,X)$ is continuous and $\Psi(t) \to \Psi(T)$ in $\mathscr{L}(Y,X)$. Then there is a constant $C_0 > 0$ such that*

$$\|\Psi(t)\|_{\mathscr{L}(Y,X)} \leq C_0, \quad t \in [0,T]. \tag{4}$$

If $\{\Psi(t)\}_{t \in [0,T]}$ is additionally assumed to be a family of isomorphisms, i.e. $\Psi(t) \in \mathscr{I}som(Y,X)$, $t \in [0,T]$, then we even have

$$\|\Psi(t)\|_{\mathscr{L}(Y,X)} \in [c_0, C_0], \quad t \in [0,T], \tag{5}$$

for certain constants $C_0 \geq c_0 > 0$. In particular, $\Psi(\cdot)^{-1} : [0,T] \to \mathscr{L}(Y,X)$ is continuous and we can choose $C_0 \geq c_0 > 0$ in a way such that also $\|\Psi(t)^{-1}\|_{\mathscr{L}(X,Y)} \in [c_0, C_0]$, $t \in [0,T]$.

PROOF.   By $\Psi(t) \to \Psi(T)$ if $t \to T$, for large t, say $t \geq T_0 > 0$, the norm of $\Psi(t)$ is close to $\|\Psi(T)\|_{\mathscr{L}(Y,X)}$. Together with the continuity of $t \mapsto \Psi(t)$ on $[0,T_0]$ this implies

(4). If $\{\Psi(t)\}_{t\in[0,T]}$ is a family of isomorphisms we have additionally $\|\Psi(t)\|_{\mathscr{L}(Y,X)} > 0$, $t \in [0,T]$. Hence, the argumentation above even yields (5). □

Before turning to the proof of the abstract maximal regularity result we list some properties of the operator

$$L(t,s) := A(t)A(s)^{-1}, \quad t,s \in [0,T],$$

where $\{A(t)\}_{t\in[0,T]}$ is a family satisfying the assumptions of Theorem 1.4. Note that L is well-defined by assumption 1. Moreover we have

LEMMA 2.2. *Let X be a Banach space, $T \in (0,\infty]$, and let $\{A(t)\}_{t\in[0,T]}$ satisfy the assumptions of Theorem 1.4 (It is not necessary to assume condition 4). Then*

1. *For each $(t,s) \in [0,T]$ the operator $L(t,s)$ is an isomorphism on X and there is a uniform $K > 0$ such that*

$$\|L(t,s)\|_{\mathscr{L}(X)} \le K, \quad t,s \in [0,T].$$

2. *The mapping $(t,s) \mapsto L(t,s)$ is continous from $[0,T)^2$ into $\mathscr{L}(X)$.*
3. *We have $L(t,s) \to I$ for $t,s \to T$ in $\mathscr{L}(X)$.*

PROOF.
1. Since $A(t)$ is assumed to be boundedly invertible, for each $t,s \in [0,T]$ the operator $L(t,s)$ is an isomorphism on X with $L(t,s)^{-1} = L(s,t)$ and $L(t,t) = I$, $t \in [0,T]$. Now assumption 2 of Theorem 1.4 implies $t \mapsto L(t,0)$ to be continuous from $[0,T)$ into $\mathscr{L}(X)$. Furthermore, assumption 3 of Theorem 1.4 yields $L(t,0) \to L(T,0)$ in $\mathscr{L}(X)$. Thus we can apply Lemma 2.1 obtaining

$$\|L(t,0)\|_{\mathscr{L}(X)}, \|L(0,t)\|_{\mathscr{L}(X)} \in [c_0, C_0], \quad t \in [0,T].$$

Consequently,

$$\|L(t,s)\|_{\mathscr{L}(X)} = \|L(t,0)L(0,s)\|_{\mathscr{L}(X)} \le C_0^2 =: K, \quad t,s \in [0,T].$$

2. First observe that $L(0,\cdot) : [0,T) \to \mathscr{L}(X)$ is continuous in view of the continuity of $t \mapsto L(t,0)$ and the identity $L(0,t) = L(t,0)^{-1}$. By the calculation

$$\begin{aligned}
\|L(t_2,s_2) &- L(t_1,s_1)\|_{\mathscr{L}(X)} \\
&\le \|L(t_2,s_2) - L(t_1,s_2)\|_{\mathscr{L}(X)} + \|L(t_1,s_2) - L(t_1,s_1)\|_{\mathscr{L}(X)} \\
&= \|(L(t_2,0) - L(t_1,0))L(0,s_2)\|_{\mathscr{L}(X)} + \|L(t_1,0)(L(0,s_2) - L(0,s_1))\|_{\mathscr{L}(X)} \\
&\le K\big(\|L(t_2,0) - L(t_1,0)\|_{\mathscr{L}(X)} + \|L(0,s_2) - L(0,s_1)\|_{\mathscr{L}(X)}\big)
\end{aligned}$$

for $t_1,t_2,s_1,s_2 \in [0,T]$, we can see that the continuity of $(t,s) \mapsto \mathscr{L}(X)$ from $[0,T)^2$ into $\mathscr{L}(X)$ can be reduced to the continuity of $t \mapsto L(t,0)$ and $t \mapsto L(0,t)$.

3. Here we compute

$$\|L(t,s) - I\|_{\mathscr{L}(X)} = \|(L(t,0) - L(s,0))L(0,s)\|_{\mathscr{L}(X)}$$
$$\leq K\|L(t,0) - L(s,0)\|_{\mathscr{L}(X)} \;\to\; 0$$

for $t, s \to T$, valid again in view of $L(t,0) \to L(T,0)$.   □

As we mentioned in the introduction we intend to show that the assumptions of Theorem 1.4 imply the assumptions of the following result:

THEOREM 2.3 [**12**, Theorem].   *Let X be a Banach space, $1 < p < \infty$, $T \in (0, \infty]$, and let $\{A(t)\}_{t \in [0,T]}$ be a family of sectorial operators in X satisfying*

(a) *$D(A(t)) = D(A(0))$, $t \in [0, T]$, where $D(A(0))$ is endowed with the graph norm.*
(b) *The operator $L(t,s) = A(t)A(s)^{-1}$ extends to a bounded operator on X and we have $\|L(t,s)\|_{\mathscr{L}(X)} \leq K$ for all $0 \leq s \leq t < T$.*
(c) *The map $(t,s) \mapsto L(t,s)$ is continuous from $\{(t,s) : 0 \leq s \leq t < T\}$ into $\mathscr{L}(X)$.*
(d) *$L(t,s) \to I$ for $t \geq s \to T$ in $\mathscr{L}(X)$.*
(e) *$A(t) \in \mathrm{MR}_p(X, C_0)$ for all $t \in [0, T)$.*

Then $A(\cdot) \in \mathrm{MR}_p(X)$.

REMARK 2.4.
(1) In [**12**] instead of (e) it is assumed that

$$\|A(t)^{is}\|_{\mathscr{L}(X)} \leq C\mathrm{e}^{\theta|s|}, \quad 0 \leq t < T, \; s \in \mathbf{R}, \tag{6}$$

for some $\theta \in (0, \pi)$ and $C > 0$, i.e. $A(t)$ has bounded imaginary powers. But by checking the proof one realizes that only the condition (e) above is used. Therefore we also can skip the assumption of the ζ-convexity of the underlying Banach space X which is needed to conclude (e) from (6).
(2) Comparing the above result with Theorem 1.4 we see that we only assume the continuity of $(t,s) \mapsto L(t,s)$ in the first component. Moreover, note that in (e) there is the condition of a uniform constant C_0, which is rather unwieldy for concrete applications. Instead, assumption 4 in Theorem 1.4 is only a pointwise condition. In that sense Theorem 1.4 is an improvement of Theorem 2.3. On the other hand here we assume $A(t)$, $t \in [0, T]$, to be boundedly invertible which is not assumed in [**12**].
(3) In [**12**] the space for the initial data is $\mathscr{I}^p = \{x \in X : (\int_0^T \|A(0)\mathrm{e}^{-tA(0)}x\|^p \mathrm{d}t)^{1/p} < \infty\}$. It is well known that our space $\mathscr{I}^p(A(0)) := (X, D(A(0)))_{1-\frac{1}{p},p}$ coincides with the space $\{x \in X : \|x\|_X + (\int_0^T \|A(0)\mathrm{e}^{-tA(0)}x\|^p \mathrm{d}t)^{1/p} < \infty\}$ if $A(0)$ is the generator of a holomorphic semigroup on X (see e.g. [**4**], [**21**]). Hence our choice of the space for initial data is compatible.

PROOF (of Theorem 1.4).   In virtue of Lemma 2.2 it remains to prove the existence of the uniform constant C_0 in condition (e) of Theorem 2.3. To this end we define the spaces

$$Y := L^p((0,T); D(A(0))) \cap W^{1,p}((0,T); X) \quad \text{and}$$

$$Z := L^p((0,T); X) \times \mathscr{I}^p(A(0))$$

equipped with their canonical norms. Note again that it is well known, that $\mathscr{I}^p(A(0))$ is exactly the trace space of Y in $t = 0$ under the assumptions $A(0) \in \mathrm{MR}_p(X)$ and $A(0)^{-1} \in \mathscr{L}(X)$ (see e.g. [**21**, Section 1.2.2]), in particular $\|u|_{t=0}\|_{\mathscr{I}^p(A(0))} \le C\|u\|_Y$, $u \in Y$. Furthermore, observe that in view of Lemma 2.2.1 the graph norms $\|\cdot\|_{D(A(0))}$ and $\|\cdot\|_{D(A(\tau))}$ are equivalent with equivalence constants that do not depend on $\tau \in [0,T]$. These two facts together with assumption 4 of Theorem 1.4 imply that for each $\tau \in [0,T]$ the operator

$$\Psi(\tau): Y \to Z, \quad \Psi(\tau)u := \begin{pmatrix} (\partial + A(\tau))u \\ u|_{t=0} \end{pmatrix}$$

is an isomorphism. Furthermore we have

$$\|(\Psi(\tau) - \Psi(s))u\|_Z = \left\| \begin{pmatrix} (A(\tau) - A(s))u \\ 0 \end{pmatrix} \right\|_Z$$

$$= \left(\int_0^T \|(A(\tau) - A(s))u(t)\|^p \mathrm{d}t \right)^{1/p}$$

$$\le \|(A(\tau) - A(s))\|_{\mathscr{L}(D(A(0)),X))} \|u\|_Y, \quad \tau, s \in [0,T].$$

This shows the continuity of $\Psi : [0,T) \to \mathscr{I}som(Y,Z)$ according to assumption 2 of Theorem 1.4. On the other hand, if we set $s = T$, we obtain $\Psi(\tau) \to \Psi(T)$ in $\mathscr{I}som(Y,Z)$ by virtue of assumption 3. Applying Lemma 2.1 we conclude

$$\|\Psi(\tau)^{-1}\|_{\mathscr{L}(Z,Y)} \le K, \quad \tau \in [0,T],$$

which yields (e). Theorem 2.3 then yields $A(\cdot) \in \mathrm{MR}_p(X)$. But since

$$\|A(t)^{-1}\|_{\mathscr{L}(X)} \le C\|A(0)A(t)^{-1}\|_{\mathscr{L}(X)} \le CK, \quad t \in [0,T],$$

it is easy to see that we even have $u \in W^{1,p}((0,T); X) \cap L^p((0,T); D(A(0)))$, hence also the stronger estimate (3). This proves Theorem 1.4. □

In our applications we will also employ the following perturbation result.

THEOREM 2.5. *Let X be a Banach space, $1 < p < \infty$, and $T \in (0, \infty]$. Let $\{A(t)\}_{t \in [0,T)}$ be a family of boundedly invertible operators such that $D(A(t)) = D(A(0))$, $\|A(t)A(s)^{-1}\|_{\mathscr{L}(X)} \le C_0$, $t, s \in [0,T)$, and $A(0) \in \mathrm{MR}_p(X)$ as well as $A(\cdot) \in \mathrm{MR}_p(X)$. Furthermore, let $\{B(t)\}_{t \in [0,T)}$ be a family of linear closed operators satisfying $D(A(0)) \subseteq D(B(t))$, $t \in [0,T)$, and for some $\kappa < 1$ assume*

$$\|B(t)x\| \le \kappa \|A(t)x\|, \quad x \in D(A(0)), \ t \in [0,T).$$

Then $D(A(t) + B(t)) = D(A(0))$, $t \in [0, T)$, and we have

$$\left\| (A(t) + B(t))(A(s) + B(s))^{-1} \right\|_{\mathscr{L}(X)} \leq C_1, \quad t, s \in [0, T). \tag{7}$$

Furthermore, if κ is small enough, for each $f \in L^p((0, T); X)$ and $u_0 \in \mathscr{I}^p(A(0))$ there is a unique solution $u \in W^{1,p}((0, T); X) \cap L^p((0, T); D(A(0)))$ of problem (2) satisfying

$$\|u\|_{W^{1,p}((0,T);X)} + \|(A + B)(\cdot)u\|_{L^p((0,T);X)}$$
$$\leq C_2 \big(\|f\|_{L^p((0,T);X)} + \|u_0\|_{\mathscr{I}^p(A(0))} \big), \tag{8}$$

i.e. $(A + B)(\cdot) \in \mathrm{MR}_p(X)$.

PROOF.   First fix $t \in [0, T)$. By the invertibility of $A(t)$ this operator is an isomorphism from $D(A(t))$ to X. The relative boundedness of $B(t)$ with $\kappa < 1$ implies that also $A(t) + B(t) : D(A(t)) \to X$ is an isomorphism. Hence $A(t) + B(t)$ is a closed operator in X with domain $D(A(t) + B(t)) = D(A(t)) = D(A(0))$. To see estimate (7) we compute

$$\|(A(t) + B(t))(A(s) + B(s))^{-1}\|_{\mathscr{L}(X)}$$
$$\leq (1 + \kappa)\|A(t)A(s)^{-1}[I + B(s)A(s)^{-1}]^{-1}\|_{\mathscr{L}(X)}.$$

Since $\|B(s)A(s)^{-1}\|_{\mathscr{L}(X)} \leq \kappa < 1$ uniformly in $s \in [0, T)$, the Neumann series yield

$$\left\| [I + B(s)A(s)^{-1}]^{-1} \right\|_{\mathscr{L}(X)} \leq C, \quad s \in [0, T).$$

Together with the assumptions on $A(\cdot)$ this implies (7).

Now let Y, Z be defined as in the proof of Theorem 1.4. We already mentioned that the trace operator $\gamma : Y \to \mathscr{I}^p(A(0))$, $\gamma u := u|_{t=0}$ is bounded by taking into account that $A(0) \in \mathrm{MR}_p(X)$. Together with the assumptions $A(0)^{-1} \in \mathscr{L}(X)$, $\|A(t)A(0)^{-1}\|_{\mathscr{L}(X)} \leq C_0$ and $A(\cdot) \in \mathrm{MR}_p(X)$ this implies

$$\Psi : Y \to Z, \quad \Psi u := \begin{pmatrix} (\partial + A(\cdot))u \\ \gamma u \end{pmatrix}$$

to be an isomorphism. Now set $\tilde{B} : Y \to Z$, $\tilde{B}u := (B(\cdot)u, 0)^T$, $u \in Y$. Then

$$(\Psi + \tilde{B})u = \begin{pmatrix} \partial + A(\cdot) + B(\cdot))u \\ \gamma u \end{pmatrix}.$$

To show invertibility of this operator we write formally

$$(\Psi + \tilde{B})^{-1} = \Psi^{-1}(I + \tilde{B}\Psi^{-1})^{-1}$$

and calculate

$$\left\| \tilde{B}\Psi^{-1}(f, u_0)^T \right\|_Z = \left\| B(\cdot)\Psi^{-1}(f, u_0)^T \right\|_{L^p((0,T),X)} + 0$$

$$= \left(\int_0^T \left\| B(t)[\Psi^{-1}(f, u_0)^T](t) \right\|^p \mathrm{d}t \right)^{1/p}$$

$$\leq \kappa \left(\int_0^T \left\| A(t)[\Psi^{-1}(f, u_0)^T](t) \right\|^p \mathrm{d}t \right)^{1/p}$$

$$\leq \kappa a \|(f, u_0)\|_Z, \quad (f, u_0) \in Z,$$

where $a := \|A(\cdot)\Psi^{-1}\|_{\mathscr{L}(L^p((0,T);X))}$. Thus, for $\kappa < 1/a$ we can employ the Neumann series obtaining

$$\left\| (\Psi + \tilde{B})^{-1} \right\|_{\mathscr{L}(Z,Y)} \leq \frac{1}{1 - \kappa a} \|\Psi^{-1}\|_{\mathscr{L}(Z,Y)}.$$

This implies $\Psi + \tilde{B} : Y \to Z$ to be an isomorphism. In combination with

$$\|(A(t) + B(t))u\|_X = \|(A(t) + B(t))(A(0) + B(0))^{-1}(A(0) + B(0))u\|_X$$

$$\leq C_1(1 + \kappa)\|A(0)u\|_X, \quad u \in D(A(0)), \ t \in [0, T),$$

this yields the assertion. $\qquad\square$

3.   The Stokes equations on noncylindrical domains.

Now we turn to the proof of Theorem 1.2. First let us list some obvious consequences of Assumption 1.1. In view of $\det \nabla\phi(\xi, t) \equiv 1$ and $\psi(\xi, t) = (\phi(\xi; t), t)$ we also have $\det \nabla\psi = 1$. Moreover, Assumption 1.1.2 implies $\psi \in C_b^{3,1}(Q_0; \boldsymbol{R}^{n+1})$. By virtue of the implicit function theorem we therefore have $\psi^{-1} \in C_b^{3,1}(Q_T; \boldsymbol{R}^{n+1})$ and since $\psi^{-1}(x, t) = (\phi^{-1}(x; t), t)$, $(x, t) \in Q_T$, also $\phi^{-1} \in C_b^{3,1}(Q_T; \boldsymbol{R}^n)$. Furthermore we calculate

$$0 = \partial_t \xi = \partial_t \phi^{-1}(\phi(\xi; t); t) = (\nabla_x \phi^{-1})(\phi(\xi; t); t)(\partial_t \phi)(\xi, t) + (\partial_t \phi^{-1})(\phi(\xi; t); t).$$

Hence

$$(\partial_t \phi^{-1})(\phi(\xi; t); t) = -(\nabla_\xi \phi)(\xi; t)^{-1}(\partial_t \phi)(\xi, t), \quad (\xi, t) \in Q_0,$$

and we see that Assumption 1.1.4 yields

$$(\partial_t \phi^{-1})(\phi(\cdot; t); t) \to 0, \quad t \to \infty, \tag{9}$$

in $C_b^1(\Omega_0)$ for the case $T = \infty$.

In order to apply the abstract results of the previous section we transform $(SE)_{f,0}^{\Omega(t)}$ to a system on a fixed domain as follows. For a function $v : Q_T \to \boldsymbol{C}^n$ we set

$$\tilde{v}(\xi, t) := v(\phi(\xi; t), t), \quad (\xi, t) \in \Omega_0 \times [0, T].$$

Then

$$(\nabla_x v)(\phi(\xi;t),t) = \left[(\nabla_\xi \phi)^{-T} \nabla_\xi \tilde{v}\right](\xi,t), \tag{10}$$

where M^{-T} denotes $(M^T)^{-1}$ and M^T stands for the transposed Matrix. Now define

$$u(\xi,t) := (\Phi(t)v)(\xi,t) := \left[(\nabla_\xi \phi)^{-1}\tilde{v}\right](\xi,t), \quad (\xi,t) \in \Omega_0 \times [0,T]. \tag{11}$$

If $T = \infty$, note that $\Phi(\infty)$ is the corresponding operator to the limit function $\phi(\cdot,\infty)$ which exists according to Assumption 1.1.4. Assumption 1.1.1, 2, and 3 on ϕ imply that $\Phi(t) : W^{k,q}(\Omega(t)) \to W^{k,q}(\Omega_0)$ and $\Phi(t) : W_0^{k,q}(\Omega(t)) \to W_0^{k,q}(\Omega_0)$ are isomorphisms for $k = 0, 1, 2$ and $t \in [0,T]$, and we even have the uniform estimates

$$\|\Phi(t)v\|_{W^{k,p}(\Omega_0)} \le C_1 \|v\|_{W^{k,p}(\Omega(t))} \le C_2 \|\Phi(t)v\|_{W^{k,p}(\Omega_0)} \tag{12}$$

for all $v \in W^{k,p}(\Omega(t))$, $t \in [0,T]$, $k = 0, 1, 2$. It is also easy to see that $\nu(x,t)$ is the outer normal at $\partial\Omega(t)$ in x if and only if $\mu(\xi,t) = (\nabla\phi)^T(\xi,t)\nu(\phi(\xi,t))$ is the outer normal at $\partial\Omega_0$ in ξ. This implies $\nu \cdot v = 0$ if and only if $\mu \cdot \Phi v = 0$. Moreover, under Assumption 1.1 (in particular 2) in [**18**, Proposition 2.4][1] it is proved that

$$\operatorname{div}_\xi u(\xi,t) = \operatorname{div}_x v(\phi(\xi;t),t), \quad (\xi,t) \in \Omega_0 \times [0,T].$$

This implies that $\Phi(t) : L_\sigma^q(\Omega(t)) \to L_\sigma^q(\Omega_0)$ is an isomorphism as well. This property of Φ, which is essential in what follows, is the reason why we have to choose the special transform given in (11). On the other hand note that this transform is responsible for the fact, that we have to assume C^3 boundary instead of C^2 only.

In view of (10) it is clear that $\Phi(t)\Delta_x\Phi(t)^{-1}$ has a representation as

$$\Phi(t)\Delta_x\Phi(t)^{-1} = \sum_{|\alpha| \le 2} a_\alpha(\cdot,t)D^\alpha \tag{13}$$

with certain matrices $a_\alpha \in C_b(\overline{\Omega_0 \times (0,T)})$ such that $\partial_t^k D^\gamma a_\alpha \in C_b(\overline{\Omega_0 \times (0,T)})^{n \times n}$ for $2k + |\gamma| \le |\alpha| \le 2$ with $k \in \boldsymbol{N}_0$, $\gamma \in \boldsymbol{N}_0^n$. Explicitly we have

$$\begin{aligned}
\left[\Phi(t)\Delta_x\Phi(t)^{-1}u\right](\xi,t) &= \left[(\nabla_\xi\phi)^{-1}(\nabla_\xi\phi)^{-T}\nabla_\xi \cdot (\nabla_\xi\phi)^{-T}\nabla_\xi(\nabla_\xi\phi)u\right](\xi,t) \\
&= \sum_{i,j,k,\ell,m=1}^n \left[(\partial_{x_k}\phi^{-1})(\partial_{x_j}\phi^{-1})^i(\partial_{x_j}\phi^{-1})^\ell\right](\phi(\xi;t),t) \\
&\quad \cdot \left[(\partial_{\xi_\ell}\partial_{\xi_i}\partial_{\xi_m}\phi^k)u^m + (\partial_{\xi_i}\partial_{\xi_m}\phi^k)\partial_{\xi_\ell}u^m \right. \\
&\quad \left. + (\partial_{\xi_\ell}\partial_{\xi_m}\phi^k)\partial_{\xi_i}u^m + (\partial_{\xi_m}\phi^k)\partial_{\xi_\ell}\partial_{\xi_i}u^m\right](\xi,t).
\end{aligned} \tag{14}$$

We also have

---

[1] Actually in [**18**] only bounded Ω_0 are treated. But since it is a pointwise condition the proof given there applies to each $\Omega \subset \boldsymbol{R}^n$.

$$\partial_t v(x,t) = \partial_t [(\nabla_\xi \phi) u](\phi^{-1}(x;t),t)$$

$$= \sum_{i,j=1}^n (\partial_t \phi^{-1})^j (x;t) \big[(\partial_{\xi_i}\partial_{\xi_j}\phi)u^i + (\partial_{\xi_i}\phi)\partial_{\xi_j}u^i\big](\phi^{-1}(x;t),t)$$

$$+ \sum_{i=1}^n \big[(\partial_{\xi_i}\partial_t\phi)u^i + (\partial_{\xi_i}\phi)\partial_t u^i\big](\phi^{-1}(x;t),t). \tag{15}$$

Thus

$$\Phi(t)\partial_t \Phi(t)^{-1} = \partial_t + \sum_{|\beta|\leq 1} b_\beta(\cdot,t)D^\beta \tag{16}$$

with certain $b_\beta \in C_b(\overline{\Omega_0 \times (0,T)})^{n\times n}$ such that $\partial_t^k D^\gamma b_\beta \in C_b(\overline{\Omega_0 \times (0,T)})$ for $2k+|\gamma| \leq 2|\beta| \leq 2$ with $k \in \boldsymbol{N}_0$, $\gamma \in \boldsymbol{N}_0^n$. If we set $F := \Phi f$ and $u_0 := \Phi(0)v_0$, as well as $\nabla^\phi(t) := (\nabla_\xi \phi(t))^{-1}(\nabla_\xi \phi(t))^{-T}\nabla_\xi$ and $\tilde{p} := p \circ \psi$, the transformed equations become

$$(TSE)_{F,u_0}^{\Omega_0} \begin{cases} u_t + \sum_{|\beta|\leq 1} b_\beta D^\beta u - \sum_{|\alpha|\leq 2} a_\alpha D^\alpha u + \nabla^\phi(\cdot)\tilde{p} = F & \text{in } \Omega_0 \times (0,T), \\ \operatorname{div} u = 0 & \text{in } \Omega_0 \times (0,T), \\ u = 0 & \text{on } \partial\Omega_0 \times (0,T), \\ u|_{t=0} = u_0 & \text{in } \Omega_0. \end{cases}$$

Since $\Phi(t)$ is an isomorphism, clearly $(u,\tilde{p})$ satisfies $(TSE)_{F,u_0}^{\Omega_0}$ if and only if (v,p) fulfills $(SE)_{f,v_0}^{\Omega(t)}$. Obviously

$$P_{\Omega_0}(t) := \Phi(t)P_{\Omega(t)}\Phi(t)^{-1} : L^q(\Omega_0) \to L^q_\sigma(\Omega_0), \quad t \in [0,T],$$

is again a projection, where $P_{\Omega(t)}$ denotes the Helmholtz projection on $L^q(\Omega(t))$. Note that

$$G_q(t) := (I - P_{\Omega_0}(t))L^q(\Omega_0) = \big\{\nabla^\phi(t)(\pi \circ \psi); \pi \in \widehat{W}^{1,q}(\Omega(t))\big\}.$$

Thus, $P_{\Omega_0}(t)$ is not the Helmholtz projection on $L^q(\Omega_0)$ in general. As $G_q(t)$ depends on t we see that also the projection $P_{\Omega_0}(t)$ does, although its range, $L^q_\sigma(\Omega_0)$, does not depend on t. Defining

$$A_{\Omega_0}(t) := -P_{\Omega_0}(t)\sum_{|\alpha|\leq 2} a_\alpha(\cdot,t)D^\alpha \quad \text{on} \tag{17}$$

$$D(A_{\Omega_0}(t)) = \Phi(t)D(A_{\Omega(t)}) = W^{2,q}(\Omega_0) \cap W_0^{1,q}(\Omega_0) \cap L^q_\sigma(\Omega_0)$$

$$= D(A_{\Omega_0}), \quad t \in [0,T],$$

and

$$B(t) := P_{\Omega_0}(t)\sum_{|\beta|\leq 1} b_\beta(\cdot,t)D^\beta, \quad t \in [0,T], \tag{18}$$

the system $(TSE)^{\Omega_0}_{F,u_0}$ can be rephrased as the nonautonomous Cauchy problem

$$
(CP)_{F,u_0} \begin{cases} u'(t) + (A_{\Omega_0}(t) + B(t))u(t) = F(t), & t \in (0,T), \\ \qquad\qquad\qquad\qquad\quad u(0) = u_0, \end{cases}
$$

on the space $L^q_\sigma(\Omega_0)$. Observe that $A_{\Omega_0}(t) = \Phi(t)A_{\Omega(t)}\Phi(t)^{-1}$, i.e. it is exactly the transformed Stokes operator on $\Omega(t)$ for $t \in [0,T]$. In particular we have $A_{\Omega_0}(\infty) = \Phi(\infty)A_{\Omega(\infty)}\Phi(\infty)^{-1}$ if $t = T = \infty$. Moreover, we see that the domain of $A_{\Omega_0}(t)$ does not depend on t and equals the domain of the Stokes operator A_{Ω_0} in $L^q_\sigma(\Omega_0)$.

We proceed by stating the maximal regularity result for the shifted propagator $\mu + A_{\Omega_0}(\cdot)$. Then the abstract perturbation result Theorem 2.5 will give the result for the shifted full operator $\mu + A_{\Omega_0}(\cdot) + B(\cdot)$ for some $\mu \geq 0$ and general Ω_0 as assumed in Assumption 1.1. On the other hand, if Ω_0 is assumed to be bounded, we will show the maximal regularity even for $\mu = 0$. This will yield Theorem 1.2.

One ingredient in the proof of the above mentioned results will be the continuity of $t \mapsto P_{\Omega_0}(t)$. This is a consequence of

LEMMA 3.1. *Let $T \in (0,\infty]$ and Ω_0, ϕ as in Assumption 1.1. We have*

$$
\|P_{\Omega_0}(t) - P_{\Omega_0}(s)\|_{\mathscr{L}(L^q(\Omega_0))} \leq C\big\|(\nabla\phi(\cdot;t))^{-1} - (\nabla\phi(\cdot;s))^{-1}\big\|_\infty, \quad t,s \in [0,T].
$$

In particular this yields $\|P_{\Omega_0}(t)\|_{\mathscr{L}(L^q(\Omega_0))} \leq C$ uniformly in $t \in [0,T]$ and, if $T = \infty$,

$$
P_{\Omega_0}(t) \longrightarrow P_{\Omega_0}(\infty) = \Phi(\infty)P_{\Omega(\infty)}\Phi(\infty)^{-1} \quad \text{in } \mathscr{L}(L^q(\Omega_0)) \text{ for } t \to \infty.
$$

PROOF.   Fix $t \in [0,T]$ and let $v \in L^q(\Omega(t))$. It is well known that $P_{\Omega(t)}v = v - \nabla\pi$, where $\pi \in \widehat{W}^{1,q}(\Omega(t))$ is the unique solution of the weak Neumann problem

$$
(\nabla\pi, \nabla\varphi) = (v, \nabla\varphi), \quad \varphi \in \widehat{W}^{1,q'}(\Omega(t)).
$$

Since $P_{\Omega_0}(t) = \Phi(t)P_{\Omega(t)}\Phi(t)^{-1}$, we therefore deduce $P_{\Omega_0}(t)u = u - \nabla^\phi(t)p$, where $p \in \big(\widehat{W}^{1,q}(\Omega_0), \|\nabla^\phi(t) \cdot \|_q\big)$ is the unique solution of the transformed weak Neumann problem

$$
(\nabla^\phi(t)p, \nabla\varphi) = (u, \nabla\varphi), \quad \varphi \in \widehat{W}^{1,q'}(\Omega_0), \tag{19}
$$

for $u \in L^q(\Omega_0)$ and as former $\nabla^\phi(t) = (\nabla\phi)^{-1}(t)(\nabla\phi)^{-T}(t)\nabla$. For $t,s \in [0,T]$ and $u \in L^q(\Omega_0)$ this allows us to write

$$
P_{\Omega_0}(t)u - P_{\Omega_0}(s)u = \nabla^\phi(t)p(t) - \nabla^\phi(s)p(s)
$$

and to prove the estimate under discussion for the solutions of the corresponding transformed Neumann problems.

To this end first note that in view of $\det \nabla \phi \equiv 1$, the family $\{\|\nabla^\phi(t) \cdot \|_q\}_{t \in [0,T]}$ is a collection of equivalent norms on the space $\widehat{W}^{1,q}(\Omega_0)$, and the equivalence constants even do not depend on t. Next we write the difference above as

$$\nabla^\phi(t)p(t) - \nabla^\phi(s)p(s) = \nabla^\phi(t)(p(t) - p(s)) + (\nabla^\phi(t) - \nabla^\phi(s))p(s). \qquad (20)$$

For the first addend of (20) observe that in view of (19),

$$
\begin{aligned}
&\left(\nabla^\phi(t)p(t) - \nabla^\phi(t)p(s), \nabla\varphi\right) \\
&= (u, \nabla\varphi) - \left((\nabla^\phi(t) - \nabla^\phi(s))p(s), \nabla\varphi\right) - \left(\nabla^\phi(s)p(s), \nabla\varphi\right) \\
&= \left((\nabla^\phi(s) - \nabla^\phi(t))p(s), \nabla\varphi\right)
\end{aligned}
\qquad (21)
$$

for $\varphi \in \widehat{W}^{1,q'}(\Omega_0)$. By the Helmholtz decomposition we know $(G_q(\Omega(t)))' = G_{q'}(\Omega(t))$. This fact implies for arbitrary $w \in \widehat{W}^{1,q}(\Omega_0)$

$$
\begin{aligned}
\|\nabla^\phi(t)w\|_{L^q(\Omega_0)} &= \left\|\Phi(t)\nabla(w \circ \phi^{-1})\right\|_{L^q(\Omega_0)} \\
&\leq C\left\|\nabla(w \circ \phi^{-1})\right\|_{L^q(\Omega(t))} \\
&= C \sup_{\substack{\eta \in \widehat{W}^{1,q'}(\Omega(t)), \\ \eta \neq 0}} \frac{\left|\left(\nabla(w \circ \phi^{-1}), \nabla\eta\right)_{\Omega(t)}\right|}{\|\nabla\eta\|_{L^q(\Omega(t))}}.
\end{aligned}
$$

By the change of variables $x \to \xi = \phi^{-1}(x;t)$ and by putting $\varphi := \eta \circ \phi$ we can continue this calculation obtaining

$$\|\nabla^\phi(t)w\|_{L^q(\Omega_0)} \leq C \sup_{\substack{\varphi \in \widehat{W}^{1,q'}(\Omega_0), \\ \varphi \neq 0}} \frac{\left|\left(\nabla^\phi(t)w, \nabla\varphi\right)_{\Omega_0}\right|}{\|\nabla\varphi\|_{L^q(\Omega_0)}},$$

where we used $\|\nabla\eta\|_{L^q(\Omega(t))} \geq C\|\Phi(t)\nabla\eta\|_{L^q(\Omega_0)} = C\|\nabla^\phi(t)\varphi\|_{L^q(\Omega_0)} \geq C\|\nabla\varphi\|_{L^q(\Omega_0)}$. If we set $w = p(t) - p(s)$ this yields by virtue of (21),

$$
\begin{aligned}
\|\nabla^\phi(t)(p(t) - p(s))\|_{L^q(\Omega_0)} &\leq C \sup_{\substack{\varphi \in \widehat{W}^{1,q'}(\Omega_0), \\ \varphi \neq 0}} \frac{\left|\left(\nabla^\phi(t)(p(t) - p(s)), \nabla\varphi\right)_{\Omega_0}\right|}{\|\nabla\varphi\|_{L^q(\Omega_0)}} \\
&= C \sup_{\substack{\varphi \in \widehat{W}^{1,q'}(\Omega_0), \\ \varphi \neq 0}} \frac{\left|\left((\nabla^\phi(t) - \nabla^\phi(s))p(s), \nabla\varphi\right)_{\Omega_0}\right|}{\|\nabla\varphi\|_{L^q(\Omega_0)}} \\
&\leq C\left\|(\nabla^\phi(t) - \nabla^\phi(s))p(s)\right\|_{L^q(\Omega_0)}.
\end{aligned}
\qquad (22)
$$

This shows that the estimate for the first addend of (20) can be reduced to the one for the second addend. Considering that addend we may estimate

$$\left\|\left(\nabla^\phi(t) - \nabla^\phi(s)\right)p(s)\right\|_q$$

$$\leq \left\|(\nabla\phi)^{-1}(t)(\nabla\phi)^{-T}(t) - (\nabla\phi)^{-1}(s)(\nabla\phi)^{-T}(s)\right\|_\infty \|\nabla p(s)\|_q$$

$$\leq C\left\|(\nabla\phi)^{-1}(t) - (\nabla\phi)^{-1}(s)\right\|_\infty \|\nabla p(s)\|_q \qquad (23)$$

due to

$$\left\|(\nabla\phi)^{-1}(t)(\nabla\phi)^{-T}(t) - (\nabla\phi)^{-1}(s)(\nabla\phi)^{-T}(s)\right\|_\infty$$

$$\leq \left(\left\|(\nabla\phi)^{-1}(t)\right\|_\infty + \left\|(\nabla\phi)^{-1}(s)\right\|_\infty\right)\left\|(\nabla\phi)^{-1}(t) - (\nabla\phi)^{-1}(s)\right\|_\infty$$

$$\leq C\left\|(\nabla\phi)^{-1}(t) - (\nabla\phi)^{-1}(s)\right\|_\infty, \qquad (24)$$

which is valid by our Assumption 1.1.3 on ϕ.

Now let $L(t) : L^q(\Omega_0) \to \widehat{W}^{1,q}(\Omega_0)$, $u \mapsto L(t)u := p(t)$ be the solution operator to the transformed weak Neumann problem (19). Since that problem is uniquely solvable, $L(t)$ is well defined and we have $L(t) \in \mathscr{L}(L^q(\Omega_0), \widehat{W}^{1,q}(\Omega_0))$, $t \in [0, T]$. Inequality (22) in combination with (23) now yields

$$\|(L(t) - L(s))u\|_{\widehat{W}^{1,q}(\Omega_0))} \leq C\|\nabla L(s)u\|_q\left\|(\nabla\phi)^{-1}(t) - (\nabla\phi)^{-1}(s)\right\|_\infty$$

for $t, s \in [0, T]$. By Assumption 1.1.3 this implies the continuity of $L : [0, T) \to \mathscr{L}(L^q(\Omega_0), \widehat{W}^{1,q}(\Omega_0))$ and, if we set $s = T$, by virtue of Assumption 1.1.4 we also obtain $L(t) \to L(T)$ in $\mathscr{L}(L^q(\Omega_0), \widehat{W}^{1,q}(\Omega_0))$, even if $T = \infty$. Employing Lemma 2.1 we deduce for the family $\{L(t)\}_{t\in[0,T]}$,

$$\|L(t)\|_{\mathscr{L}(L^q(\Omega_0), \widehat{W}^{1,q}(\Omega_0))} \leq C, \quad t \in [0, T].$$

This fact applied on estimate (23) and (22) implies in view of $p(s) = L(s)u$,

$$\left\|\left(\nabla^\phi(t) - \nabla^\phi(s)\right)p(s)\right\|_q \leq C\left\|(\nabla\phi)^{-1}(t) - (\nabla\phi)^{-1}(s)\right\|_\infty \|u\|_q$$

and

$$\|\nabla^\phi(t)(p(t) - p(s))\|_q \leq C\|(\nabla\phi)^{-1}(t) - (\nabla\phi)^{-1}(s)\|_\infty \|u\|_q$$

for $s, t \in [0, T]$ and $u \in L^q(\Omega_0)$. Combining these two estimates yields the assertion. $\square$

In order to prove the maximal regularity for $\mu + A_{\Omega_0}(\cdot)$ we will apply Theorem 1.4.

THEOREM 3.2.   *Let $T \in (0, \infty]$ and $\mu > 0$. Let Ω_0, ϕ as in Assumption 1.1 and the family $\{A_{\Omega_0}(t)\}_{t\in[0,T]}$ defined as in (17). Then $\mu + A_{\Omega_0}(\cdot) \in \mathrm{MR}(L^q_\sigma(\Omega_0), C)$. In particular, estimate (3) holds for $\mu + A_{\Omega_0}(\cdot)$. If Ω_0 is bounded the assertions are also valid for $\mu = 0$.*

PROOF.   If $T = \infty$ first observe that in view of Assumption 1.1.4 on ϕ and representation (14) we obtain for the coefficients of $A_{\Omega_0}(t)$,

$$a_\alpha(\cdot, t) \to a_\alpha(\cdot, \infty) \quad \text{in } C^{|\alpha|}(\overline{\Omega_0}), \ |\alpha| \leq 2,$$

where $a_\alpha(\cdot, \infty)$ are the corresponding coefficients to the limit variable transform $\phi(\cdot, \infty)$. Furthermore, by Lemma 3.1

$$P_{\Omega_0}(t) \to P_{\Omega_0}(\infty) = \Phi(\infty) P_{\Omega(\infty)} \Phi(\infty)^{-1} \quad \text{in } \mathscr{L}(L^q(\Omega_0)).$$

This implies $\mu + A_{\Omega_0}(t) \to \mu + A_{\Omega_0}(\infty)$ in $\mathscr{L}(D(A_{\Omega_0}), L^q(\Omega_0))$, hence condition 3 of Theorem 1.4 is satisfied for the family $\{\mu + A_{\Omega_0}(t)\}_{t \in [0,T]}$ and $\mu \geq 0$.

In order to see condition 2 we write

$$P_{\Omega_0}(t) a_\alpha(t) - P_{\Omega_0}(s) a_\alpha(s) = (P_{\Omega_0}(t) - P_{\Omega_0}(s)) a_\alpha(t) - P_{\Omega_0}(s)(a_\alpha(t) - a_\alpha(s)).$$

Note that $\|P_{\Omega_0}(s)\|_{\mathscr{L}(L^q(\Omega_0))} \leq C$, $s \in [0, T]$, and $\|a_\alpha(t)\|_\infty \leq C$, $t \in [0, T]$, $|\alpha| \leq 2$, according to Lemma 3.1 and Assumption 1.1.3, respectively. Thus, by Lemma 3.1 and again representation (14) we obtain

$$\|P_{\Omega_0}(t) a_\alpha(t) - P_{\Omega_0}(s) a_\alpha(s)\|_{\mathscr{L}(L^q(\Omega_0))}$$
$$\leq C\big(\|(\nabla \phi(t))^{-1} - (\nabla \phi(s))^{-1}\|_\infty + \|\phi(t) - \phi(s)\|_{C_b^3(\Omega_0)}\big)$$
$$\leq C\|\phi(t) - \phi(s)\|_{C_b^3(\Omega_0)}, \quad t, s \in [0, T], \ |\alpha| \leq 2.$$

We conclude

$$\big\|\big((\mu + A_{\Omega_0}(t)) - (\mu + A_{\Omega_0}(s))\big)u\big\|_q$$
$$\leq \sum_{|\alpha| \leq 2} \|P_{\Omega_0}(t) a_\alpha(t) - P_{\Omega_0}(s) a_\alpha(s)\|_{\mathscr{L}(L^q(\Omega_0))} \|D^\alpha u\|_q$$
$$\leq C\|\phi(t) - \phi(s)\|_{C_b^3(\Omega_0)} \|u\|_{D(A_{\Omega_0})}$$

for all $t, s \in [0, T]$ and $\mu \geq 0$. The continuity of $\mu + A_{\Omega_0}(\cdot) : [0, T) \to \mathscr{L}(D(A_{\Omega_0}), L^q_\sigma(\Omega_0))$ now is an immediate consequence of Assumption 1.1.3.

Note that the property of having maximal regularity is invariant under conjugation with isomorphisms. Therefore the pointwise maximal regularity, i.e. $\mu + A_{\Omega_0}(t) \in \text{MR}(L^q_\sigma(\Omega_0), C(t))$, $t \in [0, T]$, $\mu \geq 0$, is a consequence of the pointwise maximal regularity for the Stokes operator $\mu + A_{\Omega(t)}$ on $\Omega(t)$ (see e.g. [**34**], [**27**], or [**30**] for references that include all types of domains handled in this note) and the representation $\mu + A_{\Omega_0}(t) = \Phi(t)(\mu + A_{\Omega(t)})\Phi(t)^{-1}$, $t \in [0, T]$, $\mu \geq 0$.

Since $\mu + A_{\Omega_0}(t) : D(A_{\Omega_0}) \to L^q_\sigma(\Omega_0)$ is boundedly invertible, we see that the assumptions of Theorem 1.4 are fulfilled for $\{\mu + A_{\Omega_0}(t)\}_{t \in [0,T]}$ and $\mu > 0$. On the other hand, if Ω_0 is bounded, obviously the invertibility of $A_{\Omega(t)} : D(A_{\Omega(t)}) \to L^q_\sigma(\Omega(t))$ implies the invertibility of $A_{\Omega_0}(t)$ again by the representation $A_{\Omega_0}(t) = \Phi(t) A_{\Omega(t)} \Phi(t)^{-1}$, $t \in [0, T]$. Therefore, in this case the assertion remains true also for $\mu = 0$. $\square$

As a corollary we can show the maximal regularity of $A_{\Omega_0}(\cdot)$ for general Ω_0, at least

for the case of finite T.

COROLLARY 3.3.   *Assume the situation of Theorem 3.2. For $T < \infty$ we have*
$A_{\Omega_0}(\cdot) \in \mathrm{MR}(L^q_\sigma(\Omega_0), C(T))$. *In particular $A_{\Omega_0}(\cdot)$ satisfies estimate* (3).

PROOF.   Let u be the solution to the Cauchy problem $(CP)_{F,u_0}$ with right hand side (F, u_0) associated to $A_{\Omega_0}(\cdot)$. Then the shifted function $\tilde{u}(t) := \mathrm{e}^{-\mu t} u(t)$ for some $\mu > 0$ solves the Cauchy problem $(CP)_{\mathrm{e}^{-\mu t}F,u_0}$ associated to $\mu + A_{\Omega_0}(\cdot)$. Theorem 3.2 now implies

$$\int_0^T \left(\|\tilde{u}(t)\|_q^p + \|A_{\Omega_0}(t)\tilde{u}(t)\|_q^p \right) \mathrm{d}t \leq C_\mu \int_0^T \left(\|\tilde{u}(t)\|_q^p + \|(\mu + A_{\Omega_0}(t))\tilde{u}(t)\|_q^p \right) \mathrm{d}t$$

$$\leq C_\mu \left(\int_0^T \|\mathrm{e}^{-\mu t} F(t)\|_q^p \mathrm{d}t + \|u_0\|_{\mathscr{I}^p}^p \right).$$

Consequently,

$$\int_0^T \left(\|u(t)\|_q^p + \|A_{\Omega_0}(t)u(t)\|_q^p \right) \mathrm{d}t = \int_0^T \left(\|\mathrm{e}^{\mu t}\tilde{u}(t)\|_q^p + \|\mathrm{e}^{\mu t} A_{\Omega_0}(t)\tilde{u}(t)\|_q^p \right) \mathrm{d}t$$

$$\leq C_\mu \mathrm{e}^{\mu T} \left(\int_0^T \|F(t)\|_q^p \mathrm{d}t + \|u_0\|_{\mathscr{I}^p}^p \right). \qquad \square$$

In order to apply Theorem 2.5 to the full operator $\mu + A_{\Omega_0}(\cdot) + B(\cdot)$ we need a uniform relative bound κ for the evolution family $\{B(t)\}_{t \in [0,T)}$. This requires the following uniform resolvent estimates.

LEMMA 3.4.   *Let $T \in (0, \infty]$ and Ω_0, ϕ, and $\{A_{\Omega_0}(t)\}_{t \in [0,T]}$ as in Theorem 3.2. Then there is a $C > 0$ such that we have the uniform estimate*

$$\sum_{k=0}^2 \|\lambda^{(2-k)/2} \nabla^k (\lambda + A_{\Omega_0}(t))^{-1}\|_{\mathscr{L}(L^q_\sigma(\Omega_0), L^q(\Omega_0))} \leq C, \quad t \in [0,T], \ \lambda \geq 1.$$

PROOF.   First fix $t \in [0,T]$ and let $\lambda \geq 1$. From well known results for the Stokes operator $A_{\Omega(t)}$ we obtain the equivalence of the norms $\sum_{k=0}^2 \lambda^{(k-2)/2} \|\nabla^k \cdot\|_{L^q(\Omega(t))}$ and $\|\lambda \cdot\|_{L^q(\Omega(t))} + \|A_{\Omega_0}(t) \cdot\|_{L^q(\Omega(t))}$ on $D(A_{\Omega(t)})$ with equivalence constants independent of $\lambda \geq 1$. Therefore the same statement is valid for the corresponding situation in Ω_0, i.e. there are constants $C_1, C_2 > 0$ (that may depend on t) such that

$$\sum_{k=0}^2 \lambda^{(k-2)/2} \|\nabla^k u\|_{L^q(\Omega_0)} \leq C_1 \left(\|\lambda u\|_{L^q(\Omega_0)} + \|A_{\Omega_0}(t)u\|_{L^q(\Omega_0)} \right)$$

$$\leq C_2 \sum_{k=0}^2 \lambda^{(k-2)/2} \|\nabla^k u\|_{L^q(\Omega_0)}$$

for $u \in D(A_{\Omega_0})$ and $\lambda \geq 1$. Denote by $D_\lambda(A_{\Omega_0})$ the Banach space $D(A_{\Omega_0})$ equipped

with the equivalent norm $\sum_{k=0}^{2} \lambda^{(k-2)/2} \|\nabla^k \cdot\|_{L^q(\Omega_0)}$. By the observations above we see that

$$\|(\lambda + A_{\Omega_0}(t))^{-1}\|_{\mathscr{L}(L^q_\sigma(\Omega_0), D_\lambda(A_{\Omega_0}))} \leq C(t), \quad \lambda \geq 1.$$

We will show that $C(t) \leq C_0$, $t \in [0, T]$, for a certain $C_0 > 0$. To this end we write for $t, t_0 \in [0, T]$,

$$(\lambda + A_{\Omega_0}(t))^{-1} = (\lambda + A_{\Omega_0}(t_0))^{-1}\big[I + (A_{\Omega_0}(t) - A_{\Omega_0}(t_0))(\lambda + A_{\Omega_0}(t_0))^{-1}\big]^{-1}.$$

Note that for $t_0 \in [0, T]$ we have

$$\|(\lambda + A_{\Omega_0}(t_0))^{-1}\|_{\mathscr{L}(L^q_\sigma(\Omega_0), D(A_{\Omega_0}))} \leq C \sum_{j=0}^{2} \big\|\nabla^j(\lambda + A_{\Omega_0}(t_0))^{-1}\big\|_{\mathscr{L}(L^q_\sigma(\Omega_0), L^q(\Omega_0))}$$

$$\leq C\big\|(\lambda + A_{\Omega_0}(t_0))^{-1}\big\|_{\mathscr{L}(L^q_\sigma(\Omega_0), D_\lambda(A_{\Omega_0}))}$$

$$\leq C(t_0), \quad \lambda \geq 1.$$

Hence we may compute

$$\|(A_{\Omega_0}(t) - A_{\Omega_0}(t_0))(\lambda + A_{\Omega_0}(t_0))^{-1}\|_{\mathscr{L}(L^q_\sigma(\Omega_0))}$$

$$\leq \|A_{\Omega_0}(t) - A_{\Omega_0}(t_0)\|_{\mathscr{L}(D(A_{\Omega_0}), L^q_\sigma(\Omega_0))}\|(\lambda + A_{\Omega_0}(t_0))^{-1}\|_{\mathscr{L}(L^q_\sigma(\Omega_0), D(A_{\Omega_0}))}$$

$$\leq C(t_0)\|A_{\Omega_0}(t) - A_{\Omega_0}(t_0)\|_{\mathscr{L}(D(A_{\Omega_0}), L^q_\sigma(\Omega_0))}, \quad t, t_0 \in [0, T].$$

Now set $t_0 := T$. Since condition 3 of Theorem 1.4 is satisfied for $A_{\Omega_0}(\cdot)$, there is a $T_0 \in (0, T)$ such that

$$\|(A_{\Omega_0}(t) - A_{\Omega_0}(T))(\lambda + A_{\Omega_0}(T))^{-1}\|_{\mathscr{L}(L^q_\sigma(\Omega_0))} \leq C(T)\varepsilon$$

for $t \geq T_0$ and $\lambda \geq 1$. Putting $\varepsilon := 1/2C(T)$ we obtain by using the Neumann series,

$$\|(\lambda + A_{\Omega_0}(t))^{-1}\|_{\mathscr{L}(L^q_\sigma(\Omega_0), D_\lambda(A_{\Omega_0}))} \leq 2C(T), \quad t \geq T_0, \ \lambda \geq 1.$$

On the other hand, if $t_0 \in [0, T_0]$, there is a $\delta(t_0) > 0$ such that

$$\|(A_{\Omega_0}(t) - A_{\Omega_0}(t_0))(\lambda + A_{\Omega_0}(t_0))^{-1}\|_{\mathscr{L}(L^q_\sigma(\Omega_0))} \leq C(t_0)\varepsilon$$

for $t \in (t - \delta(t_0), t + \delta(t_0)) \cap [0, T)$ and $\lambda \geq 1$, since also condition 2 of Theorem 1.4 is satisfied for $A_{\Omega_0}(\cdot)$. Again by using the Neumann series this implies by choosing $\varepsilon := 1/2C(t_0)$,

$$\|(\lambda + A_{\Omega_0}(t))^{-1}\|_{\mathscr{L}(L^q_\sigma(\Omega_0), D_\lambda(A_{\Omega_0}))} \leq 2C(t_0)$$

for $t \in (t - \delta(t_0), t + \delta(t_0)) \cap [0, T]$ and $\lambda \geq 1$. Since $[0, T_0]$ is compact we can achieve $[0, T_0] \subseteq \bigcup_{j=1}^{N}(t_j - \delta(t_j), t_j + \delta(t_j))$ for certain $t_j \in [0, T_0]$, $j = 1, \ldots, N$. Setting $C_0 := 2 \max\{C(t_1), \ldots, C(t_N), C(T)\}$ then yields the assertion. □

Now we can apply the perturbation result Theorem 2.5.

THEOREM 3.5.   *Let $T \in (0, \infty]$. Let Ω_0, ϕ be as in Assumption 1.1 and the families $\{A_{\Omega_0}(t)\}_{t \in [0,T]}$ and $\{B(t)\}_{t \in [0,T]}$ be defined as in (17) and (18), respectively. For $\mu > 0$ large enough we have $\mu + A_{\Omega_0}(\cdot) + B(\cdot) \in \mathrm{MR}(L_\sigma^q(\Omega_0), C)$. In particular $\mu + A_{\Omega_0}(\cdot) + B(\cdot)$ fulfills (8).*

PROOF.   In the proof of Theorem 3.2 we observed that all assumptions of Theorem 1.4 are fulfilled for the family $\{\mu + A_{\Omega_0}(t)\}_{t \in [0,T]}$. Together with Lemma 2.2.1 this shows that $\{\mu + A_{\Omega_0}(t)\}_{t \in [0,T]}$ also satisfies the assumptions of Theorem 2.5 for each $\mu > 0$. Obviously, $D(A_{\Omega_0}(t)) \subseteq D(B(t))$, $t \in [0, T)$. Thus it suffices to verify the existence of a relative bound for $B(\cdot)$. But, since $B(\cdot)$ contains lower order terms only, we can achieve by employing Lemma 3.1 and Lemma 3.4

$$\|B(t)u\|_q \leq C \sum_{|\beta| \leq 1} \|P_{\Omega_0(t)} b_\beta(t)\|_\infty \|D^\beta(\mu + A_{\Omega_0}(t))^{-1}(\mu + A_{\Omega_0}(t))u\|_q$$

$$\leq \frac{C}{\sqrt{\mu}} \|(\mu + A_{\Omega_0}(t))u\|_q, \quad \mu \geq 1, \ t \in [0, T).$$

Hence, by choosing μ arbitrary large, we can have the relative bound arbitrary small which shows that all the assumptions of Theorem 2.5 are satisfied. Applying that result yields the assertion. □

Completely analogous to Corollary 3.3 we obtain by a shift back

COROLLARY 3.6.   *Assume the situation of Theorem 3.5. For $T < \infty$ we have $A_{\Omega_0}(\cdot) + B(\cdot) \in \mathrm{MR}(L_\sigma^q(\Omega_0), C(T))$. Moreover estimate (8) is valid for $A_{\Omega_0}(\cdot) + B(\cdot)$.*

We turn to the case of bounded Ω_0.

THEOREM 3.7.   *Assume the situation of Theorem 3.5 with bounded Ω_0. Then the assertion there is valid for $\mu = 0$, in particular $A_{\Omega_0}(\cdot) + B(\cdot) \in \mathrm{MR}(L_\sigma^q(\Omega_0), C)$.*

PROOF.   According to the latter corollary it remains to handle the case of $T = \infty$ and large t. To this end let $T_0 \in (0, \infty)$ and put

$$A_{T_0}(t) := A_{\Omega_0}(T_0 + t), \quad B_{T_0}(t) := B(T_0 + t), \quad t \geq 0.$$

Obviously $A_{T_0}(\cdot) \in \mathrm{MR}(L_\sigma^q(\Omega_0), C)$ by Theorem 3.2. From (15) and (18) we read off that each coefficient b_β of $B(\cdot)$ contains a time derivative of ϕ or ϕ^{-1}. By Assumption 1.1.4 on ϕ and (9) it follows

$$\left\|(\partial_t \phi^{-1})(\phi(\cdot; t), t)\right\|_{C_b(\Omega_0)} \to 0 \quad \text{and} \quad \|\partial_t \phi(\cdot, t)\|_{C_b^1(\Omega_0)} \to 0, \quad \text{for} \quad t \to \infty.$$

Consequently, $b_\beta(t) \to 0$ if $t \to \infty$. This allows us to choose for $\varepsilon > 0$ a $T_0 = T_0(\varepsilon) > 0$ such that $\|b_\beta(t)\|_\infty < \varepsilon$ for all $t \geq T_0$ and all coefficients b_β. Furthermore, an application of Lemma 2.1 to $\{A_{\Omega_0}(t)\}_{t \in [0,T]}$ with $Y := D(A_{\Omega_0})$ yields $\|A_{\Omega_0}(t)^{-1}\|_{\mathscr{L}(L^q_\sigma(\Omega_0), D(A_{\Omega_0}))} \leq C$, $t \geq 0$, and therefore

$$\|A_{T_0}(t)^{-1}\|_{\mathscr{L}(L^q_\sigma(\Omega_0), W^{2,q}(\Omega_0))} = \|A_{\Omega_0}(T_0 + t)^{-1}\|_{\mathscr{L}(L^q_\sigma(\Omega_0), W^{2,q}(\Omega_0))} \leq C, \quad t \geq 0.$$

Hence, by virtue of Lemma 3.1 we can calculate

$$\|B_{T_0}(t)u\|_q \leq C \sum_{|\beta| \leq 1} \|b_\beta(T_0 + t)\|_\infty \|D^\beta A_{T_0}(t)^{-1} A_{T_0}(t)u\|_q$$

$$\leq C\varepsilon \|A_{T_0}(t)u\|_q, \quad u \in D(A_{\Omega_0}), \ t \geq 0.$$

Setting $\varepsilon := \kappa/C$ we may apply Theorem 2.5 to the pair $A_{T_0}(\cdot)$ and $B_{T_0}(\cdot)$ which yields $A_{T_0}(\cdot) + B_{T_0}(\cdot) \in \mathrm{MR}(L^q_\sigma(\Omega_0), C)$.

Now let u be the solution of the Cauchy problem $(CP)_{F,u_0}$ associated to $A_{\Omega_0}(\cdot) + B(\cdot)$. Then $u_{T_0}(\cdot) := u(T_0 + \cdot)$ is the solution of $(CP)_{F_{T_0}, u(T_0)}$ associated to $A_{T_0}(\cdot) + B_{T_0}(\cdot)$, where $F_{T_0}(\cdot) := F(T_0 + \cdot)$ and T_0 as before. Applying Corollary 3.6 for $T = T_0$ and $A_{T_0}(\cdot) + B_{T_0}(\cdot) \in \mathrm{MR}(L^q_\sigma(\Omega_0), C)$ we conclude

$$\int_0^\infty (\|u(t)\|_q^p + \|(A_{\Omega_0}(t) + B(t))u(t)\|_q^p) \mathrm{d}t$$

$$= \int_0^{T_0} (\|u(t)\|_q^p + \|(A_{\Omega_0}(t) + B(t))u(t)\|_q^p) \mathrm{d}t$$

$$+ \int_0^\infty (\|u_{T_0}(t)\|_q^p + \|(A_{T_0}(t) + B(t))u_{T_0}(t)\|_q^p) \mathrm{d}t$$

$$\leq C(T_0) \left(\int_0^\infty \|F(t)\|_q^p \mathrm{d}t + \|u_0\|_{\mathscr{I}^p}^p + \|u(T_0)\|_{\mathscr{I}^p}^p \right). \tag{25}$$

It remains to show that we can omit the last addend. We have (see e.g. [**21**, Section 1.2.2])

$$\|u(T_0)\|_{\mathscr{I}^p} = \|u_{T_0}(0)\|_{\mathscr{I}^p}$$

$$\leq C \left(\int_0^{T_0} \left(\|(A_{T_0}(0) + B_{T_0}(0))u_{T_0}(t)\|_q^p + \|u'_{T_0}(t)\|_q^p \right) \mathrm{d}t \right)^{1/p}. \tag{26}$$

Since the pair $A_{T_0}(\cdot)$, $B_{T_0}(\cdot)$ satisfies the assumptions of Theorem 2.5, by estimate (7) we obtain

$$\|(A_{T_0}(0) + B_{T_0}(0))(A_{T_0}(t) + B_{T_0}(t))^{-1}\|_{\mathscr{L}(L^q_\sigma(\Omega_0))} \leq C(T_0), \quad t \in [0, \infty].$$

By this fact and by an application of Corollary 3.6 for $T = 2T_0$ we may continue the

calculation in (26) obtaining

$$\|u(T_0)\|_{\mathscr{I}^p} \le C\bigg(\int_0^{T_0} \big(\|(A_{T_0}(t) + B_{T_0}(t))u_{T_0}(t)\|_q^p + \|u_{T_0}'(t)\|_q^p \big) \mathrm{d}t \bigg)^{1/p}$$

$$= C \int_{T_0}^{2T_0} \big(\|(A_{\Omega_0}(t) + B(t))u(t)\|_q^p + \|u'(t)\|_q^p \big) \mathrm{d}t$$

$$\le C \int_0^{2T_0} \big(\|(A_{\Omega_0}(t) + B(t))u(t)\|_q^p + \|u'(t)\|_q^p \big) \mathrm{d}t$$

$$\le C(T_0) \bigg(\int_0^\infty \|F(t)\|_q^p \mathrm{d}t + \|u_0\|_{\mathscr{I}^p}^p \bigg).$$

Combining this with (25) completes the proof.   □

We conclude with the proof of the main result.

Proof.   (of Theorem 1.2)
First let Ω_0 be an arbitrary domain as described in Assumption 1.1. Estimate (7) applied on $\mu + A_{\Omega_0}(\cdot) + B(\cdot)$ implies the norms $\|\cdot\|_{D(A_{\Omega_0}(t)+B(t))}$ and $\|\cdot\|_{D(A_{\Omega_0}(0)+B(0))}$ to be equivalent with equivalence constants that do not depend on $t \in [0,\infty]$. As $\|\cdot\|_{D(A_{\Omega_0}(0)+B(0))}$ is equivalent to $\|\cdot\|_{2,q}$ we obtain

$$\|u\|_{2,q} \le C_1 \|u\|_{D(A_{\Omega_0}(t)+B(t))} \le C_2 \|u\|_{2,q}, \quad u \in D(A_{\Omega_0}), \ t \in [0,\infty].$$

Thus, Corollary 3.6 gives us for $T < \infty$

$$\int_0^T \big(\|u'(t)\|_q^p + \|u(t)\|_{2,q}^p \big) \mathrm{d}t \le C(T) \bigg(\int_0^T \|F(t)\|_q^p \mathrm{d}t + \|u_0\|_{\mathscr{I}^p}^p \bigg). \tag{27}$$

This yields

$$\int_0^T \bigg(\bigg\| \bigg(\partial_t + \sum_{|\beta| \le 1} b_\beta(t) \bigg) u(t) \bigg\|_q^p + \|u(t)\|_{2,q}^p + \|\nabla^\phi(t)\tilde{p}(t)\|_q^p \bigg) \mathrm{d}t$$

$$\le C(T) \bigg(\int_0^T \|F(t)\|_q^p \mathrm{d}t + \|u_0\|_{\mathscr{I}^p}^p \bigg).$$

for the solution (u,p) of $(TSE)_{F,u_0}^{\Omega_0}$. In view of (12), (16), and since $\{\Phi(t)\}_{t\in[0,T]}$ is a family of isomorphisms, this implies estimate (1) for the solution of the original equations $(SG)_{f,v_0}^{\Omega(t)}$. If Ω_0 is bounded Theorem 3.7 implies that (27) is even valid for $T = \infty$ with a finite constant $C > 0$. Hence the same arguments as above yield the assertion of Theorem 1.2 for bounded Ω_0. The assertion concerning the equivalence of the norms is an easy consequence of Lemma 3.4 and the properties of the isomorphism $\Phi(t)$.   □

4. An application to rotations.

Since Stokes- and Navier-Stokes equations with rotations are of greatest interest in the current research we want to close this note with a remark about the relation of that subject to our problem. Indeed the Stokes equations with rotations can be regarded as a special case of the much more general situation of Theorem 1.2.

For simplicity assume the rotation $\mathscr{O}$ in 3 dimensions around the x_3-axis. Then ϕ is given as

$$\phi(\xi; t) := \mathscr{O}(\omega t)\xi, \quad (\xi, t) \in \Omega_0 \times [0, T],$$

where $\omega = (0, 0, \omega_3)$. By (14) and (15) we see that here

$$\Phi(t)\Delta_x\Phi(t)^{-1}u = \Delta_\xi u$$

and

$$\Phi(t)\partial_t\Phi(t)^{-1}u = \partial_t u + \omega \times u - \nabla u(\omega \times \xi).$$

We immediately observe that in this special case the coefficients do not depend on t. Furthermore, since $[\Phi(t)\nabla_x p](\xi, t) = \nabla_\xi p(\mathscr{O}(\omega t)\xi, t)$, we see that also the transformed Helmholtz projection is independent of t. Indeed it is exactly the Helmholtz projection P_{Ω_0} on $L^q(\Omega_0)$. For this reason the transformed propagator $A_{\Omega_0}(\cdot)$ coincides with the Stokes operator A_{Ω_0} in $L^q_\sigma(\Omega_0)$. The full operator then is given by

$$(A_{\Omega_0} + B)u = -P_{\Omega_0}(\Delta u - \omega \times u + \nabla u(\omega \times \xi)).$$

This operator arises naturally in the studies of the Navier-Stokes flow around a rotating obstacle and is the research object of several recent works, e.g. see [16], [8], [15], [11]. For recent works related to rotating Navier-Stokes equations see also [22], [13]. If we assume $0 < T < \infty$ and Ω_0 to be bounded of class C^3 it is easy to see that the above ϕ satisfies Assumption 1.1. Thus, Theorem 3.7 implies $A_{\Omega_0} + B \in \mathrm{MR}(L^q_\sigma(\Omega_0), C)$, or Theorem 1.2 the maximal regularity for the original Stokes equations on the rotated domain $\Omega(t) = \mathscr{O}(\omega t)\Omega_0$. Note that a similar result can be obtained even if the angular velocity ω depends on t, a problem which is considered in [17].

However, we can not apply our results to the case of unbounded domains, since then $(x, t) \mapsto \partial_t\phi^{-1}(x; t) = (\partial_t\mathscr{O}(\omega t))x$ is not bounded in x, i.e. Assumption 1.1.2 is not fulfilled. Indeed, this time derivative is responsible for the occurence of the linearly growing coefficient in the drift term $\nabla u(\omega \times \xi)$, which makes the examination of $A_{\Omega_0} + B$ more delicate. On the other hand we also do not expect the maximal regularity in the case of unbounded domains Ω_0, since due to results in [16] or [15] it is well known that $A_{\Omega_0} + B$ is not the generator of a holomorphic semigroup even in $L^2_\sigma(\Omega_0)$ for Ω_0 exterior or $\Omega_0 = \mathbf{R}^n$, respectively.

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