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Geometry of reflective submanifolds in Riemannian symmetric spaces

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Abstract. The set $\mathscr{R}(B)$ of submanifolds conjugate to a given reflective submanifold B in a Riemannian symmetric space M has a structure of symmetric space. Using this structure, for a submanifold N in M we establish integral formulae which represent the integrals of the functions $C \mapsto \mathrm{vol}(N \cap C)$ on $\mathscr{R}(B)$ by some extrinsic geometric amounts of N.

1. Introduction.

A reflective submanifold in a Riemannian manifold is a connected component of the fixed point set of an involutive isometry of a complete Riemannian manifold, which was defined by Leung [**7**]. A plane of any dimension in a Euclidean space is a typical example of reflective submanifolds. We denote by G the identity component of the group of isometries of $\boldsymbol{R}^n$ and take a plane B of dimension l in $\boldsymbol{R}^n$. We consider the set $\mathscr{R}(B)$ of all planes which are conjugate to B under the action of G:

$$\mathscr{R}(B) = \{gB \mid g \in G\},$$

which is equal to the set of all planes of dimension l in $\boldsymbol{R}^n$. Then $\mathscr{R}(B)$ has a structure of symmetric space and a G-invariant measure. For an integer k with $k + l \geq n$ there exists a constant σ such that

$$\int_{\mathscr{R}(B)} \mathrm{vol}(N \cap C) d\mu(C) = \sigma \mathrm{vol}(N)$$

holds for any submanifold N of dimension k in $\boldsymbol{R}^n$. For almost every C in $\mathscr{R}(B)$ the intersection $N \cap C$ is empty or a submanifold of dimension $k + l - n$. The function $C \mapsto \mathrm{vol}(N \cap C)$ is a measurable function on $\mathscr{R}(B)$ and we can consider its integration. The above integral formula is a classical Crofton formula. Its explicit expression is stated in Corollary 4.5. One can find various versions of Crofton formulae in Santaló [**12**].

The purpose of this paper is to extend these results to the case where B is a reflective submanifold in a Riemannian symmetric space. In Section 2 we recall the definitions of symmetric spaces and reflective submanifolds. We show in Theorem 2.5 that the set $\mathscr{R}(B)$ of all submanifolds which are conjugate to one reflective submanifold B in a Riemannian

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symmetric space M has a structure of symmetric space. If M is a Riemannian symmetric space of compact type, $\mathscr{R}(B)$ is a Riemannian symmetric space of compact type. If M is a Riemannian symmetric space of noncompact type, $\mathscr{R}(B)$ is a semi-Riemannian symmetric space. Essential parts of these facts have been already proved by Naitoh [**10**], but our assumptions of the statements mentioned in Theorem 2.5 are weaker than that of [**10**]. So we shall give its proof. In order to establish Crofton formulae with respect to $\mathscr{R}(B)$ we have to consider the integration on it. In Section 3 we define a canonical measure on a semi-Riemannian manifold and in Theorem 3.7 introduce a coarea formula on it. Its proof is given in Appendix. In Section 4, Theorem 4.1 we describe the Crofton formula with respect to $\mathscr{R}(B)$ as follows:

$$\int_{\mathscr{R}(B)} \mathrm{vol}(N \cap C) d\mu(C) = \int_N \sigma_B(T_x N) d\mu(x),$$

where N is a submanifold in M. In Section 5 we consider Crofton formulae in complex space forms. The author defined the multiple Kähler angle in order to describe Poincaré formulae of submanifolds in complex projective spaces in [**13**], which is explained in Section 5. Using multiple Kähler angles, we can describe Crofton formulae in complex space forms more explicitly.

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2. Reflective submanifolds in Riemannian symmetric spaces.

We recall the definitions of symmetric spaces and reflective submanifolds in Riemannian manifolds. Elie Cartan originated the theory of symmetric spaces, but the following definition of symmetric space is due to Loos [**9**].

Definition 2.1. A differential manifold M is called a *symmetric space* if for each x in M there exists a diffeomorphism s_x which satisfies the following conditions.

(1) x is an isolated fixed point of s_x,
(2) s_x is involutive, that is, $s_x^2 = 1$,
(3) $s_x(s_y(z)) = s_{s_x(y)}(s_x(z))$ for any y and z in M.

If M is a (semi-)Riemannian manifold and if each s_x is isometric, we call M a *(semi-) Riemannian symmetric space*.

Example 2.2. Let G be a Lie group with an involutive automorphism τ. We denote by $\mathrm{Fix}(\tau, G)$ and $\mathrm{Fix}(\tau, G)_0$ the fixed point set of τ and its identity component respectively. Let K be a closed Lie subgroup of G which satisfies

$$\mathrm{Fix}(\tau, G)_0 \subset K \subset \mathrm{Fix}(\tau, G).$$

Let $M = G/K$ and denote $\bar{x} = xK$. We define an involutive diffeomorphism $s_{\bar{x}}$ of M by

$$s_{\bar{x}}(\bar{y}) = x\tau(x)^{-1}\tau(y)K \quad (y \in G).$$

Then M is a symmetric space. See Theorem 1.3 (p. 73) in Loos [**9**].

The following definition of reflective submanifolds is due to Leung [**7**].

Definition 2.3. Let M be a complete connected Riemannian manifold. A connected component of the fixed point set of an involutive isometry of M is called a *reflective submanifold*.

Lemma 2.4. *Let M be a complete connected Riemannian manifold and B be a reflective submanifold of M. Then an involutive isometry such that B is a connected component of its fixed point set is determined uniquely.*

We can prove this lemma by considering the differential of such an involutive isometry at a point of B.

There is another class of submanifolds, called symmetric submanifolds, in Riemannian manifolds, which was first defined by Ferus [**2**] in Euclidean spaces. We can define symmetric submanifolds in general Riemannian manifolds. Totally geodesic, symmetric submanifolds in Riemannian symmetric spaces are nothing but reflective submanifolds. Naitoh [**10**] showed the following theorem in his study on symmetric submanifolds in Riemannian symmetric spaces, where the ambient Riemannian symmetric spaces are simply connected and have no Euclidean factor.

Theorem 2.5. *Let M be a Riemannian symmetric space and B be a reflective submanifold of M. We denote by G the identity component of the group of all isometries on M and by $\mathscr{R}(B)$ the set of all reflective submanifolds in M which are conjugate to B under the action of G:*

$$\mathscr{R}(B) = \{gB \mid g \in G\}.$$

Then $\mathscr{R}(B)$ has a structure of symmetric space. If M is a Riemannian symmetric space of compact type, $\mathscr{R}(B)$ is a Riemannian symmetric space of compact type. If M is a Riemannian symmetric space of noncompact type, $\mathscr{R}(B)$ is a semi-Riemannian symmetric space of semisimple type.

The assumption of the theorem is slightly weaker than that of Theorem 2.3 in Naitoh [**10**], so we show its proof for completeness. Take a point o in B and define

$$K = \{g \in G \mid go = o\}.$$

Then M is diffeomorphic to G/K. We identify M with G/K in a natural way. We denote by $S(B)$ the stabilizer of B in G:

$$S(B) = \{g \in G \mid gB = B\}.$$

We shall show that $S(B)$ is a closed subgroup of G which satisfies the condition $(*)$ mentioned below. Then $\mathscr{R}(B)$ is bijective to $G/S(B)$ and $\mathscr{R}(B)$ has a manifold structure. According to Lemma 2.4 we can take a unique involutive isometry τ of M such that B is a connected component of its fixed point set. We define an automorphism $A(\tau)$ of G by

$$A(\tau) : G \to G \ ; \ g \mapsto \tau g \tau^{-1}.$$

We show the following relation between the sets $\mathrm{Fix}(A(\tau), G)$ and $S(B)$:

$$(*) \quad \mathrm{Fix}(A(\tau), G)_0 \subset S(B) \subset \mathrm{Fix}(A(\tau), G).$$

Take any element g in $S(B)$. For any point x in B, we have $gx \in B$ and $dg_x T_x B = T_{gx} B$. So $dg_x T_x^{\perp} B = T_{gx}^{\perp} B$ also holds. The isometry τ is a reflection with respect to B, hence $d\tau_x|_{T_x B} = 1$ and $d\tau_x|_{T_x^{\perp} B} = -1$ hold and

$$d(\tau g \tau^{-1})_o|_{T_o B} = d\tau_{go} dg_o d\tau^{-1}|_{T_o B} = d\tau_{go} dg_o|_{T_o B} = dg_o|_{T_o B},$$
$$d(\tau g \tau^{-1})_o|_{T_o^{\perp} B} = d\tau_{go} dg_o d\tau^{-1}|_{T_o^{\perp} B} = d\tau_o dg_o(-1)|_{T_o^{\perp} B} = dg_o|_{T_o^{\perp} B}.$$

Thus $dg_o = d(\tau g \tau^{-1})_o$. Two isometries g and $\tau g \tau^{-1}$ of M have the same target and differential at o. So $g = \tau g \tau^{-1}$, that is, g belongs to the fixed point set $\mathrm{Fix}(A(\tau), G)$ of $A(\tau)$. This shows $S(B) \subset \mathrm{Fix}(A(\tau), G)$.

Take any element g in $\mathrm{Fix}(A(\tau), G)$. Then $\tau g \tau^{-1} = g$ and $\tau g = g \tau$. This operates x in B and

$$\tau g x = g \tau x = g x,$$

which means that gx belongs to the fixed point set $\mathrm{Fix}(\tau, M)$ of τ. Thus we have $\mathrm{Fix}(A(\tau), G)x \subset \mathrm{Fix}(\tau, M)$. Since $\mathrm{Fix}(A(\tau), G)_0 x$ is connected, this is included in B. Hence we obtain $\mathrm{Fix}(A(\tau), G)_0 B \subset B$ and $\mathrm{Fix}(A(\tau), G)_0 \subset S(B)$. Therefore the relation $(*)$ holds.

Since the relation $(*)$ implies that $S(B)$ is a union of some cosets with respect to $\mathrm{Fix}(A(\tau), G)_0$, the complement of $S(B)$ in $\mathrm{Fix}(A(\tau), G)$ is also a union of some cosets. Each coset is open in $\mathrm{Fix}(A(\tau), G)$, so $S(B)$ is closed in $\mathrm{Fix}(A(\tau), G)$. Thus $S(B)$ is a closed Lie subgroup of G. By Example 2.2 $\mathscr{R}(B) = G/S(B)$ is a symmetric space.

If M is a Riemannian symmetric space of compact type, G is a compact semisimple Lie group. The Riemannian metric of M is induced from a bi-invariant Riemannian metric of G. This also induces a G-invariant Riemannian metric of $\mathscr{R}(B) = G/S(B)$. With respect to this metric, $\mathscr{R}(B)$ is a Riemannian symmetric space of compact type.

If M is a Riemannian symmetric space of noncompact type, G is a noncompact semisimple Lie group. The Riemannian metric of M is induced from a bi-invariant semi-Riemannian metric of G. We show that this bi-invariant semi-Riemannian metric of G induces a G-invariant semi-Riemannian metric of $\mathscr{R}(B) = G/S(B)$. With respect to this metric, $\mathscr{R}(B)$ is a semi-Riemannian symmetric space of semisimple type. We denote by $\mathfrak{g}$ the Lie algebra of G. The bi-invariant semi-Riemannian metric of G induces an $\mathrm{Ad}(G)$-invariant indefinite inner product $\langle \ , \ \rangle$ on $\mathfrak{g}$, which is a positive constant multiple of the Killing form of $\mathfrak{g}$ on each simple factor of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be a canonical decomposition of $\mathfrak{g}$ associated with the Riemannian symmetric space M. These subspaces $\mathfrak{k}$ and $\mathfrak{m}$ are orthogonal with respect to the inner product $\langle \ , \ \rangle$ and this inner product is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{m}$. Let s_o be the geodesic symmetry of M at o. The involutive automorphism σ associated with the Riemannian symmetric space M is given by

$$\sigma : G \to G \; ; \; g \mapsto s_o g s_o^{-1}.$$

Since τs_o and $s_o \tau$ have the same target and differential at o, we obtain $\tau s_o = s_o \tau$, which implies $A(\tau)\sigma = \sigma A(\tau)$. Hence $dA(\tau)\mathfrak{k} = \mathfrak{k}$ and $dA(\tau)\mathfrak{m} = \mathfrak{m}$. We can decompose $\mathfrak{k}$ and $\mathfrak{m}$ to the direct sums of ± 1-eigenspaces of $dA(\tau)$ as follows:

$$\mathfrak{k} = \mathfrak{k}_+ + \mathfrak{k}_-, \quad \mathfrak{m} = \mathfrak{m}_+ + \mathfrak{m}_-.$$

The Lie algebra of $S(B)$ is $\mathfrak{k}_+ + \mathfrak{m}_+$ and the tangent space of $\mathscr{R}(B) \cong G/S(B)$ is naturally isomorphic to $\mathfrak{k}_- + \mathfrak{m}_-$. The inner product $\langle \ , \ \rangle$ is negative definite on $\mathfrak{k}_-$ and positive definite on $\mathfrak{m}_-$. In particular it induces G-invariant semi-Riemannian metric on $\mathscr{R}(B)$. Thus we have proved the theorem.

Remark 2.6. We can describe the structure of symmetric space on $\mathscr{R}(B)$ in a geometric way as follows. We take x and y in G. We can write the symmetry s_{xB} of $\mathscr{R}(B)$ at xB by

$$s_{xB}(yB) = x(A(\tau)x)^{-1}A(\tau)yB = x\tau x^{-1}yB.$$

Thus $s_{xB} = x\tau x^{-1}$. On the other hand $x\tau x^{-1}$ is the involutive isometry corresponding to the reflective submanifold xB.

3. Integration on semi-Riemannian manifolds.

In the case where M is a Riemannian symmetric space of noncompact type, for a reflective submanifold B in M the manifold $\mathscr{R}(B)$ is a semi-Riemannian symmetric space of semisimple type as is shown in the previous section. In order to consider the integration on $\mathscr{R}(B)$, we prepare for a measure on a semi-Riemannian manifold.

Lemma 3.1. *Let E be a real vector space of finite dimension. A bilinear form A on E is bijectively corresponding to a linear map α from E to its dual space E^* by*

$$A(x, y) = (\alpha(x))(y) \qquad (x, y \in E).$$

This correspondence gives a linear isomorphism between the vector space $\otimes^2 E^*$ of all bilinear forms on E and the vector space $\mathrm{Hom}(E, E^*)$ *of all linear maps from E to E^*.*

Proposition 3.2. *Let E be a real vector space of finite dimension with an inner product $\langle \ , \ \rangle$. We do not always assume that $\langle \ , \ \rangle$ is positive definite. We denote by α the element in* $\mathrm{Hom}(E, E^*)$ *associated with $\langle \ , \ \rangle$ by Lemma* 3.1. *Then $\alpha : E \to E^*$ induces a linear map*

$$\wedge^p \alpha : \wedge^p E \to \wedge^p E^* \cong (\wedge^p E)^*.$$

Moreover the bilinear form in $\otimes^2(\wedge^p E)^*$ *associated with* $\wedge^p \alpha$ *by Lemma* 3.1 *gives an inner product on* $\wedge^p E$.

In the case where $\langle\ ,\ \rangle$ is positive definite the manner of Proposition 3.2 is stated in 1.7.5 of Federer [**1**]. The manner also works well in the case where $\langle\ ,\ \rangle$ is not positive definite. From now on we always consider the inner product on the exterior algebra $\wedge^p E$ mentioned above, and also denote it by the same symbol $\langle\ ,\ \rangle$. We denote the length of u by $|u| = |\langle u, u\rangle|^{1/2}$. We can show the following lemma.

Lemma 3.3. *Under the situation of Proposition* 3.2 *we have*

$$\langle u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_p\rangle = \det[\langle u_i, v_j\rangle]_{1\le i,j\le p}$$

for elements $u_1, \ldots, u_p$ *and* $v_1, \ldots, v_p$ *in* E. *If* $\{e_1, \ldots, e_n\}$ *is an orthonormal basis of* E, *then*

$$e_{i_1} \wedge \cdots \wedge e_{i_p} \qquad (1 \le i_1 < \cdots < i_p \le n)$$

is an orthonormal basis of $\wedge^p E$.

Lemma 3.4. *Let* $m \ge n$ *and let* V *and* W *be real vector spaces of dimension* m *and* n *with inner products. Let* $F : V \to W$ *be a linear map. We assume that the inner product restricted to* $\ker F$ *is nondegenerate. If* F *is not surjective, we define a constant* JF *by* $JF = 0$. *If* F *is surjective, we take a basis* $v_1, \ldots, v_n$ *of* $(\ker F)^\perp$ *and define it by*

$$JF = \frac{|F(v_1) \wedge \cdots \wedge F(v_n)|}{|v_1 \wedge \cdots \wedge v_n|}.$$

Then this is independent of the choice of $v_1, \ldots, v_n$.

From now on we assume that a manifold has a countable open base in order to consider integration.

Definition 3.5. Let (M, g) be a semi-Riemannian manifold of dimension n. We denote by $\mathscr{K}(M)$ the set of all real valued continuous functions on M with compact support. We take a local coordinate system $(U; x_1, \ldots, x_n)$ of M. For each $x \in U$ we consider the inner product on $\wedge^n T_x(M)$ induced from the semi-Riemannian metric g. For $f \in \mathscr{K}(M)$ with $\mathrm{supp} f \subset U$ we define $L(f)$ by

$$L(f) = \int_U f(x_1, \ldots, x_n)\left|\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}\right| dx_1 \cdots dx_n.$$

Here the integral on the right hand side is the Lebesgue integral on the Euclidean space. The integrand is a continuous function with compact support, so the integral is equal to the usual Riemann integral. We can see that $L(f)$ is independent of the choice of the local coordinate system by the integration formula of variable change. For a function $f \in \mathscr{K}(M)$ whose support is not included in a single coordinate neighborhood, we extend the value of L by the use of a partition of unity. This is also independent of the choice of the partition of unity. By Riesz representation theorem there exists a Radon measure μ_M on M which satisfies

$$L(f) = \int_M f d\mu_M \qquad (f \in \mathscr{K}(M)).$$

We call this measure μ_M the *canonical measure*. We will hereafter consider this canonical measure on a semi-Riemannian manifold. We write $\mathrm{vol}(M) = \mu_M(M)$ and call $\mathrm{vol}(M)$ the *volume* of M.

REMARK 3.6. In Definition 3.5

$$\left| \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} \right| = \left| \det \left[\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle \right] \right|^{1/2}.$$

In the case of a Riemannian manifold we use the density

$$\left(\det \left[\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle \right] \right)^{1/2},$$

because $\det[\langle \partial/\partial x_i, \partial/\partial x_j \rangle]$ is always positive.

THEOREM 3.7 (Coarea formula). *Let $m \geq n$ and $f : M \to N$ be a map of class C^∞ from a semi-Riemannian manifold M of dimension m to a semi-Riemannian manifold N of dimension n. We assume that the inverse images $f^{-1}(y)$ are semi-Riemannian submanifolds of M for almost all $y \in N$. Let ϕ be a μ_M-measurable function on M. Then the function which maps y in N to $\int_{f^{-1}(y)} \phi(x) d\mu_{f^{-1}(y)}(x)$ is a μ_N-measurable function on N. Moreover if $\phi J df$ is μ_M-integrable or $\phi \geq 0$, then*

$$\int_N \left(\int_{f^{-1}(y)} \phi(x) d\mu_{f^{-1}(y)}(x) \right) d\mu_N(y) = \int_M \phi(x) J df(x) d\mu_M(x).$$

We shall give a proof of this theorem in Appendix.

4. Crofton formulae by reflective submanifolds.

We retain the notations in Section 2 and for a submanifold N of M which satisfies $\dim N + \dim B \geq \dim M$, consider integral formulae of the following types:

$$\int_{\mathscr{R}(B)} \mathrm{vol}(N \cap C) d\mu(C) = \text{geometric amount of } N,$$

which is called *Crofton formula*. In our cases the geometric amount of N in the right hand side of the above formula is dependent of its embedding to M. In order to formulate this we put by definition

$$\mathscr{R}_0(B) = \{kT_oB \mid k \in K\},$$

which coincides with the set of tangent spaces T_oC of all reflective submanifolds C in $\mathscr{R}(B)$ through o. Since $\mathscr{R}_0(B) \cong K/(K \cap S(B))$, we can consider a K-invariant Riemannian metric on $\mathscr{R}_0(B)$ induced from the bi-invariant Riemannian metric on K. If a vector subspace $V \subset T_oM$ satisfies $\dim V + \dim B \geq \dim M$, we define $\sigma_B(V)$ by

$$\sigma_B(V) = \int_{\mathscr{R}_0(B)} |\vec{V}^\perp \wedge \vec{\mathfrak{c}}^\perp| d\mu(\mathfrak{c}).$$

Here $\vec{V}^\perp$ is the wedge product of an orthonormal basis of $V^\perp$ and $\vec{\mathfrak{c}}^\perp$ is similar. We consider only the norm of their wedge product, so there is no ambiguity. For a vector subspace $V \subset T_xM$ satisfying $\dim V + \dim B \geq \dim M$, we take $g \in G$ which satisfies $go = x$ and put $\sigma_B(V) = \sigma_B(dg^{-1}V)$. Since K acts isometrically on $\mathscr{R}_0(B)$, the definition of $\sigma_B(V)$ is independent of the choice of $g \in G$. Using σ_B we can give Crofton formulae by reflective submanifolds as follows.

Theorem 4.1. *Let M be a Riemannian symmetric space of compact type or non-compact type and B be a reflective submanifold of M. For a submanifold N of M which satisfies $\dim N + \dim B \geq \dim M$, the following equation holds.*

$$\int_{\mathscr{R}(B)} \mathrm{vol}(N \cap C) d\mu(C) = \int_N \sigma_B(T_xN) d\mu(x).$$

Since the amount $\sigma_B(T_xN)$ depends on the inclusion of T_xN in T_xM, the integral of the right hand side is extrinsic in general. If M is a real space form, it is determined by the volume of N as is shown in Corollary 4.5. This is the simplest Crofton formula. On the other hand, if M is a complex space form, $\sigma_B(T_xN)$ is described by the multiple Kähler angle of T_xN as is shown in Corollary 5.9. So its integral is not intrinsic but extrinsic.

Howard [**3**] touched on Crofton formulae of submanifolds in Riemannian homogeneous spaces. In our case the invariant measure on the class of submanifolds and the integrand of the right side are explicitly given.

In order to prove this theorem we put

$$I(M \times \mathscr{R}(B)) = \{(x, C) \in M \times \mathscr{R}(B) \mid x \in C\}$$

and investigate some properties about this. We define an action of G on $M \times \mathscr{R}(B)$ by

$$g(x, C) = (gx, gC) \quad (g \in G,\ x \in M,\ C \in \mathscr{R}(B)).$$

Then G is a Lie transformation group of $M \times \mathscr{R}(B)$.

Lemma 4.2.

$$G(o, B) = I(M \times \mathscr{R}(B)).$$

In particular $I(M \times \mathscr{R}(B))$ is a homogeneous submanifold of $M \times \mathscr{R}(B)$.

PROOF. For $g \in G$ we have $gB \in \mathscr{R}(B)$ and $go \in gB$, so $g(o, B) = (go, gB) \in I(M \times \mathscr{R}(B))$. Hence we get $G(o, B) \subset I(M \times \mathscr{R}(B))$.

Conversely we take any $(x, C) \in I(M \times \mathscr{R}(B))$. Since $C \in \mathscr{R}(B)$, there exists $g_1 \in G$ which satisfies $C = g_1 B$. Thus $g_1^{-1} x \in g_1^{-1} C = B$. Since $S(B)$ acts transitively on B, there exists $g_2 \in S(B)$ such that $g_2 g_1^{-1} x = o \in B$, which implies

$$g_1^{-1} x = g_2^{-1} o \in g_2^{-1} B = B,$$

thus,

$$x = g_1 g_2^{-1} o \in g_1 g_2^{-1} B = g_1 B = C.$$

This shows $(x, C) = g_1 g_2^{-1}(o, B)$ and we obtain $I(M \times \mathscr{R}(B)) \subset G(o, B)$. Therefore

$$G(o, B) = I(M \times \mathscr{R}(B)).$$

In particular $I(M \times \mathscr{R}(B))$ is a homogeneous submanifold of $M \times \mathscr{R}(B)$.

PROOF OF THEOREM 4.1. The Lie algebra of $K \cap S(B)$ is equal to $\mathfrak{k}_+$. The tangent space of $I(M \times \mathscr{R}(B)) \cong G/(K \cap S(B))$ at the origin (o, B) is identified with

$$\mathfrak{k}_- + \mathfrak{m} = \mathfrak{k}_- + \mathfrak{m}_+ + \mathfrak{m}_-.$$

If M is a Riemannian symmetric space of compact type, $\langle\ ,\ \rangle$ is positive definite on $\mathfrak{k}_- + \mathfrak{m}$, and it induces a G-invariant Riemannian metric on $I(M \times \mathscr{R}(B))$. If M is a Riemannian symmetric space of noncompact type, $\langle\ ,\ \rangle$ is negative definite on $\mathfrak{k}_-$ and positive definite on $\mathfrak{m}$, and it induces a G-invariant semi-Riemannian metric on $I(M \times \mathscr{R}(B))$. In each case the considered G-invariant metric is not equal to the metric induced on $I(M \times \mathscr{R}(B))$ from that of $M \times \mathscr{R}(B)$. We define the following maps p_B and p_M by $p_B(x, C) = C$ and $p_M(x, C) = x$.

$$\begin{array}{ccccccc}
I(M \times \mathscr{R}(B)) & \xrightarrow{p_B} & \mathscr{R}(B) & & G/(K \cap S(B)) & \xrightarrow{p_B} & G/S(B) \\
\downarrow{\scriptstyle p_M} & & & \cong & \downarrow{\scriptstyle p_M} & & \\
M & & & & G/K & &
\end{array}$$

These maps p_B and p_M are both (semi-)Riemannian submersions and projections of fiber bundles. Their differentials at the origin are given as follows:

$$\begin{array}{ccc} \mathfrak{k}_- + \mathfrak{m}_+ + \mathfrak{m}_- & \xrightarrow{dp_B} & \mathfrak{k}_- + \mathfrak{m}_- \\ \Big\downarrow {\scriptstyle dp_M} & & \\ \mathfrak{m}_+ + \mathfrak{m}_- & & \end{array}$$

Since $\ker dp_M = \mathfrak{k}_-$ at the origin, $\langle\ ,\ \rangle$ is positive definite on $\ker dp_M$ if M is of compact type, and $\langle\ ,\ \rangle$ is negative definite on $\ker dp_M$ if M is of noncompact type. Since $\ker dp_B = \mathfrak{m}_+$ at the origin, $\langle\ ,\ \rangle$ is positive definite on $\ker dp_B$ in both cases. The fiber of p_M at the origin coincides with $K/(K \cap S(B))$.

Let N be a submanifold of M which satisfies $\dim N + \dim B \geq \dim M$. We put

$$I(N) = \{(x, C) \in I(M \times \mathscr{R}(B)) \mid x \in N\} = p_M^{-1}(N).$$

We consider the metric of $I(N)$ induced from that of $I(M \times \mathscr{R}(B))$. For any $(x, C) \in I(N)$ there exists $g \in G$ which satisfies $gx = o$ and $gC = B$.

$$T_{(o,B)}(gI(N)) = T_{(o,B)}(I(gN)) = \mathfrak{k}_- + T_o(gN)$$

holds. The inner product $\langle\ ,\ \rangle$ is negative definite on $\mathfrak{k}_-$ and positive definite on $T_o(gN)$. Thus the induced metric on $I(N)$ is a semi-Riemannian metric. Since p_M is the projection of the fiber bundle, $p_M^{-1}(N)$ is a submanifold of $I(M \times \mathscr{R}(B))$. We consider the inverse image of the restriction of p_B to $I(N)$. The inner product $\langle\ ,\ \rangle$ is positive definite on the inverse images of p_B as is shown above. Thus $\langle\ ,\ \rangle$ is also positive definite on almost all inverse images of $p_B|_{I(N)}$. We can apply Theorem 3.7 to $p_B|_{I(N)} : I(N) \to \mathscr{R}(B)$ and obtain

$$\int_{\mathscr{R}(B)} \mathrm{vol}((p_B|_{I(N)})^{-1}(C)) d\mu(C) = \int_{I(N)} Jd(p_B|_{I(N)}) d\mu.$$

Here

$$(p_B|_{I(N)})^{-1}(C) = \{(x, C) \mid x \in C,\ x \in N\} = (N \cap C) \times C,$$

hence we get $\mathrm{vol}((p_B|_{I(N)})^{-1}(C)) = \mathrm{vol}(N \cap C)$ and

$$\int_{\mathscr{R}(B)} \mathrm{vol}(N \cap C) d\mu(C) = \int_{I(N)} Jd(p_B|_{I(N)}) d\mu.$$

So we calculate the right hand side. We take any $(x, C) \in I(N)$. There exists $g \in G$ which satisfies $go = x$, $gB = C$.

$$T_{(x,C)}(G/(K \cap S(B))) = dg(\mathfrak{k}_- + \mathfrak{m}_+ + \mathfrak{m}_-)$$

and we can regard $dg^{-1}T_xN \subset T_oM \cong \mathfrak{m}_+ + \mathfrak{m}_-$. We have

$$T_{(x,C)}(I(N)) = dg(\mathfrak{k}_- + dg^{-1}T_xN).$$

Since $dp_B dg(T, X_+, X_-) = dg(T, X_-)$, we get

$$\ker_{(x,C)} = \ker(dp_B : T_{(x,C)}(I(N)) \to T_C(\mathscr{R}(B))) = dg(dg^{-1}T_xN \cap \mathfrak{m}_+).$$

These imply

$$\begin{aligned}\ker^{\perp}_{(x,C)} \cap T_{(x,C)}(I(N)) &= \ker^{\perp}_{(x,C)} \cap dg(\mathfrak{k}_- + dg^{-1}T_xN) \\ &= dg(\mathfrak{k}_- + (dg^{-1}T_xN \cap \mathfrak{m}_+)^{\perp} \cap dg^{-1}T_xN).\end{aligned}$$

We take an orthonormal basis $\{T_a\}$ of $\mathfrak{k}_-$ and an orthonormal basis $\{X_b\}$ of $(dg^{-1}T_xN \cap \mathfrak{m}_+)^{\perp} \cap dg^{-1}T_xN$. Using these we can write

$$\begin{aligned}Jd(p_B|_{I(N)}) &= \left|\bigwedge_a dp_B T_a \wedge \bigwedge_b dp_B X_b\right| = \left|\bigwedge_a T_a \wedge \bigwedge_b (X_b)_-\right| \\ &= \left|\bigwedge_b (X_b)_-\right|,\end{aligned}$$

where $(X_b)_-$ is its $\mathfrak{m}_-$-component. In order to investigate this we prepare the following lemma.

Lemma 4.3. *Let E be a real vector space of finite dimension with a positive definite inner product and V and W be vector subspaces of E which satisfy $E = V + W$. We denote by $p_{W^{\perp}} : E \to W^{\perp}$ the orthogonal projection from E to $W^{\perp}$ and take an orthonormal basis $\{X_b\}$ of $(V \cap W)^{\perp} \cap V$. Then we have the following equation*:

$$\left|\bigwedge_b p_{W^{\perp}}(X_b)\right| = \left|\vec{V}^{\perp} \wedge \vec{W}^{\perp}\right|.$$

Proof. Note that

$$E = (V \cap W) \oplus ((V \cap W)^{\perp} \cap V) \oplus ((V \cap W)^{\perp} \cap W).$$

We set

$$E_V = (V \cap W)^{\perp} \cap V, \quad E_W = (V \cap W)^{\perp} \cap W.$$

Then we have $(V \cap W)^{\perp} = E_V \oplus E_W$. $\{X_b\}$ is an orthonormal basis of E_V. We take an orthonormal basis $\{Y_c\}$ of E_W.

$$
\begin{aligned}
\Big|\bigwedge_b p_{W^\perp}(X_b)\Big| &= \Big|\bigwedge_b X_b \wedge \bigwedge_c Y_c\Big| = \Big|\vec{E}_V \wedge \vec{E}_W\Big| \\
&= \big|((V\cap W)^\perp \cap V)^\rightarrow \wedge ((V\cap W)^\perp \cap W)^\rightarrow\big| \\
&= \big|((V\cap W)^\perp \cap V^\perp)^\rightarrow \wedge ((V\cap W)^\perp \cap W^\perp)^\rightarrow\big| \\
&= \big|\vec{V}^\perp \wedge \vec{W}^\perp\big|.
\end{aligned}
$$

Now we return to the proof of Theorem 4.1. For almost all $(x, C) \in I(N)$ we have $dg^{-1}T_xN + \mathfrak{m}_+ = \mathfrak{m}$ and the inner product $\langle\ ,\ \rangle$ is positive definite on $\mathfrak{m}$, so we can apply Lemma 4.3 to these vector subspaces.

$$
\begin{aligned}
Jd(p_B|_{I(N)}) &= \Big|\bigwedge_b (X_b)_-\Big| = \big|(dg^{-1}T_x^\perp N)^\rightarrow \wedge \vec{\mathfrak{m}}_+^\perp\big| \\
&= \big|(T_x^\perp N)^\rightarrow \wedge (T_x^\perp C)^\rightarrow\big|.
\end{aligned}
$$

This implies

$$
\int_{\mathscr{R}(B)} \mathrm{vol}(N\cap C)\,d\mu(C) = \int_{I(N)} \big|(T_x^\perp N)^\rightarrow \wedge (T_x^\perp C)^\rightarrow\big|\,d\mu.
$$

The projection $p_M|_{I(N)} : I(N) \to N$ is a semi-Riemannian submersion. We apply Theorem 3.7 to this semi-Riemannian submersion and obtain

$$
\begin{aligned}
&\int_{I(N)} \big|(T_x^\perp N)^\rightarrow \wedge (T_x^\perp C)^\rightarrow\big|\,d\mu \\
&\quad= \int_N \left(\int_{p_M^{-1}(x)} \big|(T_x^\perp N)^\rightarrow \wedge (T_x^\perp C)^\rightarrow\big|\,d\mu(C)\right) d\mu(x) \\
&\quad= \int_N \sigma_B(T_xN)\,d\mu(x),
\end{aligned}
$$

which implies the following equation:

$$
\int_{\mathscr{R}(B)} \mathrm{vol}(N\cap C)\,d\mu(C) = \int_N \sigma_B(T_xN)\,d\mu(x).
$$

Howard [**3**] showed that two Riemannian homogeneous spaces which have orthogonally equivalent linear isotropy representations have same Poincaré formulae of submanifolds and he called it a *transfer principle*. Typical examples are symmetric spaces of compact type and their noncompact duals. They have same Poincaré formulae of submanifolds. He also mentioned the transfer principle of Crofton formulae.

Corollary 4.4. *Let $M = G/K$ be a Riemannian symmetric space of compact* *type and B be a reflective submanifold of M. We denote by $M^* = G^*/K$ the noncompact dual of M and by B^* the reflective submanifold in M^* associated with B. We can take bi-invariant metrics on G and G^* which coincide on K. Then we have $\sigma_B(V) = \sigma_{B^*}(V)$ for any vector subspace $V \subset T_oM \cong T_oM^*$. As a consequence, for a submanifold N of M which satisfies* $\dim N + \dim B \geq \dim M$, *the following equation holds.*

$$\int_{\mathscr{R}(B)} \mathrm{vol}(N \cap C) d\mu(C) = \int_N \sigma_B(T_xN) d\mu(x).$$

On the other hand for a submanifold N^* of M^* which satisfies $\dim N^* = \dim N$, *the following equation holds.*

$$\int_{\mathscr{R}(B^*)} \mathrm{vol}(N^* \cap C^*) d\mu(C^*) = \int_{N^*} \sigma_B(T_xN^*) d\mu(x).$$

In the case where $M = G/K$ is a real space form any totally geodesic submanifold B in M is a reflective submanifold and σ_B is constant, because the linear isotropy action of K on the Grassmann manifold consisting of real vector subspaces of dimension $\dim B$ in T_oM is transitive. In the case where

$$M = S^n = SO(n+1)/SO(n), \qquad B = S^l$$

we consider a submanifold N of dimension k in S^n such that $k + l \geq n$. We have

$$\int_{\mathscr{R}(S^l)} \mathrm{vol}(N \cap C) d\mu(C) = \int_N \sigma_{S^l}(T_xN) d\mu(x) = \sigma(n; k, l)\mathrm{vol}(N),$$

where $\sigma(n; k, l)$ is a constant dependent on n, k and l. If N is equal to a great sphere S^k, then $S^k \cap C$ is isometric to a great sphere S^{k+l-n} for almost every C in $\mathscr{R}(S^l)$ and

$$\int_{\mathscr{R}(S^l)} \mathrm{vol}(N \cap C) d\mu(C) = \mathrm{vol}(S^{k+l-n})\mathrm{vol}(\mathscr{R}(S^l)).$$

Thus we get

$$\mathrm{vol}(S^{k+l-n})\mathrm{vol}(\mathscr{R}(S^l)) = \sigma(n; k, l)\mathrm{vol}(S^k),$$

that is

$$\sigma(n; k, l) = \frac{\mathrm{vol}(S^{k+l-n})}{\mathrm{vol}(S^k)}\mathrm{vol}(\mathscr{R}(S^l)).$$

Therefore we obtain the following corollary.

Corollary 4.5. *Let B be a totally geodesic submanifold of dimension l in a real* *space form $M = G/SO(n)$ of dimension n. For a submanifold N of dimension k in M such that $k + l \geq n$, the following equation holds.*

$$\int_{\mathscr{R}(B)} \mathrm{vol}(N \cap C) d\mu(C) = \frac{\mathrm{vol}(S^{k+l-n})}{\mathrm{vol}(S^k)} \mathrm{vol}(\mathscr{R}(S^l)) \mathrm{vol}(N).$$

Leung [**8**] gave a classification of all reflective submanifolds in simply connected Riemannian symmetric spaces. Among them we can give Crofton formulae for the real and the complex space forms more explicitly than Theorem 4.1. We show this for the complex space forms in the next section.

5. Complex space forms.

In order to give Crofton formulae of submanifolds in complex space forms we use the notion of multiple Kähler angle which the author introduced in [**13**]. We denote by ω the standard Kähler form of $\boldsymbol{C}^n$.

Definition 5.1. Let $1 < k \leq n$. For a real vector subspace V of dimension k in $\boldsymbol{C}^n$ we consider a canonical form of the restriction $\omega|_V$, that is, we take an orthonormal basis $\{\alpha^1, \ldots, \alpha^k\}$ of the dual space of V which satisfies

$$\omega|_V = \sum_{i=1}^{[k/2]} \cos\theta_i \alpha^{2i-1} \wedge \alpha^{2i}, \qquad 0 \leq \theta_1 \leq \cdots \leq \theta_{[k/2]} \leq \pi/2.$$

Then we put $\theta(V) = (\theta_1, \ldots, \theta_{[k/2]})$ and call it the *multiple Kähler angle* of V. In the case where $n < k \leq 2n - 1$, for a real vector subspace V of dimension k in $\boldsymbol{C}^n$ we define the multiple Kähler angle of V by $\theta(V) = \theta(V^\perp)$.

Remark 5.2. Let $1 < k \leq n$. For a real vector subspace V of dimension k in $\boldsymbol{C}^n$ the followings hold.

(1) For any $g \in U(n)$ we have $\theta(gV) = \theta(V)$.
(2) If $k = 2$, the multiple Kähler angle is nothing but the Kähler angle.
(3) $\theta(V) = (0, \ldots, 0)$ holds if and only if there exists a complex vector subspace of complex dimension $[k/2]$ in V.
(4) $\theta(V) = (\pi/2, \ldots, \pi/2)$ holds if and only if V and $\sqrt{-1}V$ are orthogonal. In this case with the condition $\dim V = n$ we call V a *Lagrangian vector subspace.*

We denote by $G_k^{\boldsymbol{R}}(\boldsymbol{C}^n)$ the real Grassmann manifold consisting of real vector subspaces of dimension k in $\boldsymbol{C}^n$. The author [**13**] has showed the following fundamental property of the multiple Kähler angle.

Proposition 5.3. *Let V and W be real vector subspaces of same dimension in $\boldsymbol{C}^n$. There exists g in $U(n)$ such that $W = gV$ if and only if $\theta(V) = \theta(W)$.*

The definition of the multiple Kähler angle depends only on the Hermitian structure of $\boldsymbol{C}^n$, so we can consider the multiple Kähler angle for any real submanifold in an almost

Hermitian manifold. Using the multiple Kähler angle the author described Poincaré formulae for real submanifolds in the complex space forms in [**13**]. In the present paper we describe Crofton formulae for real submanifolds in the complex space forms.

Before we treat general cases, we review some known Crofton formulae in complex space forms. We recall a result of Leung on reflective submanifolds in complex space forms which is stated in Theorem 7 in [**8**].

THEOREM 5.4 (Leung). *The reflective submanifolds of the complex projective space* $\boldsymbol{C}P^n$ *are* $\boldsymbol{C}P^k$ $(1 \leq k < n)$ *and the real projective space* $\boldsymbol{R}P^n$ *which is naturally embedded in* $\boldsymbol{C}P^n$. *The reflective submanifolds of the complex hyperbolic space* $\boldsymbol{C}H^n$ *are the totally geodesic submanifolds which are dual to the reflective submanifolds of* $\boldsymbol{C}P^n$.

In the case where

$$M = \boldsymbol{C}P^n = SU(n+1)/S(U(n) \times U(1)), \qquad B = \boldsymbol{C}P^l$$

we consider a complex submanifold N of complex dimension k in $\boldsymbol{C}P^n$ such that $k+l \geq n$. Since the linear isotropy action of K on the Grassmann manifold consisting of complex vector subspaces of dimension k in T_oM is transitive, σ_B is constant. We have

$$\int_{\mathscr{R}(\boldsymbol{C}P^l)} \mathrm{vol}(N \cap C) d\mu(C) = \int_N \sigma_B(T_xN) d\mu(x) = \sigma(n; k, l)\mathrm{vol}(N),$$

where $\sigma(n; k, l)$ is a constant dependent on n, k and l. If N is equal to a totally geodesic $\boldsymbol{C}P^k$, then $\boldsymbol{C}P^k \cap C$ is isometric to $\boldsymbol{C}P^{k+l-n}$ for almost every C in $\mathscr{R}(\boldsymbol{C}P^l)$ and

$$\int_{\mathscr{R}(\boldsymbol{C}P^l)} \mathrm{vol}(N \cap C) d\mu(C) = \mathrm{vol}(\boldsymbol{C}P^{k+l-n})\mathrm{vol}(\mathscr{R}(\boldsymbol{C}P^l)).$$

Thus we get

$$\mathrm{vol}(\boldsymbol{C}P^{k+l-n})\mathrm{vol}(\mathscr{R}(\boldsymbol{C}P^l)) = \sigma(n; k, l)\mathrm{vol}(\boldsymbol{C}P^k),$$

that is

$$\sigma(n; k, l) = \frac{\mathrm{vol}(\boldsymbol{C}P^{k+l-n})}{\mathrm{vol}(\boldsymbol{C}P^k)}\mathrm{vol}(\mathscr{R}(\boldsymbol{C}P^l)).$$

Therefore we obtain the following corollary.

COROLLARY 5.5. *Let* B *be a totally geodesic complex submanifold of complex dimension* l *in a complex space form* $M = G/S(U(n) \times U(1))$ *of complex dimension* n. *For a complex submanifold* N *of complex dimension* k *in* M *such that* $k + l \geq n$, *the following equation holds.*

$$\int_{\mathscr{R}(B)} \mathrm{vol}(N \cap C) d\mu(C) = \frac{\mathrm{vol}(\boldsymbol{C}P^{k+l-n})}{\mathrm{vol}(\boldsymbol{C}P^k)}\mathrm{vol}(\mathscr{R}(\boldsymbol{C}P^l))\mathrm{vol}(N).$$

Remark 5.6. We can describe similar Crofton formulae of complex submanifolds in the other Hermitian symmetric spaces, using results obtained in Kang-Sakai-Takahashi-Tasaki [**5**] and Sakai [**11**].

In the case where $M = \boldsymbol{C}P^n$ and $B = \boldsymbol{R}P^n$, we consider a Lagrangian submanifold N in $\boldsymbol{C}P^n$. Since the linear isotropy action of K on the Grassmann manifold consisting of Lagrangian vector subspaces in T_oM is transitive, σ_B is constant. We have

$$\int_{\mathscr{R}(\boldsymbol{R}P^n)} \mathrm{vol}(N \cap C) d\mu(C) = \int_N \sigma_B(T_xN) d\mu(x) = \sigma(n)\mathrm{vol}(N),$$

where $\sigma(n)$ is a constant dependent on n. If N is equal to a totally geodesic $\boldsymbol{R}P^n$, then $\boldsymbol{R}P^n \cap C$ is a set of $n+1$ points for almost every C in $\mathscr{R}(\boldsymbol{C}P^l)$ by the result of Howard [**3**] (pp. 26, 27) and

$$\int_{\mathscr{R}(\boldsymbol{R}P^n)} \mathrm{vol}(N \cap C) d\mu(C) = (n+1)\mathrm{vol}(\mathscr{R}(\boldsymbol{C}P^l)).$$

Thus we get

$$(n+1)\mathrm{vol}(\mathscr{R}(\boldsymbol{R}P^n)) = \sigma(n)\mathrm{vol}(\boldsymbol{R}P^n),$$

that is

$$\sigma(n) = \frac{n+1}{\mathrm{vol}(\boldsymbol{R}P^n)}\mathrm{vol}(\mathscr{R}(\boldsymbol{R}P^n)).$$

Therefore we obtain the following corollary.

Corollary 5.7. *Let B be a totally geodesic Lagrangian submanifold in a complex space form $M = G/S(U(n) \times U(1))$ of complex dimension n. For a Lagrangian submanifold N in M, the following equation holds.*

$$\int_{\mathscr{R}(B)} \#(N \cap C) d\mu(C) = \frac{n+1}{\mathrm{vol}(\boldsymbol{R}P^n)}\mathrm{vol}(\mathscr{R}(\boldsymbol{R}P^n))\mathrm{vol}(N),$$

where $\#X$ denote the number of the points in X.

Now we consider the case where σ_B is not constant. We use Poincaré formulae in order to describe Crofton formulae, so we briefly review Poincaré formulae. We assume that G/K is a Riemannian homogeneous space. We denote by o the origin of G/K. For any x and y in G/K and vector subspaces V and W in $T_x(G/K)$ and $T_y(G/K)$ respectively, we define $\sigma_K(V, W)$ by

$$\sigma_K(V, W) = \int_K |(dg_x)_o^{-1}\vec{V} \wedge dk_o^{-1}(dg_y)_o^{-1}\vec{W}| d\mu_K(k),$$

where g_x and g_y are elements of G such that $g_x o = x$ and $g_y o = y$. Using σ_K we can state the generalized Poincaré formula obtained by Howard.

THEOREM 5.8 (Howard [**3**]). *Assume that G is unimodular. Let N and N' be submanifolds of G/K such that* $\dim N + \dim N' \geq \dim(G/K)$. *Then the following equation holds*:

$$\int_G \mathrm{vol}(N \cap gN') d\mu_G(g) = \int_{N \times N'} \sigma_K(T_x^{\perp} N, T_y^{\perp} N') d\mu_{N \times N'}(x, y).$$

In the case where G/K is a complex projective space, according to Poincaré formulae stated in Theorem 8 in [**13**], $\sigma_K(T_x^{\perp} N, T_y^{\perp} N')$ is described by the multiple Kähler angles of $T_x N$ and $T_y N'$. If N' is a given reflective submanifold B, $\sigma_K(T_x^{\perp} N, T_y^{\perp} B)$ is described by the multiple Kähler angle of $T_x N$, that means there exists a function $\sigma(\theta)$ which satisfies $\sigma_K(T_x^{\perp} N, T_y^{\perp} B) = \sigma(\theta(T_x N))$. The function $\sigma(\theta)$ is dependent on the choice of B. The Poincaré formula mentioned above implies

$$\int_G \mathrm{vol}(N \cap gB) d\mu_G(g) = \mathrm{vol}(B) \int_N \sigma(\theta(T_x N)) d\mu_N(x).$$

The left hand side is equal to

$$\mathrm{vol}(S(B)) \int_{\mathscr{R}(B)} \mathrm{vol}(N \cap C) d\mu(C).$$

Thus we obtain the following corollary from the discussion above and Corollary 4.4.

COROLLARY 5.9. *For positive integers k, l and n which satisfy $k, 2l < 2n \leq k + 2l$, there exists a function $\sigma_{k,l}^n(\theta^{(k)})$ of variables $\theta^{(k)} \in \boldsymbol{R}^{[\min\{k, 2n-k\}/2]}$ such that the following Crofton formula holds. Let B^l be a totally geodesic complex submanifold of complex dimension l in a complex space form $M = G/S(U(n) \times U(1))$ of complex dimension n. For a real submanifold N of dimension k in M, the following equation holds.*

$$\int_{\mathscr{R}(B^l)} \mathrm{vol}(N \cap C) d\mu(C) = \int_N \sigma_{k,l}^n(\theta(T_x N)) d\mu(x),$$

where $\theta(T_x N)$ is the multiple Kähler angle of $T_x N$.

For positive integers k and n which satisfy $k < 2n \leq k + n$, there exists a function $\tau_k^n(\theta^{(k)})$ of variables $\theta^{(k)} \in \boldsymbol{R}^{[\min\{k, 2n-k\}/2]}$ such that the following Crofton formula holds. Let L be a totally geodesic Lagrangian submanifold in a complex space form M of complex dimension n. For a real submanifold N of dimension k in M, the following equation holds.

$$\int_{\mathscr{R}(L)} \mathrm{vol}(N \cap C) d\mu(C) = \int_N \tau_k^n(\theta(T_x N)) d\mu(x).$$

In some cases we can express $\sigma^n_{k,l}$ and τ^n_k mentioned above more explicitly. In the case where $M = \boldsymbol{C}P^n$ and $B = \boldsymbol{C}P^{n-1}$, we consider a real submanifold N of dimension 2 in $\boldsymbol{C}P^n$. By Theorem 4.1 we have

$$\int_{\mathscr{R}(\boldsymbol{C}P^{n-1})} \#(N \cap C) d\mu(C) = \int_N \sigma_B(T_x N) d\mu(x).$$

In this case, Theorem 1.1 in [**14**] induces

$$\sigma_B(T_x N) = a(1 + \cos^2 \theta_x) \qquad (x \in N),$$

where a is a constant and θ_x is the Kähler angle of $T_x N$. So we have

$$\int_{\mathscr{R}(\boldsymbol{C}P^{n-1})} \#(N \cap C) d\mu(C) = \int_N a(1 + \cos^2 \theta_x) d\mu(x).$$

If N is equal to a totally geodesic $\boldsymbol{C}P^1$, then $\boldsymbol{C}P^1 \cap C$ is equal to a point for almost every C in $\mathscr{R}(\boldsymbol{C}P^l)$ and

$$\int_{\mathscr{R}(\boldsymbol{C}P^{n-1})} \#(N \cap C) d\mu(C) = \mathrm{vol}(\mathscr{R}(\boldsymbol{C}P^{n-1})).$$

Since $\theta_x = 0$ in this case, we get

$$\mathrm{vol}(\mathscr{R}(\boldsymbol{C}P^{n-1})) = 2a\mathrm{vol}(\boldsymbol{C}P^1),$$

that is

$$a = \frac{\mathrm{vol}(\mathscr{R}(\boldsymbol{C}P^{n-1}))}{2\mathrm{vol}(\boldsymbol{C}P^1)}.$$

Therefore we obtain the following corollary.

COROLLARY 5.10. *Let B be a totally geodesic complex submanifold of complex dimension $n-1$ in a complex space form $M = G/S(U(n) \times U(1))$ of complex dimension n. For a real submanifold N of dimension 2 in M, the following equation holds.*

$$\int_{\mathscr{R}(B)} \#(N \cap C) d\mu(C) = \frac{\mathrm{vol}(\mathscr{R}(\boldsymbol{C}P^{n-1}))}{2\mathrm{vol}(\boldsymbol{C}P^1)} \int_N (1 + \cos^2 \theta_x) d\mu(x),$$

where θ_x is the Kähler angle of N at x.

A similar argument shows the following corollary.

COROLLARY 5.11. *Let B be a totally geodesic complex submanifold of complex dimension 1 in a complex space form $M = G/S(U(n) \times U(1))$ of complex dimension n.*

For a real submanifold N of dimension $2n-2$ in M, the following equation holds.

$$\int_{\mathscr{R}(B)} \#(N \cap C)d\mu(C) = \frac{\mathrm{vol}(\mathscr{R}(\boldsymbol{C}P^1))}{2\mathrm{vol}(\boldsymbol{C}P^{n-1})} \int_N (1+\cos^2\theta_x)d\mu(x),$$

where θ_x is the Kähler angle of N at x.

COROLLARY 5.12 (Kang-Tasaki [**6**, Theorem 1.1]). *Let B be a totally geodesic Lagrangian submanifold in a complex space form $M = G/S(U(2)\times U(1))$ of complex dimension 2. For a real submanifold N of dimension 2 in M, the following equation holds.*

$$\int_{\mathscr{R}(B)} \#(N \cap C)d\mu(C) = \frac{\mathrm{vol}(\mathscr{R}(\boldsymbol{R}P^2))}{\mathrm{vol}(\boldsymbol{R}P^2)} \int_N (3-\cos^2\theta_x)d\mu(x).$$

COROLLARY 5.13 ([**16**, Theorem 4.3]). *Let B be a totally geodesic Lagrangian submanifold in a complex space form $M = G/S(U(3)\times U(1))$ of complex dimension 3. For a real submanifold N of dimension 3 in M, the following equation holds.*

$$\int_{\mathscr{R}(B)} \#(N \cap C)d\mu(C) = \frac{4\mathrm{vol}(\mathscr{R}(\boldsymbol{R}P^3))}{3\mathrm{vol}(\boldsymbol{R}P^3)} \int_N (3-\cos^2\theta_x)d\mu(x).$$

COROLLARY 5.14 (Kang [**4**, Theorem 1.1]). *Let B be a totally geodesic complex submanifold of complex dimension 2 in a complex space form $M = G/S(U(4)\times U(1))$ of complex dimension 4. For a real submanifold N of dimension 4 in M, the following equation holds.*

$$\int_{\mathscr{R}(B)} \#(N \cap C)d\mu(C) = \frac{\mathrm{vol}(\mathscr{R}(\boldsymbol{C}P^2))}{8\mathrm{vol}(\boldsymbol{C}P^2)} \int_N (3+\cos^2\theta_1+\cos^2\theta_2+3\cos^2\theta_1\cos^2\theta_2)d\mu,$$

where (θ_1,θ_2) is the multiple Kähler angle of N.

6. Appendix.

In this appendix we give a proof of the coarea formula (Theorem 3.7). We define an open subset O of M by

$$O = \{x \in M \mid df_x \text{ is surjective}\}.$$

This is μ_M-measurable and

$$\int_M \phi(x)Jf(x)d\mu_M(x) = \int_O \phi(x)Jf(x)d\mu_M(x).$$

For each $x \in O$, by the implicit function theorem, there exists a local coordinate neighborhood U_x of x such that $f(U_x)$ is an open neighborhood of $f(x)$ and that we can regard

$f : U_x \to f(U_x)$ as a natural projection from the product of open subsets in Euclidean spaces. We first prove the theorem in the case where f itself is such a projection.

We assume the following situation. N is an open subset of $\boldsymbol{R}^n$, F is an open subset of $\boldsymbol{R}^{m-n}$, $M = N \times F$ and

$$f : M = N \times F \to N;\ (y,t) \mapsto y.$$

We denote by $\{y_1, \ldots, y_n\}$ and $\{x_1, \ldots, x_m\}$ the canonical systems of coordinates of $N \subset \boldsymbol{R}^n$ and $M = N \times F \subset \boldsymbol{R}^m$ respectively, where $y_i \circ f = x_i$ $(1 \leq i \leq n)$. Then $\{x_{n+1}, \ldots, x_m\}$ is a system of coordinates of F. Since ϕ is a measurable function on M,

$$\phi(y,t)\left|\frac{\partial}{\partial x_{n+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_m}\right|_M \left|\frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}\right|_N \circ f$$

is also measurable on M. For almost every $y \in N$ the induced metric on $f^{-1}(y)$ is nondegenerate, so we can consider integration on $f^{-1}(y)$ with respect to its semi-Riemannian metric. According to Fubini's theorem, it holds

$$\begin{aligned}
&\int_F \phi(y,t)\left|\frac{\partial}{\partial x_{n+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_m}\right|_M \left|\frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}\right|_N \circ f dx_{n+1} \cdots dx_m(t) \\
&= \left(\int_F \phi(y,t)\left|\frac{\partial}{\partial x_{n+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_m}\right|_M dx_{n+1} \cdots dx_m(t)\right)\left|\frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}\right|_N \\
&= \left(\int_{f^{-1}(y)} \phi(x) d\mu_{f^{-1}(y)}(x)\right)\left|\frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}\right|_N.
\end{aligned}$$

This function of variable y is a measurable function on N and moreover it holds

$$\begin{aligned}
&\int_M \phi(y,t)\left|\frac{\partial}{\partial x_{n+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_m}\right|_M \left|\frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}\right|_N \circ f dx_1 \cdots dx_m \\
&= \int_N \left(\int_{f^{-1}(y)} \phi(x) d\mu_{f^{-1}(y)}(x)\right)\left|\frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}\right|_N dy_1 \cdots dy_n(y) \\
&= \int_N \left(\int_{f^{-1}(y)} \phi(x) d\mu_{f^{-1}(y)}(x)\right) d\mu_N(y).
\end{aligned}$$

For almost every $y \in N$ the metric on the tangent space of F at (y,t) is nondegenerate, the tangent space of M is the direct sum of the tangent space of F and its orthogonal complement. We decompose each tangent vector $\dfrac{\partial}{\partial x_i}$ of $M = N \times F$ into the sum of its components tangent to F and orthogonal to F as follows:

$$\frac{\partial}{\partial x_i} = \left(\frac{\partial}{\partial x_i}\right)_F + \left(\frac{\partial}{\partial x_i}\right)_{F^\perp}.$$

Then we get

$$df\left(\frac{\partial}{\partial x_i}\right) = df\left(\left(\frac{\partial}{\partial x_i}\right)_{F^\perp}\right)$$

and

$$\begin{aligned}\left|\frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}\right|_N \circ f &= \left|df\left(\frac{\partial}{\partial x_1}\right) \wedge \cdots \wedge df\left(\frac{\partial}{\partial x_n}\right)\right|_N \\ &= \left|df\left(\left(\frac{\partial}{\partial x_1}\right)_{F^\perp}\right) \wedge \cdots \wedge df\left(\left(\frac{\partial}{\partial x_n}\right)_{F^\perp}\right)\right|_N.\end{aligned}$$

Since $\dfrac{\partial}{\partial x_{n+1}}, \ldots, \dfrac{\partial}{\partial x_m}$ are tangent to F,

$$\begin{aligned}\left|\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_m}\right|_M &= \left|\left(\frac{\partial}{\partial x_1}\right)_{F^\perp} \wedge \cdots \wedge \left(\frac{\partial}{\partial x_n}\right)_{F^\perp} \wedge \frac{\partial}{\partial x_{n+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_m}\right|_M \\ &= \left|\left(\frac{\partial}{\partial x_1}\right)_{F^\perp} \wedge \cdots \wedge \left(\frac{\partial}{\partial x_n}\right)_{F^\perp}\right|_M \left|\frac{\partial}{\partial x_{n+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_m}\right|_M.\end{aligned}$$

Thus we obtain

$$\begin{aligned}&\int_M \phi(y,t)\left|\frac{\partial}{\partial x_{n+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_m}\right|_M \left|\frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}\right|_N \circ f dx_1 \cdots dx_m \\ &= \int_M \phi(x) \frac{\left|df\left(\left(\frac{\partial}{\partial x_1}\right)_{F^\perp}\right) \wedge \cdots \wedge df\left(\left(\frac{\partial}{\partial x_n}\right)_{F^\perp}\right)\right|_N}{\left|\left(\frac{\partial}{\partial x_1}\right)_{F^\perp} \wedge \cdots \wedge \left(\frac{\partial}{\partial x_n}\right)_{F^\perp}\right|_M} d\mu_M(x) = \int_M \phi J f d\mu_M.\end{aligned}$$

Therefore

$$\int_N \left(\int_{f^{-1}(y)} \phi(x) d\mu_{f^{-1}(y)}(x)\right) d\mu_N(y) = \int_M \phi J f d\mu_M.$$

This is the coarea formula in the case of a natural projection from the products of open subsets on Euclidean spaces.

Now we return to the general case. For each $x \in O$, by the implicit function theorem, there exists a local coordinate neighborhood U_x of x such that $f(U_x)$ is an open neighborhood of $f(x)$ and that we can regard $f : U_x \to f(U_x)$ as a natural projection from the product of open subsets in Euclidean spaces. The collection $\{U_x\}_{x \in O}$ is an open covering of O. Since M has a countable open base, O has a countable open base. So we can select a countable subfamily $\{U_k\}$ of $\{U_x\}_{x \in O}$ which is also an open covering of O. We take a partition of unity $\{\psi_k\}$ subordinate to $\{U_k\}$. Let

$$f_k : U_k \to V_k = f(U_k)$$

be the restriction of f to U_k. We apply the local version of the coarea formula proved above to the function $\psi_k\phi$. The function

$$y \mapsto \int_{f_k^{-1}(y)} (\psi_k\phi)(x) d\mu_{f_k^{-1}(y)}(x)$$

is a μ_N-measurable function on V_k and

$$\int_{V_k} \left(\int_{f_k^{-1}(y)} (\psi_k\phi)(x) d\mu_{f_k^{-1}(y)}(x) \right) d\mu_{V_k}(y) = \int_{U_k} \psi_k \phi J f d\mu_{U_k}.$$

Hence the function

$$y \mapsto \sum_k \int_{f_k^{-1}(y)} (\psi_k\phi)(x) d\mu_{f_k^{-1}(y)}(x)$$

is a μ_N-measurable function on N. The collection $\{\psi_k|_{f^{-1}(y)}\}$ is a partition of unity on $f^{-1}(y)$ for each y, so we get

$$\sum_k \int_{f_k^{-1}(y)} (\psi_k\phi)(x) d\mu_{f_k^{-1}(y)}(x) = \int_{f^{-1}(y)} \phi(x) d\mu_{f^{-1}(y)}(x).$$

This implies that the function

$$y \mapsto \int_{f^{-1}(y)} \phi(x) d\mu_{f^{-1}(y)}(x)$$

is a μ_N-measurable function on N. We can apply Lebesgue's convergence theorem and obtain

$$\begin{aligned}
\int_M \phi J f d\mu_M &= \sum_k \int_{U_k} \psi_k \phi J f d\mu_{U_k} \\
&= \sum_k \int_{V_k} \left(\int_{f_k^{-1}(y)} (\psi_k\phi)(x) d\mu_{f_k^{-1}(y)}(x) \right) d\mu_{V_k}(y) \\
&= \int_N \left(\sum_k \int_{f_k^{-1}(y)} (\psi_k\phi)(x) d\mu_{f_k^{-1}(y)}(x) \right) d\mu_N(y) \\
&= \int_N \left(\int_{f^{-1}(y)} \phi(x) d\mu_{f^{-1}(y)}(x) \right) d\mu_N(y).
\end{aligned}$$

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