

Projective manifolds containing special curves

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Abstract. Let Y be a smooth curve embedded in a complex projective manifold X of dimension $n \geq 2$ with ample normal bundle $N_{Y|X}$. For every $p \geq 0$ let α_p denote the natural restriction maps $\text{Pic}(X) \rightarrow \text{Pic}(Y(p))$, where $Y(p)$ is the p -th infinitesimal neighbourhood of Y in X . First one proves that for every $p \geq 1$ there is an isomorphism of abelian groups $\text{Coker}(\alpha_p) \cong \text{Coker}(\alpha_0) \oplus K_p(Y, X)$, where $K_p(Y, X)$ is a quotient of the \mathbf{C} -vector space $L_p(Y, X) := \bigoplus_{i=1}^p H^1(Y, \mathbf{S}^i(N_{Y|X})^*)$ by a free subgroup of $L_p(Y, X)$ of rank strictly less than the Picard number of X . Then one shows that $L_1(Y, X) = 0$ if and only if $Y \cong \mathbf{P}^1$ and $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-1}$ (i.e. Y is a quasi-line in the terminology of [4]). The special curves in question are by definition those for which $\dim_{\mathbf{C}} L_1(Y, X) = 1$. This equality is closely related with a beautiful classical result of B. Segre [25]. It turns out that Y is special if and only if either $Y \cong \mathbf{P}^1$ and $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$, or Y is elliptic and $\deg(N_{Y|X}) = 1$. After proving some general results on manifolds of dimension $n \geq 2$ carrying special rational curves (e.g. they form a subclass of the class of rationally connected manifolds which is stable under small projective deformations), a complete birational classification of pairs (X, Y) with X surface and Y special is given. Finally, one gives several examples of special rational curves in dimension $n \geq 3$.

Introduction.

Let $X = \mathbf{P}^1 \times \mathbf{P}^1$ be a smooth quadric in \mathbf{P}^3 , and let Y be a smooth curve of bidegree $(1, 1)$ on X . Let Γ be a curve in X of bidegree (m, n) meeting transversely the conic Y in $m + n$ distinct points P_1, \dots, P_{m+n} . Let α_i, β_i be the two ruling lines of X passing through P_i , let γ_i be the tangent line of Γ at P_i , and let θ_i be the tangent line of Y at P_i . These are four lines through P_i , contained in the projective tangent plane of X at P_i . Thus it makes sense to consider the cross-ratios $(\alpha_i, \gamma_i, \theta_i, \beta_i) \in \mathbf{C}$ of the four lines through the point P_i , $i = 1, \dots, m + n$. A result of B. Segre [25], §37 asserts that

$$\sum_{i=1}^{m+n} (\alpha_i, \gamma_i, \theta_i, \beta_i) = n. \quad (1)$$

Conversely, given $m + n$ distinct points $P_1, \dots, P_{m+n} \in Y$, and a line γ_i through each point P_i contained in the tangent space of X at P_i satisfying (1), then there exists a curve Γ in X of bidegree (m, n) meeting Y transversely only at the points P_1, \dots, P_{m+n} and such that γ_i is the tangent line of Γ at P_i , $i = 1, \dots, m + n$. In modern terminology this fact can be rephrased in terms of the Picard group of the first infinitesimal neighbourhood of Y in X .

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On the other hand, the classical condition of Reiss concerning the existence of a degree d curve in \mathbf{P}^2 intersecting a given line Y in d different prescribed points, with prescribed tangents and second-order conditions, can be reinterpreted in modern language in terms of the Picard group of the second infinitesimal neighbourhood of Y in \mathbf{P}^2 (see again [25] and [13], p. 698–699).

These facts provide good motivation to study infinitesimal neighbourhoods of special curves in projective manifolds. More precisely, let X be a complex projective manifold of dimension $n \geq 2$, and let Y be a smooth connected curve of genus g embedded in X such that the normal bundle $N_{Y|X}$ of Y in X is ample. For every $p \geq 0$ we will denote by $Y(p)$ the p -th infinitesimal neighbourhood of Y in X , i.e., $Y(p)$ is the algebraic scheme over \mathbf{C} whose underlying topological space coincides with the underlying topological space of Y , and whose structural sheaf $\mathcal{O}_{Y(p)}$ is by definition $\mathcal{O}_X/\mathcal{I}^{p+1}$, where $\mathcal{I} \subset \mathcal{O}_X$ denotes the ideal sheaf of Y in X . Of course $Y = Y(0)$. For every integer $p \geq 0$ we may consider the natural restriction maps

$$\alpha_p : \text{Pic}(X) \rightarrow \text{Pic}(Y(p)). \quad (2)$$

Then by Theorem 1.1 below, for every $p \geq 1$ there exists an isomorphism

$$\text{Coker}(\alpha_p) \cong \text{Coker}(\alpha_0) \oplus K_p(Y, X),$$

where $K_p(Y, X)$ is a quotient of the \mathbf{C} -vector space $L_p(Y, X) := \bigoplus_{i=1}^p H^1(Y, \mathbf{S}^i(N_{Y|X})^*)$ by a free subgroup of $L_p(Y, X)$ of rank $\leq \rho(X) - 1$, where $\rho(X)$ is the Picard number of X . Here $\mathbf{S}^i(E)$ denotes the i -th symmetric power of a vector bundle E .

The aim of this paper is to study the maps α_p , especially α_1 and α_2 , when $L_p(Y, X)$ is of small dimension. For example, Theorem 1.4 below describes the situation when $\dim_{\mathbf{C}} L_2(Y, X)$ is minimal. If $\dim(X) \leq 3$ or if Y is not an elliptic curve this happens if and only if $Y \cong \mathbf{P}^1$ and $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-1}$, i.e. if and only if Y is a quasi-line in X in the terminology of [4] (if X is a surface this means that the embedding $Y \hookrightarrow X$ is Zariski equivalent to the embedding of a line in \mathbf{P}^2 ; this is the modern interpretation of Reiss' relation, see [13], p. 698–699).

On the other hand, Corollary 2.1 below asserts that $L_1(Y, X) = 0$ if and only if $\text{Coker}(\alpha_1)$ is finite, or if and only if Y is a quasi-line. Moreover, Theorem 2.4 below takes care of the case $\dim_{\mathbf{C}} L_1(Y, X) = 1$, in which case there are two possibilities: either

$$Y \cong \mathbf{P}^1 \text{ and } N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}, \quad (3)$$

or Y is an elliptic curve, $\deg(N_{Y|X}) = 1$ and the irregularity of X is ≤ 1 . Moreover, if we assume that X is irregular and that Y is G3 in X (see Definition 2.2 below), then the canonical morphism of Albanese varieties $\text{Alb}(Y) = Y \rightarrow \text{Alb}(X)$ is an isomorphism; in particular, the Albanese morphism $X \rightarrow \text{Alb}(X)$ yields a retraction $\pi : X \rightarrow Y$ of the inclusion $Y \hookrightarrow X$.

In the case of surfaces one can say a lot more than Theorem 2.4. In fact, Theorem 3.7 provides a very precise birational classification of pairs (X, Y) , with X a smooth projective surface and Y a smooth curve such that $(Y^2) > 0$ and $\dim_{\mathbf{C}} L_1(Y, X) = 1$.

The curves Y in X satisfying (3) are interesting from the point of view of varieties carrying quasi-lines (see [4] and [19]). Indeed by a result proved in [19] (Lemma 2.2), if Y is such a curve and if Z is a smooth two-codimensional closed subvariety of X meeting Y at just one point transversely, in the variety X' obtained by blowing up X along Z the proper transform Y' of Y (via the blowing up morphism $X' \rightarrow X$) becomes a quasi-line. In other words, any example of curves Y in X satisfying (3) provides examples of projective manifolds containing quasi-lines. In section 4 we give several examples of projective manifolds X carrying curves Y satisfying (3). One shows that the projective manifolds carrying curves satisfying (3) are rationally connected in the sense of [22], [21], and that they are stable under small projective deformations (Theorem 2.9).

Throughout this paper we shall use the standard terminology and notation in algebraic geometry. All varieties considered are defined over the field \mathbf{C} of complex numbers.

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1. General results.

Let X be a complex projective manifold of dimension $n \geq 2$, and let Y be a smooth connected curve of genus g embedded in X such that the normal bundle $N_{Y|X}$ of Y in X is ample. For a non-negative integer p , we shall denote by $Y(p)$ the p -th infinitesimal neighbourhood $(Y, \mathcal{O}_X/\mathcal{I}^{p+1})$ of Y in X as in Introduction. Clearly $Y(0) = Y$. Then for every $p \geq 1$ the truncated exponential sequence

$$0 \rightarrow \mathcal{I}^p/\mathcal{I}^{p+1} \cong \mathbf{S}^p(N_{Y|X}^*) \rightarrow \mathcal{O}_{Y(p)}^* \rightarrow \mathcal{O}_{Y(p-1)}^* \rightarrow 0,$$

(in which \mathcal{O}_Z^* denotes the sheaf of multiplicative groups of nowhere vanishing functions on a scheme Z and the first nontrivial map is the truncated exponential $u \mapsto 1+u$) yields the cohomology sequence

$$\begin{aligned} 0 \rightarrow H^0(Y, \mathbf{S}^p(N_{Y|X}^*)) \rightarrow H^0(Y(p), \mathcal{O}_{Y(p)}^*) \rightarrow H^0(Y(p-1), \mathcal{O}_{Y(p-1)}^*) \rightarrow \\ \rightarrow H^1(Y, \mathbf{S}^p(N_{Y|X}^*)) \rightarrow \text{Pic}(Y(p)) \rightarrow \text{Pic}(Y(p-1)) \rightarrow H^2(Y, \mathbf{S}^p(N_{Y|X}^*)) = 0. \end{aligned}$$

Since we work over a field of characteristic zero, $\mathbf{S}^p(N_{Y|X}^*) \cong \mathbf{S}^p(N_{Y|X})^*$ (see [16], Exercise 4.9, p. 114). Moreover, the hypothesis that $N_{Y|X}$ is ample implies that $\mathbf{S}^p(N_{Y|X})$ is also ample for every $p \geq 1$ (see [16]). From this it follows that

$$H^0(Y, \mathbf{S}^p(N_{Y|X}^*)) = 0,$$

whence the map

$$a_p: H^0(Y(p), \mathcal{O}_{Y(p)}^*) \rightarrow H^0(Y(p-1), \mathcal{O}_{Y(p-1)}^*)$$

is an injective map of \mathbf{C} -algebras for every $p \geq 1$. But $H^0(Y(0), \mathcal{O}_{Y(0)}^*) = H^0(Y, \mathcal{O}_Y^*) =$

$\mathcal{C}^* = \mathcal{C} \setminus \{0\}$, and therefore a_p is an isomorphism for every $p \geq 1$. It follows that the above cohomology sequence yields the exact sequence of abelian groups

$$0 \rightarrow H^1(Y, \mathbf{S}^p(N_{Y|X}^*)) \rightarrow \text{Pic}(Y(p)) \rightarrow \text{Pic}(Y(p-1)) \rightarrow 0, \quad \forall p \geq 1. \tag{4}$$

Since $H^1(Y, \mathbf{S}^p(N_{Y|X}^*))$ is a \mathcal{C} -vector space, the additive group $H^1(Y, \mathbf{S}^p(N_{Y|X}^*))$ is divisible (and hence injective), whence the exact sequence (4) splits for every $p \geq 1$. Then by induction we get

$$\text{Pic}(Y(p)) \cong \text{Pic}(Y) \oplus L_p(Y, X), \quad \forall p \geq 1, \tag{5}$$

where we put

$$L_p(Y, X) := \bigoplus_{i=1}^p H^1(Y, \mathbf{S}^i(N_{Y|X}^*)). \tag{6}$$

Clearly, $L_p(Y, X)$ is a finite dimensional \mathcal{C} -vector space.

THEOREM 1.1. *Let X be a complex projective manifold of dimension $n \geq 2$, and let Y be a smooth connected curve embedded in X such that the normal bundle $N_{Y|X}$ of Y in X is ample. Then, for every $p \geq 1$, there exists an isomorphism*

$$\text{Coker}(\alpha_p) \cong \text{Coker}(\alpha_0) \oplus K_p(Y, X), \tag{7}$$

where $\alpha_p: \text{Pic}(X) \rightarrow \text{Pic}(Y(p))$ is the map (2) and the abelian group $K_p(Y, X)$ is a quotient of the \mathcal{C} -vector space $L_p(Y, X)$ by a free subgroup of $L_p(Y, X)$ of rank $\leq \rho(X) - 1$, with $\rho(X)$ the rank of the Néron-Severi group of X (the Picard number of X).

PROOF. Denote by $\beta_p: \text{Pic}(Y(p)) \rightarrow \text{Pic}(Y)$ the natural restriction map and by $j: L_p(Y, X) \hookrightarrow \text{Pic}(Y(p))$ the canonical inclusion into the direct sum (via the isomorphism (5)). Now consider the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{Ker}(\alpha_0) & \xrightarrow{\alpha'_p} & L_p(Y, X) & \longrightarrow & K_p(Y, X) \longrightarrow 0 \\
 & & \downarrow i & & \downarrow j & & \downarrow \\
 & & \text{Pic}(X) & \xrightarrow{\alpha_p} & \text{Pic}(Y(p)) & \longrightarrow & \text{Coker}(\alpha_p) \longrightarrow 0 \\
 & & \downarrow & & \downarrow \beta_p & & \downarrow \bar{\beta}_p \\
 0 & \longrightarrow & \text{Pic}(X)/\text{Ker}(\alpha_0) & \xrightarrow{\bar{\alpha}_0} & \text{Pic}(Y) & \longrightarrow & \text{Coker}(\alpha_0) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array} \tag{8}$$

in which the map $\bar{\alpha}_0$ is deduced from α_0 by factorization, the map α'_p is induced by α_p , the map $\bar{\beta}_p$ is induced by β_p and $K_p(Y, X) := \text{Ker}(\bar{\beta}_p)$. Clearly, all columns and the second and the third rows are exact. Then by the snake lemma the first row is also exact. In particular, $K_p(Y, X) \cong L_p(Y, X)/\text{Im}(\alpha'_p)$. Since \mathbf{Z} is a principal ring and $L_p(Y, X)$ is an injective \mathbf{Z} -module, we infer that $K_p(Y, X)$ is also an injective \mathbf{Z} -module. This implies that the last column splits, which yields the isomorphism (7). Observe also that the subgroup $\text{Im}(\alpha'_p)$ is torsion-free since $L_p(Y, X)$ is a \mathbf{C} -vector space. Therefore $\text{Im}(\alpha'_p)$ is free as soon as we know that $\text{Im}(\alpha'_p)$ is a finitely generated group. Thus it remains to show that $\text{Im}(\alpha'_p)$ is a finitely generated abelian group of rank $\leq \rho(X) - 1$.

In view of decomposition (5), the middle column splits, i.e. there exists a map $\eta: \text{Pic}(Y(p)) \rightarrow L_p(Y, X)$ such that $\eta \circ j = \text{id}$. Clearly $\eta \circ \alpha_p \circ i = \eta \circ j \circ \alpha'_p = \alpha'_p$. Thus we get the map

$$\gamma_p := \eta \circ \alpha_p: \text{Pic}(X) \rightarrow L_p(Y, X),$$

such that $\gamma_p \circ i = \alpha'_p$.

Observe now that the Picard scheme $\text{Pic}^0(X)$ is an abelian variety since X is smooth and projective (see [14], exposés 232, 236). Therefore $\gamma_p(\text{Pic}^0(X)) = 0$, since $L_p(Y, X)$ is an (additive) linear algebraic group. Thus the map γ_p factors through a map

$$\gamma'_p: \text{NS}(X) := \text{Pic}(X)/\text{Pic}^0(X) \rightarrow L_p(Y, X).$$

By the theorem of Néron-Severi, $\text{NS}(X)$ is a finitely generated abelian group of rank $\rho(X) \geq 1$ (since X is projective). Since $L_p(Y, X)$ is a \mathbf{C} -vector space it follows that $\text{Im}(\gamma_p)$ is a free abelian group of finite rank. Thus $\text{Im}(\gamma_p) = \text{Im}(\gamma'_p)$, and therefore also $\text{Im}(\alpha'_p)$ is a finitely generated subgroup of $L_p(Y, X)$. In fact, one can say more. Since Y is a smooth projective curve, $\rho(Y) = 1$. Note that the induced map $\text{NS}(X) \rightarrow \text{NS}(Y)$ is surjective after tensoring with \mathbf{Q} . Therefore the image of $\text{Ker}(\alpha_0)$ in $\text{NS}(X)$ is a (finitely generated) subgroup of rank equal to $\rho(X) - 1$. Thus $\text{Im}(\alpha'_p)$ is a free abelian subgroup of $L_p(Y, X)$ of rank $\leq \rho(X) - 1$, which completes the proof of the theorem. \square

REMARK 1.2. From Theorem 1.1 it follows that $K_p(Y, X) = 0$ if and only if $L_p(Y, X) = 0$. Moreover, if $\rho(X) = 1$ we have $K_p(Y, X) \cong L_p(Y, X)$.

DEFINITION 1.3 ([4]). Let Y be a smooth connected curve in the projective manifold X of dimension $n \geq 2$. The curve Y is said to be a *quasi-line* in X if $Y \cong \mathbf{P}^1$ and $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-1}$.

We are going to apply Theorem 1.1 repeatedly. For instance we can easily compute the dimension of $L_2(Y, X)$ in the case when $Y \cong \mathbf{P}^1$ and $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-1}$, i.e. Y is a quasi-line in X . Since $H^1(Y, N_{Y|X}^*) \cong H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus n-1}) = 0$, we have in this case

$$\dim_{\mathbf{C}} L_2(Y, X) = \dim_{\mathbf{C}} H^1(Y, \mathbf{S}^2(\mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus n-1})) = \binom{n}{2} = \frac{n(n-1)}{2}.$$

In particular, if Y is a line in $X = \mathbf{P}^2$, then the maps α_0 and α_1 are surjective and $\dim_{\mathbf{C}} L_2(Y, X) = 1$. This is closely related to the so-called Reiss relation (see B. Segre [25], or also [13], p. 698–699).

THEOREM 1.4. *Let Y be a smooth connected curve embedded in a projective manifold X of dimension $n \geq 2$ with normal bundle $N_{Y|X}$ ample. Then the following hold:*

- i) *If the curve Y has genus $g \neq 1$, then $\dim_{\mathbf{C}} L_2(Y, X) \geq \frac{n(n-1)}{2}$. Moreover, the equality holds if and only if $Y \cong \mathbf{P}^1$ and $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-1}$ (i.e., if and only if Y is a quasi-line in X);*
- ii) *If Y is an elliptic curve, then $\dim_{\mathbf{C}} L_2(Y, X) = (n+1) \deg(N_{Y|X})$.*

PROOF. By Theorem 1.1 (with $p = 2$) we have to compute $\dim_{\mathbf{C}} H^1(Y, \mathbf{S}^i(N_{Y|X}^*))$ for $i = 1, 2$. This follows from duality, Riemann-Roch, the fact that $H^0(Y, E^*) = 0$ for every ample vector bundle E on Y , and the following formulae:

$$\deg(\mathbf{S}^2(N_{Y|X})) = n \deg(N_{Y|X}), \quad \text{and} \quad \text{rank}(\mathbf{S}^2(N_{Y|X})) = \frac{n(n-1)}{2}.$$

By the ampleness of $N_{Y|X}$ (which implies the fact that $\mathbf{S}^2(N_{Y|X})$ is also ample), and the standard formula $\mathbf{S}^2(N_{Y|X}^*) \cong \mathbf{S}^2(N_{Y|X})^*$, by duality we get

$$H^1(Y, \omega_Y \otimes N_{Y|X}) \cong H^0(Y, N_{Y|X}^*) = 0,$$

and

$$H^1(Y, \omega_Y \otimes \mathbf{S}^2(N_{Y|X})) \cong H^0(Y, \mathbf{S}^2(N_{Y|X})^*) = 0.$$

Thus by duality and Riemann-Roch we have

$$\begin{aligned} \dim_{\mathbf{C}} H^1(Y, N_{Y|X}^*) &= \dim_{\mathbf{C}} H^0(Y, \omega_Y \otimes N_{Y|X}) = \chi(Y, \omega_Y \otimes N_{Y|X}) \\ &= \deg(N_{Y|X}) + (n-1)(g-1), \end{aligned} \tag{9}$$

and

$$\begin{aligned} \dim_{\mathbf{C}} H^1(Y, \mathbf{S}^2(N_{Y|X}^*)) &= \dim_{\mathbf{C}} H^0(Y, \omega_Y \otimes \mathbf{S}^2(N_{Y|X})) = \chi(Y, \omega_Y \otimes \mathbf{S}^2(N_{Y|X})) \\ &= n \deg(N_{Y|X}) + \frac{n(n-1)}{2}(g-1). \end{aligned} \tag{10}$$

From (9) and (10) we get

$$\dim_{\mathbf{C}} L_2(Y, X) = (n+1) \deg(N_{Y|X}) + \frac{(n+2)(n-1)}{2}(g-1).$$

Thus, if $g = 1$, we get directly ii).

Notice that since $N_{Y|X}$ is ample, $\deg(N_{Y|X}) > 0$. Thus if $g > 1$ the last estimate yields $\dim_{\mathbf{C}} L_2(Y, X) > \frac{n(n-1)}{2}$. If instead $g = 0$, by a result of Grothendieck, we get

$$N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \mathcal{O}_{\mathbf{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_{n-1}),$$

with $a_1 \geq a_2 \geq \cdots \geq a_{n-1} > 0$ because $N_{Y|X}$ is ample. Then it is easily seen that the inequality of i) holds, with equality if and only if $a_1 = \cdots = a_{n-1} = 1$. This completes the proof of the theorem. \square

COROLLARY 1.5. *If in Theorem 1.4 we assume $n = 2$ or $n = 3$, then $\dim_{\mathbf{C}} L_2(Y, X) \geq \frac{n(n-1)}{2}$, with equality if and only if Y is a quasi-line in X .*

2. The first infinitesimal neighbourhood.

Now using Theorem 1.1, we proceed to analyze the map α_1 . As a direct consequence of Theorem 1.1 we get the following result (see [2], Theorem 14.2, which slightly improves Theorem (2.1) of [4]; the latter generalizes a result of d'Almeida [1] proved when X is a surface and using different methods).

COROLLARY 2.1. *Let Y be a smooth connected curve embedded in a projective manifold X of dimension $n \geq 2$ with normal bundle $N_{Y|X}$ ample. The following conditions are equivalent:*

- i) $L_1(Y, X) = 0$.
- ii) $K_1(Y, X) = 0$.
- iii) $\text{Coker}(\alpha_1)$ is a finite group.
- iv) Y is a quasi-line.

Moreover, the map α_1 is surjective if and only if Y is a quasi-line and the map α_0 is surjective.

PROOF. The equivalence i) \iff ii) follows from Remark 1.2. On the other hand, by duality we have $L_1(Y, X) = 0$ if and only if $H^0(\omega_Y \otimes N_{Y|X}) = 0$. Using Riemann-Roch and the fact that every vector bundle on \mathbf{P}^1 is the direct sum of line bundles of the form $\mathcal{O}_{\mathbf{P}^1}(a)$, with $a \in \mathbf{Z}$, it is easy to see that the latter condition holds if and only if $Y \cong \mathbf{P}^1$ and $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-1}$. In particular, $L_1(Y, X) = 0$ implies that $\text{Coker}(\alpha_1)$ is finite because $\text{Pic}(Y) \cong \mathbf{Z}$. Conversely, if $\text{Coker}(\alpha_1)$ is finite then $L_1(Y, X) = 0$ by Remark 1.2. The last statement is a direct consequence of decomposition (7). \square

To prove Theorem 2.4 below we first need a definition and a lemma.

DEFINITION 2.2. Let Y be a closed subvariety of a projective irreducible variety X . We say that Y is *G3* in X if the canonical map $K(X) \rightarrow K(X/Y)$ is an isomorphism of rings, where $K(X)$ is the field of rational functions of X , and $K(X/Y)$ is the ring of formal-rational functions of X along Y (see e.g. [18], or also [16]). In particular, if Y is *G3* in X , $K(X/Y)$ is a field. We also say that Y is *G2* in X if $K(X/Y)$ is a field and if the above map makes $K(X/Y)$ a finite field extension of $K(X)$.

By a result of Hartshorne (see [16], p. 198), if X is smooth, Y is connected and local complete intersection in X and the normal bundle $N_{Y|X}$ is ample, then Y is $G2$ in X . Thus, in our hypotheses from the beginning (i.e. Y is a smooth connected curve in the projective manifold X with ample normal bundle), Y is always $G2$ in X . Moreover in the case when X is a surface, Y is $G3$ in X whenever $N_{Y|X}$ is ample, i.e., $(Y^2) > 0$. However, if $\dim X \geq 3$ and $Y \subset X$ is a curve with ample normal bundle, Y is not necessarily $G3$ in X (see e.g. [16], Exercise 4.10, p. 209, or also [4], Example (2.7)).

LEMMA 2.3. *Let Y be an elliptic curve embedded in an irregular projective manifold X of dimension $n \geq 2$ with normal bundle $N_{Y|X}$ ample. Assume that Y is $G3$ in X (this is always the case if X is a surface). Then the canonical morphism of Albanese varieties $\text{Alb}(Y) = Y \rightarrow \text{Alb}(X)$ (induced by the inclusion $Y \hookrightarrow X$) is an isomorphism. In particular, the Albanese morphism $f : X \rightarrow \text{Alb}(X)$ yields a retraction $\pi : X \rightarrow Y$ of $Y \hookrightarrow X$.*

PROOF. By a general elementary result of Matsumura (see [16], Exercise 4.15, p. 116), the morphism $u : \text{Alb}(Y) = Y \rightarrow \text{Alb}(X) =: A$, associated to $Y \hookrightarrow X$, is surjective. In particular, u is a finite étale morphism. Let d be the degree of u . We have to prove that $d = 1$.

Since X is irregular and Y is an elliptic curve we infer that A is an elliptic curve, and since $f(X)$ generates A , the morphism $f : X \rightarrow A$ is surjective. Consider now the cartesian diagram

$$\begin{array}{ccc} X' := X \times_A Y & \xrightarrow{u'} & X \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{u} & A. \end{array}$$

The inclusion $i : Y \hookrightarrow X$ yields a morphism $i' : Y \rightarrow X'$ such that $u' \circ i' = i$ and $f' \circ i' = \text{id}_Y$; in particular, i' is a closed embedding and u' yields an isomorphism $i'(Y) \cong Y$. By a general elementary fact, $\dim(f(X)) = \dim(A) = 1$ implies that the morphism f has connected fibers (see e.g. [6], Lemma (2.4.5)). Since the above diagram is cartesian, it follows that f' has also connected fibers. Therefore X' is connected (since Y is so).

On the other hand, the morphism $u' : X' \rightarrow X$ is finite and étale of degree d , because u is so. Moreover, since X is projective and nonsingular, X' is also projective and nonsingular. In other words, X' is a projective manifold such that u' and i' define an étale neighbourhood of $i : Y \hookrightarrow X$. In particular, u' yields an isomorphism of formal completions $\widehat{u}' : X'_{/i'(Y)} \cong X_{/Y}$, whence an isomorphism of rings of formal-rational functions $\widehat{u}'^* : K(X_{/Y}) \cong K(X'_{/i'(Y)})$. Now, look at the commutative diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{u'^*} & K(X') \\ \downarrow & & \downarrow \\ K(X_{/Y}) & \xrightarrow{\widehat{u}'^*} & K(X'_{/i'(Y)}). \end{array}$$

Since Y is $G3$ in X the first vertical map is an isomorphism. This and the isomorphism \widehat{u}'^* imply that $\deg(u'^*) = 1$ (because the second vertical map is injective). But $\deg(u'^*) = \deg(u') = d$, whence $d = 1$. This completes the proof of the lemma. \square

Now we can prove the following result.

THEOREM 2.4. *Let Y be a smooth connected curve of genus g embedded in a projective manifold X of dimension $n \geq 2$ with normal bundle $N_{Y|X}$ ample. Assume that $\dim_{\mathcal{C}} L_1(Y, X) = 1$. Then $g \leq 1$.*

- i) *If $g = 0$ then $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$, and X is rationally connected (in the sense of [22], cf. also [21]).*
- ii) *If $g = 1$ then $\deg(N_{Y|X}) = 1$ and the irregularity of X is ≤ 1 . Assume moreover that X is irregular and Y is $G3$ in X . Then the canonical morphism of Albanese varieties $\text{Alb}(Y) = Y \rightarrow \text{Alb}(X)$ is an isomorphism, and in particular, the Albanese morphism $X \rightarrow \text{Alb}(X)$ yields a retraction $\pi : X \rightarrow Y$ of the inclusion $Y \hookrightarrow X$.*

PROOF. The hypothesis says that $\dim_{\mathcal{C}} H^1(Y, N_{Y|X}^*) = 1$. As in the proof of Theorem 1.4, $\dim_{\mathcal{C}} H^1(N_{Y|X}^*) = \deg(N_{Y|X}) + (n-1)(g-1)$, whence

$$1 = \deg(N_{Y|X}) + (n-1)(g-1). \quad (11)$$

Since $N_{Y|X}$ is ample, $\deg(N_{Y|X}) \geq 1$, so that (11) implies $g \leq 1$. Moreover, if $g = 1$, it follows that $\deg(N_{Y|X}) = 1$.

- i) If $g = 0$ then $Y \cong \mathbf{P}^1$, and (11) yields $\deg(N_{Y|X}) = n$; moreover we get

$$N_{Y|X} = \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \mathcal{O}_{\mathbf{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_{n-1}), \quad \text{with } a_1 \geq a_2 \geq \cdots \geq a_{n-1}.$$

It follows that $\deg(N_{Y|X}) = a_1 + a_2 + \cdots + a_{n-1}$, and since $N_{Y|X}$ is ample, $a_{n-1} > 0$. Since $\deg(N_{Y|X}) = n$ we get $a_1 + a_2 + \cdots + a_{n-1} = n$, whence $a_1 = 2$ and $a_2 = \cdots = a_{n-1} = 1$. Then it is a general fact that X is rationally connected (see [21], [22]).

- ii) When $g = 1$, the ampleness of $N_{Y|X}$ and the result of Matsumura quoted in the proof of Lemma 2.3 imply that the map $\text{Alb}(Y) = Y \rightarrow \text{Alb}(X)$ is surjective, whence the irregularity of X is ≤ 1 . Finally, if the irregularity of X is 1, Lemma 2.3 proves the remaining part of ii), thus completing the proof of the theorem. \square

REMARKS 2.5. i) The hypothesis in Theorem 2.4, ii) that Y is $G3$ in X is not very restrictive. Indeed, as we remarked above, the ampleness of $N_{Y|X}$ implies by the result of Hartshorne quoted above that Y is in any case $G2$ in X . Then by a result of Hartshorne-Gieseker (see [10, Theorem 4.3]) there is a finite morphism $f : X' \rightarrow X$ with the following properties: the inclusion $Y \hookrightarrow X$ lifts to a closed embedding $j : Y \hookrightarrow X'$ such that f is étale along $j(Y)$ (i.e. $(X', j(Y))$ is an étale neighbourhood of (X, Y)) and $j(Y)$ is $G3$ in X' . In particular, X' is nonsingular along Y and $N_{j(Y)|X'} \cong N_{Y|X}$ is ample. Desingularizing X' away $j(Y)$ we get even a projective manifold \widetilde{X} containing Y such that $N_{Y|\widetilde{X}} \cong N_{Y|X}$ is ample and Y is $G3$ in \widetilde{X} . Moreover, if X is irregular,

\tilde{X} is also irregular (both having irregularity 1 since $N_{j(Y)|\tilde{X}} \cong N_{Y|X}$ is ample and the morphism $\tilde{X} \rightarrow X$ is surjective).

ii) Let $Y \subset X$ be as in Theorem 2.4, and assume that the irregularity of X is 1 (in particular, Y is an elliptic curve). We claim that the normal exact sequence

$$0 \rightarrow T_Y = \mathcal{O}_Y \rightarrow T_X|_Y \rightarrow N_{Y|X} \rightarrow 0$$

splits. Indeed, if Y is $G3$ in X , then the retraction $X \rightarrow Y$ of $Y \hookrightarrow X$ yields the desired splitting. Otherwise, use the previous remark to lift the embedding $Y \hookrightarrow X$ to $j : Y \hookrightarrow X'$ such that $j(Y)$ is $G3$ in X' and $f : X' \rightarrow X$ is étale along $j(Y)$. Then by Lemma 2.3 there exists a retraction $X' \rightarrow Y$ for j , so that the normal exact sequence

$$0 \rightarrow T_Y = \mathcal{O}_Y \rightarrow T_{X'}|_Y \rightarrow N_{Y|X'} = N_{Y|X} \rightarrow 0$$

splits. Since f is étale along $j(Y)$ then the splitting of the latter normal sequence implies the splitting of the former normal sequence.

iii) Assume, as in Theorem 2.4, that Y is a smooth connected curve of genus g embedded in a projective manifold X of dimension $n \geq 2$ with normal bundle $N_{Y|X}$ ample. Then the arguments of the proof of Theorem 2.4, i) yield in fact the following more general statement: if $L_1(Y, X)$ is of dimension $h < n$ (respectively $h = n$) then $g \leq 1$ (respectively $g \leq 2$, and if $g = 2$ then $\deg(N_{Y|X}) = 1$). Indeed, instead of equality (11) we have

$$h = \deg(N_{Y|X}) + (n - 1)(g - 1).$$

If $h < n$, since $\deg(N_{Y|X}) \geq 1$ we cannot have $g \geq 2$. If $h = n$ then $g \leq 2$, with $\deg(N_{Y|X}) = 1$ if $g = 2$.

If $g = 1$ then $\deg(N_{Y|X}) = h$ and the irregularity of X is ≤ 1 . If instead $g = 0$ one has $\deg(N_{Y|X}) = h + n - 1$ and

$$N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_h) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-h-1},$$

with $a_1 \geq a_2 \geq \cdots \geq a_h \geq 1$ and $\sum_{i=1}^h a_i = 2h$. In particular, X is rationally connected.

Now we recall the following result of Ionescu-Naie [19], Lemma (2.2):

THEOREM 2.6 ([19]). *Let X be a projective manifold of dimension $n \geq 2$ and let Y be a smooth rational curve in X with normal bundle of the form*

$$N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \mathcal{O}_{\mathbf{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_{n-1}) \quad \text{with} \quad a_1 \geq a_2 \geq \cdots \geq a_{n-1}.$$

Let $Z \subset X$ be a general smooth 2-codimensional subvariety of X meeting Y transversely in one point. Let $f : \tilde{X} \rightarrow X$ be the blowing up of X along Z and let \tilde{Y} be the proper transform of Y via f . Then the normal bundle of \tilde{Y} in \tilde{X} is given by

$$N_{\tilde{Y}|\tilde{X}} \cong \mathcal{O}_{\mathbf{P}^1}(a_1 - 1) \oplus \mathcal{O}_{\mathbf{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_{n-1}).$$

COROLLARY 2.7. *Let X be a projective manifold of dimension $n \geq 2$ and let Y be a smooth rational curve in X with normal bundle of the form $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$. Let $Z \subset X$ be a general smooth 2-codimensional subvariety of X meeting Y transversely in one point. Let $f: \tilde{X} \rightarrow X$ be the blowing up of X along Z and let \tilde{Y} be the proper transform of Y via f . Then \tilde{Y} is a quasi-line in \tilde{X} .*

REMARK 2.8. Let Y be a smooth rational curve in the projective n -fold X (with $n \geq 2$) such that $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$. Since by Bertini there always exist smooth 2-codimensional subvarieties Z of X meeting Y transversely in one point, Corollary 2.7 shows that as soon as we start with such a pair (X, Y) we easily produce a projective n -fold \tilde{X} (dominating X) and a quasi-line \tilde{Y} in \tilde{X} .

As proved in [22], the rationally connected manifolds are stable under any projective deformation. The next result shows that the projective manifolds of dimension n carrying nonsingular rational curves with normal bundle $\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$ are stable under small (but not under global) projective deformations. A similar result holds for quasi-lines, see [4], (3.10).

THEOREM 2.9. *Any small projective deformation of a projective manifold X of dimension $n \geq 2$ containing a smooth rational curve Y such that $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$ is a projective manifold containing a smooth rational curve with normal bundle isomorphic to $\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$.*

PROOF. Let $f: \mathcal{M} \rightarrow T$ be a smooth projective morphism such that there is a point $t_0 \in T$ with the property that $f^{-1}(t_0) \cong X$. By taking an appropriate base change we may assume that T is a smooth curve. We may view Y as a curve in \mathcal{M} . The proof of the openness of the deformations of rationally connected manifolds works in our situation as well (see the first part of the proof of Proposition (2.13) of [23], p. 107). In fact in our case it becomes even simpler, working with the Hilbert scheme (instead of the Hom-scheme). In fact, consider the canonical exact sequence

$$0 \rightarrow N_{Y|X} \rightarrow N_{Y|\mathcal{M}} \rightarrow N_{X|\mathcal{M}}|_Y \cong \mathcal{O}_Y \rightarrow 0. \quad (12)$$

Since $H^1(Y, N_{Y|X}) \cong H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}) = 0$, the exact sequence splits. Therefore we get

$$H^0(Y, N_{Y|\mathcal{M}}) \cong H^0(Y, N_{Y|X}) \oplus \mathbf{C} \quad \text{and} \quad H^1(Y, N_{Y|\mathcal{M}}) = 0.$$

It follows that there exists an one parameter family of curves $\{Y_s\}_{s \in D}$ (parametrized by the unit disk D) such that $Y_0 = Y$ and Y_s is not contained in $X_{t_0} = X$ for $s \neq 0$. Since Y_0 is contained in the fiber $f^{-1}(t_0)$, for each $s \in D$ the curve Y_s is contained in some fiber $X_{t_s} = f^{-1}(t_s)$, and the morphism defined by $s \mapsto t_s$ is unramified near 0. Clearly $N_{Y|X} = \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$ is ample. Since ampleness is an open condition and $\deg(N_{Y_s|X_{t_s}}) = \deg(N_{Y|X}) = n$ for all $s \in T$, it follows that $N_{Y_s|X_{t_s}} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$ for s near

0. Finally, since T is a smooth curve, there is an open neighbourhood of t_0 in T over which X_t contains a smooth rational curve Y_t with $N_{Y_t|X_t} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$. \square

REMARK 2.10. Let X be an n -dimensional manifold X containing a smooth rational curve Y with ample normal bundle such that $\dim_{\mathbf{C}} L_1(Y, X) = 1$. Then by Theorem 2.4, i), we have $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus(n-2)}$. Let $\text{Hilb}_Y(X)$ be the Hilbert scheme of Y . Then standard considerations yield the following facts: $\text{Hilb}_Y(X)$ is smooth at the point corresponding to Y , the general embedded deformations of Y are smooth rational curves having the same normal bundle as Y , and their union is dense in X . Moreover, through any three general points of Y there pass only finitely many smooth rational curves from the given family.

3. The case of surfaces.

Now we want to look more closely to what happens in the case when $n = 2$, i.e. when X is a surface. First let us give some examples.

EXAMPLE 3.1. Let $X := \mathbf{P}^1 \times \mathbf{P}^1$ and let $Y \in |\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1, 1)|$ be any smooth curve. Then $Y \cong \mathbf{P}^1$, $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2)$, whence $\dim_{\mathbf{C}} H^1(Y, N_{Y|X}^*) = 1$. By Theorem 1.1, $\text{Coker}(\alpha_1) \cong \mathbf{C}/F$, with F a free subgroup of $(\mathbf{C}, +)$ of rank ≤ 1 . This was the case classically studied by B. Segre (see [25], §37).

EXAMPLE 3.2. Let $X := \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-2))$ be the Segre-Hirzebruch surface \mathbf{F}_2 . Let C_0 be the minimal section of the canonical projection $\pi: X \rightarrow \mathbf{P}^1$ ($(C_0^2) = -2$), and let Y be a section of π such that $(Y^2) = 2$ and $Y \cap C_0 = \emptyset$. Clearly, $Y \cong \mathbf{P}^1$, $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2)$, and since Y is a section of π the map α_0 is surjective.

Note that Examples 3.1 and 3.2 of embeddings of \mathbf{P}^1 into $\mathbf{P}^1 \times \mathbf{P}^1$ and \mathbf{F}_2 respectively are not Zariski equivalent. Indeed, if we blow down the minimal section C_0 of \mathbf{F}_2 we get the projective cone V in \mathbf{P}^3 over the conic of \mathbf{P}^2 of equation $x_1^2 = x_0x_2$. Then the conclusion follows from the facts that the image of Y in V is an ample divisor on V and, on the other hand, in Example 3.1 the curve Y is ample on $\mathbf{P}^1 \times \mathbf{P}^1$.

A result of Gieseker (see [10], Theorem 4.5) together with the fact that the embeddings of \mathbf{P}^1 of Examples 3.1, 3.2 are not Zariski equivalent implies that they are not formally equivalent either. More precisely, in both cases we have $(Y^2) > 0$, whence by [18] or by [16], Y is $G3$ in X . Then the above-mentioned result of Gieseker tells us that if the two embeddings were formally equivalent, then they would be also Zariski equivalent.

EXAMPLE 3.3. Let B be an elliptic curve and L a line bundle of degree one on B . Since $H^1(B, L^{-1}) \neq 0$ (in fact, $\dim_{\mathbf{C}} H^1(B, L^{-1}) = 1$), there exists a non splitting exact sequence of vector bundles

$$0 \rightarrow \mathcal{O}_B \rightarrow E \rightarrow L \rightarrow 0.$$

Then a result of Gieseker [11] shows that E is ample because L is so. Put $X := \mathbf{P}(E)$, and let $\pi: X \rightarrow B$ be the canonical projection. Let also Y be the section of π that

corresponds to the surjection $E \rightarrow L$. Then $Y \in |\mathcal{O}_{\mathbf{P}(E)}(1)|$, and in particular, Y is an ample Cartier divisor on X with normal bundle $N_{Y|X} = L$. Clearly, the map α_0 is surjective, $(Y^2) = \deg(E) = 1$ and $N_{Y|X} \cong L$. It follows that $\dim_{\mathcal{C}} H^1(Y, N_{Y|X}^*) = 1$.

EXAMPLE 3.4. Let B be an elliptic curve and L a line bundle of degree one on B . Set $F := \mathcal{O}_B \oplus L$ and $X := \mathbf{P}(F)$. Let Y be the section of the canonical projection $\pi: X \rightarrow B$ corresponding to the canonical map $F \rightarrow L$. Then again $N_{Y|X} \cong L$. There is another section C_0 of π (the minimal section corresponding to the map $F \rightarrow \mathcal{O}_B$) such that $(C_0^2) = -1$ and $C_0 \cap Y = \emptyset$. Then C_0 can be blown down to get the projective cone X' over the polarized curve (B, L) , i.e. $X' \cong \text{Proj}(S[T])$, where $S := \bigoplus_{i=0}^{\infty} H^0(B, L^i)$, T is an indeterminate over S and the grading of $S[T]$ is given by $\deg(sT^j) = i + j$ whenever $s \in S$ is a homogeneous element of degree i . Let $f: X \rightarrow X'$ be the blowing down morphism and set $Y' := f(Y)$. Then Y' is an elliptic curve (isomorphic to B) such that Y' is embedded in the smooth locus of X' with normal bundle isomorphic to L .

Consider Examples 3.3 and 3.4 with the same B and L . Then $Y \cong Y'$ and $N_{Y|X} \cong N_{Y'|X'}$. On the other hand, exactly as in Examples 3.1 and 3.2, one shows that these two embeddings are not formally equivalent (and hence not Zariski equivalent either).

To draw a consequence of Theorem 2.4 we need to recall two well known results.

THEOREM 3.5 ([12]). *Let X be a normal projective surface containing $Y = \mathbf{P}^1$ as an ample Cartier divisor. Then, up to isomorphism, one has one of the following cases:*

- i) $X = \mathbf{P}^2$ and Y is either a line or a conic; or
- ii) $X = \mathbf{F}_e = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-e))$ and Y is a section of the canonical projection $\pi: \mathbf{F}_e \rightarrow \mathbf{P}^1$; or
- iii) X is the projective cone in \mathbf{P}^{s+1} over the rational normal curve of degree s in \mathbf{P}^s , and Y is the intersection of X with the hyperplane at infinity.

Theorem 3.5 is classical, a modern reference for it is [12].

THEOREM 3.6 ([9], [3]). *Let X be a normal projective surface containing an elliptic curve Y as an ample Cartier divisor. Then one has one of the following cases:*

- i) X is a (possibly singular) Del Pezzo surface (i.e. a rational surface with at most rational double points as singularities and with ample anticanonical class), and $-Y$ is a canonical divisor of X ; or
- ii) There exists an elliptic curve B and an ample rank two vector bundle E on B such that $X \cong \mathbf{P}(E)$ and $Y \in |\mathcal{O}_{\mathbf{P}(E)}(1)|$ (in particular, Y is a section of the canonical projection $\mathbf{P}(E) \rightarrow B$); or
- iii) X is the projective cone over the polarized curve $(Y, N_{Y|X})$ (i.e. $X \cong \text{Proj}(S[T])$, where $S = \bigoplus_{i=0}^{\infty} H^0(Y, N_{Y|X}^{\otimes i})$, T is an indeterminate over S and the gradation of $S[T]$ is given by $\deg(sT^j) = \deg(s) + j$, whenever s is a homogeneous element of S) and Y is embedded in X as the infinite section.

Theorem 3.6 is a generalization of a classical result, see [9] if X is smooth, and [3], p. 3, if X is singular.

Now we can prove the main result of this section:

THEOREM 3.7. *Let X be a smooth projective surface and Y a smooth connected curve on X such that $(Y^2) > 0$ and $\dim_{\mathbf{C}} L_1(Y, X) = 1$. Then there exist a birational morphism $\varphi: X \rightarrow X'$ and a Zariski open neighbourhood U of Y in X such that the restriction $\varphi|_U: U \rightarrow \varphi(U)$ is a biregular isomorphism, $Y' := \varphi(Y)$ is an ample Cartier divisor on X' , and (X', Y') is one of the following pairs:*

- i) $X' \cong \mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$ and $Y' \in |\mathcal{O}(1, 1)|$; or
- ii) X' is isomorphic to the quadratic normal cone in \mathbf{P}^3 of equation $x_1^2 = x_0x_2$, and Y' is the intersection of X' with the hyperplane $x_3 = 0$; or
- iii) Y is an elliptic curve, $(Y^2) = 1$, and there exists an exact sequence of vector bundles on Y

$$0 \rightarrow \mathcal{O}_Y \rightarrow E \rightarrow N_{Y|X} \rightarrow 0$$

with E ample such that $X' \cong \mathbf{P}(E)$ and $Y' \in |\mathcal{O}_{\mathbf{P}(E)}(1)|$; or

- iv) Y is an elliptic curve such that $(Y^2) = 1$ and X' is the projective cone over the polarized curve $(Y, N_{Y|X})$ (i.e. $X \cong \text{Proj}(S[T])$, where $S = \bigoplus_{i=0}^{\infty} H^0(Y, N_{Y|X}^{\otimes i})$, T is an indeterminate over S and the gradation of $S[T]$ is given by $\deg(sT^j) = \deg(s) + j$, whenever s is a homogeneous element of S) and Y' is embedded in X' as the infinite section (i.e. $Y' = D_+(T)$); or
- v) Y is an elliptic curve such that $(Y^2)_{X'} = 1$ and X' is a (possibly singular) Del Pezzo surface of degree 1 and $-Y$ is a canonical divisor of X' . (These surfaces are classified in [8].)

PROOF. If X is a surface then Y is a divisor on X . Since $N_{Y|X}$ is ample, by [16], Theorem 4.2, p. 110, there exists a birational isomorphism $\varphi: X \rightarrow X'$ with the following properties:

- X' is a normal projective surface,
- there is a Zariski open neighbourhood U of Y in X such that the restriction $\varphi|_U: U \rightarrow \varphi(U)$ is a biregular isomorphism, and
- $Y' := \varphi(Y)$ is an ample Cartier divisor on X' .

Note that in loc. cit. one first proves that the linear system $|mY|$ is base point free for $m \gg 0$. Then φ is gotten from the morphism associated to $|mY|$, for $m \gg 0$, by passing to the Stein factorization.

Now, by Theorem 2.4, $g \leq 1$; moreover, $(Y^2) = 2$ if $g = 0$, and $(Y^2) = 1$ if $g = 1$. Now the classification of the normal projective surfaces X' supporting a smooth rational or a smooth elliptic curve Y' as an ample Cartier divisor is given by Theorems 3.5 and 3.6 above.

If $g = 0$, $(Y'^2)_{X'} = (Y^2)_X = 2$, and then we apply Theorem 3.5. In case i) of Theorem 3.5 we have that $(Y'^2)_{X'}$ is 1 or 4, whence this case is ruled out. Moreover, $(Y'^2)_{X'} = 2$ can be realized in cases ii) or iii) of Theorem 3.5 either if $X' \cong \mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$ and $Y' \in |\mathcal{O}(1, 1)|$ (and this corresponds to Example 3.1 above), or if Y' is isomorphic to the quadratic normal cone in \mathbf{P}^3 of equation $x_1^2 = x_0x_2$ (which corresponds to Example 3.2 above).

If $g = 1$ and the surface X is rational, by Theorem 3.6, i), X' is a (possibly singular) Del Pezzo surface and $-Y$ is a canonical divisor of X' . Moreover, since $(Y^2)_{X'} = 1$, the degree of X' is 1.

Assume now X not rational. Then X' is a surface as in each of cases ii) or iii) of Theorem 3.6. In both cases, by Theorem 2.4, we have $\deg(N_{Y|X}) = 1$, whence $(Y'^2)_{X'} = 1$.

If we are in case ii) (of Theorem 3.6), then $X' \cong \mathbf{P}(E)$, with E an ample rank two vector bundle over an elliptic curve B , and $Y' \in |\mathcal{O}_{\mathbf{P}(E)}(1)|$. Let $\pi: \mathbf{P}(E) \rightarrow B$ be the canonical projection. Then the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}(E)} \rightarrow \mathcal{O}_{\mathbf{P}(E)}(1) \cong \mathcal{O}_X(Y) \rightarrow N_{Y'|X'} \cong N_{Y|X} \rightarrow 0$$

yields the cohomology sequence

$$0 \rightarrow \pi_*(\mathcal{O}_{\mathbf{P}(E)}) \rightarrow \pi_*(\mathcal{O}_{\mathbf{P}(E)}(1)) \rightarrow \pi_*(N_{Y'|X'}) \rightarrow R^1\pi_*(\mathcal{O}_{\mathbf{P}(E)}) = 0,$$

or else,

$$0 \rightarrow \mathcal{O}_Y \rightarrow E \rightarrow N_{Y|X} \rightarrow 0.$$

So we get case iii) of our statement.

Finally, case iii) of Theorem 3.6 yields case iv). □

REMARK 3.8. In Theorem 3.7, the cokernel of the restriction map $\alpha_0: \text{Pic}(X) \rightarrow \text{Pic}(Y)$ is finite (and in fact the map α_0 is surjective) if and only if (X, Y) is in one of cases i)–iv) (see [2], §14).

4. Examples in higher dimension.

To give the first examples of curves $Y \cong \mathbf{P}^1$ on projective n -folds X , such that $n \geq 3$ and $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$, we need the following simple lemma.

LEMMA 4.1. *Let X be a smooth projective variety of dimension $n \geq 3$, and let X' be a smooth irreducible hypersurface of X such that:*

- i) X' contains a quasi-line Y , i.e. there is a curve Y on X' such that $Y \cong \mathbf{P}^1$ and $N_{Y|X'} \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$; and
- ii) $(N_{X'|X} \cdot Y) = 2$.

Then $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$.

PROOF. Consider the canonical exact sequence of normal bundles

$$0 \rightarrow N_{Y|X'} \rightarrow N_{Y|X} \rightarrow N_{X'|X}|_Y \rightarrow 0,$$

in which $N_{Y|X'} \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$ and $N_{X'|X}|_Y \cong \mathcal{O}_{\mathbf{P}^1}(2)$ by conditions i) and ii) respectively. Since $H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus n-2}) = 0$, this exact sequence splits to give the conclusion. □

EXAMPLE 4.2. Let $v_2: \mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^m$ be the 2-fold Veronese embedding of \mathbf{P}^{n-1} (with $m = (n^2 + n - 2)/2$). In Lemma 4.1 we take $X' = v_2(\mathbf{P}^{n-1})$ and $Y = v_2(L)$, with

L a line in \mathbf{P}^{n-1} . Then Y is a smooth conic in \mathbf{P}^m , contained in X' . In particular, $N_{Y|X'} \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$. Let now $Z \subset \mathbf{P}^{m+1}$ be the cone over X' with vertex a point $z \in \mathbf{P}^{m+1} \setminus \mathbf{P}^m$. Notice that Z is isomorphic to the weighted projective space $\mathbf{P}^n(1, \dots, 1, 2)$. Then Z contains $X' \cong \mathbf{P}^{n-1}$ as an ample Cartier divisor such that $N_{X'|Z} \cong \mathcal{O}_{\mathbf{P}^{n-1}}(2)$. Let X be the blowing up of Z at z . Then $X \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbf{P}^{n-1}})$, X still contains X' , and $N_{X'|X} \cong N_{X'|Z} \cong \mathcal{O}_{\mathbf{P}^{n-1}}(2)$, whence $(N_{X'|X} \cdot Y) = 2$. Therefore by Lemma 4.1 we get

$$N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}. \tag{13}$$

We call this example *the standard example* of a smooth rational curve Y in an n -fold X satisfying (13). Example 4.2 is a higher dimensional analogue of Example 3.2. Note that X is also isomorphic to the projective closure $\mathbf{P}(\mathcal{O}_{\mathbf{P}^{n-1}}(-2) \oplus \mathcal{O}_{\mathbf{P}^{n-1}})$ of the geometric vector bundle $\mathbf{V}(\mathcal{O}_{\mathbf{P}^{n-1}}(-2))$.

EXAMPLE 4.3. Consider the projective bundle $X' := \mathbf{P}(T_{\mathbf{P}^d})$ associated to the tangent bundle $T_{\mathbf{P}^d}$ of \mathbf{P}^d , with $d \geq 2$. In particular, $\dim(X') = 2d - 1$. It is well known that X' contains quasi-lines Y (see e.g. [24] if $d = 2$ and [2], Example 13.1 in general). Then

$$X' \cong \{([x_0, \dots, x_d], [y_0, \dots, y_d]) \in \mathbf{P}^d \times \mathbf{P}^d \mid x_0y_0 + \dots + x_dy_d = 0\}.$$

In case $d = 2$ the threefold X' is sometimes called *Hitchin's flag manifold* (see [24]). It follows that

$$\mathcal{O}_{\mathbf{P}^d \times \mathbf{P}^d}(1, 1)|_{X'} \cong \mathcal{O}_{\mathbf{P}(T_{\mathbf{P}^d})}(1).$$

Now take $X = \mathbf{P}^d \times \mathbf{P}^d$. Then $(\mathcal{O}_{\mathbf{P}^d \times \mathbf{P}^d}(1, 1) \cdot Y)_X = (\mathcal{O}_{\mathbf{P}(T_{\mathbf{P}^d})}(1) \cdot Y)_{X'} = 2$. Then Lemma 4.1 applies to this situation to show that

$$N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus 2d-2}. \tag{14}$$

In conclusion, $X = \mathbf{P}^d \times \mathbf{P}^d$ contains smooth rational curves Y with the normal bundle given by (14). This example is a higher-dimensional analogue of Example 3.1.

As in the case of surfaces we have the following result:

PROPOSITION 4.4. Consider the projective variety $X \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbf{P}^{n-1}})$, with $n = 2d$ and $d \geq 2$, and let Y be the smooth rational curve in X with $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus 2d-2}$ constructed in Example 4.2. Set also $X' := \mathbf{P}^d \times \mathbf{P}^d$, and let Y' be the smooth rational curve in X' with $N_{Y'|X'} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus 2d-2}$ constructed in Example 4.3. Then the pairs (X, Y) and (X', Y') are not formally equivalent.

PROOF. First we claim that Y is G3 in X . To see this, clearly we may replace X by the cone Z , which is isomorphic to the weighted projective space $\mathbf{P}^n(1, \dots, 1, 2)$. Then the assertion follows from [2], Corollary 13.3. It also follows that Y meets every

hypersurface of Z .

On the other hand, since $N_{Y'|X'} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus 2d-2}$, $N_{Y'|X'}$ is ample, so by a result of Hartshorne Y' is $G2$ in $X' = \mathbf{P}^d \times \mathbf{P}^d$. Since X' is a rational homogeneous space, it follows that Y' is $G3$ in X (see [5], Theorem (4.5), (ii)). Moreover, Y' generates the homogeneous space in the sense of Chow [7]; see also [5]. Then by Proposition (4.3) of [5] it also follows that Y' meets every hypersurface of X' . Now assume that the formal completions X/Y and X'/Y' are isomorphic. Then by a result of Gieseker (see [10] and also [2], Corollary 9.20) this implies that there are Zariski open neighbourhoods U in X containing Y , and U' in X' containing Y' and a biregular isomorphism $f: U \rightarrow U'$ such that $f(Y) = Y'$ and f induces the given formal isomorphism. Again we may replace X by the cone $Z = \mathbf{P}^n(1, \dots, 1, 2)$, which has the advantage that it is a normal \mathbf{Q} -Fano variety. Then the complements $Z \setminus U$ and $X' \setminus U'$ are both of codimension ≥ 2 (since Y meets every hypersurface of Z and Y' meets every hypersurface of X'). The isomorphism f yields an isomorphism between the anticanonical classes $-K_U$ and $-K_{U'}$, and since $\text{codim}_Z(Z \setminus Y) \geq 2$, Z is a normal \mathbf{Q} -Fano variety, $\text{codim}_{X'}(X' \setminus Y') \geq 2$ and X' is a Fano variety, it follows that f extends to an isomorphism $Z \cong X'$. But this is absurd because X' is smooth and Z is singular. \square

In dimension $n \geq 3$ there are many more examples of smooth rational curves Y lying on an n -fold X with $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$ than in dimension 2, as the following examples show.

EXAMPLE 4.5. Let X' be a smooth Fano threefold of index 2 such that $\text{Pic}(X') = \mathbf{Z}[H]$, with H very ample. By a result of Oxbury [24] (see also [4], Theorem (3.2), for another proof), X' contains a quasi-line Y which is a conic with respect to the projective embedding $X' \hookrightarrow \mathbf{P}^m$ given by $|H|$, i.e. such that $(H \cdot Y) = 2$. By Fano-Iskovskih classification (see [20]), X' is one of the following:

- a cubic hypersurface in \mathbf{P}^4 (with $m = 4$); or
- a complete intersection of two hyperquadrics in \mathbf{P}^5 (with $m = 5$); or
- a section of the Plücker embedding of the Grassmannian $\mathbf{G}(1, 4)$, of lines in \mathbf{P}^4 , in \mathbf{P}^9 with three general hyperplanes of \mathbf{P}^9 (with $m = 6$).

Let now X be a smooth projective fourfold in \mathbf{P}^{m+1} such that X' is a hyperplane section of X . For example, in the first case, X can be an arbitrary cubic fourfold in \mathbf{P}^5 . Clearly, $N_{X'|X} = \mathcal{O}_{X'}(1) = H$. Then Lemma 4.1 can be applied in this case to get

$$N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1). \tag{15}$$

In particular, every cubic fourfold X in \mathbf{P}^5 contains smooth rational curves Y with the normal bundle given by (15).

EXAMPLE 4.6. Start with the curve $Y = \mathbf{P}^1 \in |\mathcal{O}(1, 1)|$ in $X' := \mathbf{P}^1 \times \mathbf{P}^1$ of Example 3.1, and with two linear embeddings $i: \mathbf{P}^1 \hookrightarrow \mathbf{P}^m$, and $j: \mathbf{P}^1 \hookrightarrow \mathbf{P}^n$, where $m, n \geq 1$, and $m + n \geq 3$. Consider the embedding

$$i \times j: X' = \mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow X := \mathbf{P}^m \times \mathbf{P}^n.$$

Then $N_{X'|X} \cong \mathcal{O}_{\mathbf{P}^m \times \mathbf{P}^n}(1, 0)^{\oplus m-1} \oplus \mathcal{O}_{\mathbf{P}^m \times \mathbf{P}^n}(0, 1)^{\oplus n-1}$. In particular, $N_{X'|X}|Y \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus m+n-2}$. Thus the exact sequence

$$0 \rightarrow N_{Y|X'} = \mathcal{O}_{\mathbf{P}^1}(2) \rightarrow N_{Y|X} \rightarrow N_{X'|X}|Y \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus m+n-2} \rightarrow 0$$

splits, to give

$$N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus m+n-2}.$$

If for example we take $m = 3$ and $n = 1$ we get an embedding $\alpha: \mathbf{P}^1 \hookrightarrow \mathbf{P}^3 \times \mathbf{P}^1$ whose image has the normal bundle isomorphic to $\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$. Since the Fano fourfolds $\mathbf{P}^2 \times \mathbf{P}^2$ and $\mathbf{P}^3 \times \mathbf{P}^1$ (which are both homogeneous spaces) cannot be isomorphic (because $\mathbf{P}^2 \times \mathbf{P}^2$ has index 3 and $\mathbf{P}^3 \times \mathbf{P}^1$ has index 2), the proof of Proposition 4.4 can be applied to yield the fact that this latter embedding cannot be formally equivalent to the embedding $\mathbf{P}^1 \hookrightarrow \mathbf{P}^2 \times \mathbf{P}^2$ of Example 4.3 (or to the embedding $\beta: \mathbf{P}^1 \hookrightarrow \mathbf{P}^2 \times \mathbf{P}^2$ obtained by the above procedure when $m = n = 2$).

EXAMPLE 4.7 (Hypercubic in \mathbf{P}^{n+1}). Let X' be a cubic fourfold in \mathbf{P}^5 and $Y \cong \mathbf{P}^1 \subset X'$ with $N_{Y|X'} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$ as in Example 4.5. Let X be the five dimensional cubic in \mathbf{P}^6 having X' as hyperplane section. Then $-K_X \cong 4H$ and $X' \in |H|$. Moreover $N_{X'|X} = H$ and $(Y \cdot H) = 2$. Thus arguing as in the proof of Lemma 4.1, we see that the normal bundle of Y in X is

$$N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1).$$

That is, $N_{Y|X}$ is of the form as in Remark 2.5, iii), with $h := \dim_{\mathbf{C}} L_1(Y, X) = \deg(N_{Y|X}) - 4 = 2$.

More generally, we see that an arbitrary hypercubic X in \mathbf{P}^{n+1} contains a curve $Y \cong \mathbf{P}^1$ with $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2)^{\oplus n-3} \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$, so that $\deg(N_{Y|X}) = 2n - 4$. Therefore

$$h := \dim_{\mathbf{C}} L_1(Y, X) = \deg(N_{Y|X}) - n + 1 = n - 3,$$

as in Remark 2.5, iii).

Similar conclusions by taking as (X, H) any Del Pezzo n -fold with $\text{Pic}(X) \cong \mathbf{Z}[H]$. That is X is the complete intersection of two hyperquadrics in \mathbf{P}^{n+2} , or a linear section of the Grassmannian $\mathbf{G}(1, 4)$ (of lines in \mathbf{P}^4) of dimension $\dim X = 4, 5$.

REMARK 4.8. Let X be an n dimensional Fano manifold containing a smooth rational curve Y with $N_{Y|X} \cong \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-2}$. Thus the index, r , of X satisfies the condition

$$r \leq \frac{n+2}{2}. \tag{16}$$

Indeed, by adjunction formula, $-(K_X \cdot Y) = n+2$. Let $-K_X \cong rH$, H ample line bundle

on X . Thus $r(H \cdot Y) = n + 2$, giving $(H \cdot Y) \geq 2$ and hence the claimed inequality.

In particular, X is neither \mathbf{P}^n nor a hyperquadric. If X is a Del Pezzo manifold (case $r = n - 1$) then (16) yields $n \leq 4$ and therefore $r = 3$, $n = 4$, as in the case of the cubic fourfold X in \mathbf{P}^5 discussed in Example 4.5.

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