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The stable Calabi-Yau dimension of tame symmetric algebras

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Abstract. We determine the Calabi-Yau dimension of the stable module categories of all symmetric algebras of tame representation type over an algebraically closed field, and derive some consequences.

Introduction.

One of the goals of noncommutative algebraic geometry is to obtain an understanding of triangulated K-linear categories over an algebraically closed field K which have properties close to the derived categories $D^b(\mathrm{coh}X)$ of bounded complexes of coherent sheaves on projective varieties X over K. Following Bondal and Kapranov [**11**] a triangulated K-linear category $\mathscr{A}$ is said to have a Serre duality if there is a triangle autoequivalence $F : \mathscr{A} \to \mathscr{A}$, called a Serre functor, such that there are natural K-linear automorphisms

$$\mathrm{Hom}_{\mathscr{A}}(A, B) \cong D\mathrm{Hom}_{\mathscr{A}}(B, F(A))$$

for all objects A and B in $\mathscr{A}$, where $D = \mathrm{Hom}_K(-, K)$. Moreover, if F and F' are two Serre functors of $\mathscr{A}$, then they are naturally isomorphic (see [**11**], [**40**]). Then, for a nonsingular projective variety X of dimension n and the canonical sheaf $\omega_X = \bigwedge^n \Omega_X$, the classical Serre duality [**31**]

$$H^i(X, \mathscr{F}) \cong D\mathrm{Ext}^{n-i}_{\mathrm{coh}(X)}(\mathscr{F}, \omega_X),$$

for $\mathscr{F} \in \mathrm{coh}(X)$, is just the statement that the functor $F = - \otimes \omega_X[n]$ defines a Serre duality on the triangulated K-linear category $D^b(\mathrm{coh}X)$. Important examples of triangulated K-linear categories with Serre duality are provided by the derived categories $D^b(\mathrm{mod}A)$ of bounded complexes over the category $\mathrm{mod}A$ of finite dimensional modules of finite dimensional K-algebras A of finite global dimension (see [**27**]). We refer also to [**40**] for a complete classification of the noetherian hereditary abelian categories $\mathscr{C}$ such that the derived category $D^b(\mathscr{C})$ has Serre duality. Recently triangulated K-linear categories with more subtle Serre dualities came to be of interest. Following Kontsevich [**36**] (see also [**35**]), a triangulated K-linear category $\mathscr{A}$, with shift functor T, is said to

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be Calabi-Yau if an iterated shift $T^n = [n]$ is a Serre duality of $\mathscr{A}$. If so then the minimal $n \geq 0$ having this property is called the Calabi-Yau dimension of $\mathscr{A}$, and is denoted by CYdim$\mathscr{A}$. If $\mathscr{A}$ is not Calabi-Yau, we set CYdim$\mathscr{A} = \infty$.

An important class of triangulated K-linear categories of algebraic nature is formed by the stable module categories $\underline{\mathrm{mod}}A$ of finite dimensional selfinjective K-algebras A, where the shift is given by the inverse Ω_A^{-1} of Heller's syzygy functor Ω_A. Then $\underline{\mathrm{mod}}A$ is Calabi-Yau if and only if an iterated shift Ω_A^{-n} (for some $n \geq 1$) is isomorphic to the Nakayama functor $\nu_A = D\mathrm{Hom}_A(-, A)$. If A is selfinjective then we define the stable Calabi-Yau dimension of A to be CYdim$\underline{\mathrm{mod}}A$, and we write briefly $\underline{\mathrm{CY}}\mathrm{dim}A$. In particular, for a symmetric algebra A, we have $\underline{\mathrm{CY}}\mathrm{dim}A < \infty$ if and only if $\Omega_A^n \cong 1_{\underline{\mathrm{mod}}A}$ for some $n \geq 1$. We also note that the stable module category $\underline{\mathrm{mod}}A$ of a selfinjective algebra A is equivalent (as a triangulated category) to the quotient category $D^b(\mathrm{mod}A)/K^b(\mathscr{P}_A)$ of $D^b(\mathrm{mod}A)$ by the homotopy category $K^b(\mathscr{P}_A)$ of bounded complexes of finite dimensional projective A-modules [**41**, Theorem 2.1], and that $D^b(\mathrm{mod}A)$ has no Serre duality (see [**29**, Corollary 1.5], [**40**, Theorem A]).

The finite dimensional algebras over an algebraically closed field K may be divided into two disjoint classes (see [**15**], [**17**]). One class consists of the tame algebras for which the indecomposable modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras; the representation theory of any wild algebra comprises the representation theories of all finite dimensional algebras over K. Accordingly, we may realistically hope to classify the indecomposable finite dimensional modules only for the tame algebras. We note that the class of tame algebras contains all representation-finite algebras, that is algebras which have only finitely many indecomposable modules up to isomorphism. It also contains all algebras of polynomial growth. Recall that an algebra is of polynomial growth if there is a natural number m such that the indecomposable modules occur, in each dimension d, in a finite number of discrete and at most d^m one-parameter families [**47**].

We are concerned with the problem of describing the stable Calabi-Yau dimension of (tame) selfinjective algebras. Recently, the authors completed in [**22**] the classification of the connected tame symmetric algebras all of whose indecomposable nonprojective modules are periodic. This class of algebras consists of algebras which are Morita equivalent to the algebras of the following three types:

- socle deformations of the symmetric algebras of Dynkin type (described in [**14**], [**33**], [**43**], [**44**], [**52**], [**53**]);
- socle deformations of the symmetric algebras of tubular type (described in [**8**], [**9**], [**10**], [**37**], [**38**], [**48**]);
- algebras of quaternion type (investigated in [**19**], [**20**], [**21**]).

We refer also to [**1**], [**6**], [**7**], [**26**], [**30**], [**32**] and [**39**] for the derived and stable equivalence classification of these tame symmetric algebras. We prove in this paper that these algebras are precisely all tame symmetric algebras of finite stable Calabi-Yau dimension, and determine their stable Calabi-Yau dimension.

We now describe the content of this paper in detail. In Section 1 we present necessary and sufficient conditions for a stable module category of a selfinjective algebra to be Calabi-Yau. In Section 2 we describe all selfinjective algebras of stable Calabi-Yau dimensions 0 and 1. In Section 3 we provide information on the selfinjective algebras of stable Calabi-Yau dimension 2. In particular, all connected tame selfinjective algebras of stable Calabi-Yau dimension 2 are completely described. In Section 4 we determine the stable Calabi-Yau dimension of all symmetric algebras of finite representation type. In particular, we prove that every nonnegative integer occurs as the stable Calabi-Yau dimension of a connected representation-finite symmetric algebra. Moreover, we exhibit representation-finite selfinjective algebras of infinite stable Calabi-Yau dimension. In Section 5 we prove that all algebras of expected quaternion type, that is the algebras listed in [**21**, pp. 303–306], have stable Calabi-Yau dimension 3, and hence are in fact of quaternion type. As a consequence, we also obtain that all connected tame symmetric algebras of nonpolynomial growth and finite stable Calabi-Yau dimension have stable Calabi-Yau dimension 3. In the final Section 6 we determine the stable Calabi-Yau dimension of connected representation-infinite symmetric algebras of polynomial growth and finite stable Calabi-Yau dimension. We show that the stable Calabi-Yau dimensions of these algebras are precisely the prime numbers 2, 3, 5, 7 and 11.

For basic background on the representation theory of algebras we refer to [**2**], [**4**], [**21**], [**24**], [**27**], [**46**] and [**54**].

1. Stable module categories.

Throughout this paper K will denote a fixed algebraically closed field. By an algebra we mean a finite dimensional K-algebra (associative, with identity). For an algebra A, we denote by $\operatorname{mod}A$ the category of finite-dimensional right A-modules, and by $D : \operatorname{mod}A \to \operatorname{mod}A^{\mathrm{op}}$ the standard duality $\operatorname{Hom}_K(-,K)$. We write $\varGamma_A$ for the Auslander-Reiten quiver of A, then $\varGamma_A^s$ is the stable Auslander-Reiten quiver of A, which is obtained from $\varGamma_A$ by removing the nonstable vertices and arrows attached to them; and we write τ_A and τ_A^{-1} for the Auslander-Reiten translations $D\mathrm{Tr}$ and $\mathrm{Tr}D$, respectively. We shall identify an indecomposable module from $\operatorname{mod}A$ with the corresponding vertex of $\varGamma_A$.

An algebra A is called selfinjective if $A \cong D(A)$ in $\operatorname{mod}A$, that is, the projective A-modules are injective. Further, A is called symmetric if A and $D(A)$ are isomorphic as A-A-bimodules. The classical examples of selfinjective algebras are provided by the blocks of group algebras of finite groups, or more generally Hopf algebras. For a selfinjective algebra A, we denote by $\underline{\operatorname{mod}}A$ the stable category of $\operatorname{mod}A$. Recall that the objects of $\underline{\operatorname{mod}}A$ are the objects of $\operatorname{mod}A$ without nonzero projective direct summands, and for any two objects M and N of $\underline{\operatorname{mod}}A$ the space of morphisms from M to N in $\underline{\operatorname{mod}}A$ is the quotient $\underline{\operatorname{Hom}}_A(M,N) = \operatorname{Hom}_A(M,N)/P(M,N)$, where $P(M,N)$ is the subspace of $\operatorname{Hom}_A(M,N)$ consisting of all morphisms which factor through projective A-modules. Then the Auslander-Reiten translations induce two mutually inverse functors τ_A and $\tau_A^{-1} : \underline{\operatorname{mod}}A \xrightarrow{\sim} \underline{\operatorname{mod}}A$. We shall also consider two (mutually inverse) Heller's loop and suspension functors $\varOmega_A$, $\varOmega_A^{-1} : \underline{\operatorname{mod}}A \xrightarrow{\sim} \underline{\operatorname{mod}}A$. Recall that $\varOmega_A$ (respectively, $\varOmega_A^{-1}$) assigns to any object M of $\underline{\operatorname{mod}}A$ the kernel of its projective cover $P_A(M) \to M$ (respectively, the cokernel of its injective envelope $M \to I_A(M)$) in $\operatorname{mod}A$. Finally, denote by $\nu_A : \underline{\operatorname{mod}}A \xrightarrow{\sim} \underline{\operatorname{mod}}A$ the Nakayama functor $D\operatorname{Hom}_A(-,A)$. By general theory [**4**, (IV.3.7)] we have $\tau_A = \varOmega_A^2\nu_A = \nu_A\varOmega_A^2$ and $\tau_A^{-1} = \varOmega_A^{-2}\nu_A^{-1} = \nu_A^{-1}\varOmega_A^{-2}$. In particular, $\tau_A = \varOmega_A^2$ and $\tau_A^{-1} = \varOmega_A^{-2}$ if A is symmetric. Two selfinjective algebras A and $\varLambda$

are said to be stably equivalent if their stable module categories $\underline{\mathrm{mod}}A$ and $\underline{\mathrm{mod}}\Lambda$ are equivalent. Further, two selfinjective algebras A and Λ are said to be derived equivalent if the derived categories $D^b(\mathrm{mod}A)$ and $D^b(\mathrm{mod}\Lambda)$ are equivalent as triangulated categories. It is known that derived equivalent selfinjective algebras are stably equivalent [**41**, Corollary 2.2]. Finally, a selfinjective algebra Λ is said to be a socle deformation of a selfinjective algebra A if $\Lambda/\mathrm{soc}\Lambda \cong A/\mathrm{soc}A$.

The following known facts (see [**4**, Chapter IV]) will be crucial for our investigations.

PROPOSITION 1.1. *Let A be a selfinjective algebra and M, N be modules from $\underline{\mathrm{mod}}A$. Then there are natural K-linear isomorphisms*

(i) $D\underline{\mathrm{Hom}}_A(\tau_A^{-1}N, M) \cong \mathrm{Ext}_A^1(M,N) \cong D\underline{\mathrm{Hom}}_A(N, \tau_A M)$;
(ii) $\underline{\mathrm{Hom}}_A(\Omega_A^n M, N) \cong \mathrm{Ext}_A^n(M,N) \cong \underline{\mathrm{Hom}}_A(M, \Omega_A^{-n} N)$.

PROPOSITION 1.2. *Let A be a selfinjective algebra. Then*

(i) *$\underline{\mathrm{mod}}A$ is a triangulated category whose shift is the suspension functor Ω_A^{-1}.*
(ii) *$\Omega_A \nu_A$ is a unique (up to natural equivalence of functors) Serre duality of $\underline{\mathrm{mod}}A$.*

PROOF.
(i) This is proved in [**27**, (2.6)].
(ii) Invoking Proposition 1.1, for X and Y in $\underline{\mathrm{mod}}A$, we have natural K-linear isomorphisms

$$\begin{aligned}\underline{\mathrm{Hom}}_A(X,Y) &\cong \underline{\mathrm{Hom}}_A(\tau_A^{-1}(\tau_A X), Y) \cong D\mathrm{Ext}_A^1(Y, \tau_A X)\\ &\cong D\underline{\mathrm{Hom}}_A(Y, \Omega_A^{-1}\tau_A X) \cong D\underline{\mathrm{Hom}}_A(Y, \Omega_A^{-1}\Omega_A^2 \nu_A X)\\ &\cong D\underline{\mathrm{Hom}}_A(Y, \Omega_A \nu_A X).\end{aligned}$$

Therefore, $\Omega_A\nu_A$ is a Serre duality of $\underline{\mathrm{mod}}A$. Moreover, it is a unique Serre duality of $\underline{\mathrm{mod}}A$ (see [**40**, (I.1.3) and (I.1.5)]). □

COROLLARY 1.3. *Let A be a selfinjective algebra. Then the following statements are equivalent*:

(i) *$\underline{\mathrm{mod}}A$ is Calabi-Yau.*
(ii) *$\nu_A \cong \Omega_A^{-d-1}$ for some $d \geq 0$.*
(iii) *$\tau_A \cong \Omega_A^{-d+1}$ for some $d \geq 0$.*

PROOF. It follows from Proposition 1.2 that $\underline{\mathrm{mod}}A$ is Calabi-Yau if and only if $\Omega_A\nu_A \cong \Omega_A^{-d}$ for some $d \geq 0$. Then the equivalence of (i), (ii) and (iii) follows from the formula $\tau_A = \Omega_A^2\nu_A = \nu_A\Omega_A^2$. □

As an immediate consequence we obtain the following.

COROLLARY 1.4. *Let A be a symmetric algebra. Then $\underline{\mathrm{mod}}A$ is Calabi-Yau if and only if $\Omega_A^n \cong 1_{\underline{\mathrm{mod}}A}$ for some $n \geq 1$.*

Therefore, the stable Calabi-Yau dimension $\underline{\mathrm{CY}}\mathrm{dim}A$ of a selfinjective algebra A is the minimal nonnegative integer d such that $\nu_A \cong \Omega_A^{-d-1}$ (equivalently, $\tau_A \cong \Omega_A^{-d+1}$)

on $\underline{\mathrm{mod}}A$. Observe that the stable Calabi-Yau dimension of selfinjective algebras is invariant under stable equivalence and then under derived equivalence. The next proposition exhibits an Ext-symmetry of the Calabi-Yau stable module categories. Here, $\mathrm{Ext}^0_A(X,Y) = \underline{\mathrm{Hom}}_A(X,Y)$.

PROPOSITION 1.5. *Let A be a selfinjective algebra. Then the following statements are equivalent*:

(i) *$\underline{\mathrm{mod}}A$ is Calabi-Yau.*

(ii) *There is a nonnegative integer d such that*

$$\mathrm{Ext}^i_A(X,Y) \cong D\mathrm{Ext}^j_A(Y,X)$$

for all modules X and Y from $\underline{\mathrm{mod}}A$ and nonnegative integers i, j with $i+j=d$.

(iii) *There exist nonnegative integers i and j such that*

$$\mathrm{Ext}^i_A(X,Y) \cong D\mathrm{Ext}^j_A(Y,X)$$

for all modules X and Y from $\underline{\mathrm{mod}}A$.

PROOF. (i) $\Rightarrow$ (ii). Assume $\underline{\mathrm{mod}}A$ is Calabi-Yau. Then $\tau_A \cong \Omega_A^{-d+1}$ for some nonnegative integer d, by Corollary 1.3. Let X, Y be modules from $\underline{\mathrm{mod}}A$ and i, j nonnegative integers with $i+j=d$. Then, applying Proposition 1.1, we have canonical isomorphisms of K-vector spaces

$$\begin{aligned}\mathrm{Ext}^i_A(X,Y) &\cong \underline{\mathrm{Hom}}_A(X, \Omega_A^{-i}Y) \cong \underline{\mathrm{Hom}}_A(\Omega_A^{-j+1}X, \Omega_A^{1-d}Y)\\ &\cong \underline{\mathrm{Hom}}_A(\Omega_A^{-(j-1)}X, \tau_A Y)\\ &\cong D\mathrm{Ext}^1_A(Y, \Omega_A^{-(j-1)}X) \cong D\underline{\mathrm{Hom}}_A(Y, \Omega_A^{-j}X)\\ &\cong D\mathrm{Ext}^j_A(Y,X).\end{aligned}$$

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). Assume $\mathrm{Ext}^i_A(X,Y) \cong D\mathrm{Ext}^j_A(Y,X)$ for some nonnegative integers i, j and all modules X, Y from $\underline{\mathrm{mod}}A$. Then, for $d=i+j$ and X, Y from $\underline{\mathrm{mod}}A$, we obtain canonical isomorphisms of K-vector spaces

$$\begin{aligned}\underline{\mathrm{Hom}}_A(X, \tau_A Y) &\cong D\mathrm{Ext}^1_A(Y,X) \cong D\underline{\mathrm{Hom}}_A(Y, \Omega_A^{-1}X)\\ &\cong D\underline{\mathrm{Hom}}_A(\Omega_A^{i}(\Omega_A^{-i}Y), \Omega_A^{-1}X)\\ &\cong D\mathrm{Ext}^i_A(\Omega_A^{-i}Y, \Omega_A^{-1}X) \cong \mathrm{Ext}^j_A(\Omega_A^{-1}X, \Omega_A^{-i}Y)\\ &\cong \underline{\mathrm{Hom}}_A(\Omega_A^{-1}X, \Omega_A^{-i-j}Y) \cong \underline{\mathrm{Hom}}_A(X, \Omega_A^{1-d}Y),\end{aligned}$$

and consequently $\tau_A Y \cong \Omega_A^{1-d}Y$. Thus $\tau_A \cong \Omega_A^{1-d}$ and $\underline{\mathrm{mod}}A$ is Calabi-Yau. $\square$

Corollary 1.6. *Let A be a selfinjective algebra. Then $\underline{\mathrm{mod}}A$ is Calabi-Yau of dimension d if and only if d is a minimal nonnegative integer such that $\mathrm{Ext}^i_A(X,Y) \cong D\mathrm{Ext}^j_A(Y,X)$ for all nonnegative integers i, j with $i+j=d$ and all modules X, Y from $\underline{\mathrm{mod}}A$.*

Corollary 1.7. *Let A be a selfinjective algebra. Then $\underline{\mathrm{mod}}A$ is Calabi-Yau of even dimension $2e$ if and only if e is a minimal nonnegative integer such that $\mathrm{Ext}^e_A(X,Y) \cong D\mathrm{Ext}^e_A(Y,X)$ for all modules X, Y from $\underline{\mathrm{mod}}A$.*

2. Selfinjective algebras of stable Calabi-Yau dimension 0 and 1.

In this section we describe the selfinjective algebras of the stable Calabi-Yau dimensions 0 and 1.

Proposition 2.1. *Let A be a selfinjective algebra. Then the following statements are equivalent*:

(i) $\underline{\mathrm{CY}}\mathrm{dim}A = 0$.
(ii) $\underline{\mathrm{Hom}}_A(X,Y) \cong D\underline{\mathrm{Hom}}_A(Y,X)$ *for all modules X, Y from $\underline{\mathrm{mod}}A$.*
(iii) $\underline{\mathrm{Hom}}_A(X,Y) = 0$ *for all nonisomorphic indecomposable modules X, Y from $\underline{\mathrm{mod}}A$.*
(iv) *A is a Nakayama algebra of Loewy length at most* 2.

Proof. The equivalence (i) and (ii) follows from Corollary 1.6. The equivalence (iii) and (iv) follows from the fact that A is a Nakayama algebra of Loewy length at most 2 if and only if every indecomposable A-module is either projective or simple. The implication (iii) $\Rightarrow$ (ii) is trivial. Finally, assume that A is of stable Calabi-Yau dimension 0. Then $\nu_A^{-1} \cong \Omega_A$. Suppose S is a simple nonprojective A-module. Then we have an exact sequence

$$0 \longrightarrow \Omega_A S \longrightarrow P_A(S) \longrightarrow S \longrightarrow 0.$$

On the other hand, $\nu_A^{-1}(S)$ is the socle of the projective cover $P_A(S)$ of S. Hence $\nu_A^{-1}(S) \cong \Omega_A(S)$ implies that $\mathrm{rad}P_A(S) = \mathrm{soc}P_A(S)$, and consequently $P_A(S)$ is uniserial of Loewy length 2. Therefore, A is a Nakayama algebra of Loewy length at most 2. □

Proposition 2.2. *Let A be a connected selfinjective algebra. Then the following statements are equivalent*:

(i) $\underline{\mathrm{CY}}\mathrm{dim}A = 1$.
(ii) *A is isomorphic to a matrix algebra $\mathrm{Mat}_m(K[X]/(X^n))$ for some $m \geq 1$ and $n \geq 3$.*
(iii) *A is Morita equivalent to a local Nakayama algebra $K[X]/(X^n)$ of Loewy length $n \geq 3$.*

Proof. It follows from Corollary 1.3 and Proposition 2.1 that $\underline{\mathrm{CY}}\mathrm{dim}A = 1$ if and only if $\tau_A = 1_{\underline{\mathrm{mod}}A}$ and A is not Nakayama of Loewy length at most 2. The equivalence of (ii) and (iii) follows from the connectedness of A and the known structure of the module category of a local Nakayama algebra $K[X]/(X^n)$. Further, for A Morita equivalent to $K[X]/(X^n)$, we have $\tau_A = 1_{\underline{\mathrm{mod}}A}$, and hence (iii) implies (i). In order to prove the implication (i) $\Rightarrow$ (iii), we may assume that A is basic. Since A has Loewy length at least 3, there is an indecomposable projective A-module P with $\mathrm{rad}P \neq \mathrm{soc}P$. We have an Auslander-Reiten sequence of the form (see [**4**, (V.5.5)])

$$0 \longrightarrow \mathrm{rad}P \longrightarrow (\mathrm{rad}P/\mathrm{soc}P) \oplus P \longrightarrow P/\mathrm{soc}P \longrightarrow 0.$$

Then $\mathrm{rad}P = \tau_A(P/\mathrm{soc}P) \cong P/\mathrm{soc}P$, and consequently we obtain $\mathrm{rad}^i P \cong \mathrm{rad}^{i-1}P/\mathrm{soc}P$ for all $i \geq 1$. This implies that P is uniserial with all composition factors isomorphic to $S = P/\mathrm{rad}P$. Since A is basic and connected, we conclude that $A = P$ and A is a local Nakayama algebra, $A \cong K[X]/(X^n)$ for some $n \geq 3$. This shows (i) $\Rightarrow$ (iii). $\square$

As an immediate consequence of the above propositions we obtain the following corollaries.

Corollary 2.3. *Let A be a selfinjective algebra with $\underline{\mathrm{CY}}\mathrm{dim}A \leq 1$. Then A is representation-finite.*

Corollary 2.4. *Let A be a connected symmetric algebra. Then $\underline{\mathrm{CY}}\mathrm{dim}A \leq 1$ if and only if A is Morita equivalent to $K[X]/(X^n)$ for some $n \geq 1$.*

3. Selfinjective algebras of stable Calabi-Yau dimension 2.

In [**5**] we proved that if Λ is an arbitrary connected selfinjective algebra of stable Calabi-Yau dimension 2 then Λ is a deformed preprojective algebra of generalized Dynkin type. We will now introduce these algebras, recalling [**5**, Section 3], and obtain a classification of all tame selfinjective algebras of stable Calabi-Yau dimension 2.

Let Δ be a generalized Dynkin graph: $\boldsymbol{A}_n (n \geq 1)$, $\boldsymbol{D}_n (n \geq 4)$, $\boldsymbol{E}_n (n = 6,7,8)$, and $\boldsymbol{L}_n (n \geq 1)$. Then the Gabriel quiver $Q_{P(\Delta)}$ of the ordinary preprojective algebra $P(\Delta)$ of type Δ is of the form

$Q_{P(\boldsymbol{A}_n)}$:
$(n \geq 1)$

$$0 \underset{\bar{a}_0}{\overset{a_0}{\rightleftarrows}} 1 \underset{\bar{a}_1}{\overset{a_1}{\rightleftarrows}} 2 \cdots n-2 \underset{\bar{a}_{n-2}}{\overset{a_{n-2}}{\rightleftarrows}} n-1$$

$Q_{P(\boldsymbol{D}_n)}$:
$(n \geq 4)$

$$\begin{tikzcd}
0 \arrow[dr, shift left, "a_0"] & & & & \\
& 2 \arrow[ul, shift left, "\bar{a}_0"] \arrow[dl, shift left, "\bar{a}_1"] \arrow[r, shift left, "a_2"] & 3 \arrow[l, shift left, "\bar{a}_2"] \cdots n-2 \arrow[r, shift left, "a_{n-2}"] & n-1 \arrow[l, shift left, "\bar{a}_{n-2}"] \\
1 \arrow[ur, shift left, "a_1"] & & & &
\end{tikzcd}$$

$Q_{P(\boldsymbol{E}_n)}$:
$(n = 6,7,8)$

$$\begin{tikzcd}
& & 0 \arrow[d, shift left, "a_0"] & & \\
1 \arrow[r, shift left, "a_1"] & 2 \arrow[l, shift left, "\bar{a}_1"] \arrow[r, shift left, "a_2"] & 3 \arrow[l, shift left, "\bar{a}_2"] \arrow[u, shift left, "\bar{a}_0"] \arrow[r, shift left, "a_3"] & 4 \arrow[l, shift left, "\bar{a}_3"] \cdots n-2 \arrow[r, shift left, "a_{n-2}"] & n-1 \arrow[l, shift left, "\bar{a}_{n-2}"]
\end{tikzcd}$$

$$Q_{P(\boldsymbol{L}_n)}: \quad (n \geq 1) \qquad \varepsilon = \bar{\varepsilon} \circlearrowright 0 \underset{\bar{a}_0}{\overset{a_0}{\rightleftarrows}} 1 \underset{\bar{a}_1}{\overset{a_1}{\rightleftarrows}} 2 \;\cdot\;\cdot\;\cdot\; n-2 \underset{\bar{a}_{n-2}}{\overset{a_{n-2}}{\rightleftarrows}} n-1\,.$$

We choose the exceptional vertex in the Gabriel quiver $Q_{P(\Delta)}$ of $P(\Delta)$ as: 0, 2, 3 and 0 if $\Delta = \boldsymbol{A}_n, \boldsymbol{D}_n, \boldsymbol{E}_n$ and $\boldsymbol{L}_n$, respectively. To this exceptional vertex we associate a local algebra $R(\Delta)$ as follows:

$$R(\boldsymbol{A}_n) = K;$$
$$R(\boldsymbol{D}_n) = K\langle x, y\rangle/(x^2, y^2, (x+y)^{n-2});$$
$$R(\boldsymbol{E}_n) = K\langle x, y\rangle/(x^2, y^3, (x+y)^{n-3});$$
$$R(\boldsymbol{L}_n) = K[x]/(x^{2n}).$$

Here, $K\langle x, y\rangle$ denotes the polynomial algebra in two noncommuting variables x and y over K and $K[x]$ the polynomial algebra in one variable x over K. Then $R(\Delta)$ is isomorphic to the finite dimensional local selfinjective algebra $eP(\Delta)e$, where e is the primitive idempotent of $P(\Delta)$ corresponding to the exceptional vertex of $Q = Q_{P(\Delta)}$ (see [**5**, Lemma 3.1]). We call an element f admissible if it belongs to the square of the radical of $R(\Delta)$. Note that $f = 0$ is the unique admissible element of $R(\boldsymbol{A}_n)$. Finally, we denote by ia the starting vertex of an arrow a of the quiver $Q_{P(\Delta)}$.

For an admissible element f of $R(\Delta)$, the deformed preprojective algebra $P^f(\Delta)$ of type Δ, with respect to f, is defined to be the bound quiver algebra KQ/I^f, where $Q = Q_{P(\Delta)}$ and I^f is the ideal in the path algebra KQ of Q generated by the elements of the form

$$\sum_{a,\, ia=v} a\bar{a}, \quad \text{if } v \text{ is an ordinary vertex of } Q,$$

together with the elements:

$a_0\bar{a}_0$	$\Delta = \boldsymbol{A}_n\ (n \geq 1)$;
$\bar{a}_0 a_0 + \bar{a}_1 a_1 + a_2\bar{a}_2 + f(\bar{a}_0 a_0, \bar{a}_1 a_1), (\bar{a}_0 a_0 + \bar{a}_1 a_1)^{n-2}$	$\Delta = \boldsymbol{D}_n\ (n \geq 4)$;
$\bar{a}_0 a_0 + \bar{a}_2 a_2 + a_3\bar{a}_3 + f(\bar{a}_0 a_0, \bar{a}_2 a_2), (\bar{a}_0 a_0 + \bar{a}_2 a_2)^{n-3}$	$\Delta = \boldsymbol{E}_n\ (n = 6, 7, 8)$;
$\varepsilon^2 + a_0\bar{a}_0 + \varepsilon f(\varepsilon),\ \varepsilon^{2n}$	$\Delta = \boldsymbol{L}_n\ (n \geq 1)$.

Hence, for $f = 0$ we have $P^f(\Delta)$ is $P(\Delta)$, the ordinary preprojective algebra of type Δ. The following theorem, proved in [**5**, Theorem 1.1], describes the basic homological properties of the algebras $P^f(\Delta)$.

THEOREM 3.1. *Let $\Lambda = P^f(\Delta)$ be a deformed preprojective algebra of a generalized Dynkin type Δ. Then the following statements hold.*

(i) *Λ is a finite dimensional selfinjective algebra with* $\dim_K \Lambda = \dim_K P(\Delta)$.

(ii) *The Nakayama permutation of Λ is the identity for $\Delta = \boldsymbol{A}_1$, $\boldsymbol{D}_n$ (n even), $\boldsymbol{E}_7$, $\boldsymbol{E}_8$, $\boldsymbol{L}_n$, and of order 2 for $\Delta = \boldsymbol{A}_n$ ($n \geq 2$), $\boldsymbol{D}_n$ (n odd), $\boldsymbol{E}_6$.*
(iii) *$\Omega^3_{\Lambda^e}(\Lambda) \cong {}_1\Lambda_\varrho$ for an automorphism ϱ of Λ of finite order.*
(iv) *There is a positive integer $m = m_\Lambda$ such that $\Omega^{3m}_\Lambda(M) \cong M$ for any nonprojective indecomposable module M in* $\mathrm{mod}\Lambda$.

The most relevant for this paper is the following theorem proved in [**5**, Theorem 1.2].

THEOREM 3.2. *Let Λ be a basic, connected finite dimensional selfinjective algebra. Then the following statements are equivalent*:

(i) *Λ is isomorphic to a deformed preprojective algebra $P^f(\Delta)$ of a generalized Dynkin type Δ.*
(ii) *$\Omega^3_\Lambda S \cong \nu_\Lambda^{-1} S$ for any nonprojective simple Λ-module S.*

As an immediate consequence of Corollary 1.3 and Theorem 3.2 we obtain the following fact.

COROLLARY 3.3. *Let A be a connected selfinjective algebra of stable Calabi-Yau dimension 2. Then A is Morita equivalent to a deformed preprojective algebra $P^f(\Delta)$ of a generalized Dynkin type Δ.*

We note that, by Propositions 2.1 and 2.2, for a deformed preprojective algebra $\Lambda = P^f(\Delta)$ of a generalized Dynkin type Δ, we have $\underline{\mathrm{CY}}\mathrm{dim}\Lambda \leq 1$ if and only if Λ is the ordinary preprojective algebra $P(\boldsymbol{A}_1)$, $P(\boldsymbol{A}_2)$, or $P(\boldsymbol{L}_1)$. Moreover, $P(\boldsymbol{A}_1)$, $P(\boldsymbol{A}_2)$ and $P(\boldsymbol{L}_1)$ have stable Calabi-Yau dimension 0. We do not know whether any deformed preprojective algebra $P^f(\Delta)$ of generalized Dynkin type Δ other than $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{L}_1$ has stable Calabi-Yau dimension 2. On the other hand, we have the following direct consequence of [**3**, (3.1)–(3.3)] and [**5**, (2.6)].

PROPOSITION 3.4. *Let $\Lambda = P(\Delta)$ be a preprojective algebra of a generalized Dynkin type $\Delta \neq \boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{L}_1$. Then $\underline{\mathrm{CY}}\mathrm{dim}\Lambda = 2$.*

To identify the tame algebras $P^f(\Delta)$ we need two preliminary lemmas.

LEMMA 3.5. *Let D be the bound quiver algebra $K\Sigma/J$ where Σ is the quiver*

$$\begin{tikzcd}
\bullet \arrow[dr, "\varphi"'] & & \bullet \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & & \bullet \arrow[dl, "\xi"'] \arrow[dr, "\varrho"] & \\
& \bullet \arrow[dr, "\beta"'] & & \bullet \arrow[dl, "\sigma"'] \arrow[dr, "\eta"'] & & \bullet \arrow[dl, "\omega"'] \\
& & \bullet & & \bullet &
\end{tikzcd}$$

and J is the ideal in the path algebra $K\Sigma$ of Σ generated by $\alpha\beta - \gamma\sigma$ and $\xi\eta - \varrho\omega$. Then D is wild.

PROOF. Let C be the bound quiver algebra $K\Sigma'/J'$, where Σ' is the full subquiver of Σ given by all arrows and vertices of Σ except φ and its starting vertex, and J' is the ideal in $K\Sigma'$ generated by $\alpha\beta - \gamma\sigma$ and $\xi\eta - \varrho\omega$. Observe that C is a tame concealed algebra of Euclidean type $\widetilde{\boldsymbol{D}}_6$ (see [**34**]) and D is the one-point extension $C[P]$ of C by the indecomposable projective C-module P of dimension 3 whose top is the simple C-module corresponding to the starting vertex of β. Then $D = C[P]$ is wild, by [**45**, Lemma 2.5.3], because the projective C-modules are preprojective. □

Lemma 3.6. *Let E be the bound quiver algebra $K\Theta/L$ where Θ is the quiver*

$$\begin{tikzcd}
 & \bullet \arrow[dl, "\alpha"'] \arrow[d, "\gamma"] \arrow[dr, "\xi"] & & \bullet \arrow[dl, "\varphi"] \\
\bullet \arrow[dr, "\beta"'] & \bullet \arrow[d, "\sigma"] & \bullet \arrow[dl, "\eta"] & \\
 & \bullet & &
\end{tikzcd}$$

and L is the ideal in the path algebra $K\Theta$ of Θ generated by $\alpha\beta + \gamma\sigma + \xi\eta$. Then E is wild.

Proof. Similarly as above, E is the one-point extension $F[R]$ of the tame concealed algebra $F = K\Theta'/L'$ of Euclidean type $\widetilde{\boldsymbol{D}}_4$, for the subquiver Θ' of Θ given by all arrows and vertices except φ and its source, L' is generated by $\alpha\beta + \gamma\sigma + \xi\eta$, and R is the indecomposable projective F-module whose top is the simple module given by the starting vertex of η. Therefore, E is (as above) wild. □

Denote by $P^*(\boldsymbol{D}_4)$ the deformed preprojective algebra $P^f(\boldsymbol{D}_4)$ with $f = xy$, and by $P^*(\boldsymbol{L}_2)$ the deformed preprojective algebra $P^f(\boldsymbol{L}_2)$ with $f = \varepsilon^2$. Observe that $P^*(\boldsymbol{D}_4)$ is a socle deformation of $P(\boldsymbol{D}_4)$ and $P^*(\boldsymbol{L}_2)$ is a socle deformation of $P(\boldsymbol{L}_2)$. The following theorem shows that there are only very few tame deformed preprojective algebras of generalized Dynkin type.

Theorem 3.7. *Let $\Lambda = P^f(\Delta)$ be a deformed preprojective algebra of a generalized Dynkin type Δ. The following statements are equivalent*:

(i) *Λ is tame.*
(ii) *Λ is isomorphic to one of the algebras*:
   (1) $P(\boldsymbol{A}_n)$, $1 \leq n \leq 5$;
   (2) $P(\boldsymbol{D}_4)$;
   (3) $P(\boldsymbol{L}_n)$, $1 \leq n \leq 2$;
   (4) $P^*(\boldsymbol{D}_4)$, $P^*(\boldsymbol{L}_2)$, *for K of characteristic* 2.

Proof. We will analyse the representation type of the algebra $\Lambda = P^f(\Delta)$ case by case.

(a) $\Delta = \boldsymbol{A}_n$. Then $P^f(\Delta) = P(\Delta)$, and it is well known that $P(\boldsymbol{A}_n)$ is wild for $n \geq 6$. In fact, for $n \geq 6$, $P(\boldsymbol{A}_n)$ is a factor algebra of $P(\boldsymbol{A}_6)$, and the universal Galois covering $\widetilde{P(\boldsymbol{A}_6)}$ of $P(\boldsymbol{A}_6)$ contains, as a convex subalgebra, the wild algebra Σ described in Lemma 3.5. Then, invoking [**16**, Proposition 2], we conclude that $P(\boldsymbol{A}_6)$ is wild. Clearly, then all algebras $P(\boldsymbol{A}_n)$, $n \geq 6$, are wild. Further, $P(\boldsymbol{A}_1) = K$, $P(\boldsymbol{A}_2)$ is a

Nakayama algebra of Loewy length 2, $P(\boldsymbol{A}_3)$ is a representation-finite (special biserial) selfinjective algebra of Dynkin type $\boldsymbol{A}_3$, and $P(\boldsymbol{A}_4)$ is a representation-finite selfinjective algebra of Dynkin type $\boldsymbol{D}_6$ (see [**42**], [**43**]). Finally, $P(\boldsymbol{A}_5)$ is a representation-infinite tame selfinjective algebra of tubular type, isomorphic to the algebra A_{30} in [**10**, p. 696]. In particular, the algebras $P(\boldsymbol{A}_n)$, $1 \leq n \leq 5$, are tame.

(b) $\Delta = \boldsymbol{D}_n$. Let $P(\boldsymbol{D}_5)'$ be the factor algebra of $P(\boldsymbol{D}_5)$ by the ideal $\operatorname{rad}^2 P(\boldsymbol{D}_5) e \operatorname{rad}^2 P(\boldsymbol{D}_5)$, where e is the primitive idempotent of $P(\boldsymbol{D}_5)$ corresponding to the exceptional vertex of $Q_{P(\boldsymbol{D}_5)}$. Then the universal Galois covering $\widetilde{P(\boldsymbol{D}_5)}'$ of $P(\boldsymbol{D}_5)'$ contains, as a convex subalgebra, the wild algebra E described in Lemma 3.6, and consequently $P(\boldsymbol{D}_5)'$ is wild. Further, it is easy to see that $P(\boldsymbol{D}_5)'$ is a factor algebra of any deformed preprojective algebra $P^f(\boldsymbol{D}_n)$ with $n \geq 5$. Hence, the algebras $P^f(\boldsymbol{D}_n)$, $n \geq 5$, are wild. Finally, consider a deformed preprojective algebra $\Lambda = P^f(\boldsymbol{D}_4)$ of type $\boldsymbol{D}_4$ with f nonzero. Any such algebra Λ is a socle deformation of the preprojective algebra $P(\boldsymbol{D}_4)$ since the square of the radical of the local algebra $R(\boldsymbol{D}_4)$ at the exceptional vertex is already in the socle. It was proved in [**10**, Lemma 5.10] that there is, up to isomorphism, exactly one selfinjective algebra socle equivalent but not isomorphic to $P(\boldsymbol{D}_4)$, and then necessarily the field K has characteristic 2. Hence, one can take $P^*(\boldsymbol{D}_4)$. Moreover, $P(\boldsymbol{D}_4)$ is a representation-infinite tame selfinjective (even symmetric) algebra of tubular type $(3,3,3)$ isomorphic to the algebra A_3 in [**9**, Theorem]. Hence, the algebras $P(\boldsymbol{D}_4)$ and $P^*(\boldsymbol{D}_4)$ are tame.

(c) $\Delta = \boldsymbol{E}_n$. Then any deformed projective algebra $P^f(\boldsymbol{E}_n)$, $n = 6,7,8$, admits a factor algebra isomorphic to the algebra $P(\boldsymbol{D}_5)'$ considered in (b), and hence is wild.

(d) $\Delta = \boldsymbol{L}_n$. Let $P(\boldsymbol{L}_3)'$ be the factor algebra of $P(\boldsymbol{L}_3)$ by the ideal generated by ε^2. Then the universal Galois covering $\widetilde{P(\boldsymbol{L}_3)}'$ of $P(\boldsymbol{L}_3)'$ contains, as a convex subalgebra, the wild algebra D described in Lemma 3.5, and consequently $P(\boldsymbol{L}_3)'$ is wild. Further, any deformed preprojective algebra $P^f(\boldsymbol{L}_n)$ with $n \geq 3$ has a factor algebra isomorphic to $P(\boldsymbol{L}_3)'$, and hence is wild. Observe also that the preprojective algebra $P(\boldsymbol{L}_1) = K[\varepsilon]/(\varepsilon^2)$ is the unique deformed preprojective algebra of type $\boldsymbol{L}_1$. Finally, consider a deformed preprojective algebra $\Lambda = P^f(\boldsymbol{L}_2)$ of type $\boldsymbol{L}_2$ with f nonzero. Then any such algebra Λ is a socle deformation of the preprojective algebra $P(\boldsymbol{L}_2)$. Moreover, it follows from [**44**] (see also [**52**]) that $P(\boldsymbol{L}_2)$ is a representation-finite selfinjective (even symmetric) algebra of Dynkin type $\boldsymbol{D}_6$, and there is, up to isomorphism, exactly one selfinjective algebra socle equivalent but not isomorphic to $P(\boldsymbol{D}_4)$, and then necessarily the field K is of characteristic 2. Hence, one can take $P^*(\boldsymbol{D}_4)$. In particular, $P(\boldsymbol{L}_2)$ and $P^*(\boldsymbol{L}_2)$ are representation-finite, hence tame, algebras.

This finishes the proof. □

We may now present the classification of all connected tame selfinjective algebras of stable Calabi-Yau dimension 2.

THEOREM 3.8. *Let A be a connected selfinjective algebra. The following statements are equivalent*:

(i) *A is tame and $\underline{\mathrm{CY}}\dim A = 2$.*
(ii) *A is Morita equivalent to one of the algebras $P(\boldsymbol{A}_3)$, $P(\boldsymbol{A}_4)$, $P(\boldsymbol{A}_5)$, $P(\boldsymbol{D}_4)$, $P(\boldsymbol{L}_2)$, or to $P^*(\boldsymbol{D}_4)$, $P^*(\boldsymbol{L}_2)$ if K has characteristic 2.*

Proof. We proved in [**5**, Example 3.6] that $\Omega^3 M \cong M \cong \nu^{-1} M$ for any nonprojective indecomposable module M in mod$P^*(\boldsymbol{D}_4)$, and hence $\underline{\mathrm{CY}}\mathrm{dim}P^*(\boldsymbol{D}_4) = 2$. Further, $P^*(\boldsymbol{L}_2)$ is a symmetric representation-finite algebra whose stable Auslander-Reiten quiver is isomorphic to $\boldsymbol{Z}\boldsymbol{D}_6/(\tau^3)$. Moreover, a direct checking shows that $\Omega_{P^*(\boldsymbol{L}_2)} \cong \tau^2_{P^*(\boldsymbol{L}_2)}$ on $\underline{\mathrm{mod}}P^*(\boldsymbol{L}_2)$, and hence $\Omega^3_{P^*(\boldsymbol{L}_2)} \cong 1_{\underline{\mathrm{mod}}P^*(\boldsymbol{L}_2)}$. This shows that $\underline{\mathrm{CY}}\mathrm{dim}P^*(\boldsymbol{L}_2) = 2$ (see also Theorem 4.3). Now, the theorem is a consequence of Propositions 2.1, 2.2, 3.4, Corollary 3.3 and Theorem 3.7. □

Corollary 3.9. *Let A be a connected selfinjective algebra. The following statements are equivalent*:

(i) *A is tame, symmetric and $\underline{\mathrm{CY}}\mathrm{dim}A = 2$.*
(ii) *A is Morita equivalent to one of the algebras $P(\boldsymbol{D}_4)$, $P(\boldsymbol{L}_2)$, or $P^*(\boldsymbol{D}_4)$, $P^*(\boldsymbol{L}_2)$ if K is of characteristic* 2.

Corollary 3.10. *The algebras $P(\boldsymbol{A}_3)$, $P(\boldsymbol{A}_4)$, $P(\boldsymbol{L}_2)$, $P^*(\boldsymbol{L}_2)$ (if* char$K = 2$) *are unique (up to Morita equivalence) connected representation-finite selfinjective algebras of stable Calabi-Yau dimension* 2.

Corollary 3.11. *The algebras $P(\boldsymbol{A}_5)$, $P(\boldsymbol{D}_4)$, and $P^*(\boldsymbol{D}_4)$ (if* char$K = 2$) *are unique (up to Morita equivalence) connected representation-infinite tame selfinjective algebras of stable Calabi-Yau dimension* 2.

4. Symmetric algebras of finite type.

In this section we describe the stable Calabi-Yau dimension of all connected symmetric algebras of finite representation type.

An important class of selfinjective algebras is formed by the orbit algebras $\widehat{B}/G$, where $\widehat{B}$ is the repetitive algebra of B (see [**33**]), locally bounded without identity, and G is an admissible group of automorphisms of $\widehat{B}$. Recall that

$$\widehat{B} = \bigoplus_{k \in \boldsymbol{Z}} (B_k \oplus D(B)_k)$$

with $B_k = B$ and $D(B)_k = D(B)$ for all $k \in \boldsymbol{Z}$ and the multiplication in $\widehat{B}$ is defined by

$$(a_k, f_k) \cdot (b_k, g_k) = (a_k b_k, a_k g_k + f_k b_{k-1})_k$$

for $a_k, b_k \in B_k$, $f_k, g_k \in D(B)_k$. For a fixed set $\mathscr{E} = \{e_i | 1 \le i \le n\}$ of orthogonal primitive idempotents of B with $1_B = e_1 + \cdots + e_n$, consider the canonical set $\widehat{\mathscr{E}} = \{e_{j,k} | 1 \le j \le n, k \in \boldsymbol{Z}\}$ of orthogonal primitive idempotents of $\widehat{B}$ such that $1_{B_k} = e_{1,k} + \cdots + e_{n,k}$. By an automorphism of $\widehat{B}$ we mean a K-algebra automorphism of $\widehat{B}$ which fixes the chosen set $\widehat{\mathscr{E}}$ of orthogonal primitive idempotents of $\widehat{B}$. A group G of automorphisms of $\widehat{B}$ is said to be admissible if the induced action of G on $\widehat{\mathscr{E}}$ is free and has finitely many orbits. Then the orbit algebra $\widehat{B}/G$ (see [**25**]) is a selfinjective algebra and the G-orbits in $\widehat{\mathscr{E}}$ form a canonical set of orthogonal primitive idempotents of $\widehat{B}/G$ whose sum is the identity of $\widehat{B}/G$. We denote by $\nu_{\widehat{B}}$ the Nakayama automorphism of $\widehat{B}$ whose restriction to each copy $B_k \oplus D(B)_k$ is the identity map $B_k \oplus D(B)_k \to B_{k+1} \oplus D(B)_{k+1}$. Then the infinite cyclic group $(\nu_{\widehat{B}})$ generated by $\nu_{\widehat{B}}$ is admissible and $\widehat{B}/(\nu_{\widehat{B}})$ is the trivial extension $\mathrm{T}(B) = B \ltimes D(B)$ of B by $D(B)$, and it is a symmetric algebra. An automorphism φ of $\widehat{B}$ is said to be positive (respectively, rigid) if $\varphi(B_k) \subseteq \sum_{i \geq k}(B_i)$ (respectively, $\varphi(B_k) = B_k$) for any $k \in \boldsymbol{Z}$. Moreover, φ is said to be strictly positive if φ is positive but not rigid.

Let Δ be a Dynkin graph $\boldsymbol{A}_n (n \geq 1)$, $\boldsymbol{D}_n (n \geq 4)$, $\boldsymbol{E}_n (n = 6, 7, 8)$. By a tilted algebra of Dynkin type Δ we mean an algebra $B = \mathrm{End}_H(T)$ where H is the path algebra $K\vec{\Delta}$ of a quiver $\vec{\Delta}$ with the underlying graph Δ and T is a multiplicity-free tilting H-module. Then a selfinjective algebra of Dynkin type Δ is defined to be an algebra of the form $\widehat{B}/G$, where B is a tilted algebra of type Δ and G is an admissible group of automorphisms of $\widehat{B}$. It is known that such an admissible group G is always infinite cyclic generated by a strictly positive automorphism of $\widehat{B}$. By general theory (see [**33**], [**25**], [**42**]) every selfinjective algebra $A = \widehat{B}/G$ of Dynkin type Δ is of finite representation type and its stable Auslander-Reiten quiver Γ_A^s is the translation quiver $\boldsymbol{Z}\vec{\Delta}/G$, for an (arbitrarily chosen) orientation $\vec{\Delta}$ of Δ. In fact, the selfinjective algebras of Dynkin type exhaust all basic connected selfinjective algebras of finite representation type having simply connected Galois coverings. Moreover, the remaining (nonstandard) selfinjective algebras of finite representation type are socle deformations of selfinjective algebras of Dynkin type (see [**14**], [**33**], [**43**], [**44**], [**52**], [**53**]).

In particular, we have the following description of the symmetric algebras of finite representation type.

THEOREM 4.1. *Let A be a basic connected selfinjective algebra nonisomorphic to K. Then A is symmetric of finite representation type if and only if A is isomorphic to an algebra of one of the following types*:

- (i) $\mathrm{T}(B)$, *for a tilted algebra B of Dynkin type $\boldsymbol{A}_n$, $\boldsymbol{D}_n$, $\boldsymbol{E}_6$, $\boldsymbol{E}_7$, or $\boldsymbol{E}_8$.*
- (ii) $\widehat{B}/(\varphi)$, *where B is a tilted algebra of Dynkin type $\boldsymbol{A}_n$ and φ is a proper root of the Nakayama automorphism $\nu_{\widehat{B}}$.*
- (iii) *a socle deformation of an algebra $\widehat{B}/(\varphi)$, where B is a tilted algebra of Dynkin type $\boldsymbol{D}_{3s}$ and φ is a root of order 3 of the Nakayama automorphism $\nu_{\widehat{B}}$.*

It is known that the class of algebras described in (ii) coincides with the class of Brauer tree algebras $A(T_S^m)$ given by Brauer trees T_S^m with $e \geq 1$ edges and one exceptional vertex S of multiplicity $m \geq 2$, and then the Dynkin type $\boldsymbol{A}_n$ is given by $n = m \cdot e$. In fact, the class of the trivial extension algebras $\mathrm{T}(B)$ of tilted algebras B type $\boldsymbol{A}_n$ coincides with the class of Brauer tree algebras $A(T_S^1) = A(T)$ of Brauer trees T with $n = e$ edges and the exceptional vertex S of multiplicity $m = 1$ (see [**33**], [**43**]). Further, the class of tilted algebras of type $\widehat{B}/(\varphi)$, with B tilted of type $\boldsymbol{D}_{3m}$ and φ a cube root of $\nu_{\widehat{B}}$, coincides with the class of modified Brauer tree algebras $D(T_S)$ given by Brauer trees with $e \geq 2$ edges and an extreme exceptional vertex S, and then $m = e$. Moreover, for such an algebra $D(T_S)$, there exists exactly one (up to isomorphism) symmetric socle deformation $D(T_S)'$ with $D(T_S) \ncong D(T_S)'$ only for K of characteristic 2 (see [**44**], [**52**], [**53**]).

For a Dynkin graph Δ, denote by h_Δ the Coxeter number of Δ. Recall that h_Δ is the order of the Coxeter element of the Coxeter (Weyl) group of Δ, so that

$$h_{\boldsymbol{A}_n} = n+1, \quad h_{\boldsymbol{D}_n} = 2n-2, \quad h_{\boldsymbol{E}_6} = 12, \quad h_{\boldsymbol{E}_7} = 18, \quad h_{\boldsymbol{E}_8} = 30.$$

Moreover, we put $m_\Delta = h_\Delta - 1$.

The following proposition describes the action of the syzygy functors $\Omega_{\widehat{B}}$ on the stable module categories $\underline{\mathrm{mod}}\widehat{B}$ of the repetitive algebras $\widehat{B}$ of tilted algebras B of Dynkin types (see also [**13**]).

Proposition 4.2. *Let B be a tilted algebra of Dynkin type Δ. Then we have equivalences of functors on the category $\underline{\mathrm{mod}}\widehat{B}$:*

(i) $\Omega_{\widehat{B}} \cong \tau_{\widehat{B}}^{h_\Delta/2}$, *for $\Delta = \boldsymbol{A}_1$, $\boldsymbol{D}_n$ (n even), $\boldsymbol{E}_7$, $\boldsymbol{E}_8$.*
(ii) $\Omega_{\widehat{B}} \cong \sigma\tau_{\widehat{B}}^{h_\Delta/2}$, *for $\Delta = \boldsymbol{A}_n$ ($n \geq 3$ odd), $\boldsymbol{D}_n$ (n odd), $\boldsymbol{E}_6$ and an automorphism σ of order 2.*
(iii) $\Omega_{\widehat{B}} \cong \varrho\tau_{\widehat{B}}^{m_\Delta/2}$, *for $\Delta = \boldsymbol{A}_n$ (n even) and an automorphism ϱ with $\varrho^2 = \tau_{\widehat{B}}$.*

Proof. It is known (see [**51**]) that $\underline{\mathrm{mod}}\widehat{B} \cong \underline{\mathrm{mod}}\widehat{H}$ for the path algebra $H = K\vec{\Delta}$ of the Dynkin quiver $\vec{\Delta}$ given by a bipartite (sink-source) orientation of Δ. Moreover, any equivalence $\underline{\mathrm{mod}}\widehat{B} \cong \underline{\mathrm{mod}}\widehat{H}$ commutes with the syzygy and Auslander-Reiten functors (see [**4**, Section X]). Then the required equivalences follow by direct checking of the action of $\Omega_{\widehat{H}}$ on the indecomposable nonprojective modules over the radical cube zero selfinjective algebra $\widehat{H}$. We note that (in this case) it is sufficient to check the action of $\Omega_{\widehat{H}}$ on the simple $\widehat{H}$-modules. □

We will determine now the stable Calabi-Yau dimension of the symmetric algebras of finite representation type.

Theorem 4.3. *Let A be a basic connected symmetric algebra of finite representation type. Assume A is a socle deformation of a selfinjective algebra $\widehat{B}/(\varphi)$, where B is a tilted algebra of Dynkin type Δ and φ is a root of order r of the Nakayama automorphism $\nu_{\widehat{B}}$. Then*

$$\underline{\mathrm{CY}}\mathrm{dim} A = \begin{cases} \dfrac{m_\Delta}{r} - 1 & \text{for } \Delta = \boldsymbol{A}_1,\ \boldsymbol{D}_n\ (n \text{ even}),\ \boldsymbol{E}_7,\ \boldsymbol{E}_8. \\[2ex] \dfrac{2m_\Delta}{r} - 1 & \text{for } \Delta = \boldsymbol{A}_n\ (n \geq 2),\ \boldsymbol{D}_n\ (n \text{ odd}),\ \boldsymbol{E}_6. \end{cases}$$

Proof. We assume first that $A = \widehat{B}/(\varphi)$ and that A is a symmetric algebra of Dynkin type. Then we have a canonical Galois covering $F : \widehat{B} \to \widehat{B}/G = A$ with Galois group $G = (\varphi)$ and the associated push-down functor $F_\lambda : \mathrm{mod}\widehat{B} \to \mathrm{mod}A$. By general theory (see [**25**], [**33**]) the functor F_λ is dense and $\Omega_A F_\lambda \cong F_\lambda \Omega_{\widehat{B}}$, $\tau_A F_\lambda \cong F_\lambda \tau_{\widehat{B}}$, as functor on $\underline{\mathrm{mod}}\widehat{B}$. We have several cases to consider.

(1) Assume $A = \mathrm{T}(B)$. Hence $\varphi = \nu_{\widehat{B}}$ and $r = 1$. Then A is stably equivalent to the trivial extension $\mathrm{T}(H)$ of the path algebra $H = K\vec{\Delta}$ of a bipartite oriented quiver $\vec{\Delta}$

of type Δ, and consequently m_Δ is the order of τ_A on $\underline{\mathrm{mod}}A$ (see [**49**], [**50**], [**27**]).

Let Δ be one of the graphs $\boldsymbol{A}_1, \boldsymbol{D}_n$ (n even), $\boldsymbol{E}_7$, or $\boldsymbol{E}_8$. It follows from Proposition 4.2 that $\Omega_A \cong \tau_A^{h_\Delta/2}$ on $\underline{\mathrm{mod}}A$. Let d be a natural number such that $\Omega_A^d \cong 1_{\underline{\mathrm{mod}}A}$. Then $\tau_A^{dh_\Delta/2} \cong 1_{\underline{\mathrm{mod}}A}$, and hence $dh_\Delta/2 = d(m_\Delta+1)/2$ is divisible by m_Δ. Since $\gcd((m_\Delta+1)/2, m_\Delta) = 1$, we obtain $m_\Delta | d$. On the other hand, $\nu_A \cong 1_{\underline{\mathrm{mod}}A}$ and $\Omega_A^{m_\Delta} \cong \tau_A^{m_\Delta h_\Delta/2} \cong 1_{\underline{\mathrm{mod}}A}$. Then m_Δ is the minimal nonnegative integer such that $\nu_A \cong \Omega_A^{-m_\Delta}$, and consequently $\underline{\mathrm{CY}}\mathrm{dim}\mathrm{T}(B) = m_\Delta - 1$.

Let Δ be one of the graphs $\boldsymbol{A}_n$ ($n \geq 3$ odd), $\boldsymbol{D}_n$ (n odd), or $\boldsymbol{E}_6$. Let d be a natural number such that $\Omega_A^d \cong 1_{\underline{\mathrm{mod}}A}$. Then we have $\Omega_{\widehat{B}}^d \cong \tau_{\widehat{B}}^{lm_\Delta}$ for some natural number $l \geq 1$. Invoking Proposition 4.2, we obtain $\sigma^d \tau_{\widehat{B}}^{dh_\Delta/2} \cong \tau_{\widehat{B}}^{lm_\Delta}$ for an automorphism σ of order 2, and consequently d is even. Let $d = 2e$. Then $\tau_{\widehat{B}}^{eh_\Delta} \cong \tau_{\widehat{B}}^{lm_\Delta}$. Since $\gcd(m_\Delta, h_\Delta) = 1$, we conclude that $m_\Delta | e$. Clearly, $\Omega_A^{2m_\Delta} \cong \tau_A^{m_\Delta} \cong 1_{\underline{\mathrm{mod}}A}$. Therefore, we have $\underline{\mathrm{CY}}\mathrm{dim}\mathrm{T}(B) = 2m_\Delta - 1$.

Finally, assume that $\Delta = \boldsymbol{A}_n (n$ even$)$. Let d be a natural number such that $\Omega_A^d \cong 1_{\underline{\mathrm{mod}}A}$. Then $\Omega_{\widehat{B}}^d \cong \tau_{\widehat{B}}^{lm_\Delta}$ for some $l \geq 1$, and, applying Proposition 4.2, we conclude that $\varrho^d \tau_{\widehat{B}}^{dm_\Delta/2} \cong \tau_{\widehat{B}}^{lm_\Delta}$ for some natural number $l \geq 1$, and an automorphism ϱ of $\widehat{B}$ with $\varrho^2 = \tau_{\widehat{B}}$. Then d is even, say $d = 2e$, and we obtain $\tau_{\widehat{B}}^{eh_\Delta} = \tau_{\widehat{B}}^{e}\tau_{\widehat{B}}^{em_\Delta} \cong \tau_{\widehat{B}}^{lm_\Delta}$. Again, $\gcd(m_\Delta, h_\Delta) = 1$ forces $m_\Delta | e$. Since $\Omega_A^{2m_\Delta} \cong \tau_A^{m_\Delta} \cong 1_{\underline{\mathrm{mod}}A}$, we conclude that $\underline{\mathrm{CY}}\mathrm{dim}\mathrm{T}(B) = 2m_\Delta - 1$.

(2) Assume $A = \widehat{B}/(\varphi)$, where B is a tilted algebra of type $\boldsymbol{A}_n$ and φ is an automorphism of $\widehat{B}$ with $\varphi^r = \nu_{\widehat{B}}$ for some $r \geq 2$. Then the Galois covering $F : \widehat{B} \to \widehat{B}/(\varphi) = A$ is the composition of the canonical Galois covering $\widehat{B} \to \widehat{B}(\nu_{\widehat{B}}) = \mathrm{T}(B)$ with Galois group $(\nu_{\widehat{B}})$ and a Galois covering $\mathrm{T}(B) \to A$ with Galois group cyclic of order r. In particular, we have $r|n$, $n = m_\Delta$, and $\tau_A^{m_\Delta/r} \cong 1_{\underline{\mathrm{mod}}A}$. Using arguments as in (1) and applying Proposition 4.2, we conclude that $d = 2m_\Delta/r$ is the minimal nonnegative integer with $\Omega_A^d \cong 1_{\underline{\mathrm{mod}}A}$, and consequently $\underline{\mathrm{CY}}\mathrm{dim}\widehat{B}/(\varphi) = \frac{2m_\Delta}{r} - 1$.

(3) Assume $A = \widehat{B}/(\varphi)$, where B is a tilted algebra of type $\boldsymbol{D}_{3s}$ and φ is an automorphism of $\widehat{B}$ with $\varphi^3 = \nu_{\widehat{B}}$. Then as above the Galois covering $F : \widehat{B} \to \widehat{B}/(\varphi) = A$ is the composition of the canonical Galois covering $\widehat{B} \to \widehat{B}/(\nu_{\widehat{B}}) = \mathrm{T}(B)$ with Galois group $(\nu_{\widehat{B}})$ and a Galois covering $\mathrm{T}(B) \to A$ whose Galois group is cyclic of order 3. Observe also that $m_{\boldsymbol{D}_{3s}} = 2(3s-1) - 1 = 3(2s-1)$ and hence $\frac{1}{3}m_{\boldsymbol{D}_{3s}} = 2s - 1$. In particular, we obtain $\tau_A^{m_\Delta/3} = \tau_A^{2s-1} \cong 1_{\underline{\mathrm{mod}}A}$. Let d be a minimal nonnegative integer such that $\Omega_A^d \cong 1_{\underline{\mathrm{mod}}A}$. By arguments as above and by Proposition 4.2, we conclude that $d = m_\Delta/3$ for s even and $d = 2m_\Delta/3$ for s odd, because $3s$ and s have the same parity. Therefore, $\underline{\mathrm{CY}}\mathrm{dim}A = m_\Delta/3 - 1$ if $\Delta = \boldsymbol{D}_{3s}$ with s even, and $\underline{\mathrm{CY}}\mathrm{dim}A = 2m_\Delta/3 - 1$ if $\Delta = \boldsymbol{D}_{3s}$ with s odd.

(4) Assume A is not a symmetric algebra of Dynkin type. Then, by Theorem 4.1, A is a socle deformation of $\Lambda = \widehat{B}/(\varphi)$, where B is a tilted algebra of Dynkin type $\boldsymbol{D}_{3s}$ and φ is an automorphism of $\widehat{B}$ with $\varphi^3 = \nu_{\widehat{B}}$. In fact, since $A \not\cong \Lambda$, the field K has characteristic 2. Observe also that $A/\mathrm{soc}A \cong \Lambda/\mathrm{soc}\Lambda$ forces $\Gamma_A \cong \Gamma_\Lambda$. In particular, $m_\Delta/3$ is the order of τ_A and τ_Λ.

Moreover, it follows from [**33**] and [**42**] that the full subcategory $\mathrm{ind}\widehat{B}$ of $\mathrm{mod}\widehat{B}$ formed by chosen representations of indecomposable modules is equivalent to the meshcategory $K(\widetilde{\Gamma}_\Lambda) \cong K(\widetilde{\Gamma}_A)$ of the universal cover $\widetilde{\Gamma}_\Lambda \cong \widetilde{\Gamma}_A$ of $\Gamma_\Lambda \cong \Gamma_A$. Further we have the push-down functor

$$F_\lambda : \mathrm{mod}\widehat{B} \to \mathrm{mod}\Lambda$$

associated to the canonical Galois covering $F : \widehat{B} \to \widehat{B}/(\varphi) = \Lambda$ and a well-behaved functor (see [**12**, (3.1)], [**43**, (1.5)])

$$F^* : \mathrm{mod}\widehat{B} \to \mathrm{mod}A,$$

and the both functors F_λ and F^* are dense, exact, and preserve indecomposable modules, projective-injective modules and Auslander-Reiten sequences. Therefore, we have the equivalences of functors on $\underline{\mathrm{mod}}\widehat{B}$

$$F_\lambda \Omega_{\widehat{B}} \cong \Omega_\Lambda F_\lambda, \quad F_\lambda \tau_{\widehat{B}} \cong \tau_\Lambda F_\lambda$$

and

$$F^* \Omega_{\widehat{B}} \cong \Omega_A F^*, \quad F^* \tau_{\widehat{B}} \cong \tau_A F^*.$$

Therefore, we obtain $\underline{\mathrm{CY}}\mathrm{dim}A = \underline{\mathrm{CY}}\mathrm{dim}\Lambda$, and the required formula for $\underline{\mathrm{CY}}\mathrm{dim}A$ follows from (3). □

COROLLARY 4.4. *Let d be a natural number. Then there exists a connected symmetric algebra A of finite representation type with $\underline{\mathrm{CY}}\mathrm{dim}A = d$.*

PROOF. It follows from Propositions 2.1 and 2.2 that $\underline{\mathrm{CY}}\mathrm{dim}K[X]/(X^2) = 0$ and $\underline{\mathrm{CY}}\mathrm{dim}K[X]/(X^3) = 1$. Therefore, we may assume $d \geq 2$.

Let d be an odd number. Then $d = 2n - 1$ for some $n \geq 2$. Taking A the trivial extension algebra $\mathrm{T}(B)$ of an arbitrary tilted algebra B of type $\boldsymbol{A}_n$, we obtain from Theorem 4.3 that $\underline{\mathrm{CY}}\mathrm{dim}\mathrm{T}(B) = 2n - 1 = d$.

Let d be an even number $2n$, $n \geq 1$. For n even, take $s = n + 2$, B a tilted algebra of type $\boldsymbol{D}_s$ and $A = \mathrm{T}(B)$. Then, applying Theorem 4.3, we obtain

$$\underline{\mathrm{CY}}\mathrm{dim}A = m_{\boldsymbol{D}_s} - 1 = (2s - 2) - 2 = 2(n + 2) - 4 = 2n = d.$$

For n odd, take $s = n + 1$ and $A = \widehat{B}/(\varphi)$, where B is a tilted algebra of type $\boldsymbol{D}_{3s}$ and φ is a cube root of $\nu_{\widehat{B}}$. Then, applying Theorem 4.3, we obtain

$$\underline{\mathrm{CY}}\mathrm{dim}A = \frac{1}{3} m_{\boldsymbol{D}_{3s}} - 1 = (2s - 1) - 1 = 2(s - 1) = 2n = d. \qquad \square$$

The final result of this section shows that there are selfinjective algebras of finite representation type and of infinite stable Calabi-Yau dimension.

COROLLARY 4.5. *Let B be a tilted algebra of type Δ where Δ is a Dynkin graph* *different from* $\boldsymbol{A}_1$*, and let* $A = \widehat{B}/(\nu_{\widehat{B}}^{h_\Delta})$*. Then* $\underline{\mathrm{CY}}\mathrm{dim} A = \infty$*.*

Proof. Observe that we have a Galois covering $A \to A/(\nu_A) = \mathrm{T}(B)$ with cyclic Galois group (ν_A) of order h_Δ generated by ν_A. Since $m_\Delta = h_\Delta - 1$ is the order of $\tau_{\mathrm{T}(B)}$ on $\underline{\mathrm{mod}}\mathrm{T}(B)$, we conclude that $h_\Delta m_\Delta$ is the order of τ_A on $\underline{\mathrm{mod}}A$. Further, it follows from Proposition 4.2 that $\Omega_{\widehat{B}}^2 \cong \tau_{\widehat{B}}^{h_\Delta}$, and hence $\Omega_A^2 \cong \tau_A^{h_\Delta}$. Suppose $\underline{\mathrm{mod}}A$ is Calabi-Yau. Observe that A is neither a Nakayama algebra of Loewy length 2 nor a local Nakayama algebra, and hence $\underline{\mathrm{CY}}\mathrm{dim} A \geq 2$, by Propositions 2.1 and 2.2. Thus we have $\tau_A^{-1} \cong \Omega_A^r$ for some $r \geq 1$. Then $\tau_A^{rh_\Delta+2} = \tau_A^{rh_\Delta}\tau_A^2 \cong \Omega_A^{2r}\tau_A^2 \cong 1_{\underline{\mathrm{mod}}A}$ implies that $h_\Delta m_\Delta$ divides $rh_\Delta + 2$, and consequently h_Δ divides 2. On the other hand, $\Delta \neq \boldsymbol{A}_1$ implies $h_\Delta \geq 3$. Therefore $\underline{\mathrm{mod}}A$ is not Calabi-Yau, and hence $\underline{\mathrm{CY}}\mathrm{dim} A = \infty$. □

5. Algebras of quaternion type.

Following [**21**] an algebra A is said to be of quaternion type if A is connected, symmetric, tame, the Cartan matrix of A is nonsingular, and the stable Auslander-Reiten quiver Γ_A^s of A consists only of tubes of rank at most 2. These include all blocks of group algebras of finite groups with generalized quaternion defect groups. It was proved in [**18**] that an algebra of quaternion type has at most three simple modules. Further, the first named author proved in [**19**], [**20**], [**21**] that any algebra of quaternion type is Morita equivalent to one of 12 types of symmetric representation-infinite algebras defined by quivers and relations (see [**21**, Tables]). Moreover, T. Holm has classified in [**32**] these algebras up to derived equivalence, and proved that they are in fact tame.

The main aim of this section is to prove the following result.

Theorem 5.1. *Let A be an algebra of quaternion type. Then for all indecomposable nonprojective A-modules M we have $\Omega_A^4 M \cong M$. In particular,* $\underline{\mathrm{CY}}\mathrm{dim} A = 3$*.*

Since for a symmetric algebra A we have $\tau_A \cong \Omega_A^2$, we obtain the following immediate consequence of the theorem and the results described above, solving the problem raised in [**21**, VII.9].

Corollary 5.2. *Let A be an algebra. Then the following statements are equivalent*:

(i) *A is of quaternion type.*
(ii) *A is Morita equivalent to one of the algebras in the list given in* [**21**, pp. 303–306].

For the proof of Theorem 5.1 we use the derived equivalence classification from [**32**], and we show that for most algebras in [**32**, (5.1) and (5.9)] the bimodule resolution of the algebra is periodic of period ≤ 4. Algebras with small parameters behave differently, and we deal with these first. Our notation for the algebras is based on [**32**].

For $a \in K \setminus \{0, 1\}$ and $c \in K$, let $\mathscr{Q}(2\mathscr{B})_3^3(a, c)$ be the bound quiver algebra KQ/I, where Q is the quiver

$$\begin{tikzcd} 0 \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & 1 \arrow[l, shift left, "\gamma"] \arrow[loop right, "\eta"] \end{tikzcd}$$

and I is the ideal in KQ generated by the elements

$$\alpha\beta - \beta\gamma, \quad \eta\gamma - \gamma\alpha, \quad \beta\gamma - \alpha^2, \quad \gamma\beta - a\eta^2 - c\eta^3,$$
$$\alpha^4, \quad \eta^4, \quad \gamma\alpha^2, \quad \alpha^2\beta$$

(see [**21**, Tables] and [**32**, (5.1), (5.8)]).

Lemma 5.3. *Let $A = \mathscr{Q}(2\mathscr{B})_3^3(a,c)$. Then the following statements hold.*

(i) *If* $\operatorname{char}K \neq 2$ *then* $A \cong \mathscr{Q}(2\mathscr{B})_3^3(a,0)$.
(ii) *If* $\operatorname{char}K = 2$ *then* $A \cong \mathscr{Q}(2\mathscr{B})_3^3(a,0)$ *or* $A \cong \mathscr{Q}(2\mathscr{B})_3^3(a,1)$.

Proof. Observe that $A = \mathscr{Q}(2\mathscr{B})_3^3(a,c)$ is a socle deformation of $\mathscr{Q}(2\mathscr{B})_3^3(a,0)$. It follows from [**9**, Theorem 1] that $\mathscr{Q}(2\mathscr{B})_3^3(a,0)$ is a symmetric algebra of tubular type $(2,2,2,2)$, isomorphic to the algebra $A_2(a^{-1})$ (described in [**9**]). Then, applying [**10**, Lemma 5.4], we obtain that $A \cong \mathscr{Q}(2\mathscr{B})_3^3(a,0)$ or $A \cong \mathscr{Q}(2\mathscr{B})_3^3(a,1)$. Moreover, $\mathscr{Q}(2\mathscr{B})_3^3(a,0) \cong \mathscr{Q}(2\mathscr{B})_3^3(a,1)$ if and only if $\operatorname{char}K \neq 2$. □

For $a \in K \setminus \{0,1\}$, let $\mathscr{Q}(3\mathscr{A})_1^{2,2}(a)$ be the bound quiver algebra KQ/I, where Q is the quiver

$$1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 0 \underset{\eta}{\overset{\delta}{\rightleftarrows}} 2$$

and I is the ideal in KQ generated by the elements

$$\beta\delta\eta - \beta\gamma\beta, \quad \delta\eta\gamma - \gamma\beta\gamma, \quad \eta\gamma\beta - a\eta\delta\eta,$$
$$\gamma\beta\delta - a\delta\eta\delta, \quad \beta\delta\eta\delta, \quad \eta\gamma\beta\delta.$$

We note that $\mathscr{Q}(3\mathscr{A})_1^{2,2}(a)$ is a symmetric algebra of tubular type $(2,2,2,2)$, and isomorphic to the algebra $A_1(a)$ in [**9**, Theorem 1].

Proposition 5.4. *Let A be a selfinjective algebra. Then the following statements are equivalent*:

(i) *A is of quaternion type and of polynomial growth.*
(ii) *A is Morita equivalent to one of the algebras $\mathscr{Q}(2\mathscr{B})_3^3(a,0)$, $\mathscr{Q}(2\mathscr{B})_3^3(a,1)$, or $\mathscr{Q}(3\mathscr{A})_1^{2,2}(a)$.*

Proof. It is a direct consequence of the description of the Morita equivalence classes of algebras of quaternion type, given in [**19**], [**20**], [**21**], and the classification of the symmetric algebras of tubular types with nonsingular Cartan matrices and their socle deformations, obtained in [**9**], [**10**]. □

Corollary 5.5. *Let A be an algebra of quaternion type which is of polynomial growth. Then for any indecomposable nonprojective A-module M we have $\Omega_A^4 M \cong M$. In particular, $\underline{\mathrm{CY}}\dim A = 3$.*

Proof. This follows from Proposition 5.4 and the fact that $\mathscr{Q}(2\mathscr{B})_3^3(a,0)$, $\mathscr{Q}(2\mathscr{B})_3^3(a,1)$, $\mathscr{Q}(3\mathscr{A})_1^{2,2}(a)$, $a \in K \setminus \{0,1\}$, are representation-infinite symmetric algebras of tubular type $(2,2,2,2)$ (see [**9**], [**10**], [**48**]), and hence stable Auslander-Reiten quiver consists only of tubes of rank at most 2. Then for any indecomposable nonprojective A-module M, we have $\Omega_A^4 M \cong \tau_A^2 M \cong M$. Since A is representation-infinite, we then have $\underline{\mathrm{CY}}\mathrm{dim} A = 3$ (see Corollary 2.3). □

In the proof of Theorem 5.1 for the remaining algebras of quaternion type, we apply the derived equivalence classification of these algebras [**32**], construct the first part of minimal bimodule resolutions for one algebra from each derived equivalence class, and use these to show that they have periodic bimodule resolutions of period 4. We need three families of bound quiver algebras.

For $k \geq 2$ and $a, b \in K$, let $\mathscr{Q}^k(a,b)$ be the local bound quiver algebra KQ/I, where Q is the quiver

$$\begin{tikzcd} 0 \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

and I is the ideal in KQ generated by the elements

$$\alpha^2 - (\beta\alpha)^{k-1}\beta - a(\beta\alpha)^k, \quad \beta^2 - (\alpha\beta)^{k-1}\alpha - b(\alpha\beta)^k,$$
$$(\alpha\beta)^k - (\beta\alpha)^k, \quad (\alpha\beta)^k\alpha, \quad (\beta\alpha)^k\beta.$$

Lemma 5.6. *Let A be a local algebra of quaternion type. Then A is Morita equivalent to an algebra $\mathscr{Q}^k(a,b)$. Moreover, if* $\mathrm{char} K \neq 2$, *then $\mathscr{Q}^k(a,b) \cong \mathscr{Q}(a,0)$.*

Proof. See [**21**, Theorem III.1]. □

For $k \geq 1$ and $s \geq 3$ with $k+s > 4$, $a \in K \setminus \{0\}$, $c \in K$, let $\mathscr{Q}(2\mathscr{B})_1^{k,s}(a,c)$ be the bound quiver algebra KQ/I, where Q is the quiver

$$\begin{tikzcd} 0 \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & 1 \arrow[l, shift left, "\gamma"] \arrow[loop right, "\eta"] \end{tikzcd}$$

and I is the ideal in KQ generated by the elements

$$\gamma\beta - \eta^{s-1}, \quad \beta\eta - (\alpha\beta\gamma)^{k-1}\alpha\beta, \quad \eta\gamma - (\gamma\alpha\beta)^{k-1}\gamma\alpha,$$
$$\alpha^2 - a(\beta\gamma\alpha)^{k-1}\beta\gamma + c(\beta\gamma\alpha)^k, \quad \alpha^2\beta.$$

Lemma 5.7. *Let $A = \mathscr{Q}(2\mathscr{B})_1^{k,s}(a,c)$ with $k \geq 1$, $s \geq 3$ and $k+s > 4$, and where $a \in K \setminus \{0\}$ and $c \in K$. Then we have*

(i) *$\mathscr{Q}(2\mathscr{B})_1^{k,s}(a,c) \cong \mathscr{Q}(2\mathscr{B})_1^{k,s}(1,c')$ for some $c' \in K$.*
(ii) *If* $\mathrm{char} K \neq 2$ *then $\mathscr{Q}(2\mathscr{B})_1^{k,s}(a,c) \cong \mathscr{Q}(2\mathscr{B})_1^{k,s}(a,0)$.*

Proof. We identify an element of KQ with its residue class in $A = KQ/I$. For (ii), if char$K \neq 2$, we replace α by $\alpha' = \alpha - (c/2)(\beta\gamma\alpha)^{k-1}\beta\gamma$ and obtain the required isomorphism of bound quiver algebras.

(i) We set $\alpha = x\alpha'$, $\beta = b\beta'$, $\gamma = g\gamma'$, $\eta = m\eta'$, where x, b, g and m are nonzero scalars. We substitute these into the relations and we want to replace α by α', β by β', γ by γ' and η by η'. If we can find solutions for the scalar identities

$$x^2 = a(bgx)^{k-1}bg, \quad bg = m^{s-1}, \quad (xbg)^{k-1}xb = bm, \quad mg = (bgx)^{k-1}gx$$

then we will have replaced the scalar a by 1 (and the scalar c if nonzero by some other scalar) but have kept the other relations unchanged.

The last three equations above are equivalent with

$$bg = m^{s-1}, \quad m = x^k(bg)^{k-1}. \tag{$*$}$$

The scalar m does not play any role and we can eliminate it, that is replace $(*)$ by

$$bg = (x^k(bg)^{k-1})^{s-1}. \tag{$**$}$$

Since b and g always come with the same exponent we can assume $g = 1$. Then we have to solve

$$b = (x^k b^{k-1})^{s-1} \text{ and } x^2 = ax^{k-1}b^k.$$

Equivalently,

$$1 = x^{k(s-1)} \cdot b^{(k-1)(s-1)-1} \text{ and } 1 = ax^{k-3} \cdot b^k.$$

From the first equation (note that $k(s-1) \neq 0$) we obtain

$$x = b^{\frac{1-(k-1)(s-1)}{k(s-1)}}.$$

Substituting this into the second equation we get $1 = ab^r$, where

$$r = \left[(k-3)\frac{1-(k-1)(s-1)}{k(s-1)}\right] + k.$$

If we know that $r \neq 0$ then we take for b a root of $t^r - a^{-1}$. Then this determines x (and then also m). In case $r < 0$, write this as $(t^{-1})^{-r} - a^{-1}$.

Suppose we would have $r = 0$, that is

$$k^2(s-1) + (k-3)(1-(k-1)(s-1)) = 0.$$

Equivalently,

$$0 = (s-1)(k^2 - (k-1)(k-3)) + (k-3) = (s-1)(4k-3) + (k-3).$$

But, for the integers $k \geq 1$ and $s \geq 3$, this equality hold only for $k = 1$ and $s = 3$. Since $k + s > 4$, we obtain $r \neq 0$, as required. $\square$

For $a, b, c \geq 1$ (at most one parameter equal 1) let $\mathscr{Q}(3\mathscr{K})^{a,b,c}$ be the bound quiver algebra KQ/I, where Q is the quiver

$$\begin{tikzcd}
0 \arrow[rr, shift left, "\beta"] \arrow[dr, shift left, "\kappa"] & & 1 \arrow[ll, shift left, "\gamma"] \arrow[dl, shift left, "\delta"] \\
& 2 \arrow[ul, shift left, "\lambda"] \arrow[ur, shift left, "\eta"] &
\end{tikzcd}$$

and I is the ideal in KQ generated by the elements

$$\begin{gathered}
\beta\delta - (\kappa\lambda)^{a-1}\kappa, \quad \eta\gamma - (\lambda\kappa)^{a-1}\lambda, \\
\delta\lambda - (\gamma\beta)^{b-1}\gamma, \quad \kappa\eta - (\beta\gamma)^{b-1}\beta, \\
\lambda\beta - (\eta\delta)^{c-1}\eta, \quad \gamma\kappa - (\delta\eta)^{c-1}\delta, \\
\gamma\beta\delta, \quad \delta\eta\gamma, \quad \lambda\kappa\eta.
\end{gathered}$$

PROPOSITION 5.8. *Let A be an algebra of quaternion type which is not of polynomial growth. Then A is derived equivalent to one of the algebras*:

(i) $\mathscr{Q}_1^k(a,b)$, $k \geq 2$, $a, b \in K$;
(ii) $\mathscr{Q}(2\mathscr{B})_1^{k,s}(1,c)$, $k \geq 1$, $s \geq 1$, $k+s > 4$, $c \in K$;
(iii) $\mathscr{Q}(3\mathscr{K})^{a,b,c}$, $a, b, c \geq 1$ (*at most one equal* 1).

PROOF. This is a direct consequence of [**19**], [**20**], [**21**, Section VII], [**32**, Section 5], Proposition 5.4 and Lemma 5.7. $\square$

THEOREM 5.9. *Let A be one of the algebras $\mathscr{Q}_1^k(a,b)$, $\mathscr{Q}(2\mathscr{B})_1^{k,s}(1,c)$, $\mathscr{Q}(3\mathscr{K})^{a,b,c}$ listed in the above proposition. Then $\Omega^4_{A^e}(A) \cong A$ as A-A-bimodules.*

PROOF. Let $A = KQ/I$ be the bound quiver presentation of A, as described above. Denote by Q_0 and Q_1 the set of vertices and arrows of the quiver Q, respectively. For each arrow α of Q, we denote by $i\alpha$ the initial vertex of α and by $t\alpha$ the target of α. Moreover, by $\otimes$ we mean the tensor product $\otimes_K$. We divide the proof into several steps.

(1) A minimal projective resolution for any simple module in mod$A$ is easily calculated. Using [**28**, Lemma 1.5] this shows that the projectives in the first few terms of a minimal projective bimodule resolution

$$P_3 \xrightarrow{S} P_2 \xrightarrow{R} P_1 \xrightarrow{d} P_0 \xrightarrow{u} A \to 0 \qquad (*)$$

are as follows

$$P_0 = \oplus_{i \in Q_0}(Ae_i \otimes e_i A), \quad P_1 = \oplus_{\alpha \in Q_1}(Ae_{i\alpha} \otimes e_{t\alpha} A), \quad P_2 \cong P_1, \quad P_3 \cong P_0.$$

We must define the maps.

(1a) The first two maps are standard. Take u to be the multiplication map, its kernel has minimal generators

$$x_\alpha := \alpha \otimes e_{t\alpha} - e_{i\alpha} \otimes \alpha, \quad \alpha \in Q_1.$$

Then we define the bimodule homomorphism d by sending $e_{i\alpha} \otimes e_{t\alpha}$ to x_α, for each arrow α of the quiver Q; and then $\mathrm{Im}(d) = \mathrm{Ker}(u)$.

(1b) The map R will be defined in terms of the minimal relations as follows. Let $\mu = \alpha_1\alpha_2 \dots \alpha_k$ be a monomial in the arrows. Define $\rho(\mu) \in \oplus A e_{i\alpha} \otimes e_{t\alpha} A$ by setting

$$\rho(\mu) := \sum_{j=1}^{k} \alpha_1\alpha_2 \dots \alpha_{j-1} \otimes \alpha_{j+1} \dots \alpha_k$$

where the j-th term lies in $Ae_{i\alpha_j} \otimes e_{t\alpha_j} A$. Then for a certain minimal relation of the form $\sum c_\mu \mu$ from e_i to e_j, where $c_\mu \in K$, we define

$$R(e_i \otimes e_j) = \sum_\mu c_\mu \rho(\mu).$$

For calculations we will use the following observation. Suppose α is an arrow, then

$$\alpha\rho(\mu) = \rho(\alpha\mu) - e_{i\alpha} \otimes \mu.$$

We will later specify the map R explicitly for each of the three cases. Then it will be straightforward to show that $d \circ R = 0$ and we will leave this as an exercise without further comments.

(1c) We will write down the map S in each case later, and show $R \circ S = 0$.

(1d) We will show that the kernel of the map S is isomorphic to A. To do so, we will define a one-to-one bimodule homomorphism $j : A \to P_3$ and show that $S \circ j = 0$.

Once this is done, the proof of Theorem 5.9 will be complete. Namely, we will know that all compositions of the maps in $(*)$ are zero, and also that $S \circ j = 0$. Furthermore, from the definition of R and S it will be clear that the respective image contains generators of the previous kernel. This will then imply that the sequence $(*)$ is exact and that the kernel of S is equal to the image of j and hence isomorphic to A, as required.

(2) We start by constructing an injective hull $j : A \to \oplus_i A(e_i \otimes e_i)A$ of A as a bimodule. Here A can be an arbitrary symmetric algebra; this is a variation of [**23**]. Let $(-,-)$ be a symmetrizing bilinear form on A, that is, a nonsingular associative symmetric bilinear form. We fix a full set of orthogonal primitive idempotents e_i, $i \in Q_0$, of A such that $\sum e_i = 1_A$. For each i, take a vector space basis $\mathscr{B}_i$ of e_iA, and let $\mathscr{B}_i^*$ be the corresponding dual basis, with respect to the given form. Define

$$\zeta_i := \sum_{b \in \mathscr{B}_i} b \otimes b^* \quad \in \oplus_j Ae_j \otimes e_j A.$$

(2a) We claim that ζ_i is independent of the basis. Namely, let $\mathscr{B}_i = \{b_1, \ldots, b_n\}$; if one takes some other basis of e_iA and writes $b'_j = \sum_{k=1}^n c_{jk}b_k$ with coefficient matrix $C = [c_{jk}]$, then the matrix C^{-T} is the coefficient matrix expressing $(b'_j)^*$ in terms of the b_i^*. If one substitutes this into $\sum_{j=1}^n b'_j \otimes (b'_j)^*$ one obtains precisely ζ_i.

(2b) Suppose $a \in e_xAe_y$ and a is in the radical but not in the square of the radical. We claim that then

$$a\zeta_y = \zeta_x a.$$

To see this, take a basis of e_yA which contains a basis $\{b_i\}$ of the kernel of the linear map $m \to am$ from e_yA to e_xA. Say $\mathscr{B}_y = \{b_i\} \cup \{c_i\}$. Then we can take $\mathscr{B}_x = \{ac_i\} \cup \{d_i\}$. By (2a) we can use these bases and their dual bases to write down ζ_x and ζ_y. Then

$$a\zeta_y = \sum_i ac_i \otimes c_i^*, \quad \zeta_x a = \sum ac_i \otimes (ac_i)^* a + \sum_i d_i \otimes d_i^* a.$$

We claim that $d_i^* a = 0$ and that $(ac_i)^* a = c_i^*$. Write

$$d_i^* a = \sum r_j (ac_j)^* + \sum s_j d_j^* \quad \text{with } r_j \text{ and } s_j \text{ in } K.$$

Then

$$r_j = (ac_j, d_i^* a) = (d_i^* a, ac_j) = (d_i^*, a^2 c_j) = (a^2 c_j, d_i^*).$$

Since $a^2 c_j$ is in the span of the $\{ac_s\}$ this is zero. Similarly one shows that $s_j = 0$ for all j. Now write

$$(ac_i)^* a = \sum_j r_j b_j^* + \sum_j s_j c_j^* \quad \text{with } r_j \text{ and } s_j \text{ in } K.$$

Then

$$r_j = (b_j, (ac_i)^* a) = ((ac_i)^* a, b_j) = (\pi^{-1}(ac_i)^*, ab_j) = 0,$$

since $ab_j = 0$. Moreover

$$s_j = (c_j, (ac_i)^* a) = ((ac_i)^* a, c_j) = (ac_j, (ac_i)^*) = \delta_{ij},$$

as required. Now it follows as in [**23**] that we have a bimodule map $j : A \to \oplus(Ae_i \otimes e_iA)$ such that $j(e_i) = \zeta_i$ and that j is one-to-one.

(3) We continue with the setup as in (2). Take any element $a = e_x a e_y$ which belongs to the radical (it will later be an arrow). Then we take bases compatible with idempotents; we claim that for $r, s \geq 0$

$$\sum_{b \in \mathscr{E}} b(a^r \otimes a^s)b^* = \sum_{b \in \mathscr{F}} b(e_x \otimes a^{r+s})b^*,$$

where $\mathscr{E}$ and $\mathscr{F}$ are suitable bases for the module e_iA.

We use a similar argument as in (2). Namely, we take for $\mathscr{E}$ a basis which contains a basis $\{b_i\}$ of the kernel of $m \to ma^r$, say $\mathscr{E} = \{b_i\} \cup \{c_i\}$. For $\mathscr{F}$ we take a basis $\{c_ia^r\} \cup \{d_i\}$. Now we must show that

$$\sum_i c_ia^r \otimes a^s(c_i^*) = \sum_i (c_ia^r) \otimes a^{r+s}(c_ia^r)^* + \sum d_i \otimes a^{r+s}(d_i^*).$$

That is, we must show $a^{r+s}d_i^* = 0$ and $a^r(c_ia^r)^* = c_i^*$. This is done by the argument in (2).

(4) Now let $A = \mathscr{Q}(3\mathscr{K})^{a,b,c}$, $a, b, c \geq 1$ (at most one parameter equal 1). We may assume $a \geq b \geq c$. Consider first the case when $c \geq 2$.

We will now define the map R in this case.

(4a) Take any arrow α of the quiver Q. Then there is a unique minimal relation between $e_{i\alpha}$ and $e_{t\alpha}$ which involves a monomial of degree two. We use this to define $R(e_{i\alpha} \otimes e_{t\alpha})$. For $\alpha = \kappa$, we define

$$R(e_0 \otimes e_2) := \rho(\beta\delta) - \rho((\kappa\lambda)^{a-1}\kappa).$$

For $\alpha = \lambda$, we define

$$R(e_2 \otimes e_0) := \rho(\eta\gamma) - \rho((\lambda\kappa)^{a-1}\lambda).$$

By the same principle we define $R(e_i \otimes e_j)$ for the other vertices $i \neq j$.

(4b) Next we define the map $S : P_3 \to P_2$. Set

$$\begin{aligned} S(e_0 \otimes e_0) &:= \beta \otimes e_0 - e_0 \otimes \gamma + \kappa \otimes e_0 - e_0 \otimes \lambda, \\ S(e_1 \otimes e_1) &:= \gamma \otimes e_1 - e_1 \otimes \beta + \delta \otimes e_1 - e_1 \otimes \eta, \\ S(e_2 \otimes e_2) &:= \lambda \otimes e_2 - e_2 \otimes \kappa + \eta \otimes e_2 - e_2 \otimes \delta. \end{aligned}$$

As we have explained, it suffices to show that $R \circ S = 0$ and that $S \circ j = 0$.

(4c) The composition $R \circ S$ is zero: As a typical case we calculate $R \circ S(e_0 \otimes e_0)$. This is equal to

$$\beta R(e_1 \otimes e_0) - R(e_0 \otimes e_1)\gamma + \kappa R(e_2 \otimes e_0) - R(e_0 \otimes e_2)\lambda.$$

The first two terms are

$$\beta[\rho(\delta\lambda) - \rho((\gamma\beta)^{b-1}\gamma)] - [\rho(\kappa\eta) - \rho((\beta\gamma)^{b-1}\beta)]\gamma.$$

Using the observation in (1) and cancelling, invoking the relations of the algebra, this becomes

$$\rho(\beta\delta\lambda) - \rho(\kappa\eta\gamma).$$

Similarly, we calculate

$$\kappa R(e_2 \otimes e_0) - R(e_0 \otimes e_2)\lambda = \rho(\kappa\eta\gamma) - \rho(\beta\delta\lambda),$$

and in total we get $R \circ S(e_0 \otimes e_0) = 0$.

(4d) The composition $S \circ j$ is zero. Write

$$S \circ j(e_i) \ = \sum_{b \in \mathscr{B}} bS(e_0 \otimes e_0)b^* + \sum_b bS(e_1 \otimes e_1)b^* + \sum_b bS(e_2 \otimes e_2)b^*,$$

where every time we sum over a basis of e_iA which is compatible with the idempotents. We substitute, and by (3), for any arrow α, the sum $\sum_b b\alpha \otimes b^*$ cancels against $\sum_b b \otimes \alpha b^*$. So we get zero.

This completes now the proof of the theorem for algebras with three simple modules when $c \geq 2$.

(4e) It remains to deals with the case when $c = 1$. Then δ and η lie in the square of the radical, that is we must delete these arrows from the quiver. Now the minimal relations take the form

$$\beta\gamma\kappa = (\kappa\lambda)^{a-1}\kappa, \quad \gamma\kappa\lambda = (\gamma\beta)^{b-1}\gamma,$$
$$\lambda\beta\gamma = (\lambda\kappa)^{a-1}\lambda, \quad \kappa\lambda\beta = (\beta\gamma)^{b-1}\beta,$$

and some zero relations which we will not need.

We define the map R by the same principle as described in (1), so for example

$$R(e_0 \otimes e_2) = \rho(\beta\gamma\kappa) - \rho((\kappa\lambda)^{a-1}\kappa).$$

We define the map S as in (4b) but omitting the two terms in which δ occurs, and also the two terms in which η occurs. It is clear from (3) that $S \circ j = 0$, and it is straightforward to check that $R \circ S = 0$. This completes now the proof of the theorem for $A = \mathscr{Q}(3\mathscr{K})^{a,b,c,}$.

(5) We consider now $A = \mathscr{Q}(2\mathscr{B})_1^{k,s}(1,c)$. By Lemma 5.7 we assume K has characteristic 2 when $c \neq 0$.

(5a) We start by defining the map R in this case. We use the four minimal relations which involve paths of length two. That is we define

$$R(e_1 \otimes e_1) := \rho(\gamma\beta) - \rho(\eta^{s-1}),$$
$$R(e_0 \otimes e_1) := \rho(\beta\eta) - \rho((\alpha\beta\gamma)^{k-1}\alpha\beta),$$
$$R(e_1 \otimes e_0) := \rho(\eta\gamma) - \rho((\gamma\alpha\beta)^{k-1}\gamma\alpha),$$
$$R(e_0 \otimes e_0) := \rho(\alpha^2) - \rho((\beta\gamma\alpha)^{k-1}\beta\gamma) + c\rho((\beta\gamma\alpha)^k).$$

Note that we made a choice in the definition of $R(e_0 \otimes e_0)$.

(5b) We define the bimodule homomorphism $S : P_3 \to P_2$ as follows. Set

$$\begin{aligned} S(e_0 \otimes e_0) &:= (\beta \otimes e_0 - e_0 \otimes \gamma) + (\alpha \otimes e_0 - e_0 \otimes \alpha) \\ &\quad + c(\alpha \otimes \alpha + e_0 \otimes \alpha^2) + c^2(\alpha \otimes \alpha^2 + e_0 \otimes \alpha^3), \\ S(e_1 \otimes e_1) &:= (\gamma \otimes e_1 - e_1 \otimes \beta) + (\eta \otimes e_1 - e_1 \otimes \eta). \end{aligned}$$

As before, it suffices to show that $R \circ S = 0$ and that $S \circ j = 0$.

(5c) Consider the composition $R \circ S$. It is easy to check that $R \circ S(e_1 \otimes e_1) = 0$. We will give details for $R \circ S(e_0 \otimes e_0)$. Using the remark in (1) and cancelling the scalar multiples of $\beta \otimes \gamma$ we have

$$\begin{aligned} R(\beta \otimes e_0 - e_0 \otimes \gamma) &= \beta R(e_1 \otimes e_0) - R(e_0 \otimes e_1)\gamma \\ &= \beta\eta \otimes e_0 - \rho\big((\beta\gamma\alpha)^k\big) + e_0 \otimes (\gamma\alpha\beta)^{k-1}\gamma\alpha \\ &\quad - \ e_0 \otimes \eta\gamma - \rho((\alpha\beta\gamma)^k) + (\alpha\beta\gamma)^{k-1}\alpha\beta \otimes e_0. \end{aligned}$$

Since $\beta\eta = (\alpha\beta\gamma)^{k-1}\alpha\beta$, the first and the last term cancel. Similarly two terms cancel using the relation for $\eta\gamma$. We are left with

$$-\rho(\beta\gamma\alpha)^k + \rho(\alpha\beta\gamma)^k. \tag{x}$$

Consider similarly $R(\alpha \otimes e_0 - e_0 \otimes \alpha)$. After cancellation this leaves

$$-\rho((\alpha\beta\gamma)^k) - c\rho((\alpha\beta\gamma)^k\alpha) + \rho((\beta\gamma\alpha)^k) + c\rho((\beta\gamma\alpha)^k\alpha). \tag{y}$$

The first and the third term in (y) cancel against (x) and we are left with

$$-c\rho[(\alpha\beta\gamma)^k\alpha] + c\rho[(\beta\gamma\alpha)^k\alpha]. \tag{z}$$

If $c = 0$ (which happens always for characteristic $\neq 2$) we see already now that $R \circ S(e_0 \otimes e_0) = 0$. So assume now $c \neq 0$; and then we have $\mathrm{char}(K) = 2$, which we will use freely.

Using the relation for α^2 and also the fact that $\alpha^3 = (\alpha\beta\gamma)^k = (\beta\gamma\alpha)^k$, we calculate

$$\begin{aligned} R(\alpha \otimes \alpha) &= \alpha R(e_0 \otimes e_0)\alpha \\ &= \alpha^2 \otimes \alpha + \alpha \otimes \alpha^2 + e_0 \otimes \alpha^3 + \alpha^3 \otimes e_0 + \rho((\alpha\beta\gamma)^k\alpha) + c\rho((\alpha\beta\gamma)^k\alpha^2), \\ R(e_0 \otimes \alpha^2) &= R(e_0 \otimes e_0)\alpha^2 \\ &= \alpha \otimes \alpha^2 + \alpha^2 \otimes \alpha + e_0 \otimes \alpha^3 + \alpha^3 \otimes e_0 + \rho((\beta\gamma\alpha)^k\alpha) + c\rho((\beta\gamma\alpha)^k\alpha^2), \end{aligned}$$

and similarly

$$\begin{aligned} R(\alpha \otimes \alpha^2) &= \alpha R(e_0 \otimes e_0)\alpha^2 \\ &= \alpha^2 \otimes \alpha^2 + \alpha \otimes \alpha^3 + \alpha^3 \otimes \alpha + \rho((\alpha\beta\gamma)^k\alpha^2) + c\rho((\alpha\beta\gamma)^k\alpha^3), \end{aligned}$$

$$\begin{aligned} R(e_0 \otimes \alpha^3) &= R(e_0 \otimes e_0)\alpha^3 \\ &= \alpha \otimes \alpha^3 + \alpha^2 \otimes \alpha^2 + \alpha^3 \otimes \alpha + \rho((\beta\gamma\alpha)^k\alpha^2) + c\rho((\beta\gamma\alpha)^k\alpha^3). \end{aligned}$$

Now adding (z) together with $c[R(\alpha \otimes \alpha) + R(e_0 \otimes \alpha^2)]$ and $c^2[R(\alpha \otimes \alpha^2) + R(e_0 \otimes \alpha^3)]$ leaves just

$$c^3[\rho((\alpha\beta\gamma)^k\alpha^3) + \rho((\beta\gamma\alpha)^k\alpha^3)].$$

Observe that

$$\rho((\alpha\beta\gamma)^k\alpha^3) = (\alpha\beta\gamma)^k \otimes \alpha^3 = (\beta\gamma\alpha)^k \otimes \alpha^3 = \rho((\beta\gamma\alpha)^k\alpha^3),$$

and we get zero, because $\mathrm{char} K = 2$.

One shows that $S \circ j = 0$ by applying (3); this is similar to (3d) and we omit details. This completes the proof of the theorem in this case.

(6) Finally we consider the local algebras $A = \mathscr{Q}^k(a, b)$, $k \geq 2$, $a, b \in K$. Here P_1 and P_2 are direct sums of two copies of the indecomposable projective A^e-module. To distinguish between the copies, we label them as

$$P_1 = P_2 = \ (A \otimes_\alpha A) \oplus (A \otimes_\beta A).$$

We make the convention that if we define $\rho(\mu)$ of some monomial μ then we replace α by $\otimes_\alpha$ and we replace β by $\otimes_\beta$.

(6a) We define

$$\begin{aligned} R(1 \otimes_\alpha 1) &:= \rho(\alpha^2) - \rho((\beta\alpha)^{k-1}\beta) - a\rho((\beta\alpha)^k), \\ R(1 \otimes_\beta 1) &:= \rho(\beta^2) - \rho((\alpha\beta)^{k-1}\alpha) - b\rho((\alpha\beta)^k). \end{aligned}$$

(6b) We define the bimodule homomorphism $S : P_3 \to P_2$ as follows. Set first

$$\begin{aligned} \xi_\alpha &:= a[(\alpha \otimes_\alpha \alpha) + (1 \otimes_\alpha \alpha^2)] + a^2[(\alpha \otimes_\alpha \alpha^2) + (1 \otimes_\alpha \alpha^3)], \\ \xi_\beta &:= +b[(\beta \otimes_\beta \beta) + (1 \otimes_\beta \beta^2)] + b^2[(\beta \otimes_\beta \beta^2) + (1 \otimes_\beta \beta^3)]. \end{aligned}$$

Now we define

$$S(1 \otimes 1) := (\alpha \otimes_\alpha 1 - 1 \otimes_\alpha \alpha) + \xi_\alpha + (\beta \otimes_\beta 1 - 1 \otimes_\beta \beta) + \xi_\beta.$$

As before, it suffices to show that $R \circ S = 0$ and that $S \circ j = 0$. The details are similar to the case of two simple modules and we omit them.

These completes the proof of the theorem. □

The following corollary completes the proof of Theorem 5.1.

Corollary 5.10. *Let A be an algebra of quaternion type which is not of polynomial growth. Then for any indecomposable nonprojective A-module M we have $\Omega_A^4 M \cong M$. In particular, $\underline{\mathrm{CY}}\mathrm{dim} A = 3$.*

Proof. The properties are invariant under stable equivalences and then under derived equivalences. Therefore, by Proposition 5.8, it is sufficient to prove the corollary for A occuring in this proposition. Then, applying Theorem 5.9, for any indecomposable nonprojective A-module M we have that $\Omega_A^4 M$ is isomorphic to the nonprojective direct summand of $\Omega_{A^e}^4(A) \otimes_A M \cong A \otimes_A M \cong M$, and hence $\Omega_A^4 M \cong M$. Moreover, it follows from Corollary 2.3 that $\underline{\mathrm{CY}}\mathrm{dim} A = 3$. □

Since all connected tame symmetric algebras of nonpolynomial growth and with the stable Auslander-Reiten quiver consisting only of stable tubes are of quaternion type [**22**] we obtain also the following fact.

Corollary 5.11. *Let A be a connected tame symmetric algebra of nonpolynomial growth and finite stable Calabi-Yau dimension. Then $\underline{\mathrm{CY}}\mathrm{dim} A = 3$.*

6. Symmetric algebras of polynomial growth.

In this section we describe the stable Calabi-Yau dimension of all connected representation-infinite symmetric algebras of polynomial growth and finite stable Calabi-Yau dimension.

Following [**46**] by a tubular algebra we mean a tubular extension (equivalently, tubular coextension) B of a tame concealed algebra C of tubular type $n_B \in \{(2,2,2,2),(3,3,3),(2,4,4),(2,3,6)\}$. Then B is of global dimension 2 and the rank of the Grothendick group $K_0(B)$ of B is equal 6, 8, 9, or 10, respectively. By a self-injective algebra of tubular type we mean an algebra of the form $\widehat{B}/G$, where B is a tubular algebra and G is an admissible group of automorphisms of $\widehat{B}$. In fact, such an admissible group G is infinite cyclic, generated by a strictly positive automorphism of $\widehat{B}$. It has been shown in [**8**, Theorem 3.1], that a basic connected algebra A is selfinjective of tubular type if and only if A is tame, admits a simply connected Galois covering, and the stable Auslander-Reiten quiver of A consists of only of tubes. By general theory (see [**48**, Section 3]), if $A = \widehat{B}/G$ is a selfinjective algebra of tubular type, then the stable Auslander-Reiten quiver Γ_A^s of A is of the form

$$\Gamma_A^s = \bigvee_{q \in S^1(\boldsymbol{Q})} \mathscr{T}^q,$$

where $S^1(\boldsymbol{Q})$ is the set of rational points of the unit circle, and, for each $q \in S^1(\boldsymbol{Q})$, $\mathscr{T}^q = (\mathscr{T}_\lambda^q)_{\lambda \in \boldsymbol{P}_1(K)}$ is a $\boldsymbol{P}_1(K)$-family of stable tubes of tubular type n_B. Here, by a $\boldsymbol{P}_1(K)$-family of stable tubes of tubular type $(n_1,\dots,n_r)$, with $n_1,\dots,n_r \geq 2$ integers, we mean a family $\mathscr{T} = (\mathscr{T}_\lambda)_{\lambda \in \boldsymbol{P}_1(K)}$ of stable tubes having tubes $\mathscr{T}_{\lambda_1},\dots,\mathscr{T}_{\lambda_r}$ (for some $\lambda_1,\dots,\lambda_r \in \boldsymbol{P}_1(K)$) of ranks respectively $n_1,\dots,n_r$, and the remaining tubes of rank 1. Therefore, to any selfinjective algebra $A = \widehat{B}/G$ of tubular type we may assign its tubular type $n_A = n_B$ (the tubular type of B), describing the ranks of stable tubes of Γ_A^s.

The class of symmetric algebras of tubular type may be divided into two disjoint classes (see [**9**], [**37**], [**38**]). One class consists of the trivial extensions $\mathrm{T}(B)$ of tubular algebras B, for which the Cartan matrices are singular. The second class is formed by the symmetric algebras of the form $\widehat{B}/(\varphi)$, with φ a proper root of the Nakayama automorphism $\nu_{\widehat{B}}$ of $\widehat{B}$, described in [**9**, Theorem 1] by quivers and relations, for which the Cartan matrices are nonsingular. Further, in [**10**, Theorem 1.1] all proper socle deformations of the symmetric algebras of tubular type have been also described by quivers and relations.

The most relevant for us is the following consequence of the main result of [**22**].

Proposition 6.1. *Let A be a connected symmetric algebra. The following statements are equivalent*:

(i) *A is representation-infinite of polynomial growth and the stable Auslander-Reiten quiver of A consists only of tubes.*
(ii) *A is representation-infinite of polynomial growth and finite stable Calabi-Yau dimension.*
(iii) *A is Morita equivalent to a socle deformation of a symmetric algebra of tubular type.*

Let A be a symmetric algebra which is a socle deformation of a selfinjective algebra $\Lambda = \widehat{B}/G$ of tubular type. Since the stable Auslander-Reiten quivers Γ_A^s and Γ_Λ^s are isomorphic, we may call the tubular type $n_\Lambda = n_B$ of Λ also the tubular type of A, and denote by n_A.

For the proof of the main result of this section, we need a preliminary result.

Lemma 6.2. *Let $A = \mathrm{T}(B)$ be the trivial extension of a tubular algebra B and m_A the least common multiple of the numbers in the tubular type n_A of A. Then there exists an indecomposable nonprojective A-module M such that $\Omega_A^{2m_A} M \cong M$ but $\Omega_A^r M \ncong M$ for r with $1 \le r < 2m_A$.*

Proof. Let $m = m_A$. It is known (see [**30**], [**41**], [**50**]) that $A = \mathrm{T}(B)$ is stably equivalent to the trivial extension $\mathrm{T}(C)$ of the canonical algebra C of tubular type $n_C = n_B$. Further, it follows from [**30**] that $\Gamma_{\mathrm{T}(C)}$ admits a stable tube $\mathscr{T}$ of rank m whose mouth is formed by $m-1$ simple modules $S_1, \dots, S_{m-1}$, with $\tau_{\mathrm{T}(C)} S_i \cong S_{i+1}$ for $1 \le i \le m-2$, and the module $\tau^-_{\mathrm{T}(C)} S_1 \cong \tau_{\mathrm{T}(C)} S_{m-1}$. Since $\mathrm{T}(C)$ is a symmetric algebra, we have $\tau_{\mathrm{T}(C)} = \Omega^2_{\mathrm{T}(C)}$. Moreover, the projective covers of the simple modules $S_1, \dots, S_{m-1}$ in $\mathrm{mod}\,\mathrm{T}(C)$ are uniserial projective module of length at least 4, and consequently the modules $\Omega_{\mathrm{T}(C)} S_i$, $1 \le i \le m-1$, are not simple. Therefore, for any simple module $S = S_i$, $1 \le i \le m-1$, we have $\Omega^{2m}_{\mathrm{T}(C)} S \cong S$ but $\Omega^r_{\mathrm{T}(C)} S \ncong S$ if $1 \le r < 2m$. Since $\underline{\mathrm{mod}}\,A \cong \underline{\mathrm{mod}}\,\mathrm{T}(B) \cong \underline{\mathrm{mod}}\,\mathrm{T}(C)$ and the syzygy functors commute with the stable equivalences of connected selfinjective algebras of Loewy length at least 3 (see [**4**, Proposition X.1.12]), the required claim for A follows. $\square$

We may now prove the main result of this section.

Theorem 6.3. *Let A be a basic, connected, representation-infinite symmetric algebra of polynomial growth and finite stable Calabi-Yau dimension. Then the following statements hold*:

(i) $\underline{\mathrm{CY}}\mathrm{dim}A = 2$, *if A is isomorphic to a deformed preprojective algebra $P(\boldsymbol{D}_4)$ or $P^*(\boldsymbol{D}_4)$.*
(ii) $\underline{\mathrm{CY}}\mathrm{dim}A = 3$, *if $n_A = (2,2,2,2)$.*
(iii) $\underline{\mathrm{CY}}\mathrm{dim}A = 5$, *if $n_A = (3,3,3)$ and A is nonisomorphic to $P(\boldsymbol{D}_4)$ or $P^*(\boldsymbol{D}_4)$.*
(iv) $\underline{\mathrm{CY}}\mathrm{dim}A = 7$, *if $n_A = (2,4,4)$.*
(v) $\underline{\mathrm{CY}}\mathrm{dim}A = 11$, *if $n_A = (2,3,6)$.*

Proof. (i) This is a consequence of Corollary 3.9.

(ii) Assume $n_A = (2,2,2,2)$. Then Γ_A^s consists of tubes of rank at most 2, and hence $\tau_A^2 \cong 1_{\underline{\mathrm{mod}}A}$. Hence, $\Omega_A^4 \cong 1_{\underline{\mathrm{mod}}A}$, because A is symmetric. Therefore, invoking Corollary 2.3, we obtain $\underline{\mathrm{CY}}\mathrm{dim}A = 3$.

(iii) Assume $n_A = (3,3,3)$. Then Γ_A^s consists of tubes of rank 1 and 3, and hence $\Omega_A^6 \cong \tau_A^3 \cong 1_{\underline{\mathrm{mod}}A}$. Therefore, we have $\underline{\mathrm{CY}}\mathrm{dim}A = 5$ or $\underline{\mathrm{CY}}\mathrm{dim}A = 2$. But it follows from Corollaries 3.9 and 3.10 that $\underline{\mathrm{CY}}\mathrm{dim}A = 2$ if and only if $A \cong P(\boldsymbol{D}_4)$ or $P^*(\boldsymbol{D}_4)$. Thus the claim follows.

(iv) Assume $n_A = (2,4,4)$. Then Γ_A^s consists of tubes of rank 1, 2 and 4, and hence $\Omega_A^8 \cong \tau_A^4 \cong 1_{\underline{\mathrm{mod}}A}$. Since A is representation-infinite, we have then $\underline{\mathrm{CY}}\mathrm{dim}A = 3$ or $\underline{\mathrm{CY}}\mathrm{dim}A = 7$. We claim that $\underline{\mathrm{CY}}\mathrm{dim}A = 7$. We have two cases to consider. Assume first that the Cartan matrix of A is nonsingular. Since A is of polynomial growth, invoking Proposition 5.4, we conclude that A is not of quaternion type. Then $\Omega_A^4 \not\cong 1_{\underline{\mathrm{mod}}A}$, and consequently $\underline{\mathrm{CY}}\mathrm{dim}A = 7$. Assume now that the Cartan matrix of A is singular. Then it follows from [**9**] and [**10**] that A is isomorphic to the trivial extension $\mathrm{T}(B)$ of a tubular algebra B of type $(2,4,4)$. Applying Lemma 6.2, we conclude that $\underline{\mathrm{CY}}\mathrm{dim}A = 7$.

(v) Assume $n_A = (2,3,6)$. Then it follows from [**9**, Theorem] and [**37**, Corollary 5.5] that A is isomorphic to the trivial extension $\mathrm{T}(B)$ of a tubular algebra B of type $(2,3,6)$. Since Γ_A^s consists of tubes of rank 1, 2, 3 and 6, we have $\Omega_A^{12} \cong 1_{\underline{\mathrm{mod}}A}$. Moreover, it follows from Lemma 6.2 that $\Omega_A^6 \not\cong 1_{\underline{\mathrm{mod}}A}$. Therefore, $\underline{\mathrm{CY}}\mathrm{dim}A = 11$. □

References

[1] H. Asashiba, The derived equivalence classification of representation-finite selfinjective algebras, J. Algebra, **214** (1999), 182–221.

[2] I. Assem, D. Simson and A. Skowroński, Elements of Representation Theory of Associative Algebras I: Techniques of Representation Theory, London Math. Soc. Student Texts, **65**, Cambridge Univ. Press, 2005, in press.

[3] M. Auslander and I. Reiten, DTr-periodic modules and functors, In: Representation Theory of Algebras, CMS Conf. Proc., **18**, Amer. Math. Soc., 1996, 39–50.

[4] M. Auslander, I. Reiten and S. O. Smalø, Representation Theory of Artin Algebras, Cambridge Stud. Adv. Math., **36**, Cambridge Univ. Press, 1995.

[5] J. Białkowski, K. Erdmann and A. Skowroński, Deformed preprojective algebras of generalized Dynkin type, Trans. Amer. Math. Soc., in press.

[6] J. Białkowski, T. Holm and A. Skowroński, Derived equivalences for tame weakly symmetric algebras having only periodic modules, J. Algebra, **269** (2003), 652–668.

[7] J. Białkowski, T. Holm and A. Skowroński, On nonstandard tame selfinjective algebras having only periodic modules, Colloq. Math., **97** (2003), 33–47.

[8] J. Białkowski and A. Skowroński, Selfinjective algebras of tubular type, Colloq. Math., **94** (2002), 175–194.

[9] J. Białkowski and A. Skowroński, On tame weakly symmetric algebras having only periodic modules, Arch. Math. (Basel), **81** (2003), 142–154.

[10] J. Białkowski and A. Skowroński, Socle deformations of selfinjective algebras of tubular type, J. Math. Soc. Japan, **56** (2004), 687–716.

[11] A. I. Bondal and M. M. Kapranov, Representable functors, Serre functors, and reconstructions, Izv. Akad. Nauk SSSR Ser. Mat., **53** (1989), 1183–1205.

[12] K. Bongartz and P. Gabriel, Covering spaces in representation theory, Invent. Mat., **65** (1982), 331–378.

[13] S. Brenner, M. C. R. Butler and A. D. King, Periodic algebras which are almost Koszul, Algebr. Represent. Theory, **5** (2002), 331–367.

[14] O. Bretscher, C. Läser and C. Riedtmann, Selfinjective and simply connected algebras, Manuscr. Math., **36** (1982), 253–307.

[15] W. W. Crawley-Boevey, On tame algebras and bocses, Proc. London Math. Soc., **56** (1988), 451–483.

[16] P. Dowbor and A. Skowroński, On Galois coverings of tame algebras, Arch. Math. (Basel), **44** (1985), 522–529.

[17] Yu. Drozd, Tame and wild matrix problems, In: Representation Theory II, Lecture Notes in Math., **832**, Springer, 1980, 242–258.

[18] K. Erdmann, On the number of simple modules of certain tame blocks and algebras, Arch. Math. (Basel), **51** (1988), 34–38.

[19] K. Erdmann, Algebras of quaternion defect groups I, Math. Ann., **281** (1988), 545–560.

[20] K. Erdmann, Algebras of quaternion defect groups II, Math. Ann., **281** (1988), 561–582.

[21] K. Erdmann, Blocks of Tame Representation Type and Related Algebras, Lecture Notes in Math., **1428**, Springer, 1990.

[22] K. Erdmann and A. Skowroński, Classification of tame symmetric algebras with periodic modules, Preprint, Toruń 2005.

[23] K. Erdmann and N. Snashall, On Hochschild cohomology of preprojective algebras I, J. Algebra, **205** (1998), 391–412.

[24] P. Gabriel, Auslander-Reiten sequences and representation-finite algebras, In: Representation Theory I, Lecture Notes in Math., **831**, Springer, 1980, 1–71.

[25] P. Gabriel, The universal cover of a representation-finite algebra, In: Representation of Algebras, Lecture Notes in Math., **903**, Springer, 1981, 68–105.

[26] P. Gabriel and C. Riedtmann, Group representations without groups, Comment. Math. Helv., **54** (1979), 240–287.

[27] D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, London Math. Soc. Lecture Note Series, **119**, Cambridge Univ. Press, 1988.

[28] D. Happel, Hochschild cohomology of finite-dimensional algebras, In: Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, Lecture Notes in Math., **1404**, Springer, 1989, 108–126.

[29] D. Happel, Auslander-Reiten triangles in derived categories of finite-dimensional algebras, Proc. Amer. Math. Soc., **112** (1991), 641–648.

[30] D. Happel and C. M. Ringel, The derived category of a tubular algebra, In: Representation Theory I. Finite Dimensional Algebras, Lecture Notes in Math., **1177**, Springer, 1986, 156–180.

[31] R. Hartshorne, Algebraic Geometry, Graduate Texts in Math., **52**, Springer, 1977.

[32] T. Holm, Derived equivalence classification of algebras of dihedral, semidihedral, and quaternion type, J. Algebra, **211** (1999), 159–205.

[33] D. Hughes and J. Waschbüsch, Trivial extensions of tilted algebras, Proc. London Math. Soc., **46** (1983), 347–364.

[34] D. Happel and D. Vossieck, Minimal algebras of infinite representation type with preprojective component, Manuscripta Math., **42** (1983), 221–243.

[35] B. Keller, On triangulated orbit categories, Preprint, Paris, 2005.

[36] M. Kontsevich, Triangulated categories and geometry, Course at the École Normale Supérieure, Paris, Notes taken by J. Bellaiche, J.-F. Dat, I. Marin, G. Racinet and H. Randriambololona, 1998.

[37] H. Lenzing and A. Skowroński, Roots of Nakayama and Auslander-Reiten translations, Colloq. Math., **86** (2000), 209–230.

[38] J. Nehring and A. Skowroński, Polynomial growth trivial extensions of simply connected algebras, Fund. Math., **132** (1989), 117–134.

[39] Z. Pogorzały and A. Skowroński, Symmetric algebras stably equivalent to the trivial extensions of tubular algebras, J. Algebra, **1981** (1996), 95–111.
[40] I. Reiten and M. van den Bergh, Noetherian hereditary abelian categories satisfying Serre duality, J. Amer. Math. Soc., **15** (2002), 295–366.
[41] J. Rickard, Derived categories and stable equivalence, J. Pure Appl. Algebra, **61** (1989), 303–317.
[42] C. Riedtmann, Algebren, Darstellungsköcher, Überlagerungen und zurück, Comment. Math. Helv., **55** (1980), 199–224.
[43] C. Riedtmann, Representation-finite selfinjective algebras of class $\boldsymbol{A}_n$, In: Representation Theory II, Lecture Notes in Math., **832**, Springer, 1980, 449–520.
[44] C. Riedtmann, Representation-finite selfinjective algebras of class $\boldsymbol{D}_n$, Compositio Math., **49** (1983), 231–282.
[45] C. M. Ringel, Tame algebras, In: Representation Theory I, Lecture Notes in Math., **831**, Springer, 1980, 137–287.
[46] C. M. Ringel, Tame Algebras and Integral Quadratic Forms, Lecture Notes in Math., **1099**, Springer, 1984.
[47] A. Skowroński, Group algebras of polynomial growth, Manuscripta Math., **59** (1987), 499–516.
[48] A. Skowroński, Selfinjective algebras of polynomial growth, Math. Ann., **285** (1989), 177–199.
[49] H. Tachikawa, Representations of trivial extensions of hereditary algebras, In: Representation Theory II, Lecture Notes in Math., **832**, Springer, 1980, 579–599.
[50] H. Tachikawa and T. Wakamatsu, Tilting functors and stable equivalences for selfinjective algebras, J. Algebra, **109** (1987), 138–165.
[51] T. Wakamatsu, Stable equivalence between universal covers of trivial extensions of self-injective algebras, Tsukuba J. Math., **9** (1985), 299–316.
[52] J. Waschbüsch, Symmetrische Algebren vom endlichen Modultyp, J. Reine Angew Math., **321** (1981), 78–98.
[53] J. Waschbüsch, On selfinjective algebras of finite representation type, Monographs of Institute of Mathematics, **14**, UNAM, Mexico, 1983.
[54] K. Yamagata, Frobenius algebras, In: Handbook of Algebra, **1**, Elsevier Science B.V., 1996, 841–887.

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