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On the first homology of the group of equivariant Lipschitz homeomorphisms

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Abstract. We study the structure of the group of equivariant Lipschitz homeomorphisms of a smooth G-manifold M which are isotopic to the identity through equivariant Lipschitz homeomorphisms with compact support. First we show that the group is perfect when M is a smooth free G-manifold. Secondly in the case of $\boldsymbol{C}^n$ with the canonical $U(n)$-action, we show that the first homology group admits continuous moduli. Thirdly we apply the result to the case of the group $L(\boldsymbol{C}, 0)$ of Lipschitz homeomorphisms of $\boldsymbol{C}$ fixing the origin.

1. Introduction and statement of the results.

Let G be a compact Lie group. Let $L_G(M)$ denote the group of equivariant Lipschitz homeomorphisms of a smooth G-manifold M which are isotopic to the identity through equivariant Lipschitz homeomorphisms with compact support. The purpose of this paper is to calculate the first homology of the group $L_G(M)$ which is defined as the quotient of $L_G(M)$ by its commutator subgroup.

In the previous papers [**3**], [**4**], we treated the subgroup $\mathscr{H}_{LIP,G}(M)$ of $L_G(M)$ whose elements are isotopic to the identity with respect to the compact open Lipschitz topology, and proved that $\mathscr{H}_{LIP,G}(M)$ is perfect when M is a Lipschitz principal G-manifold or M is a smooth G-manifold for a finite group G.

In this paper first we shall prove that $L_G(M)$ is perfect if M is a smooth principal G-manifold. In the case of $\mathscr{H}_{LIP,G}(M)$, the point of the proof is to construct a Lipschitz homeomorphism of the orbit space M/G depending on the compact open Lipschitz topology which plays a key role in investigating the orbit preserving equivariant Lipschitz homeomorphisms of M. For the case of $L_G(M)$ we shall construct it by a quite different way which depends on the compact open topology (c.f. §2).

Secondly we consider the case of $\boldsymbol{C}^n$ with the canonical $U(n)$-action. We shall prove that the group $L_{U(n)}(\boldsymbol{C}^n)$ is not perfect by calculating the first homology group $H_1(L_{U(n)}(\boldsymbol{C}^n))$.

Let $\mathscr{C}((0,1])$ be the set of real valued functions f on $(0,1]$ such that there exists a positive number K satisfying

$$|f(x) - f(y)| \leq \frac{K}{x}(y - x) \quad \text{for } 0 < x \leq y \leq 1.$$

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Then $\mathscr{C}((0,1])$ is a vector space over $\boldsymbol{R}$. Let $\mathscr{C}_0((0,1])$ denote the subspace of those $f \in \mathscr{C}((0,1])$ with f bounded on $(0,1]$. Then we shall prove that $H_1(L_{U(n)}(\boldsymbol{C}^n))$ is isomorphic to $\mathscr{C}((0,1])/\mathscr{C}_0((0,1])$. The isomorphism is induced from the map assigning each $h \in L_{U(n)}(\boldsymbol{C}^n)$ a function $\hat{a}_h \in \mathscr{C}((0,1])$ which stands for the degree of rotation of h as the point tends to zero (see §3). We note that the group $\mathscr{C}((0,1])/\mathscr{C}_0((0,1])$ is a fairly large group since it contains a linearly independent family of elements parameterized by $(0,1]$. Therefore $H_1(L_{U(n)}(\boldsymbol{C}^n))$ admits continuous moduli.

The situation is quite different in smooth category. Let $D_{U(n)}(\boldsymbol{C}^n)$ denote the group of equivariant diffeomorphisms of $\boldsymbol{C}^n$ which are equivariantly isotopic to the identity through compactly supported isotopies. By [**2**], Theorem 3.2, we have that there exists an isomorphism $H_1(D_{U(n)}(\boldsymbol{C}^n)) \cong \boldsymbol{R} \times U(1)$ induced from the map assigning each $h \in D_{U(n)}(\boldsymbol{C}^n)$ the differential of h at 0. Then it follows from the above result the group $D_{U(n)}(\boldsymbol{C}^n)$ is contained in the commutator subgroup of $L_{U(n)}(\boldsymbol{C}^n)$, which implies that the first homology group of $D_{U(n)}(\boldsymbol{C}^n)$ detects an absolutely different geometric property.

Thirdly we consider the group $L(\boldsymbol{C},0)$ of Lipschitz homeomorphisms of $\boldsymbol{C}$ which are isotopic to the identity through compactly supported Lipschitz homeomorphisms fixing the origin. Applying the above calculation of $H_1(L_{U(1)}(\boldsymbol{C}))$, we can prove that $H_1(L(\boldsymbol{C},0))$ admits continuous moduli.

By [**4**] the group $\mathscr{H}_{LIP}(\boldsymbol{C},0)$ is perfect. Then the above result implies that the group $L(\boldsymbol{C},0)$ is a fairly big group compared to its subgroup $\mathscr{H}_{LIP}(\boldsymbol{C},0)$. It is interesting to see if $H_1(L(\boldsymbol{C}^n,0))$ admits continuous moduli. If we consider the problem classifying Lipschitz manifolds, the first homology group will give a relevant geometric invariant. Therefore the group $\mathscr{H}_{LIP}(M)$ is an intriguing object in Lipschitz category.

The paper is organized as follows. In §2 we prove that $L_G(M)$ is perfect if M is a smooth principal G-manifold. §3 is devoted to investigate some basic properties of the group $L_{U(n)}(\boldsymbol{C}^n)$. In §4 we define the fundamental group homomorphism from $L_{U(n)}(\boldsymbol{C}^n)$ to $\mathscr{C}((0,1])/\mathscr{C}_0((0,1])$. In §5 we calculate $H_1(L_{U(n)}(\boldsymbol{C}^n))$. In §6 we prove that the first homology of the group $L(\boldsymbol{C},0)$ admits continuous moduli.

2. Equivariant Lipschitz homeomorphisms of principal G-manifolds.

Let G be a compact Lie group. Let $\pi : M \to X$ be a smooth principal G-bundle over an n-dimensional smooth manifold X. In this section we shall prove the following.

Theorem 2.1. *If $n > 0$, then $L_G(M)$ is perfect.*

Let $B_r(p)$ denote the closed ball in $\boldsymbol{R}^n$ of radius r centered at p. The following lemma plays a key role in the proof of Theorem 2.1.

Lemma 2.2. *Let $u : \boldsymbol{R}^n \to \boldsymbol{R}$ $(n \geq 1)$ be a Lipschitz function supported in $B_\delta(2\delta, 0, \dots, 0)$. Assume that $K < \frac{4}{81\delta}$ and $|u(x)| \leq \log\frac{3}{2}$ for $x \in \boldsymbol{R}^n$, where K is the Lipschitz constant of u. Then there exist a real valued Lipschitz function $v : \boldsymbol{R}^n \to \boldsymbol{R}$ and a Lipschitz homeomorphism $\varphi : \boldsymbol{R}^n \to \boldsymbol{R}^n$ such that*

(1) $\mathrm{supp}(v)$ *is contained in* $B_{4\delta}(3\delta, 0, \dots, 0)$.

(2) $\mathrm{supp}(\varphi)$ *is contained in* $B_\delta(2\delta, 0, \dots, 0)$.

(3) $v \circ \varphi - v = u$.

PROOF. Let $\xi : \boldsymbol{R} \to \boldsymbol{R}$ be a smooth real valued function such that

$$\xi(t) = \begin{cases} \log t & \left(\frac{2}{3}\delta \le t \le \frac{9}{2}\delta\right) \\ 0 & (t \le 0,\ t \ge 5\delta). \end{cases}$$

Let $\mu : \boldsymbol{R}^{n-1} \to \boldsymbol{R}$ be a smooth function such that, for $x = (x_1, \dots, x_{n-1}) \in \boldsymbol{R}^{n-1}$, $0 \le \mu(x) \le 1$ and

$$\mu(x_1, \dots, x_{n-1}) = \begin{cases} 1 & \left(x_1^2 + \cdots + x_{n-1}^2 \le \delta^2\right), \\ 0 & \left(x_1^2 + \cdots + x_{n-1}^2 \ge 3\delta^2\right). \end{cases}$$

Then define a map $v : \boldsymbol{R}^n \to \boldsymbol{R}$ by $v(x_1, \dots, x_n) = \xi(x_1) \cdot \mu(x_2, \dots, x_n)$ if $n \ge 2$ and $v(x_1) = \xi(x_1)$ if $n = 1$.

Let $\varphi : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be a map defined by

$$\varphi(x_1, \dots, x_n) = \left(x_1 e^{u(x_1, \dots, x_n)}, x_2, \dots, x_n\right).$$

Then for any points $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ of $B_\delta(2\delta, 0, \dots, 0)$, we have

$$\begin{aligned} |(\varphi - 1_{\boldsymbol{R}^n})(x) - (\varphi - 1_{\boldsymbol{R}^n})(y)| &\le \left|(x_1 - y_1)\left(e^{u(x)} - 1\right)\right| + |y_1|\left|e^{u(x)} - e^{u(y)}\right| \\ &\le \left(\left|e^{u(x)} - 1\right| + |y_1| K e^{u(y) + \theta(u(x) - u(y))}\right)|x - y|. \end{aligned}$$

Here θ is a real number satisfying $e^{u(x)} - e^{u(y)} = e^{u(y)+\theta(u(x)-u(y))}(u(x) - u(y))$, $0 < \theta < 1$. We have

$$|e^{u(x)} - 1| + |y_1| K e^{u(y)+\theta(u(x)-u(y))} \le e^{\log\frac{3}{2}} - 1 + 3\delta K e^{3\log\frac{3}{2}} < 1.$$

Since the map φ is the identity outside of $B_\delta(2\delta, 0, \dots, 0)$, it follows from [**3**], Lemma 4.1 that φ is a Lipschitz homeomorphism of $\boldsymbol{R}^n$.

If $x = (x_1, \dots, x_n) \in B_\delta(2\delta, 0, \dots, 0)$, then $\frac{2}{3}\delta \le x_1 e^{u(x)} \le \frac{9}{2}\delta$, and we have

$$v(\varphi(x)) - v(x) = \log\left(x_1 e^{u(x)}\right) - \log x_1 = u(x).$$

Since $\mathrm{supp}(u)$ is contained in $B_\delta(2\delta, 0, \dots, 0)$, we have $v \circ \varphi - v = u$. This completes the proof of Lemma 2.2. □

By the same argument to [**3**], Corollary 5.5 using the result in Siebenmann-Sullivan [**6**], Appendix B, we can prove the following.

LEMMA 2.3 (equivariant fragmentation lemma). *Let $f \in L_G(M)$. For any open ball covering U_i in B, there exist $f_i \in L_G(M)$ $(i = 1, 2, \dots, k)$ such that*

(1) *$f = f_k \circ f_{k-1} \circ \cdots \circ f_1$ and*

(2) *each f_i is equivariantly isotopic to the identity through an equivariant Lipschitz homeomorphism supported in $\pi^{-1}(U_i)$.*

Proof of Theorem 2.1. By Lemma 2.3, we can assume that $M = \boldsymbol{R}^n \times G$. Let $P : L_G(M) \to L(\boldsymbol{R}^n)$ be the natural group homomorphism. Here $L(\boldsymbol{R}^n)$ denotes the group of Lipschitz homeomorphisms of $\boldsymbol{R}^n$ which are isotopic to the identity through Lipschitz homeomorphisms with compact support. Let $\Psi : L(\boldsymbol{R}^n) \to L_G(M)$ be a map defined by $\Psi(f)(x,g) = (f(x), g)$ for $f \in L(\boldsymbol{R}^n)$, $x \in \boldsymbol{R}^n$, $g \in G$. Then Ψ is a group homomorphism which is the right inverse of P.

Let $\mathfrak{g}$ denote the Lie algebra of G and let $\{X_1, \dots, X_l\}$ be a basis of $\mathfrak{g}$. Define the map $\Phi : \mathfrak{g} \to G$ by $\Phi(\sum_{i=1}^{l} c_i X_i) = (\exp c_1 X_1) \cdots (\exp c_l X_l)$. Then there are neighborhoods $\hat{W}$ of 0 in $\mathfrak{g}$ and W of 1 in G such that the restricted map $\Phi|_{\hat{W}} : \hat{W} \to W$ is diffeomorphic.

Let $h \in \mathrm{Ker}P$. We shall prove that $h \in [\mathrm{Ker}P,\ L_G(M)]$. Let $a : \boldsymbol{R}^n \to G$ be the map given by $h(x,g) = (x, ga(x))$ for $x \in \boldsymbol{R}^n$, $g \in G$. Then a is a Lipschitz map. Since the homomorphism P has the right inverse Ψ, there exists a homotopy $\{a_t | 0 \le t \le 1\}$ with $a_0 = 1$, $a_1 = a$. For any integer N, we can write

$$a = a_1 = \left(a_1 \cdot a_{(N-1)/N}^{-1}\right) \cdot \left(a_{(N-1)/N} \cdot a_{(N-2)/N}^{-1}\right) \cdots \left(a_{2/N} \cdot a_{1/N}^{-1}\right) \cdot \left(a_{1/N} \cdot a_0^{-1}\right).$$

We can take N large enough such that the images of $a_{(N-i)/N} \cdot a_{(N-i-1)/N}^{-1}$ $(1 \le i \le l)$ are contained in W. Thus we can assume that the image of a is contained in W. Set $\hat{a} = \Phi^{-1} \circ a$. Then $\hat{a}$ is a Lipschitz map.

Since $\mathrm{supp}(h)$ is compact, there exists a positive number δ such that $\mathrm{supp}(a)$ is contained in D_δ, where $\mathrm{supp}(a) = \overline{\{x \in \boldsymbol{R}^n | a(x) \ne 1\}}$ and $D_\delta = \{x \in \boldsymbol{R}^n | |x| \le \delta\}$. Let $\alpha_i : \boldsymbol{R}^n \to \boldsymbol{R}$ $(1 \le i \le l)$ be the maps given by $\hat{a}(x) = \sum_{i=1}^{l} \alpha_i(x) X_i$. Then α_i $(1 \le i \le l)$ are Lipschitz maps. Let K_i be the Lipschitz constant of the map α_i. Set $K = \max\{K_i | 1 \le i \le l\}$. Let k be a positive integer satisfying $\dfrac{1}{k}|\alpha_i(x)| \le \log \dfrac{3}{2}$, $1 \le i \le l$, for $x \in \boldsymbol{R}^n$ and $\dfrac{K}{k} < \dfrac{4}{81\delta}$. Let $u_i : \boldsymbol{R}^n \to \boldsymbol{R}$ be a map defined by

$$u_i(x_1, \dots, x_n) = \frac{1}{k}\, \alpha_i(x_1 - 2\delta, x_2, \dots, x_n) \quad \text{for } (x_1, \dots, x_n) \in \boldsymbol{R}^n.$$

Since the map u_i satisfies the condition of Lemma 2.2, there exist a real valued Lipschitz function $v_i : \boldsymbol{R}^n \to \boldsymbol{R}$ and a Lipschitz homeomorphism $\varphi_i : \boldsymbol{R}^n \to \boldsymbol{R}^n$ which satisfy the conditions (1), (2) and (3) in Lemma 2.2. Let $H_{u_i}(x,g) = (x,\ g\exp(u_i(x)X_i))$ for $(x,g) \in M$. Then $H_{u_i} \in L_G(M)$ and we have

$$H_{v_i}^{-1} \circ \Psi(\varphi_i)^{-1} \circ H_{v_i} \circ \Psi(\varphi_i) = H_{u_i}.$$

Thus $H_{u_i} \in [\mathrm{Ker}P,\ L_G(M)]$.

Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be a diffeomorphism satisfying

$$f(t) = \begin{cases} t + 2\delta & (|t| \le \delta), \\ t & (|t| \ge 4\delta). \end{cases}$$

Let ψ be an equivariant diffeomorphism defined by

$$\psi((x_1,\dots,x_n),g) = ((\mu(x_2,\dots,x_n)f(x_1) + (1-\mu(x_2,\dots,x_n))x_1, x_2,\dots,x_n),\ g)$$

for $((x_1,\dots,x_n),g) \in M$, where μ is the function defined in the proof of Lemma 2.2. Then for $(x,g) \in D_\delta \times G$ we have

$$\begin{aligned}(\psi^{-1}\circ H_{u_i}\circ\psi)(x,g) &= (x,\ g\exp(u_i(x_1+2\delta, x_2,\dots,x_n)X_i)) \\ &= \left(x,\ g\exp\left(\frac{1}{k}\alpha_i(x)X_i\right)\right) = H_{\frac{1}{k}\alpha_i}(x,g).\end{aligned}$$

Since $\mathrm{supp}(\alpha_i)$ is contained in D_δ, we have $\psi^{-1}\circ H_{u_i}\circ\psi = H_{\frac{1}{k}\alpha_i}$. Thus $H_{\frac{1}{k}\alpha_i} \in [\mathrm{Ker}P,\ L_G(M)]$. Since $H_{\alpha_i} = (H_{\frac{1}{k}\alpha_i})^k$, it follows that $H_{\alpha_i} \in [\mathrm{Ker}P,\ L_G(M)]$. Note that by definition $h = H_{\alpha_l}\circ\cdots\circ H_{\alpha_1}$. Thus $h \in [\mathrm{Ker}P,\ L_G(M)]$, and we have $\mathrm{Ker}P = [\mathrm{Ker}P,\ L_G(M)]$.

Now consider the following exact sequence

$$\mathrm{Ker}P/[\mathrm{Ker}P,\ L_G(M)] \to H_1(L_G(M)) \to H_1(L(\boldsymbol{R}^n)) \to 0.$$

By [**3**] Corollary 2.4, $H_1(L(\boldsymbol{R}^n)) = 0$. Therefore $H_1(L_G(M)) = 0$, and this completes the proof of Theorem 2.1. □

COROLLARY 2.4. *Let M be a smooth G-manifold with one orbit type. If $\dim M/G > 0$, then $L_G(M)$ is perfect.*

PROOF. Let H be an isotropy subgroup of a point of M. Set $M^H = \{x \in M; h\cdot x = x \text{ for } h \in H\}$. Let $N(H)$ denote the normalizer of H in G. Then $N(H)/H$ acts freely on M^H and M is G-diffeomorphic to $G/H \times_{N(H)/H} M^H$. It is easy to see that $L_G(M) \cong L_{N(H)/H}(M^H)$. Therefore Corollary 2.4 follows from Theorem 2.1. □

3. Basic properties of $L_{U(n)}(C)$.

Let D denote the unit disk in $\boldsymbol{C}^n$ and $L_{U(n)}(D,\partial D)$ denote the group of $U(n)$-equivariant Lipschitz homeomorphisms of D which are isotopic to the identity through $U(n)$-equivariant Lipschitz homeomorphisms with identity on the boundary ∂D. Since $\boldsymbol{C}^n \setminus \{0\}$ has one orbit type, by combining Lemma 2.3 with Corollary 2.4, the group $H_1(L_{U(n)}(\boldsymbol{C}^n))$ is isomorphic to $H_1(L_{U(n)}(D,\partial D))$.

Let $e_1 = (1,0,\dots,0) \in D$. Then we have the natural group homomorphism $P : L_{U(n)}(D,\partial D) \to L([0,1])$ given by

$$P(h)(x) = |h(xe_1)| \quad \text{for} \quad h \in L_{U(n)}(D,\partial D),\ 0 \le x \le 1.$$

There exists the right inverse $\Psi : L([0,1]) \to L_{U(n)}(D,\partial D)$ of P defined by

$$\Psi(f)(xg\cdot e_1) = f(x)g\cdot e_1 \quad \text{for} \quad f \in L([0,1]),\ 0 \le x \le 1,\ g \in U(n).$$

Note that the kernel Ker$P$ of P coincides with the set of those $h \in L_{U(n)}(D, \partial D)$ which are orbit preserving and fixing the boundary. Next we shall investigate a relation between the groups Ker$P$ and $\mathscr{C}((0,1])$. Let $h \in$ Ker$P$. If $v \in D$ with $v \neq 0$, then the orbit $U(n) \cdot v$ is diffeomorphic to $U(n)/U(n-1)$. Let $N(U(n-1))$ denote the normalizer of $U(n-1)$ in $U(n)$. Then the group of $U(n)$-equivariant diffeomorphisms of $U(n)/U(n-1)$ is isomorphic to $N(U(n-1))/U(n-1) \cong U(1)$. We have a map $a_h : (0,1] \to U(1)$ satisfying

$$h(xg \cdot e_1) = xga_h(x) \cdot e_1 \quad \text{for} \quad 0 < x \leq 1,\ g \in U(n).$$

Here $U(1)$ acts on D as the scalar multiplication. We investigate the properties of those maps a_h.

For a map $\alpha : (0,1] \to U(1) \subset \boldsymbol{C}$, we define maps $\bar{\alpha} : [0,1] \to D$ and $F_\alpha : D \to D$ as follows.

$$\bar{\alpha}(x) = \begin{cases} x\alpha(x)e_1 & (0 < x \leq 1) \\ 0 & (x = 0) \end{cases},$$

$$F_\alpha(xg \cdot e_1) = g\bar{\alpha}(x) \cdot e_1 \qquad (0 \leq x \leq 1,\ g \in U(n)).$$

Lemma 3.1. *The following conditions* (1), (2) *and* (3) *are equivalent.*
(1) *There exists a positive number K such that*

$$|\alpha(x) - \alpha(y)| \leq \frac{K}{x}(y - x) \quad \textit{for} \quad 0 < x \leq y \leq 1.$$

(2) *$\bar{\alpha}$ is a Lipschitz map.*
(3) *F_α is a Lipschitz map.*

Proof. First assume the condition (1). Then, for $0 < x \leq y \leq 1$, we have

$$|\bar{\alpha}(x) - \bar{\alpha}(y)| \leq x|\alpha(x) - \alpha(y)| + |\alpha(y)||x - y| \leq (K+1)|x - y|.$$

Since $|\bar{\alpha}(x)| \leq x$ for $0 < x \leq 1$, the condition (2) is satisfied.

Secondly assume the condition (2). Then, for $0 < x \leq y \leq 1$, g_1, $g_2 \in U(n)$,

$$\begin{aligned} |F_\alpha(xg_1 \cdot e_1) - F_\alpha(yg_2 \cdot e_1)| &\leq |(\bar{\alpha}(x) - \bar{\alpha}(y))g_1 \cdot e_1| + |\bar{\alpha}(y)(g_1 \cdot e_1 - g_2 \cdot e_1)| \\ &\leq L(|x - y| + |(y - x)g_1 \cdot e_1| + |xg_1 \cdot e_1 - yg_2 \cdot e_1|) \\ &\leq 3L|xg_1 \cdot e_1 - yg_2 \cdot e_1|, \end{aligned}$$

where L is the Lipschitz constant of $\bar{\alpha}$. Since $|F_\alpha(xg_1 \cdot e_1)| \leq x$, the condition (3) is satisfied.

Finally assume the condition (3). Then, for $0 < x \leq y \leq 1$, we have

$$
\begin{aligned}
|\alpha(x)-\alpha(y)| &\le \frac{1}{x}(|x\alpha(x)\cdot e_1 - y\alpha(y)\cdot e_1| + |(y-x)\alpha(y)|)\\
&= \frac{1}{x}(|F_\alpha(xe_1)-F_\alpha(ye_1)| + |y-x|) \le \frac{L+1}{x}\ |y-x|,
\end{aligned}
$$

where L is the Lipschitz constant of F_α. Thus the condition (1) is satisfied and Lemma 3.1 follows. □

Let $E : \boldsymbol{R} \to U(1)$ denote the exponential map given by $E(x) = e^{\sqrt{-1}x}$. Let $h \in \mathrm{Ker}P$. Since h is the identity on ∂D, $a_h(1) = 1$. Let $\hat{a}_h : (0,1] \to \boldsymbol{R}$ be the lifting of a_h for E with $\hat{a}_h(1) = 0$. Then $E \circ \hat{a}_h = a_h$. Let $\mathscr{C}((0,1])$ be the set of real valued functions f on $(0,1]$ such that there exists a positive number K satisfying

$$|f(x)-f(y)| \le \frac{K}{x}(y-x) \quad \text{for} \quad 0 < x \le y \le 1.$$

Let $\mathscr{C}_0((0,1])$ denote the subspace of those $f \in \mathscr{C}((0,1])$ with f bounded on $(0,1]$.

Lemma 3.2. *$\hat{a}_h$ is an element of $\mathscr{C}((0,1])$. Conversely if $\hat{\alpha} \in \mathscr{C}((0,1])$, then $E\circ\hat{\alpha}$ satisfies the condition* (1) *in* Lemma 3.1.

Proof. By Lemma 3.1, there exists a positive number K such that

$$|a_h(x)-a_h(y)| \le \frac{K}{x}(y-x) \quad \text{for} \quad 0 < x \le y \le 1.$$

Note that, for each $x, y \in (0,1]$ with $x < y$, the restriction $a_h|_{[x,y]}$ is Lipschitz. Then we can choose an increasing series of points $x = x_0 < x_1 < \cdots < x_{n-1} < x_n = y$ such that

$$|a_h(x_{i-1}) - a_h(x_i)| \le \sqrt{3} \quad (i = 1, \ldots, n).$$

It follows that

$$|\hat{a}_h(x_{i-1}) - \hat{a}_h(x_i)| \le \frac{2\pi}{3} \quad (i = 1, \ldots, n).$$

Then we have

$$
\begin{aligned}
|a_h(x_{i-1}) - a_h(x_i)| &= \left|e^{\sqrt{-1}\,\hat{a}(x_{i-1})} - e^{\sqrt{-1}\,\hat{a}(x_i)}\right|\\
&= 2\left|\sin\frac{\hat{a}_h(x_{i-1}) - \hat{a}_h(x_i)}{2}\right|\\
&= \left|\cos\frac{\theta(\hat{a}_h(x_{i-1}) - \hat{a}_h(x_i))}{2}\right| |\hat{a}_h(x_{i-1}) - \hat{a}_h(x_i)|,
\end{aligned}
$$

for some $0 < \theta < 1$. Thus

$$|\hat{a}_h(x_{i-1}) - \hat{a}_h(x_i)| \leq 2|a_h(x_{i-1}) - a_h(x_i)| \leq \frac{2K}{x_{i-1}}|x_{i-1} - x_i|.$$

Therefore we have

$$|\hat{a}_h(x) - \hat{a}_h(y)| \leq \sum_{i=1}^{n} \frac{2K}{x_{i-1}}|x_{i-1} - x_i| \leq \frac{2K}{x}(y - x),$$

and then we have that $\hat{a}_h \in \mathscr{C}((0,1])$.

Since

$$|E(x) - E(y)| = \left|e^{\sqrt{-1}x} - e^{\sqrt{-1}y}\right| \leq y - x \quad \text{for} \quad 0 < x \leq y \leq 1,$$

it is clear that, for each $\hat{\alpha} \in \mathscr{C}((0,1])$, $E \circ \hat{\alpha}$ satisfies the condition (1) in Lemma 3.1. This completes the proof of Lemma 3.2. □

4. The fundamental homomorphism.

By Lemma 3.2 we can define a homomorphism

$$T : \mathrm{Ker}P \to \mathscr{C}((0,1])/\mathscr{C}_0((0,1]), \qquad T(h) = \hat{a}_h \mod \mathscr{C}_0((0,1]).$$

Now we have a map

$$\Theta : L_{U(n)}(D, \partial D) \to L([0,1]) \times \mathscr{C}((0,1])/\mathscr{C}_0((0,1])$$

defined by

$$\Theta(h) = (P(h), \quad T(\Psi(P(h))^{-1} \circ h)).$$

Proposition 4.1. *Θ is an onto group homomorphism.*

Proof. First we prove that Θ is a group homomorphism. For each $h \in L_{U(n)}(D, \partial D)$, we set $\tilde{h} = \Psi(P(h))^{-1} \circ h$. Let $h_i \in L_{U(n)}(D, \partial D)$ $(i = 1, 2)$. Since P is a group homomorphism, in order for the map Θ to be a group homomorphism it is sufficient to prove that

$$\hat{a}_{\widetilde{h_1 \circ h_2}} = \hat{a}_{\tilde{h}_1} + \hat{a}_{\tilde{h}_2} \quad \mod \mathscr{C}_0((0,1]).$$

For $0 < x \leq 1$, $g \in U(n)$, we have

$$h_i(xg \cdot e_1) = P(h_i)(x)\, g a_{\tilde{h}_i}(x)^{-1} \cdot e_1 \quad (i = 1, 2),$$

and

$$(h_1 \circ h_2)(xg \cdot e_1) = P(h_1 \circ h_2)(x)\, g a_{\widetilde{h_1 \circ h_2}}(x)^{-1} \cdot e_1.$$

On the other hand we have

$$(h_1 \circ h_2)(xg \cdot e_1) = P(h_1 \circ h_2)(x)\, g a_{\tilde{h}_2}(x)^{-1} a_{\tilde{h}_1}(P(h_2)(x))^{-1} \cdot e_1.$$

Then

$$a_{\widetilde{h_1 \circ h_2}} = (a_{\tilde{h}_1} \circ P(h_2)) \cdot a_{\tilde{h}_2}.$$

Thus

$$\hat{a}_{\widetilde{h_1 \circ h_2}} = \hat{a}_{\tilde{h}_1} \circ P(h_2) + \hat{a}_{\tilde{h}_2}.$$

Let L and L' be the Lipschitz constants of $P(h_2)$ and $P(h_2)^{-1}$, respectively. Let $x \in (0,1]$. For the case $x \le P(h_2)(x)$, by Lemma 3.2 there exists a positive number K such that

$$\left|\hat{a}_{\tilde{h}_1}(P(h_2)(x)) - \hat{a}_{\tilde{h}_1}(x)\right| \le \frac{K}{x}|P(h_2)(x) - x| \le K(L+1).$$

By definition $x \le L' P(h_2)(x)$. Then, for the case $P(h_2)(x) < x$, we have

$$\left|\hat{a}_{\tilde{h}_1}(P(h_2)(x)) - \hat{a}_{\tilde{h}_1}(x)\right| \le \frac{K}{P(h_2)(x)}|P(h_2)(x) - x| \le K(1+L').$$

Then

$$\hat{a}_{\tilde{h}_1} \circ P(h_2) - \hat{a}_{\tilde{h}_1} \in \mathscr{C}_0((0,1]).$$

Thus

$$\hat{a}_{\widetilde{h_1 \circ h_2}} = \hat{a}_{\tilde{h}_1} + \hat{a}_{\tilde{h}_2} \mod \mathscr{C}_0((0,1]).$$

Therefore Θ is a group homomorphism.

Let $f \in L([0,1])$, $\hat{\alpha} \in \mathscr{C}((0,1])$. Combining Lemma 3.1 with Lemma 3.2, we have that $F_{E \circ \hat{\alpha}} \in \operatorname{Ker} P$. Set

$$h(xg \cdot e_1) = f(x) F_{E\circ\hat{\alpha}}(xg \cdot e_1) \quad \text{for} \quad 0 \le x \le 1,\ g \in U(n).$$

Then we see that $h \in L_{U(n)}(D, \partial D)$ and $\Theta(h) = (f,\ \hat{\alpha} \mod \mathscr{C}_0((0,1]))$. Thus Θ is onto. This completes the proof of Proposition 4.1. □

5. The first homology of $L_{U(n)}(C^n)$.

PROPOSITION 5.1. *$\operatorname{Ker}\Theta$ is contained in the commutator subgroup of $L_{U(n)}(D, \partial D)$.*

Proof. If $h \in \operatorname{Ker} \Theta$, then $h \in \operatorname{Ker} P$ and $\hat{a}_h \in \mathscr{C}_0((0,1])$. Thus, for any positive number ε, there exists an integer n such that $\left|\dfrac{\hat{a}_h(x)}{n}\right| \leq \varepsilon$ for $0 < x \leq 1$ and

$$\left|\frac{\hat{a}_h(x)}{n} - \frac{\hat{a}_h(y)}{n}\right| \leq \frac{\varepsilon}{x}(y-x) \quad \text{for} \quad 0 < x \leq y \leq 1.$$

Note that $a_h = E(n\hat{a}_h) = E(\hat{a}_h)^n$. Then, for a sufficiently small positive number ε, we can assume that $|\hat{a}_h(x)| \leq \varepsilon$ for $0 < x \leq 1$ and

$$|\hat{a}_h(x) - \hat{a}_h(y)| \leq \frac{\varepsilon}{x}(y-x) \quad \text{for} \quad 0 < x \leq y \leq 1.$$

Let v be a real valued smooth monotone increasing function on $(0,1]$ such that

$$v(x) = \begin{cases} \log x & (0 < x \leq 1/2), \\ 0 & (3/4 \leq x \leq 1). \end{cases}$$

Then it is easy to see $v \in \mathscr{C}((0,1])$. Let f be a real valued function on $[0,1]$ defined by

$$f(x) = \begin{cases} xe^{\hat{a}_h(x)} & (0 < x \leq 1), \\ 0 & (x = 0). \end{cases}$$

Note that $f(1) = 1$. We shall prove that $f \in L([0,1])$ for sufficiently small ε. If $0 < x \leq y \leq 1$, then we have

$$\begin{aligned} &|(f(y) - y) - (f(x) - x)| \\ &\quad = \left|(y-x)(e^{\hat{a}_h(y)} - 1) + x(e^{\hat{a}_h(y)} - e^{\hat{a}_h(x)})\right| \\ &\quad \leq (y-x)\left|e^{|\hat{a}_h(y)|} - 1\right| + x|\hat{a}_h(y) - \hat{a}_h(x)|e^{\hat{a}_h(x) + \theta(\hat{a}_h(y) - \hat{a}_h(x))} \\ &\quad \leq ((e^{\varepsilon} - 1) + \varepsilon e^{3\varepsilon})(y-x), \end{aligned}$$

for some $0 < \theta < 1$. Here we take a positive number ε satisfying

$$(e^{\varepsilon} - 1) + \varepsilon e^{3\varepsilon} < 1.$$

Then it follows from [**3**], Lemma 4.1 that the function f is a Lipschitz homeomorphism of $[0,1]$ which is isotopic to the identity through Lipschitz homeomorphisms.

If $0 < x \leq \dfrac{1}{2e^{\varepsilon}}$, then we have

$$v(f(x)) - v(x) = \log\left(xe^{\hat{a}_h(x)}\right) - \log x = \hat{a}_h(x).$$

Then, for $0 < x \le \dfrac{1}{2e^{\varepsilon}}$, $g \in U(n)$ we have

$$\begin{aligned}\big(F_{E\circ v}^{-1} \circ \Psi(f)^{-1} \circ F_{E\circ v} \circ \Psi(f)\big)(xg \cdot e_1) &= \big(F_{E\circ v}^{-1} \circ \Psi(f)^{-1} \circ F_{E\circ v}\big)(f(x)g \cdot e_1)\\ &= \big(F_{E\circ v}^{-1} \circ \Psi(f)^{-1}\big)\big(f(x)ge^{\sqrt{-1}\,v(f(x))} \cdot e_1\big)\\ &= F_{E\circ v}^{-1}\big(xge^{\sqrt{-1}\,v(f(x))} \cdot e_1\big)\\ &= xge^{\sqrt{-1}\,v(f(x))}e^{-\sqrt{-1}\,v(x)} \cdot e_1\\ &= h(xg \cdot e_1).\end{aligned}$$

Set

$$h_1 = h \circ \Psi(f)^{-1} \circ F_{E\circ v}^{-1} \circ \Psi(f) \circ F_{E\circ v}.$$

Then

$$h_1(xg \cdot e_1) = xg \cdot e_1 \quad \text{for} \quad 0 \le x \le \frac{1}{2e^{\varepsilon}},\ g \in U(n).$$

Thus $\mathrm{supp}(h_1)$ is contained in $D \backslash \{0\}$. It follows from Corollary 2.4 that g is contained in the commutator subgroup of $L_{U(n)}(D, \partial D)$. Hence h is also contained in the commutator subgroup. This completes the proof of Proposition 5.1. □

THEOREM 5.2.

$$H_1(L_{U(n)}(\boldsymbol{C}^n)) \cong \mathscr{C}((0,1])/\mathscr{C}_0((0,1]).$$

PROOF. Let $\iota : \mathrm{Ker}\Theta \to L_{U(n)}(D, \partial D)$ denote the inclusion. By Proposition 4.1 we have the following exact sequence.

$$\begin{aligned}\mathrm{Ker}\Theta/[\mathrm{Ker}\Theta, L_{U(n)}(D, \partial D)] &\xrightarrow{\iota_*} H_1(L_{U(n)}(D, \partial D))\\ &\xrightarrow{\Theta_*} H_1(L([0,1]) \times \mathscr{C}((0,1])/\mathscr{C}_0((0,1])) \to 1.\end{aligned}$$

Since $\iota_* = 0$ by Proposition 5.1, Θ_* is isomorphic. By Tsuboi [**7**], Theorem 3.2 or [**4**], Remark 2.6, the group $L([0,1])$ is perfect. Thus we have

$$H_1(L_{U(n)}(D, \partial D)) \cong \mathscr{C}((0,1])/\mathscr{C}_0((0,1]).$$

Since $H_1(L_{U(n)}(D, \partial D)) \cong H_1(L_{U(n)}(\boldsymbol{C}^n))$, Theorem 5.2 follows. □

REMARK. (1) Let v_c $(0 < c \le 1)$ be real valued smooth functions on $(0,1]$ such that

$$v_c(x) = \begin{cases} (-\log x)^c & (0 < x \le 1/2), \\ 0 & (3/4 \le x \le 1). \end{cases}$$

Then $v_c \in \mathscr{C}((0,1])$. Thus the group $\mathscr{C}((0,1])/\mathscr{C}_0((0,1])$ contains a linearly independent family $\{v_c \mod \mathscr{C}_0((0,1])\ ;\ 0 < c \le 1\}$.

(2) By using the integration by parts, we can prove that $\mathscr{C}((0,1])$ is a subspace of the function space $L^1((0,1])$. We expect that the quotient space $\mathscr{C}((0,1])/\mathscr{C}_0((0,1])$ has some analytic meaning.

Let $S(\boldsymbol{C}^n \oplus \boldsymbol{R})$ be the unit sphere in $\boldsymbol{C}^n \oplus \boldsymbol{R}$ with the canonical $U(n)$-action. Combining Corollary 2.4 with Theorem 5.2 we have

Corollary 5.3.

$$H_1(L_{U(n)}(S(\boldsymbol{C}^n \oplus \boldsymbol{R}))) \cong \mathscr{C}((0,1])/\mathscr{C}_0((0,1]) \times \mathscr{C}((0,1])/\mathscr{C}_0((0,1]).$$

6. The first homology of $L(\boldsymbol{C}, 0)$.

Let $L(\boldsymbol{C}, 0)$ denote the group of Lipschitz homeomorphisms of $\boldsymbol{C}$ which are isotopic to the identity through compactly supported Lipschitz homeomorphisms fixing the origin. Set $D^* = D \setminus \{0\}$. For $h \in L(\boldsymbol{C}, 0)$ let $c_h :\ D^* \to S^1$ be a map defined by

$$c_h(rz) = \frac{h(rz)}{|h(rz)|}\, z^{-1} \quad \text{for } 0 < r \le 1,\ z \in S^1.$$

There exists a unique Lipschitz map $\hat{c}_h : D^* \to \boldsymbol{R}$ such that $E \circ \hat{c}_h = c_h$ and $\hat{c}_h = 0$ on ∂D^*. Let $\mathscr{C}(D^*)$ be the set of real valued functions f on D^* such that there exists a positive number K satisfying

$$|f(x) - f(y)| \le \frac{K}{|x|}|y - x| \qquad \text{for } x, y \in D^* \text{ with } 0 < |x| \le |y| \le 1.$$

Lemma 6.1. $\hat{c}_h \in \mathscr{C}(D^*)$.

Proof. Let $b_h : D^* \to S^1$ be a map defined by $b_h(x) = \dfrac{h(x)}{|h(x)|}$ for $x \in D^*$. Let L and L' be the Lipschitz constants of h and h^{-1}, respectively. Assume $0 < |x| \le |y| \le 1$ for $x, y \in D^*$. Then

$$\begin{aligned} |b_h(x) - b_h(y)| &= \frac{1}{|h(x)||h(y)|}|(|h(y)| - |h(x)|)h(x) + |h(x)|(h(x) - h(y))| \\ &\le \frac{2}{|h(y)|}|h(x) - h(y)| \le \frac{2LL'}{|x|}|x - y|. \end{aligned}$$

Thus we have

$$\begin{aligned}|c_h(x) - c_h(y)| &= \left| b_h(x)\frac{\bar{x}}{|x|} - b_h(y)\frac{\bar{y}}{|y|} \right| \\ &\le |b_h(x)|\left|\frac{\bar{x}}{|x|} - \frac{\bar{y}}{|y|}\right| + |b_h(x) - b_h(y)|\left|\frac{\bar{y}}{|y|}\right| \\ &\le \frac{2LL' + 1}{|x|}|x - y|.\end{aligned}$$

Since $|\hat{c}_h(x) - \hat{c}_h(y)| \le 2|c_h(x) - c_h(y)|$, it follows that $\hat{c}_h \in \mathscr{C}(D^*)$ and Lemma 6.1 follows.

Let $\mathscr{C}_0(D^*)$ denote the subspace of those $f \in \mathscr{C}(D^*)$ with f bounded on D^*. Let $\bar{T}: L(\boldsymbol{C}, 0) \to \mathscr{C}(D^*)/\mathscr{C}_0(D^*)$ be a map defined by $\bar{T}(h) = \hat{c}_h \bmod \mathscr{C}_0(D^*)$.

Proposition 6.2. *$\bar{T}$ is a group homomorphism.*

Proof. Let $g, h \in L(\boldsymbol{C}, 0)$. Since

$$g(x) = |g(x)|\frac{x}{|x|}\, c_g(x) \quad \text{for} \quad x \in D^*,$$

we have

$$g(h(x)) = |g(h(x))|\, \frac{h(x)}{|h(x)|}\, c_g(h(x)).$$

On the other hand

$$g(h(x)) = |g(h(x))|\, \frac{x}{|x|}\, c_{g\circ h}(x).$$

Then

$$c_{g\circ h}(x) = c_h(x)c_g(h(x)).$$

Thus

$$\hat{c}_{g\circ h} = \hat{c}_h + \hat{c}_g \circ h.$$

Let L and L' be the Lipschitz constants of h and h^{-1} respectively. Let $x \in D^*$. For the case $|x| \le |h(x)|$, by Lemma 6.1 there exists a positive number K such that

$$|\hat{c}_g(h(x)) - \hat{c}_g(x)| \le \frac{K}{|x|}\, |h(x) - x| \le K(L + 1).$$

By definition $|x| \le L'|h(x)|$. Then for the case $|x| > |h(x)|$,

$$|\hat{c}_g(h(x)) - \hat{c}_g(x)| \leq \frac{K}{|h(x)|}\,|h(x) - x| \leq KL'(L+1).$$

Then

$$\hat{c}_g \circ h - \hat{c}_g \in \mathscr{C}_0(D^*).$$

Thus

$$\hat{c}_{g\circ h} = \hat{c}_h + \hat{c}_g \quad \mod \ \mathscr{C}_0(D^*),$$

which completes the proof of Proposition 6.2. □

Let $j : \mathscr{C}((0,1]) \hookrightarrow \mathscr{C}(D^*)$ be a map defined by $j(\alpha)(x) = \alpha(|x|)$ for $x \in D^*$.

Lemma 6.3. *The map j induces the isomorphism*

$$j_* : \mathscr{C}((0,1])/\mathscr{C}_0((0,1]) \cong \mathscr{C}(D^*)/\mathscr{C}_0(D^*).$$

Proof. Let $\alpha \in \mathscr{C}((0,1])$. By definition $\alpha(r) = j(\alpha)(re_1)$ for $0 < r \leq 1$. If $j(\alpha)$ is bounded, then α is also bounded. Thus j_* is injective.

For $\gamma \in \mathscr{C}(D^*)$, let $\alpha(r) = \gamma(re_1)$. Then $\alpha \in \mathscr{C}((0,1])$. If $x \in D^*$, then

$$|\gamma(x) - j(\alpha)(x)| = |\gamma(x) - \gamma(|x|e_1)| \leq \frac{K}{|x|}\,|x - |x|e_1| \leq 2K,$$

where K is a positive number such that

$$|\gamma(x) - \gamma(y)| \leq \frac{K}{|x|}|y - x| \qquad \text{for } x, y \in D^* \text{ with } 0 < |x| \leq |y| \leq 1.$$

Then $\gamma - j(\alpha) \in \mathscr{C}_0(D^*)$. Thus $j_*(\alpha \mod \mathscr{C}_0((0,1])) = \gamma \mod \mathscr{C}_0(D^*)$, which completes the proof of Lemma 6.3. □

Let $i : L_{U(1)}(D) \hookrightarrow L(\boldsymbol{C}, 0)$ be the inclusion. Then

Theorem 6.4. *The induced homomorphism $i_* : H_1(L_{U(1)}(\boldsymbol{C})) \to H_1(L(\boldsymbol{C}, 0))$ is injective.*

Proof. We have the following diagram

$$\begin{array}{ccc}
H_1(L_{U(1)}(D)) & \xrightarrow[\cong]{T_*} & \mathscr{C}((0,1])/\mathscr{C}_0((0,1]) \\
\Big\downarrow {\scriptstyle i_*} & & {\scriptstyle \cong}\Big\downarrow {\scriptstyle j_*} \\
H_1(L(\boldsymbol{C}, 0)) & \xrightarrow{\bar{T}_*} & \mathscr{C}(D^*)/\mathscr{C}_0(D^*).
\end{array}$$

By Theorem 5.2 and Lemma 6.3, the maps T_* and j_* are isomorphisms. Then the map i_* is injective. □

Corollary 6.5. *The first homology of the group* $L(\boldsymbol{C}, 0)$ *admits continuous moduli.*

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