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Spacelike Dupin hypersurfaces in Lorentzian space forms

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Abstract. Similar to the definition in Riemannian space forms, we define the spacelike Dupin hypersurface in Lorentzian space forms. As conformal invariant objects, spacelike Dupin hypersurfaces are studied in this paper using the framework of the conformal geometry of spacelike hypersurfaces. Further we classify the spacelike Dupin hypersurfaces with constant Möbius curvatures, which are also called conformal isoparametric hypersurface.

1. Introduction.

Since Dupin surfaces were first studied by Dupin in 1822, the study of Dupin hypersurfaces in $\mathbb{R}^{n+1}$ has been a topic of increasing interest, (see [**2**], [**3**], [**4**], [**8**], [**9**], [**10**], [**13**], [**14**]), especially recently. In this paper we study spacelike Dupin hypersurfaces in the Lorentzian space form $M_1^{n+1}(c)$.

Let $\mathbb{R}_s^{n+2}$ be the real vector space $\mathbb{R}^{n+2}$ with the Lorentzian product $\langle , \rangle_s$ given by

$$\langle X, Y\rangle_s = -\sum_{i=1}^{s} x_i y_i + \sum_{j=s+1}^{n+2} x_j y_j.$$

For any $a > 0$, the standard sphere $\mathbb{S}^{n+1}(a)$, the hyperbolic space $\mathbb{H}^{n+1}(-a)$, the de sitter space $\mathbb{S}_1^{n+1}(a)$ and the anti-de sitter space $\mathbb{H}_1^{n+1}(-a)$ are defined by

$$\mathbb{S}^{n+1}(a) = \{x \in \mathbb{R}^{n+2} | x \cdot x = a^2\},\ \mathbb{H}^{n+1}(-a) = \{x \in \mathbb{R}_1^{n+2} | \langle x, x\rangle_1 = -a^2\},$$
$$\mathbb{S}_1^{n+1}(a) = \{x \in \mathbb{R}_1^{n+2} | \langle x, x\rangle_1 = a^2\},\ \mathbb{H}_1^{n+1}(-a) = \{x \in \mathbb{R}_2^{n+2} | \langle x, x\rangle_2 = -a^2\}.$$

Let $M_1^{n+1}(c)$ be a Lorentz space form. When $c = 0$, $M_1^{n+1}(c) = \mathbb{R}_1^{n+1}$. When $c = 1$, $M_1^{n+1}(c) = \mathbb{S}_1^{n+1}(1)$. When $c = -1$, $M_1^{n+1}(c) = \mathbb{H}_1^{n+1}(-1)$.

For Lorentz space form $M_1^{n+1}(c)$, there exists a united conformal compactification $\mathbb{Q}_1^{n+1}$, which is the projectivized light cone in $\mathbb{R}P^{n+2}$ induced from $\mathbb{R}_2^{n+3}$. Using the conformal compactification $\mathbb{Q}_1^{n+1}$, we study the conformal geometry of spacelike hypersurfaces in $M_1^{n+1}(c)$. We define the conformal metric g and the conformal second fundamental form B on a spacelike hypersurface, which determine the spacelike hypersurface up to a conformal transformation of $M_1^{n+1}(c)$. By these conformal invariants, it is clear that the Möbius curvatures of a spacelike hypersurface are invariant under the conformal transformations of $M_1^{n+1}(c)$ (see section 2). The Möbius curvatures of a spacelike

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hypersurface are defined by

$$\mathbb{M}_{ijs} = \frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_s}, \ 1 \leq i, j, k \leq n,$$

where $\lambda_1, \cdots, \lambda_r$ are the principal curvatures of the spacelike hypersurface.

Similar to the Dupin hypersurfaces in Riemannian space forms, we define the spacelike Dupin hypersurface in a Lorentzian space form. Let $x : M^n \to M_1^{n+1}(c)$ be a spacelike hypersurface in the Lorentzian space form $M_1^{n+1}(c)$. A curvature surface of M^n is a smooth connected submanifold S such that for each point $p \in S$, the tangent space T_pS is equal to a principal space of the shape operator $\mathcal{A}$ of the hypersurface M^n at p. The spacelike hypersurface M^n is called a spacelike Dupin hypersurface if, along each curvature surface, the associated principal curvature is constant. The simple examples of the spacelike Dupin hypersurface are the spacelike isoparametric hypersurfaces in $M_1^{n+1}(c)$, which are completely classified (see [**5**], [**6**], [**7**], [**16**]).

Using the conformal geometry of spacelike hypersurfaces in $M_1^{n+1}(c)$, we can prove that the spacelike Dupin hypersurfaces in $M_1^{n+1}(c)$ are invariant under the conformal transformations of $M_1^{n+1}(c)$. Like Pinkall's method of constructed Dupin hypersurface in $\mathbb{R}^{n+1}$ ([**14**]), we can use the basic constructions of building cylinders and cones over a spacelike Dupin hypersurface W^{n-1} in $\mathbb{R}_1^n$ with $r-1$ principal curvatures to get a spacelike Dupin hypersurface W^{n-1+k} in $\mathbb{R}_1^{n+k}$ with r principal curvatures. In general, these constructions are local. Therefore we have the following result.

Theorem 1.1. *Given positive integers $v_1, v_2, \ldots, v_r$ with*

$$v_1 + v_2 + \cdots + v_r = n,$$

there exists a spacelike Dupin hypersurface in $\mathbb{R}_1^{n+1}$ with r distinct principal curvatures having respective multiplicities $v_1, v_2, \ldots, v_r$.

For some special spacelike Dupin hypersurfaces, we have the following results.

Theorem 1.2. *Let $x : M^n \to M_1^{n+1}(c)$ be a spacelike Dupin hypersurface in $M_1^{n+1}(c)$ with r distinct principal curvatures. If $r = 2$, then locally x is conformally equivalent to one of the following hypersurfaces.*
(1) $\mathbb{S}^k(\sqrt{a^2+1}) \times \mathbb{H}^{n-k}(-a) \subset \mathbb{S}_1^{n+1}(1), \ a > 0, \ 1 \leq k \leq n-1$;
(2) $\mathbb{H}^k(-a) \times \mathbb{H}^{n-k}(-\sqrt{1-a^2}) \subset \mathbb{H}_1^{n+1}(-1), \ 0 < a < 1, \ 1 \leq k \leq n-1$;
(3) $\mathbb{H}^k(-a) \times \mathbb{R}^{n-k} \subset \mathbb{R}_1^{n+1}, \ a > 0, \ 1 \leq k \leq n-1$.

Theorem 1.3. *Let $x : M^n \to M_1^{n+1}(c)$ be a spacelike Dupin hypersurface in $M_1^{n+1}(c)$ with r distinct principal curvatures. If $r \geq 3$ and the Möbius curvatures are constant, then $r = 3$, and locally x is conformally equivalent to the following hypersurface,*

$$x : \mathbb{H}^q(-\sqrt{a^2-1}) \times \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \to \mathbb{R}_1^{n+1},$$

defined by

$$x(u', u'', t, u''') = (tu', tu'', u'''),$$

where $u' \in \mathbb{H}^q(-\sqrt{a^2-1}), u'' \in \mathbb{S}^p(a), u''' \in \mathbb{R}^{n-p-q-1},\ a > 1.$

Remark 1.1. A spacelike hypersurface with constant conformal principal curvatures and vanishing conformal 1-form is called a conformal isoparametric hypersurfaces (see [**11**]). In section 3, we prove that the spacelike Dupin hypersurfaces with constant Möbius curvatures are in fact the conformal isoparametric hypersurfaces. Thus Theorem 1.2 and 1.3 give a classification of conformal isoparametric hypersurfaces.

This paper is organized as follows. In section 2, we study the conformal geometry of spacelike hypersurfaces in $M_1^{n+1}(c)$. In section 3, we study the spacelike Dupin hypersurfaces in the framework of conformal geometry. In section 4 and section 5, we give the proof of Theorem 1.2 and Theorem 1.3, respectively.

2. Conformal geometry of spacelike hypersurfaces in $M_1^{n+1}(c)$.

In this section, following Wang's idea in paper [**15**], we define some conformal invariants on a spacelike hypersurface and give a congruent theorem of the spacelike hypersurfaces under the conformal group of $M_1^{n+1}(c)$.

We denote by C^{n+2} the cone in $\mathbb{R}_2^{n+3}$ and by $\mathbb{Q}_1^{n+1}$ the conformal compactification space in $\mathbb{R}P^{n+2}$,

$$C^{n+2} = \{X \in \mathbb{R}_2^{n+3} | \langle X, X\rangle_2 = 0, X \neq 0\},$$
$$\mathbb{Q}_1^{n+1} = \{[X] \in \mathbb{R}P^{n+2} | \langle X, X\rangle_2 = 0\}.$$

Let $O(n+3,2)$ be the Lorentzian group of $\mathbb{R}_2^{n+3}$ keeping the Lorentzian product $\langle X, Y\rangle_2$ invariant. Then $O(n+3,2)$ is a transformation group on $\mathbb{Q}_1^{n+1}$ defined by

$$T([X]) = [XT],\ X \in C^{n+2},\ T \in O(n+3,2).$$

Topologically $\mathbb{Q}_1^{n+1}$ is identified with the compact space $S^n \times S^1/S^0$, which is endowed by a standard Lorentzian metric $h = g_{S^n} \oplus (-g_{S^1})$, where g_{S^k} denotes the standard metric of the k-dimensional sphere S^k. Then $\mathbb{Q}_1^{n+1}$ has conformal metric

$$[h] = \{e^\tau h | \tau \in C^\infty(\mathbb{Q}_1^{n+1})\}$$

and $[O(n+3,2)]$ is the conformal transformation group of $\mathbb{Q}_1^{n+1}$(see[**1**], [**12**]).

Denote $P = \{[X] \in \mathbb{Q}_1^{n+1} | x_1 = x_{n+2}\},\ P_- = \{[X] \in \mathbb{Q}_1^{n+1} | x_{n+2} = 0\},\ P_+ = \{[X] \in \mathbb{Q}_1^{n+1} | x_1 = 0\}$, we can define the following conformal diffeomorphisms,

$$\sigma_0 : \mathbb{R}_1^{n+1} \to \mathbb{Q}_1^{n+1}\backslash P, \quad u \mapsto \left[\left(\frac{<u,u>_1 + 1}{2}, u, \frac{<u,u>_1 - 1}{2}\right)\right],$$
$$\sigma_1 : \mathbb{S}_1^{n+1}(1) \to \mathbb{Q}_1^{n+1}\backslash P_+, \quad u \mapsto [(1,u)],$$
$$\sigma_{-1} : \mathbb{H}_1^{n+1}(-1) \to \mathbb{Q}_1^{n+1}\backslash P_-, \quad u \mapsto [(u,1)].$$

We may regard $\mathbb{Q}_1^{n+1}$ as the common compactification of $\mathbb{R}_1^{n+1}, \mathbb{S}_1^{n+1}(1), \mathbb{H}_1^{n+1}(-1)$.

Let $x : M^n \to M_1^{n+1}(c)$ be a spacelike hypersurface. Using σ_c, we obtain the hypersurface in $\mathbb{Q}_1^{n+1}$, $\sigma_c \circ x : M^n \to \mathbb{Q}_1^{n+1}$. From [**1**], we have the following theorem.

Theorem 2.1. *Two hypersurfaces $x, \bar{x} : M^n \to M_1^{n+1}(c)$ are conformally equivalent if and only if there exists $T \in O(n+3,2)$ such that $\sigma_c \circ x = T(\sigma_c \circ \bar{x}) : M^n \to \mathbb{Q}_1^{n+1}$.*

Since $x : M^n \to M_1^{n+1}(c)$ is a spacelike hypersurface, $(\sigma_c \circ x)_*(TM^n)$ is a positive definite subbundle of $T\mathbb{Q}_1^{n+1}$. For any local lift Z of the standard projection $\pi : C^{n+2} \to \mathbb{Q}_1^{n+1}$, we get a local lift $y = Z \circ \sigma_c \circ x : U \to C^{n+1}$ of $\sigma_c \circ x : M \to \mathbb{Q}_1^{n+1}$ in an open subset U of M^n. Thus $\langle \mathrm{d}y, \mathrm{d}y\rangle_2 = \rho^2 \langle dx, dx\rangle_s$ is a local metric, where $\rho \in C^\infty(U)$. We denote by Δ and κ the Laplacian operator and the normalized scalar curvature with respect to the local positive definite metric $\langle \mathrm{d}y, \mathrm{d}y\rangle$, respectively. Similar to Wang's proof of Theorem 1.2 in [**15**], we can get the following theorem.

Theorem 2.2. *Let $x : M^n \to M_1^{n+1}(c)$ be a spacelike hypersurface, then the 2-form $g = -(\langle \Delta y, \Delta y\rangle_2 - n^2\kappa)\langle dy, dy\rangle_2$ is a globally defined conformal invariant. Moreover, g is positive definite at any non-umbilical point of M^n.*

We call g the conformal metric of the spacelike hypersurface M^n. There exists a unique lift

$$Y : M \to C^{n+2}$$

such that $g = \langle \mathrm{d}Y, \mathrm{d}Y\rangle_2$. We call Y the conformal position vector of the spacelike hypersurface M^n. Theorem 2.2 implies the following theorem.

Theorem 2.3. *Two spacelike hypersurfaces $x, \bar{x} : M^n \to M_1^{n+1}(c)$ are conformally equivalent if and only if there exists $T \in O(n+3,2)$ such that $\bar{Y} = YT$, where $Y, \bar{Y}$ are the conformal position vector of $x, \bar{x}$, respectively.*

Let $\{E_1, \cdots, E_n\}$ be a local orthonormal basis of M^n with respect to g with dual basis $\{\omega_1, \cdots, \omega_n\}$. Denote $Y_i = E_i(Y)$ and define

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle \Delta Y, \Delta Y\rangle_2 Y,$$

where Δ is the Laplace operator of g, then we have

$$\langle N, Y\rangle_2 = 1,\ \langle N, N\rangle_2 = 0,\ \langle N, Y_k\rangle_2 = 0,\ \langle Y_i, Y_j\rangle_2 = \delta_{ij}, \quad 1 \le i, j, k \le n.$$

We may decompose $\mathbb{R}_2^{n+3}$ such that

$$\mathbb{R}_2^{n+3} = \mathrm{span}\{Y, N\} \oplus \mathrm{span}\{Y_1, \cdots, Y_n\} \oplus \mathbb{V},$$

where $\mathbb{V} \perp \mathrm{span}\{Y, N, Y_1, \cdots, Y_n\}$. We call $\mathbb{V}$ the conformal normal bundle of x, which is linear bundle. Let ξ be a local section of $\mathbb{V}$ and $<\xi, \xi>_2 = -1$, then $\{Y, N, Y_1, \cdots, Y_n, \xi\}$ forms a moving frame in $\mathbb{R}_2^{n+3}$ along M^n. We write the structure equations as follows,

$$
\begin{aligned}
\mathrm{d}Y &= \sum_i \omega_i Y_i,\\
\mathrm{d}N &= \sum_{ij} A_{ij}\omega_j Y_i + \sum_i C_i\omega_i\xi,\\
\mathrm{d}Y_i &= -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_j B_{ij}\omega_j\xi,\\
\mathrm{d}\xi &= \sum_i C_i\omega_i Y + \sum_{ij} B_{ij}\omega_j Y_i,
\end{aligned}
\tag{2.1}
$$

where $\omega_{ij}(= -\omega_{ij})$ are the connection 1-forms on M^n with respect to $\{\omega_1, \cdots, \omega_n\}$. It is clear that $A = \sum_{ij} A_{ij}\omega_j \otimes \omega_i,\ B = \sum_{ij} B_{ij}\omega_j \otimes \omega_i,\ C = \sum_i C_i\omega_i$ are globally defined conformal invariants. We call $A,\ B$ and C the conformal 2-tensor, the conformal second fundamental form and the conformal 1-form, respectively. The covariant derivatives of these tensors with respect to ω_{ij} are defined by:

$$
\begin{aligned}
\sum_j C_{i,j}\omega_j &= dC_i + \sum_k C_k\omega_{kj},\\
\sum_k A_{ij,k}\omega_k &= dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki},\\
\sum_k B_{ij,k}\omega_k &= dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}.
\end{aligned}
$$

By exterior differentiation of structure equations (2.1), we can get the integrable conditions of the structure equations

$$A_{ij} = A_{ji},\ B_{ij} = B_{ji},$$

$$A_{ij,k} - A_{ik,j} = B_{ij}C_k - B_{ik}C_j, \tag{2.2}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{2.3}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}A_{kj} - B_{jk}A_{ki}), \tag{2.4}$$

$$R_{ijkl} = B_{il}B_{jk} - B_{ik}B_{jl} + A_{ik}\delta_{jl} + A_{jl}\delta_{ik} - A_{il}\delta_{jk} - A_{jk}\delta_{il}. \tag{2.5}$$

Furthermore, we have

$$
\begin{aligned}
&\mathrm{tr}(A) = \frac{1}{2n}(n^2\kappa - 1), \quad R_{ij} = \mathrm{tr}(A)\delta_{ij} + (n-2)A_{ij} + \sum_k B_{ik}B_{kj},\\
&(1-n)C_i = \sum_j B_{ij,j}, \quad \sum_{ij} B_{ij}^2 = \frac{n-1}{n}, \quad \sum_i B_{ii} = 0,
\end{aligned}
\tag{2.6}
$$

where κ is the normalized scalar curvature of g. From (2.6), we see that when $n \geq 3$, all coefficients in the structure equations are determined by the conformal metric g and the conformal second fundamental form B, thus we get the following conformal congruent theorem.

THEOREM 2.4. *Two spacelike hypersurfaces $x, \bar{x} : M^n \to M_1^{n+1}(c)(n \geq 3)$ are* *conformally equivalent if and only if there exists a diffeomorphism $\varphi : M^n \to M^n$ which preserves the conformal metric and the conformal second fundamental form.*

Next we give the relations between the conformal invariants and the isometric invariants of a spacelike hypersurface in $M_1^{n+1}(c)$.

First we consider the spacelike hypersurface $x : M^n \to \mathbb{R}_1^{n+1}$ in $\mathbb{R}_1^{n+1}$. Let $\{e_1, \cdots, e_n\}$ be an orthonormal local basis with respect to the induced metric $I = <dx, dx>_1$ with dual basis $\{\theta_1, \cdots, \theta_n\}$. Let e_{n+1} be a normal vector field of x, $<e_{n+1}, e_{n+1}>_1 = -1$. Let $II = \sum_{ij} h_{ij}\theta_i \otimes \theta_j$ denote the second fundamental form, the mean curvature $H = \sum_i h_{ii}/n$. Denote by Δ_M the Laplacian operator and κ_M the normalized scalar curvature for I. By structure equation of $x : M^n \to \mathbb{R}_1^{n+1}$ we get that

$$\Delta_M x = nHe_{n+1}. \tag{2.7}$$

There is a local lift of x

$$y : M^n \to C^{n+2}, \quad y = (\frac{<x,x>_1 + 1}{2}, x, \frac{<x,x>_1 - 1}{2}).$$

It follows from (2.7) that

$$\langle \Delta y, \Delta y \rangle_2 - n^2 \kappa_M = \frac{n}{n-1}(-|II|^2 + n|H|^2) = -e^{2\tau}.$$

Therefore the conformal metric g, conformal position vector of x and ξ have the following expression,

$$\begin{aligned} &g = \frac{n}{n-1}(|II|^2 - n|H|^2)<\mathrm{d}x, \mathrm{d}x>_1 := e^{2\tau} I, \ Y = e^{\tau} y, \\ &\xi = -Hy + (<x, e_{n+1}>_1, e_{n+1}, <x, e_{n+1}>_1). \end{aligned} \tag{2.8}$$

By a direct calculation we get the following expression of the conformal invariants,

$$\begin{aligned} &A_{ij} = e^{-2\tau}[\tau_i\tau_j - h_{ij}H - \tau_{i,j} + \frac{1}{2}(-|\nabla\tau|^2 + |H|^2)\delta_{ij}], \\ &B_{ij} = e^{-\tau}(h_{ij} - H\delta_{ij}), \ C_i = e^{-2\tau}(H\tau_i - H_i - \sum_j h_{ij}\tau_j), \end{aligned} \tag{2.9}$$

where $\tau_i = e_i(\tau)$ and $|\nabla\tau|^2 = \sum_i \tau_i^2$, and $\tau_{i,j}$ is the Hessian of τ for I and $H_i = e_i(H)$.

For a spacelike hypersurface $x : M^n \to \mathbb{S}_1^{n+1}(1)$, the conformal metric g, conformal position vector of x and ξ have the following expression,

$$\begin{aligned} &g = \frac{n}{n-1}(|II|^2 - n|H|^2)<\mathrm{d}x, \mathrm{d}x>_1 := e^{2\tau} I, \\ &Y = e^{\tau}(1, x) = e^{\tau} y, \ \xi = -Hy + (0, e_{n+1}). \end{aligned} \tag{2.10}$$

For a spacelike hypersurface $x : M^n \to \mathbb{H}_1^{n+1}(-1)$, the conformal metric g, conformal position vector of x and ξ have the following expression,

$$\begin{aligned} g &= \frac{n}{n-1}(|II|^2 - n|H|^2)\langle \mathrm{d}x, \mathrm{d}x\rangle_2 := e^{2\tau} I, \\ Y &= e^{\tau}(x,1) = e^{\tau} y,\ \xi = -Hy + (e_{n+1}, 0). \end{aligned} \tag{2.11}$$

Using the same calculation from (2.10) and (2.11), we have the following united expression of the conformal invariants,

$$\begin{aligned} A_{ij} &= e^{-2\tau}[\tau_i\tau_j - \tau_{i,j} - h_{ij}H + \frac{1}{2}(-|\nabla\tau|^2 + |H|^2 + c)\delta_{ij}], \\ B_{ij} &= e^{-\tau}(h_{ij} - H\delta_{ij}),\ C_i = e^{-2\tau}(H\tau_i - H_i - \sum_j h_{ij}\tau_j), \end{aligned} \tag{2.12}$$

where $c = 1$ for $x : M^n \to S_1^{n+1}(1)$, and $c = -1$ for $x : M^n \to H_1^{n+1}(-1)$.

Let $\{b_1, \cdots, b_n\}$ be the eigenvalues of the conformal second fundamental form B, which are called conformal principal curvatures of x. Let $\{\lambda_1, \cdots, \lambda_n\}$ be the principal curvatures of x. From (2.9) and (2.12), we have

$$b_i = e^{-\tau}(\lambda_i - H),\ i = 1, \cdots, n. \tag{2.13}$$

Clearly the number of distinct conformal principal curvatures is the same as that of principal curvatures of x. Further, from equations (2.13), the Möbius curvatures

$$\mathbb{M}_{ijk} = \frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_k} = \frac{b_i - b_j}{b_i - b_k}. \tag{2.14}$$

Combining equations (2.9), (2.12) and (2.14), we have the following.

Proposition 2.1. *Let $x : M^n \to M_1^{n+1}(c)$ be a spacelike hypersurface. Then the principal vectors and the conformal principal curvatures are invariant under the conformal transformations of $M_1^{n+1}(c)$. In particular, the Möbius curvatures are invariant under the conformal transformations of $M_1^{n+1}(c)$.*

3. Spacelike Dupin hypersurfaces in Lorentzian space forms.

Let $x : M^n \to M_1^{n+1}(c)$ be a spacelike hypersurface in $M_1^{n+1}(c)$. For a principal curvature λ, we have the principal space $\mathbb{D}_\lambda = \{X \in TM^n | \mathcal{A}X = \lambda X\}$. Then the spacelike hypersurface is Dupin if and only if $X(\lambda) = 0, X \in \mathbb{D}_\lambda$ for every principal curvature λ. The simple examples of spacelike Dupin hypersurface are the following spacelike isoparametric hypersurfaces in $M_1^{n+1}(c)$.

Example 3.1. $\mathbb{H}^k(-a) \times \mathbb{R}^{n-k} \subset \mathbb{R}_1^{n+1},\ a > 0,\ 0 \le k \le n.$

Example 3.2. $\mathbb{S}^k(\sqrt{1+a^2}) \times \mathbb{H}^{n-k}(-a) \subset \mathbb{S}_1^{n+1}(1),\ a > 0,\ 1 \le k \le n.$

Example 3.3. $\mathbb{H}^k(-a) \times \mathbb{H}^{n-k}(-\sqrt{1-a^2}) \subset \mathbb{H}_1^{n+1}(-1),\ 0 < a < 1,\ 1 \le k \le n.$

In fact, these spacelike isoparametric hypersurfaces are all spacelike isoparametric hypersurfaces in $M_1^{n+1}(c)$ (see [**5**], [**7**], [**16**]). The following theorem confirms that the spacelike Dupin hypersurface is conformally invariant.

THEOREM 3.1. *Let $x : M^n \to M_1^{n+1}(c)$ be a spacelike Dupin hypersurface, and $\phi : M_1^{n+1}(c) \to M_1^{n+1}(c)$ a conformal transformation. Then $\phi \circ x : M^n \to M_1^{n+1}(c)$ is a spacelike Dupin hypersurface.*

PROOF. Let $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ denote its principal curvatures, and $\{e_1, e_2, \cdots, e_n\}$ be the orthonormal basis for TM^n with respect to the induced metric I, consisting of unit principal vectors. Therefore $\{E_1 = e^{\tau}e_1, E_2 = e^{\tau}e_2, \cdots, E_n = e^{\tau}e_n\}$ is the orthonormal basis for TM^n with respect to the conformal metric $g = e^{2\tau}I$, and $\{b_1 = e^{-\tau}(\lambda_1 - H), \cdots, b_n = e^{-\tau}(\lambda_n - H)\}$ are the conformal principal curvatures. From (2.9) and (2.12), we have

$$\begin{aligned} C_i &= e^{-\tau}(-e^{-\tau}H_i + \sum_j (h_{ij} - H\delta_{ij})(e^{-\tau})_j) \\ &= e^{-\tau}(-e^{-\tau}H_i + \sum_j e_j((h_{ij} - H\delta_{ij})e^{-\tau}) - e^{-\tau}\sum_j e_j(h_{ij} - H\delta_{ij})) \\ &= e^{-\tau}(\sum_j e_j(B_{ij}) - \sum_j e^{-\tau}He_j(h_{ij})) \\ &= E_i(b_i) - e^{-\tau}E_i(\lambda_i). \end{aligned} \tag{3.15}$$

Noting that the principal vectors are conformal invariants, therefore x is Dupin if and only if $C_i = E_i(b_i)$, which is invariant under the conformal transformation of $M_1^{n+1}(c)$ from Proposition 2.1. □

The spacelike Dupin hypersurfaces with constant Möbius curvatures can be characterized in terms of the conformal invariants.

THEOREM 3.2. *Let $x : M^n \to M_1^{n+1}(c)$ be a spacelike Dupin hypersurface with $r(\geq 3)$ distinct principal curvatures. Then the Möbius curvatures are constant if and only if the conformal 1-form vanishes and the conformal principal curvatures are constant.*

PROOF. It suffices to prove that the Möbius curvatures $\mathbb{M}_{ijk}$ are constant implies all conformal principal curvatures b_i are constant and the conformal 1-form vanishes. First, for any tangent vector $X \in TM^n$, it is not hard to calculate that

$$\frac{X(b_i) - X(b_j)}{b_i - b_j} = \frac{X(b_i) - X(b_k)}{b_i - b_k} = \frac{X(b_j) - X(b_k)}{b_j - b_k}$$

from $\mathbb{M}_{ijk}$ being constant for all $1 \leq i, j, k \leq n$. Hence there exist μ and ε such that

$$X(b_j) = \mu b_j + \varepsilon \text{ for } j = 1, \cdots, n. \tag{3.16}$$

It is then immediate that (2.6) implies $\varepsilon = 0$ and $b_1X(b_1) + \cdots + b_nX(b_n) = 0$, which implies $\mu = 0$. Thus all $b_1, \cdots, b_n$ are constant. The conformal 1-form vanishes, $C = 0$ from the equation (3.15). □

Like as Pinkall's method in [**14**], we construct a new spacelike Dupin hypersurface from a spacelike Dupin hypersurface.

Proposition 3.1. *Let $u : M^k \to \mathbb{R}_1^{k+1}$ be an immersed spacelike hypersurface. The cylinder over u is defined as follows*:

$$x : M^k \times \mathbb{R}^{n-k} \to \mathbb{R}_1^{k+1} \times \mathbb{R}^{n-k} = \mathbb{R}_1^{n+1},\ x(p,y) = (u(p), y).$$

If u is a Dupin hypersurface, then cylinder x is a spacelike Dupin hypersurface.

Proposition 3.2. *Let $u : M^k \to \mathbb{S}_1^{k+1}$ be an immersed spacelike hypersurface and R^+ the half line of positive real numbers. The cone over u is defined as follows*:

$$x : M^k \times R^+ \times \mathbb{R}^{n-k-1} \to \mathbb{R}_1^{n+1},\ x(p,t,y) = (tu(p), y).$$

If u is a Dupin hypersurface, then cone x is a spacelike Dupin hypersurface.

In general, these constructions introduce a new principal curvature of multiplicity $n-k$ which is constant along its curvature surface. The other principal curvatures are determined by the principal curvatures of M^k, and the Dupin property is preserved for these principal curvatures. It is easy to prove Theorem 1.1 using these constructions.

Next we give a spacelike Dupin hypersurface which is a cone over a spacelike isoparametric hypersurface in $\mathbb{S}_1^{n+1}(1)$, which is a spacelike Dupin hypersurface with three constant conformal principal curvatures.

Example 3.4. Let p, q be any two given natural numbers with $p+q<n$ and a real number $a>1$, consider the spacelike hypersurface of warped product embedding

$$x : \mathbb{H}^q(-\sqrt{a^2-1}) \times \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \to \mathbb{R}_1^{n+1},$$

defined by

$$x(u', u'', t, u''') = (tu', tu'', u'''),$$

where $u' \in \mathbb{H}^q(-\sqrt{a^2-1}), u'' \in \mathbb{S}^p(a), u''' \in \mathbb{R}^{n-p-q-1}$.

Next we give some conformal invariants of the spacelike Dupin hypersurface x. Let $b = \sqrt{a^2-1}$. One of the normal vector of x can be taken as

$$e_{n+1} = (\frac{a}{b}u', \frac{b}{a}u'', 0).$$

The first and second fundamental form of x are given by

$$I = t^2(<du', du'>_1 + du'' \cdot du'') + dt \cdot dt + du''' \cdot du''',$$
$$II = -<dx, de_{n+1}>_1 = -t(\frac{a}{b}<du', du'>_1 + \frac{b}{a} du'' \cdot du'').$$

Thus the mean curvature of x

$$H = \frac{-pb^2 - qa^2}{nabt},$$

and

$$e^{2\tau} = \frac{n}{n-1}\left[\sum_{ij} h_{ij}^2 - nH^2\right] = \frac{p(n-p)b^4 - 2pqa^2b^2 + q(n-q)a^4}{(n-1)t^2} := \frac{\alpha^2}{t^2}.$$

From (2.8) and (2.12), the conformal 1-form $C = 0$, and the conformal metric and the conformal second fundamental form of x are given by

$$\begin{aligned} g &= \alpha^2 \langle du', du'\rangle + \alpha^2 du'' \cdot du'' + \frac{\alpha^2}{t^2}(dt \cdot dt + du''' \cdot du''') = \tilde{g}_1 + \tilde{g}_2 + \tilde{g}_3, \\ B &= \sum_{ij} B_{ij}\omega_i \otimes \omega_j,\ (B_{ij}) = (\underbrace{b_1, \cdots, b_1}_{q}, \underbrace{b_2, \cdots, b_2}_{p}, \underbrace{b_3, \cdots, b_3}_{n-p-q}), \end{aligned} \tag{3.17}$$

where

$$b_1 = \frac{pb^2 - (n-q)a^2}{nab\alpha},\ b_2 = \frac{qa^2 - (n-p)b^2}{nab\alpha},\ b_3 = \frac{pb^2 + qa^2}{nab\alpha}.$$

Furthermore, from (3.17), we have the following facts:
(1) If $q \geq 2$, then $(\mathbb{H}^q(-\sqrt{a^2-1}), \tilde{g}_1)$ has constant sectional curvature $-1/b^2\alpha^2$.
(2) If $p \geq 2$, then $(\mathbb{S}^p(a), \tilde{g}_2)$ has constant sectional curvature $1/a^2\alpha^2$.
(3) If $n - q - p \geq 2$, then $(\mathbb{R}^+ \times \mathbb{R}^{n-p-q-1}, \tilde{g}_3)$ has constant sectional curvature $-1/\alpha^2$.

4. The proof of Theorem 1.2.

To prove Theorem 1.2, we need the following Lemma.

Lemma 4.1. *Let $x : M^n \to M_1^{n+1}(c)$ be a spacelike hypersurface without umbilical points. If the conformal invariants of x satisfy $C = 0$ and $A = \mu B + \lambda g$ for some constant μ, λ, then x is conformally equivalent to the spacelike hypersurface with constant mean curvature and constant scalar curvature.*

Proof. Since $C = 0$ and $A = \mu B + \lambda g$, from structure equations (2.1) we get that

$$dN - \lambda dY - \mu d\xi = 0$$

and

$$d(N - \lambda Y - \mu\xi) = 0.$$

Therefore we can find a constant vector $e \in \mathbb{R}_2^{n+3}$ such that

$$N - \lambda Y - \mu\xi = e. \tag{4.18}$$

Therefore

$$\langle e, e\rangle_2 = -\mu^2 - 2\lambda,\ \langle Y, e\rangle_2 = 1.$$

From (2.6) and $A = \mu B + \lambda g$, we get $tr(A) = n\lambda = (n^2\kappa - 1)/2n$, therefore κ is constant.

To prove the Lemma we consider the following three cases,
Case 1: e is lightlike, i.e., $\mu^2 + 2\lambda = 0$,
Case 2: e is spacelike, i.e., $\mu^2 + 2\lambda < 0$,
Case 3: e is timelike, i.e., $\mu^2 + 2\lambda > 0$.

First we consider Case 1, e is lightlike, i.e., $\mu^2 + 2\lambda = 0$. Then there exists a $T \in O(n+3,2)$ such that

$$\bar{e} = (-1, \vec{0}, -1) = eT = (N - \lambda Y - \mu\xi)T.$$

Let $\bar{x} : M^n \to \mathbb{R}^{n+1}_1$ be a spacelike hypersurface whose conformal position vector is $\bar{Y} = YT$, then $\bar{N} = NT, \bar{\xi} = \xi T$, and

$$\bar{e} = \bar{N} - \lambda\bar{Y} - \mu\bar{\xi},\ <\bar{Y}, \bar{e}>_2 = 1,\ <\bar{\xi}, \bar{e}>_2 = \mu. \tag{4.19}$$

Writing

$$\bar{Y} = e^{\bar{\tau}}(\frac{<\bar{x}, \bar{x}>_1 + 1}{2}, \bar{x}, \frac{<\bar{x}, \bar{x}>_1 - 1}{2}) = e^{\bar{\tau}}\bar{y},\ \bar{\xi} = -\bar{H}\bar{y} + \bar{y}_{n+1},$$

then from (2.8) and (4.19), we obtain that

$$e^{\bar{\tau}} = 1,\ \bar{H} = -\mu.$$

Since $\bar{Y} = ((<\bar{x}, \bar{x}>_1 + 1)/2, \bar{x}, (<\bar{x}, \bar{x}>_1 - 1)/2)$, then $g = <d\bar{x}, d\bar{x}>_1 = \bar{I}$ and the normalized scalar curvature of $\bar{I}$, $\kappa_M = \kappa$. Therefore the mean curvature and the scalar curvature of the hypersurface $\bar{x}$ are constant.

Next we consider Case 2, e is spacelike, i.e., $\mu^2 + 2\lambda < 0$. Then there exists a $T \in O(n+3,2)$ such that

$$\bar{e} = (\vec{0}, \sqrt{-\mu^2 - 2\lambda}) = eT = (N - \lambda Y - \mu\xi)T.$$

Let $\bar{x} : M^n \to \mathbb{H}^{n+1}_1(-1)$ be a spacelike hypersurface whose conformal position vector is $\bar{Y} = YT$, then $\bar{N} = NT, \bar{\xi} = \xi T$, and

$$\bar{e} = \bar{N} - \lambda\bar{Y} - \mu\bar{\xi},\ <\bar{Y}, \bar{e}>_2 = 1,\ <\bar{\xi}, \bar{e}>_2 = \mu. \tag{4.20}$$

Writing $\bar{Y} = e^{\bar{\tau}}(\bar{x}, 1)$, $\bar{\xi} = -\bar{H}(\bar{x}, 1) + (e_{n+1}, 0)$, then from (2.11) and (4.20), we obtain that

$$e^{\bar{\tau}} = \frac{1}{\sqrt{-\mu^2 - 2\lambda}},\ \bar{H} = \frac{-\mu}{\sqrt{-\mu^2 - 2\lambda}}.$$

Since $<d\bar{x}, d\bar{x}>_2 = -(\mu^2 + 2\lambda)g$, the normalized scalar curvature of $\bar{I}$, $\kappa_M = \kappa/(-\mu^2 - 2\lambda)$. Therefore the mean curvature and the scalar curvature of the hypersurface $\bar{x}$ are constant.

Finally we consider Case 3, e is timelike, i.e., $\mu^2 + 2\lambda > 0$. Then there exists a $T \in O(n+3,2)$ such that

$$\bar{e} = (-\sqrt{2\lambda + \mu^2}, \vec{0}) = eT = (N - \lambda Y - \mu\xi)T.$$

Let $\bar{x} : M^n \to \mathbb{S}_1^{n+1}(1)$ be a spacelike hypersurface whose conformal position vector is $\bar{Y} = YT$, then $\bar{N} = NT, \bar{\xi} = \xi T$, and

$$\bar{e} = \bar{N} - \lambda \bar{Y} - \mu \bar{\xi}, \ <\bar{Y}, \bar{e}>_2 = 1, \ <\bar{\xi}, \bar{e}>_2 = \mu. \tag{4.21}$$

Writing $\bar{Y} = e^{\bar{\tau}}(1, \bar{x}), \ \bar{\xi} = -\bar{H}(1, \bar{x}) + (0, e_{n+1})$, then from (2.10) and (4.21), we obtain that

$$e^{\bar{\tau}} = \frac{1}{\sqrt{2\lambda + \mu^2}}, \ \bar{H} = \frac{-\mu}{\sqrt{2\lambda + \mu^2}}.$$

Since $<d\bar{x}, d\bar{x}>_1 = (2\lambda + \mu^2)g$, the normalized scalar curvature of $\bar{I}$, $\kappa_M = \kappa/(2\lambda + \mu^2)$. Therefore the mean curvature and the scalar curvature of the hypersurface $\bar{x}$ are constant. $\square$

Now we prove Theorem 1.2. Let $x : M^n \to M_1^{n+1}(c)$ be a spacelike Dupin hypersurface with two distinct principal curvatures. We take a local orthonormal basis $\{E_1, \cdots, E_n\}$ with respect to g such that under the basis

$$(B_{ij}) = diag(\underbrace{b_1, \cdots, b_1}_{k}, \underbrace{b_2, \cdots, b_2}_{n-k}).$$

Using the equation (2.6), we have

$$b_1 = \frac{1}{n}\sqrt{\frac{(n-1)(n-k)}{k}}, \ b_2 = \frac{-1}{n}\sqrt{\frac{(n-1)k}{n-k}}.$$

From (3.15), we can obtain that

$$C = 0. \tag{4.22}$$

From equation (2.4), we know that $[A, B] = 0$. Thus we can take a local orthonormal basis $\{E_1, \cdots, E_n\}$ with respect to g such that under the basis

$$(B_{ij}) = diag(\underbrace{b_1, \cdots, b_1}_{k}, \underbrace{b_2, \cdots, b_2}_{n-k}), \ (A_{ij}) = diag(a_1, a_2, \cdots, a_n). \tag{4.23}$$

Since b_1, b_2 are constant, using the covariant derivatives of B, (2.3) and (4.22) we can obtain

$$B_{ij,l} = 0, \ 1 \le i, j, l \le n, \ \omega_{i\alpha} = 0, \ 1 \le i \le k, \ k+1 \le \alpha \le n,$$

which implies that

$$R_{i\alpha i\alpha} = 0, \ 1 \le i \le k, \ k+1 \le \alpha \le n.$$

Combining the equation (2.5), we have

$$-b_1 b_2 + a_i + a_\alpha = 0, \ 1 \le i \le k, \ k+1 \le \alpha \le n,$$

thus

$$a_1 = \cdots = a_k,\ a_{k+1} = \cdots = a_n.$$

Using the covariant derivatives of A, we can get

$$A_{ij,\alpha} = 0,\ A_{\alpha\beta,i} = 0,\ 1 \le i, j \le k,\ k+1 \le \alpha, \beta \le n. \tag{4.24}$$

Since $E_\alpha(a_1) = A_{ii,\alpha} = 0,\ E_i(a_n) = A_{\alpha\alpha,i} = 0$, combining $b_1 b_2 + a_i + a_\alpha = 0$ we know that $a_1 = \cdots = a_k,\ a_{k+1} = \cdots = a_n$ are constant. Thus

$$(A_{ij}) = diag(\underbrace{a_1, \cdots, a_1}_{k}, \underbrace{a_2, \cdots, a_2}_{n-k}).$$

Let $\mu = (a_1 - a_2)/(b_1 - b_2)$ and $\lambda = tr(A)/n$, then

$$A = \mu B + \lambda g.$$

From Lemma 4.1, up to a conformal transformation, we know that e^τ is constant. Combining (2.9), we know that the principal curvatures of x are constant. From the classification of spacelike isoparametric hypersurfaces (see [**5**], [**7**], [**16**]), the Dupin hypersurface x is a spacelike isoparametric hypersurface in $M_1^{n+1}(c)$ up to a conformal transformation of $M_1^{n+1}(c)$. We finish the proof of Theorem 1.2.

5. The proof of Theorem 1.3.

Let M^n be a spacelike Dupin hypersurface in $M_1^{n+1}(c)$ with $r(\ge 3)$ distinct principal curvatures. If the Möbius curvatures are constant, then $C = 0$, which implies $[A, B] = 0$. Therefore we can choose a local orthonormal basis $\{E_1, \cdots, E_n\}$ with respect to the conformal metric g such that

$$\begin{aligned} &(A_{ij}) = diag(a_1, \cdots, a_n), \\ &(B_{ij}) = diag(b_1, \cdots, b_n) = diag(b_{\bar 1}, \cdots, b_{\bar 1}, b_{\bar 2}, \cdots, b_{\bar 2}, \cdots, b_{\bar r}, \cdots, b_{\bar r}). \end{aligned} \tag{5.25}$$

Using the covariant derivative of B, we have

$$(b_i - b_j)\omega_{ij} = \sum_k B_{ij,k}\omega_k. \tag{5.26}$$

For some b_i, in this section we define the index set

$$[b_i] := \{m | b_m = b_i\}.$$

Since the conformal principal curvatures $\{b_1, b_2, \cdots, b_n\}$ are constant, we have the following results,

$$\begin{cases} B_{ij,k}=0 \text{ when } \ [b_i]=[b_j] \text{ or } [b_i]=[b_k], \\ \omega_{ij}=\sum_k \dfrac{B_{ij,k}}{b_i-b_j}\omega_k \text{ when } \ [b_i]\neq[b_j]. \end{cases} \tag{5.27}$$

Using (5.27) and the second covariant derivative of B_{ij} defined by

$$\sum_l B_{ij,kl}\omega_l = dB_{ij,k}+\sum_l B_{lj,k}\omega_{li}+\sum_l B_{il,k}\omega_{lj}+\sum_l B_{ij,l}\omega_{lk},$$

and the following Ricci identities

$$B_{ij,ij}-B_{ij,ji}=\sum_l B_{lj}R_{liij}+\sum_l B_{il}R_{ljji},$$

we have

$$R_{ijij}=\sum_{k\notin[b_i],[b_j]}\frac{2B^2_{ij,k}}{(b_i-b_k)(b_j-b_k)} \text{ when } \ [b_i]\neq[b_j]. \tag{5.28}$$

Under the basis $\{E_1,\cdots,E_n\}$, $(A_{ij})=diag(a_1,\cdots,a_n)$. Using the covariant derivative of A, we have

$$(a_i-a_j)\omega_{ij}=\sum_k A_{ij,k}\omega_k.$$

From the second formula in (5.27), we obtain the following equation,

$$\frac{a_i-a_j}{b_i-b_j}B_{ij,k}=A_{ij,k}, \text{ when } [b_i]\neq[b_j]. \tag{5.29}$$

To prove Theorem 1.3, we need the following lemmas.

Lemma 5.1. *Let $\rho_1,\cdots,\rho_r$, be $r(\geq 3)$ distinct real numbers, and ε a real number. Then there does not exist any real coefficients $\{F_{ijk}\}$ satisfying*

$$\begin{aligned} &\text{(i) } F_{ijk}=F_{jik}=F_{ikj}, \\ &\text{(ii) } \varepsilon-\rho_i\rho_j=\sum_{k\neq i,j}\frac{(F_{ijk})^2}{(\rho_i-\rho_k)(\rho_j-\rho_k)}, \ \rho_i\neq\rho_j. \end{aligned} \tag{5.30}$$

Proof. We assume that there exists a group of real coefficients $\{F_{ijk}\}$ satisfying (5.30). We will find a contradiction to prove the lemma.

We can assume that $\rho_1<\rho_2<\cdots<\rho_r$. The equation (5.30) implies that

$$\varepsilon-\rho_1\rho_2\geq 0, \ \varepsilon-\rho_2\rho_3\geq 0, \ \cdots, \ \varepsilon-\rho_k\rho_{k+1}\geq 0, \ \cdots, \ \varepsilon-\rho_{r-1}\rho_r\geq 0. \tag{5.31}$$

For fixed induce i, the matrix

$$\mathfrak{F}_{jk}:=\frac{(F_{ijk})^2}{(\rho_i-\rho_k)(\rho_j-\rho_k)(\rho_i-\rho_j)}$$

is antisymmetric for indices j, k, thus

$$\sum_{j,\rho_j\neq\rho_i}\frac{\varepsilon-\rho_i\rho_j}{\rho_i-\rho_j}=\sum_{j,k,\rho_j\neq\rho_i}\frac{(F_{ijk})^2}{(\rho_i-\rho_k)(\rho_j-\rho_k)(\rho_i-\rho_j)}=0. \tag{5.32}$$

The proof of the lemma is divided into two cases: (1), $\rho_1<0$, (2), $\rho_1\geq 0$.

For case (1), $\rho_1<0$, we have $\rho_1\rho_2>\rho_1\rho_3>\cdots>\rho_1\rho_r$. Combining (5.31), we have

$$\varepsilon-\rho_1\rho_2\geq 0,\ \varepsilon-\rho_1\rho_3>0,\ \cdots,\varepsilon-\rho_1\rho_r>0.$$

Thus

$$\frac{\varepsilon-\rho_1\rho_j}{\rho_1-\rho_j}\leq 0,\ \rho_j\neq\rho_1,$$

which is a contradiction with the equation (5.32) for $i=1$.

For case (2), $\rho_1\geq 0$. Then $\rho_r>\rho_{r-1}>\cdots>\rho_1\geq 0$. Combining (5.31) we have $\varepsilon\geq\rho_r\rho_{r-1}>\rho_r\rho_{r-2}>\cdots>\rho_r\rho_1$, that is

$$\varepsilon-\rho_r\rho_{r-1}\geq 0,\ \epsilon-\rho_r\rho_{r-1}>0,\cdots,\epsilon-\rho_r\rho_1>0.$$

Thus

$$\frac{\varepsilon-\rho_r\rho_j}{\rho_r-\rho_j}\geq 0,\ \rho_j\neq\rho_r,$$

which is a contradiction with the equation (5.32) for $i=r$. Thus we finish the proof of the lemma. □

LEMMA 5.2. *Let M^n be a spacelike Dupin hypersurface in $M_1^{n+1}(c)$ with r distinct principal curvatures. If $r\geq 3$ and the Möbius curvatures are constant. Then the conformal second fundamental form is parallel, that is $B_{ij,k}=0,\ 1\leq i,j,k\leq n$.*

PROOF. We assume that there exists a $B_{i_0j_0k}\neq 0$, we will find a contradiction to prove the lemma.

We consider the pair (a_i,b_i) and let W denote the set of all of the pairs, that is,

$$W=\{(a_1,b_1),(a_2,b_2),\cdots,(a_n,b_n)\}.$$

For a number μ (including ∞) and an index i fixed, we define the set of pairs

$$S_i(\mu):=\{(a_k,b_k)\in W|\ \frac{a_i-a_k}{b_i-b_k}=\mu,\ b_k\neq b_i\}\bigcup\{(a_i,b_i)\}.$$

Since $B_{i_0j_0k}\neq 0$, from (5.27), we know that $b_{i_0}\neq b_{j_0}\neq b_k$. Using (5.29), we have

$$\frac{a_{i_0}-a_{j_0}}{b_{i_0}-b_{j_0}}=\frac{A_{i_0j_0,k}}{B_{i_0j_0,k}}=\frac{A_{i_0k,j_0}}{B_{i_0k,j_0}}=\frac{a_{i_0}-a_k}{b_{i_0}-b_k}.$$

Let $A_{i_0j_0,k}/B_{i_0j_0,k}=\mu_0$. For $(a_i,b_i),(a_j,b_j)\in S_{i_0}(\mu_0)$, we have $(a_i-a_j)/(b_i-b_j)=\mu_0$, thus there exists a constant ε such that

$$a_i = \mu_0 b_i + \varepsilon,\ (a_i, b_i) \in S_{i_0}(\mu_0).$$

Thus

$$R_{ijij} = -b_i b_j + a_i + a_j = -(b_i - \mu_0)(b_j - \mu_0) + \mu_0^2 + 2\varepsilon,\ (a_i, b_i), (a_j, b_j) \in S_{i_0}(\mu_0).$$

Let $\tilde{b}_i = b_i - \mu_0$ and $\epsilon = \mu_0^2 + 2\varepsilon$. From (2.5) and (5.28), we have

$$R_{ijij} = 2\sum_k \frac{(B_{ij,k})^2}{(\tilde{b}_i - \tilde{b}_k)(\tilde{b}_j - \tilde{b}_k)} = 2\varepsilon + \mu_0^2 - \tilde{b}_i\tilde{b}_j = \epsilon - \tilde{b}_i\tilde{b}_j. \tag{5.33}$$

Since $b_{i_0} \neq b_{j_0} \neq b_k$, the number of distinct pairs in $S_{i_0}(\mu_0)$ must be equal or more than three. Thus there exist r (≥ 3) real numbers $\tilde{b}_{i_0}, \tilde{b}_{j_0}, \tilde{b}_k, \cdots, \tilde{b}_l$ satisfying (5.33)

$$\epsilon - \tilde{b}_i\tilde{b}_j = \sum_{k, b_k \neq b_i, b_j} \frac{2(B_{ij,k})^2}{(\tilde{b}_i - \tilde{b}_k)(\tilde{b}_j - \tilde{b}_k)},$$

which is a contradiction with Lemma 5.1. Thus we finish the proof of the Lemma. $\square$

Next we give the proof of Theorem 1.3. From the equation (5.28) and lemma 5.2, we have

$$R_{ijij} = \sum_{k \notin [b_i], [b_j]} \frac{2B_{ij,k}^2}{(b_i - b_k)(b_j - b_k)} = 0,\ b_i \neq b_j. \tag{5.34}$$

Claim 1. *The number of distinct principal curvatures $r = 3$.*

We assume that $r > 3$, we can take four distinct conformal principal curvatures b_1, b_2, b_3, b_4. Using (5.34) and (2.5), we have

$$-b_1b_2 + a_1 + a_2 = 0,\ -b_1b_3 + a_1 + a_3 = 0,$$
$$-b_2b_4 + a_2 + a_4 = 0,\ -b_3b_4 + a_3 + a_4 = 0,$$

which implies $(b_1 - b_4)(b_2 - b_3) = 0$. This is a contradiction, thus the number of the distinct principal curvatures $r = 3$.

Now we assume that

$$(B_{ij}) = diag(b_1, \cdots, b_1, b_2, \cdots, b_2, b_3, \cdots, b_3),\ b_1 < b_2 < b_3.$$

From (5.34), we have $a_i = a_j,\ [b_i] = [b_j]$, and

$$-b_1b_2 + a_1 + a_2 = 0,\ -b_1b_3 + a_1 + a_3 = 0,\ -b_2b_3 + a_2 + a_3 = 0.$$

Thus we can get

$$a_1 = \frac{b_1b_2 + b_1b_3 - b_2b_3}{2},\ a_2 = \frac{b_1b_2 + b_2b_3 - b_1b_3}{2},\ a_3 = \frac{b_3b_2 + b_1b_3 - b_1b_2}{2}. \tag{5.35}$$

Using the covariant derivative of B and $B_{ij,k} = 0$, we have

$$\omega_{ij} = 0,\ [b_i] \neq [b_j],$$

which implies

$$d\omega_i = \sum_{j \in [b_i]} \omega_{ij} \wedge \omega_j. \tag{5.36}$$

Let $V_{b_i} = span\{E_j | j \in [b_i]\}$. The equations (5.36) imply that the distributions V_{b_1}, V_{b_2} and V_{b_3} are integrable. Let M_1, M_2, M_3 be integral submanifolds of $V_{b_1}, V_{b_2}, V_{b_3}$, respectively. Locally we can write

$$M^n = M_1 \times M_2 \times M_3.$$

Let

$$g_1 = \sum_i \omega_i^2,\ i \in [b_1],\ g_2 = \sum_i \omega_i^2,\ i \in [b_2],\ g_3 = \sum_i \omega_i^2,\ i \in [b_3].$$

Then we have

$$(M^n, g) = (M_1, g_1) \times (M_2, g_2) \times (M_3, g_3).$$

From (2.5) and (5.35), we have the following results:
(1) If $dim M_1 \geq 2$, then (M_1, g_1) has constant sectional curvature $(b_2 - b_1)(b_1 - b_3) < 0$.
(2) If $dim M_2 \geq 2$, then (M_2, g_2) has constant sectional curvature $(b_2 - b_1)(b_3 - b_2) > 0$.
(3) If $dim M_3 \geq 2$, then (M_3, g_3) has constant sectional curvature $(b_2 - b_3)(b_3 - b_1) < 0$.

Let $q = dim M_1$, $p = dim M_2$ and $n - p - q = dim M_3$. From example 3.4, we can find local isometries:

$$\phi_1 : (M_1, g_1) \to (\mathbb{H}^q(-\sqrt{a^2 - 1}), \tilde{g}_1),$$
$$\phi_2 : (M_2, g_2) \to (\mathbb{S}^p(a), \tilde{g}_2),\ \phi_3 : (M_3, g_3) \to (\mathbb{R}^+ \times \mathbb{R}^{n-p-q-1}, \tilde{g}_3).$$

Therefore, we obtain a local diffeomerphism

$$\phi = (\phi_1, \phi_2, \phi_3) : M^n \to \mathbb{H}^q(-\sqrt{a^2 - 1}) \times \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1}.$$

From (3.17), we see that the diffeomorphism preserves the conformal metric and the conformal second fundamental form. we know that M^n is conformally equivalent to the hypersurface given by example 3.4.

References

[1] M. Cahen and Y. Kerbrat, Domaines symmétriques des quadriques projectives, J. Math. Pures Appl., **62** (1983), 327–348.

[2] T. Cecil, Q. S. Chi and G. Jensen, Dupin hypersurfaces with four principal curvatures, II, Geom. Dedicata, **128** (2007), 55–95.

[3] T. Cecil and G. Jensen, Dupin hypersurfaces with three principal curvatures, Invent. Math., **132**

(1998), 121–178.
[4] T. Cecil and G. Jensen, Dupin hypersurfaces with four principal curvatures, Geom. Dedicata, **79** (2000), 1–49.
[5] Z. Li and X. Xie, Spacelike isoparametric hypersurfaces in Lorentzian space form, Front. Math. China, **1** (2006), 130–137.
[6] K. Nomizu, On isoparametric hypersurfaces in the Lorentzian space forms, Japan. J. Math., **7** (1981), 217–226.
[7] M. A. Magid, Lorentzian isoparametric hypersurface, Pacific J. Math., **118** (1985), 165–197.
[8] R. Miyaoka, Compact Dupin hypersurfaces with three principal curvatures, Math. Z., **187** (1984), 433–452.
[9] R. Miyaoka, Dupin hypersurfaces and a Lie invariant, Kodai Math. J., **12** (1989), 228–256.
[10] R. Miyaoka, Dupin hypersurfaces with six principal curvatures, Kodai Math. J., **12** (1989), 308–315.
[11] C. X. Nie, T. Z. Li, Y. J. He, et al., Conformal isoparametric hypersurfaces with two distinct conformal principal curvatures in conformal space, Sci. China Ser. A, **53** (2010), 953–965.
[12] B. O'Neil, Semi-Riemannian Geometry, Academic Press, New York, 1983.
[13] U. Pinkall, Dupinsche Hyperflachen in E^4, Manuscripta Math., **51** (1985), 89–119.
[14] U. Pinkall, Dupin hypersurfaces, Math. Ann., **270** (1985), 427–440.
[15] C. P. Wang, Möbius geometry of submanifolds in $\mathbb{S}^n$, Manuscripta Math., **96** (1998), 517–534.
[16] L. Xiao, Lorentzian isoparametric hypersurfaces in $\mathbb{H}_1^{n+1}$, Pacific J. Math., **189** (1999), 377–397.

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