

On Arakawa–Kaneko zeta-functions associated with $GL_2(\mathbb{C})$ and their functional relations

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Abstract. We construct a certain class of Arakawa–Kaneko zeta-functions associated with $GL_2(\mathbb{C})$, which includes the ordinary Arakawa–Kaneko zeta-function. We also define poly-Bernoulli polynomials associated with $GL_2(\mathbb{C})$ which appear in their special values of these zeta-functions. We prove some functional relations for these zeta-functions, which are regarded as interpolation formulas of various relations among poly-Bernoulli numbers. Considering their special values, we prove difference relations and duality relations for poly-Bernoulli polynomials associated with $GL_2(\mathbb{C})$.

1. Introduction.

For $k \in \mathbb{Z}$, two types of poly-Bernoulli numbers $\{B_n^{(k)}\}$ and $\{C_n^{(k)}\}$ are defined by Kaneko as follows:

$$\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \quad (1.1)$$

$$\frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!}, \quad (1.2)$$

where $\text{Li}_k(z)$ is the polylogarithm defined by

$$\text{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k} \quad (|z| < 1)$$

(see Kaneko [7] and Arakawa–Kaneko [2], also Arakawa–Ibukiyama–Kaneko [1]). Since $\text{Li}_1(x) = -\log(1 - x)$, we see that $B_n^{(1)}$ coincides with the ordinary Bernoulli number.

In this decade, these numbers have been actively investigated (see, for example, Kaneko [8]). The most remarkable formulas for them are the following ‘duality relations’:

$$B_m^{(-k)} = B_k^{(-m)}, \quad (1.3)$$

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$$C_m^{(-k-1)} = C_k^{(-m-1)} \tag{1.4}$$

for $k, m \in \mathbb{Z}_{\geq 0}$ (see [7, Theorem 2] and [8, Section 2]). Recently Kaneko and the second-named author [10] showed (1.3), (1.4) and their generalization by investigating the zeta-function of Arakawa–Kaneko type (defined below). Also it is known that

$$B_m^{(k)} = C_m^{(k)} + C_{m-1}^{(k-1)} \tag{1.5}$$

for $k \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 1}$ (see [2, Equation (9)]).

Corresponding to these numbers, Arakawa and Kaneko defined the zeta-function

$$\xi(k; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} dt \quad (\text{Re } s > 0) \tag{1.6}$$

for $k \in \mathbb{Z}_{\geq 1}$, which can be continued to \mathbb{C} as an entire function (see [2, Section 3]). Further they considered multiple versions of (1.6). Note that $\xi(k; s)$ can be regarded as generalizations of the Riemann zeta-function because $\xi(1; s) = s\zeta(s + 1)$. They also showed that

$$\xi(k; -m) = (-1)^m C_m^{(k)} \quad (m \in \mathbb{Z}_{\geq 0}). \tag{1.7}$$

From the observation of $\xi(k; s)$ and its multiple versions, they gave several relation formulas among the multiple zeta values defined by

$$\zeta(l_1, \dots, l_r) = \sum_{1 < m_1 < \dots < m_r} \frac{1}{m_1^{l_1} \dots m_r^{l_r}}$$

for $l_1, \dots, l_r \in \mathbb{Z}_{\geq 1}$ with $l_r \geq 2$ (see [2, Corollary 11]).

As a generalization of $\xi(k; s)$, Coppo and Candelpergher [5] defined

$$\xi(k; s; w) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-wt} \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} dt$$

for $k \in \mathbb{Z}_{\geq 1}$ and $w > 0$, and studied its property. Note that $\xi(k; s; 1) = \xi(k; s)$.

As a twin sibling of (1.6), Kaneko and the second-named author [10] recently defined

$$\eta(k; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_k(1 - e^t)}{1 - e^t} dt \tag{1.8}$$

for $s \in \mathbb{C}$ and for ‘any’ $k \in \mathbb{Z}$, which interpolates the poly-Bernoulli numbers of B -type, that is,

$$\eta(k; -m) = B_m^{(k)} \quad (k \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0}). \tag{1.9}$$

More generally, they defined the multi-variable version of (1.8) denoted by $\eta((-k_j); (s_j))$ for each $k_j \in \mathbb{Z}_{\geq 0}$, and showed certain duality relations for multi-indexed poly-Bernoulli numbers (see [10, Theorems 5.7 and 5.10]).

More recently, Yamamoto [12] considered $\eta(u; s)$ (where u and s are variables) and its multi-variable versions $\eta((u_j); (s_j))$ and proved functional duality relations for them.

In particular, for the case of single zeta-function, he proved

$$\eta(u; s) = \eta(s; u) \quad (u, s \in \mathbb{C}), \tag{1.10}$$

which interpolates (1.3) at non-positive integer points by (1.9).

In this paper, we consider, as generalizations of $\xi(k; s)$, $\eta(k; s)$ and $\xi(k; s; w)$, the Arakawa–Kaneko zeta-functions associated with $GL_2(\mathbb{C})$ defined as follows. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$, we let

$$gz = \frac{az + b}{cz + d}, \quad j_D(g, z) = cz + d, \quad j_N(g, z) = az + b.$$

Note that $j_D(g, z)$ coincides with the factor of automorphy for $g \in SL_2(\mathbb{Z})$ (see [6, Section 1.2]). Let

$$\Phi(z, u, y) = \sum_{m=0}^{\infty} \frac{z^m}{(m + y)^u}$$

be the Lerch transcendent for $z, u, y \in \mathbb{C}$ with $|z| < 1$ or ($z = 1$ and $\operatorname{Re} u > 1$), and $\operatorname{Re} y > 0$ (see [3, Section 1.11]). For $y, w \in \mathbb{C}$, we define

$$\xi_D(u, s; y, w; g) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-wt} \frac{\Phi(ge^t, u, y)}{j_D(g, e^t)} dt, \tag{1.11}$$

which is the main object in this paper. We construct interpolation formulas of the well-known relations among poly-Bernoulli numbers by use of $\xi_D(u, s; y, w; g)$.

In Section 2, we define the Lerch transcendent and study its properties and related results.

In Section 3, we define (1.11) (see Definition 3.1) and determine its domain (see Theorem 3.6). We confirm that $\xi(k; s)$, $\eta(k; s)$ and $\xi(k; s; w)$ can be regarded as special cases of (1.11) (see Example 3.7).

In Section 4, we give two types of functional relations among (1.11) which include (1.10) as a special case (see Theorems 4.1 and 4.3). Combining these formulas, we give interpolation formulas of the well-known relations including (1.3)–(1.5) (see Example 4.5).

In Section 5, we consider the analytic continuation for (1.11) (see Theorems 5.3, 5.5 and 5.6), and introduce several examples of duality relations (see Examples 5.8 and 5.9).

In Section 6, we define the poly-Bernoulli polynomials associated with $GL_2(\mathbb{C})$ (see Definition 6.1). From the results in Sections 4 and 5, we give general forms of difference relations and duality relations for them (see Theorems 6.7 and 6.9). These include (1.3)–(1.5) and also the duality relations for poly-Bernoulli polynomials (see Example 6.10) given by Kaneko, Sakurai and the second-named author (see [9]). Furthermore, we give new duality relations for certain sums of $C_m^{(-k)}$ (see (6.19), (6.20) and (6.21) in Example 6.11).

2. Preliminaries.

For $z, u, y \in \mathbb{C}$ with $|z| < 1$ or ($z = 1$ and $\operatorname{Re} u > 1$), and $\operatorname{Re} y > 0$, the Lerch transcendent is defined by

$$\Phi(z, u, y) = \sum_{n=0}^{\infty} \frac{z^n}{(y+n)^u},$$

which is a generalization of the polylogarithm defined by

$$\operatorname{Li}_u(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^u},$$

and is related as

$$z\Phi(z, u, 1) = \operatorname{Li}_u(z). \tag{2.1}$$

For $k \in \mathbb{Z}_{\geq 0}$, the Lerch transcendent satisfies the following.

$$\Phi(z, u, y) = z^k \Phi(z, u, y+k) + \sum_{n=0}^{k-1} \frac{z^n}{(y+n)^u} \tag{2.2}$$

$$= z^{-k} \Phi(z, u, y-k) - \sum_{n=1}^k \frac{z^{-n}}{(y-n)^u}. \tag{2.3}$$

LEMMA 2.1. *For ($\operatorname{Re} u > 0$ and $|z| < 1$) or ($\operatorname{Re} u > 1$ and $z = 1$), and $\operatorname{Re} y > 0$, $\Phi(z, u, y)$ has the integral representation*

$$\Phi(z, u, y) = \frac{1}{\Gamma(u)} \int_0^{\infty} x^{u-1} e^{-yx} \frac{1}{1 - ze^{-x}} dx.$$

This expression gives the analytic continuation of $\Phi(z, u, y)$ for $z \in \mathbb{C} \setminus [1, +\infty)$, $\operatorname{Re} u > 0$ and $\operatorname{Re} y > 0$.

PROOF. First we assume $|z| < 1$ or $z = 1$. By an integral representation of the gamma function $\Gamma(u)$ for $\operatorname{Re} u > 0$, we have

$$\frac{1}{a^u} = \frac{1}{\Gamma(u)} \int_0^{\infty} e^{-ax} x^{u-1} dx$$

for $\operatorname{Re} a > 0$. For ($\operatorname{Re} u > 0$ and $|z| < 1$) or ($\operatorname{Re} u > 1$ and $z = 1$), by substituting this into the series expression, we obtain

$$\begin{aligned} \Phi(z, u, y) &= \frac{1}{\Gamma(u)} \sum_{n=0}^{\infty} \int_0^{\infty} z^n e^{-nx} e^{-yx} x^{u-1} dx \\ &= \frac{1}{\Gamma(u)} \int_0^{\infty} x^{u-1} e^{-yx} \frac{1}{1 - ze^{-x}} dx. \end{aligned}$$

By this integral representation, $\Phi(z, u, y)$ is analytically continued for $z \in \mathbb{C} \setminus [1, +\infty)$, $\operatorname{Re} u > 0$ and $\operatorname{Re} y > 0$. \square

For a variable u , we define a difference operator D_u by

$$D_u f(u) = f(u + 1).$$

We also define the Euler operator

$$\vartheta_z = z \frac{\partial}{\partial z}.$$

LEMMA 2.2.

$$(D_u^{-1} - y)\Phi(z, u, y) = \vartheta_z \Phi(z, u, y).$$

PROOF. By the series expression, we have

$$\Phi(z, u - 1, y) = \left(y + z \frac{\partial}{\partial z}\right) \Phi(z, u, y),$$

which is rewritten in terms of the difference operator D_u . \square

LEMMA 2.3. For $n \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} \left(\prod_{k=1}^n \left(1 + \frac{1}{k} \vartheta_z\right)\right) \frac{1}{1 - ze^{-x}} &= \frac{1}{(1 - ze^{-x})^{n+1}}, \\ \left(\prod_{k=1}^n \left(-1 + \frac{1}{k} \vartheta_z\right)\right) \frac{ze^{-x}}{1 - ze^{-x}} &= \left(\frac{ze^{-x}}{1 - ze^{-x}}\right)^{n+1}. \end{aligned}$$

PROOF. Since

$$\begin{aligned} \vartheta_z \frac{1}{(1 - ze^{-x})^k} &= \frac{kze^{-x}}{(1 - ze^{-x})^{k+1}} \\ &= \frac{k}{(1 - ze^{-x})^{k+1}} - \frac{k}{(1 - ze^{-x})^k}, \end{aligned}$$

we have

$$\left(1 + \frac{1}{k} \vartheta_z\right) \frac{1}{(1 - ze^{-x})^k} = \frac{1}{(1 - ze^{-x})^{k+1}},$$

which yields the first equation.

Similarly

$$\vartheta_z \left(\frac{ze^{-x}}{1 - ze^{-x}}\right)^k = k \left(\frac{ze^{-x}}{1 - ze^{-x}}\right)^{k+1} + k \left(\frac{ze^{-x}}{1 - ze^{-x}}\right)^k$$

implies

$$\left(-1 + \frac{1}{k} \vartheta_z\right) \left(\frac{ze^{-x}}{1 - ze^{-x}}\right)^k = \left(\frac{ze^{-x}}{1 - ze^{-x}}\right)^{k+1}$$

and the second equation. □

LEMMA 2.4. For $n \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} \frac{1}{n!} \left(\prod_{k=1}^n (D_u^{-1} - y + k)\right) \Phi(z, u, y) &= \frac{1}{\Gamma(u)} \int_0^\infty x^{u-1} e^{-yx} \frac{1}{(1 - ze^{-x})^{n+1}} dx, \\ \frac{1}{n!} \left(\prod_{k=1}^n (D_u^{-1} - y - k)\right) (\Phi(z, u, y) - y^{-u}) &= \frac{1}{\Gamma(u)} \int_0^\infty x^{u-1} e^{-yx} \left(\frac{ze^{-x}}{1 - ze^{-x}}\right)^{n+1} dx. \end{aligned}$$

PROOF. The results follow from Lemmas 2.2 and 2.3. □

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$, we define the Möbius transformation

$$gz = \frac{az + b}{cz + d}$$

for $z \in \widehat{\mathbb{C}}$. Note that it is well known that Möbius transformations are conformal and map circular arcs to circular arcs, where circular arcs include line segments. Let

$$V(g) = \{g1, g\infty\} \cap \{1, \infty\}$$

be the intersection of the extremal points of the two circular arcs $g([1, +\infty])$ and $[1, +\infty]$. Let

$$j_D(g, z) = cz + d, \quad j_N(g, z) = az + b$$

for $z \in \mathbb{C}$. Then for $g, h \in GL_2(\mathbb{C})$, we have

$$\begin{aligned} j_D(gh, T) &= j_D(g, hT)j_D(h, T), \\ j_N(gh, T) &= j_N(g, hT)j_D(h, T). \end{aligned}$$

If two circular arcs intersect at their extremal points, we call such point a vertex. Moreover if the vertex angle is zero, then we call the vertex a cusp.

For $Z \in \{1, \infty\}$, we denote $\tilde{Z} = 1/Z \in \{0, 1\}$. Let

$$W_{a,\epsilon,R} = \{z \in \mathbb{C} \mid 0 < |z - a| < \epsilon\} \cup \{z \in \mathbb{R} \mid a < z < R\}$$

for $a \geq 0, \epsilon, R > 0$. We abbreviate $W_{a,\epsilon} = W_{a,\epsilon,+\infty}$.

The following lemmas give certain inequalities under the assumption that the two circular arcs $g([1, +\infty])$ and $[1, +\infty]$ intersect each other possibly only at their extremal points. See Figure 1 for typical configurations. These estimations play important roles when the domains of the main objects are determined. Their proofs will be given in Section 7.

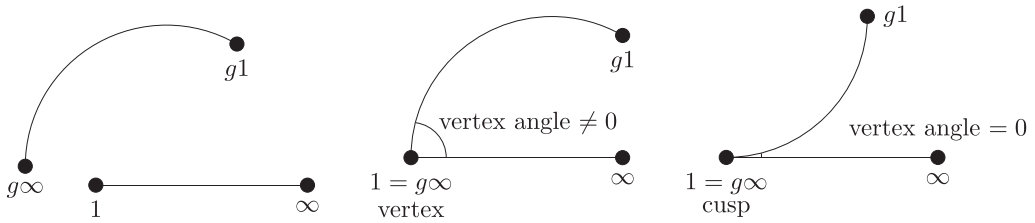


Figure 1. typical configurations.

LEMMA 2.5. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$ and $T_0, X_0 \in \{1, \infty\}$. Assume that $gT = X$ for only $(T, X) = (T_0, X_0)$ in its neighborhood in $[1, +\infty]^2$.

1. For $0 \leq q \leq 1$, there exists $M > 0$ such that

$$\left| \frac{1}{j_D(g, T)} \frac{1}{(1 - (gT)X^{-1})} \right| \leq \begin{cases} \frac{M}{T} \left| \frac{T}{\tilde{T}_0 T - 1} \right|^{1-q} \left| \frac{X}{\tilde{X}_0 X - 1} \right|^q & \text{if the vertex is not a cusp,} \\ \frac{M}{T} \left| \frac{T}{\tilde{T}_0 T - 1} \right|^{2(1-q)} \left| \frac{X}{\tilde{X}_0 X - 1} \right|^{2q} & \text{if the vertex is a cusp} \end{cases}$$

in a sufficiently small neighborhood of (T_0, X_0) in $(1, +\infty)^2$.

2. There exists $\epsilon > 0$ such that

$$\frac{1}{\epsilon} \left| \frac{\tilde{X}_0 X - 1}{X} \right| > \left| \frac{\tilde{T}_0 T - 1}{T} \right| > \epsilon \left| \frac{\tilde{X}_0 X - 1}{X} \right| \tag{2.4}$$

for any pair (T, X) satisfying $gT = X$ in a sufficiently small neighborhood of (T_0, X_0) in \mathbb{C}^2 .

LEMMA 2.6. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$ be such that

$$g([1, +\infty]) \cap [1, +\infty] \subset \{g1, g\infty\} \cap \{1, \infty\} = V(g).$$

Let N be a neighborhood of $\{(T_0, X_0) \mid X_0 \in V(g), T_0 = g^{-1}X_0\}$ in $\widehat{\mathbb{C}}^2$. Then there exist $\epsilon > 0$ and $M > 0$ such that

$$\left| \frac{1}{j_D(g, T)} \frac{1}{(1 - (gT)X^{-1})} \right| \leq \frac{M}{|T|}$$

for all $(T, X) \in W_{1, \epsilon}^2 \setminus N$.

3. Arakawa–Kaneko zeta-functions associated with $GL_2(\mathbb{C})$.

Here and hereafter we only consider $g \in GL_2(\mathbb{C})$ which satisfies that

$$g([1, +\infty]) \cap [1, +\infty] \subset \{g1, g\infty\} \cap \{1, \infty\} = V(g). \tag{3.1}$$

In this section, we give the definition of generalizations of the Arakawa–Kaneko zeta-function. The domains of the functions will be given later, which depend on the configuration of the three points $\{g_0, g_1, g_\infty\}$ on the Riemann sphere $\widehat{\mathbb{C}}$.

DEFINITION 3.1. For $g \in GL_2(\mathbb{C})$ satisfying (3.1), we define the Arakawa–Kaneko zeta-function associated with g by

$$\xi_D(u, s; y, w; g) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-wt} \frac{\Phi(ge^t, u, y)}{j_D(g, e^t)} dt. \tag{3.2}$$

We define an auxiliary function

$$\xi_N(u, s; y, w; g) = \xi_D(u, s; y + 1, w; g). \tag{3.3}$$

We have the following integral representation of $\xi_N(u, s; y, w; g)$, which clarifies the meaning of the subscripts “ D ” and “ N ”.

LEMMA 3.2.

$$\xi_N(u, s; y, w; g) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-wt} \frac{(\Phi(ge^t, u, y) - y^{-u})}{j_N(g, e^t)} dt. \tag{3.4}$$

PROOF. Since

$$z\Phi(z, u, y + 1) = \Phi(z, u, y) - y^{-u},$$

we have

$$\begin{aligned} \xi_N(u, s; y, w; g) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-wt} \frac{ge^t \Phi(ge^t, u, y + 1)}{j_N(g, e^t)} dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-wt} \frac{(\Phi(ge^t, u, y) - y^{-u})}{j_N(g, e^t)} dt. \quad \square \end{aligned}$$

By the integral representation of the Lerch transcendent in Lemma 2.1, we have double integral representations of the Arakawa–Kaneko zeta-functions.

LEMMA 3.3.

$$\xi_D(u, s; y, w; g) = \frac{1}{\Gamma(s)\Gamma(u)} \int_0^\infty \int_0^\infty \frac{t^{s-1} x^{u-1} e^{-wt} e^{-yx}}{j_D(g, e^t)} \frac{1}{1 - (ge^t)e^{-x}} dt dx, \tag{3.5}$$

$$\xi_N(u, s; y, w; g) = \frac{1}{\Gamma(s)\Gamma(u)} \int_0^\infty \int_0^\infty \frac{t^{s-1} x^{u-1} e^{-wt} e^{-yx}}{j_N(g, e^t)} \frac{(ge^t)e^{-x}}{1 - (ge^t)e^{-x}} dt dx. \tag{3.6}$$

To determine the domain of $\xi_D(u, s; y, w; g)$, we need to study when the integral (3.5) (hence (3.6)) is convergent. Here we give a sufficient condition below. It should be noted that generally the domain is wider and is dependent on g . To describe the domain, we define the following constants.

DEFINITION 3.4. Consider the two circular arcs $g([1, +\infty])$ and $[1, +\infty]$ on the Riemann sphere $\widehat{\mathbb{C}}$. Then for $T_0, X_0 \in \{1, \infty\}$, we fix $\mu_{T_0}, \nu_{X_0} \geq 0$ as follows. For

$gT_0 \notin V(g)$ (resp. $X_0 \notin V(g)$), we set $\mu_{T_0} = 0$ (resp. $\nu_{X_0} = 0$). Further for a pair (T_0, X_0) such that $gT_0 = X_0 \in V(g)$, we set

$$\mu_{T_0} + \nu_{X_0} = \begin{cases} 1 & \text{if } gT_0 = X_0 \text{ is not a cusp,} \\ 2 & \text{if } gT_0 = X_0 \text{ is a cusp.} \end{cases}$$

LEMMA 3.5. 1. *There exists $M > 0$ such that for all $(t, x) \in (0, +\infty)^2$,*

$$\left| \frac{1}{j_D(g, e^t)} \frac{1}{1 - (ge^t)e^{-x}} \right| \leq Mt^{-\mu_1} x^{-\nu_1} e^{(\mu_\infty - 1)t} e^{\nu_\infty x} (t + 1)^{\mu_1} (x + 1)^{\nu_1}. \quad (3.7)$$

2. *Let Z be a neighborhood of $\{(\log T_0, \log X_0) \mid X_0 \in V(g), T_0 = g^{-1}X_0\}$ in \mathbb{C}^2 . Then there exist $M > 0$ and $\epsilon > 0$ such that for all $(t, x) \in W_{0, \epsilon}^2 \setminus Z$,*

$$\left| \frac{1}{j_D(g, e^t)} \frac{1}{1 - (ge^t)e^{-x}} \right| \leq Me^{-\operatorname{Re} t}. \quad (3.8)$$

3. *If $g1 \neq 1$, then for any sufficiently large $R > 0$, there exist $M > 0$ and $\epsilon > 0$ such that for all $(t, x) \in W_{0, \epsilon, R}^2$,*

$$\left| \frac{1}{j_D(g, e^t)} \frac{1}{1 - (ge^t)e^{-x}} \right| \leq M. \quad (3.9)$$

4. *If $g1 = \infty$, then there exists $\epsilon > 0$ such that*

$$|t| > \epsilon e^{-x} \quad (3.10)$$

for any pair (t, x) satisfying $ge^t = e^x$ in a sufficiently small neighborhood of $(0, +\infty)$ in $\mathbb{C} \times \mathbb{R}$.

5. *If $g\infty = 1$, then there exists $\epsilon > 0$ such that*

$$|x| > \epsilon e^{-t} \quad (3.11)$$

for any pair (t, x) satisfying $ge^t = e^x$ in a sufficiently small neighborhood of $(+\infty, 0)$ in $\mathbb{R} \times \mathbb{C}$.

PROOF. Let Z be a neighborhood of $\{(\log T_0, \log X_0) \mid X_0 \in V(g), T_0 = g^{-1}X_0\}$ in \mathbb{C}^2 . If $V(g) \neq \emptyset$, then for each $X_0 \in V(g)$, consider a sufficiently small neighborhood $N'(X_0)$ of (T_0, X_0) in $(1, +\infty)^2$ such that $J'(X_0) = \{(\log T, \log X) \mid (T, X) \in N'(X_0)\} \subset Z$. By Lemma 2.5, there exists $M > 0$ such that

$$\left| \frac{1}{j_D(g, e^t)} \frac{1}{1 - (ge^t)e^{-x}} \right| \leq Me^{-t} \left| \frac{e^t}{\tilde{T}_0 e^t - 1} \right|^{\mu_{T_0}} \left| \frac{e^x}{\tilde{X}_0 e^x - 1} \right|^{\nu_{X_0}}$$

for all $(t, x) \in J'(X_0)$.

Let N be a sufficiently small neighborhood of $\{(T_0, X_0) \mid X_0 \in V(g), T_0 = g^{-1}X_0\}$ in $\widehat{\mathbb{C}}^2$ such that $N \cap (1, +\infty)^2$ is contained in the union of the neighborhoods $N'(X_0)$

taken in the previous paragraph for each $X_0 \in V(g)$. By Lemma 2.6, there exist $\epsilon > 0$ and $M > 0$ such that

$$\left| \frac{1}{j_D(g, e^t)} \frac{1}{1 - (ge^t)e^{-x}} \right| \leq M e^{-\operatorname{Re} t} \tag{3.12}$$

for all $(t, x) \in J$, where $I = W_{1,\epsilon}^2 \setminus N$ and $J = \{(\log T, \log X) \mid (T, X) \in I\}$. Let $\epsilon' > 0$ be sufficiently small such that $e^z \in W_{1,\epsilon}$ for all $z \in W_{0,\epsilon'}$. This implies (3.8). In particular, if $g1 \neq 1$, then $W_{0,\epsilon',R}^2 \subset J$ for any sufficiently large $R > 0$. Thus (3.12) implies (3.9).

Since for all $z > 0$,

$$1 \leq \left| \frac{e^z}{\tilde{Z}_0 e^z - 1} \right| \leq \begin{cases} \frac{z+1}{z} & (Z_0 = 1), \\ e^z & (Z_0 = \infty), \end{cases}$$

we have

$$\begin{aligned} 1, \left| \frac{e^t}{\tilde{T}_0 e^t - 1} \right|^{\mu_{T_0}} &\leq t^{-\mu_1} e^{\mu_\infty t} (t+1)^{\mu_1}, \\ 1, \left| \frac{e^x}{\tilde{X}_0 e^x - 1} \right|^{\nu_{X_0}} &\leq x^{-\nu_1} e^{\nu_\infty x} (x+1)^{\nu_1} \end{aligned}$$

and hence for all $(t, x) \in (0, +\infty)^2$,

$$\left| \frac{1}{j_D(g, e^t)} \frac{1}{1 - (ge^t)e^{-x}} \right| \leq M' t^{-\mu_1} x^{-\nu_1} e^{(\mu_\infty - 1)t} e^{\nu_\infty x} (t+1)^{\mu_1} (x+1)^{\nu_1}$$

for some $M' > 0$, which implies (3.7).

Inequalities (3.10) and (3.11) follow from (2.4). □

THEOREM 3.6. For $\operatorname{Re} u > \nu_1$, $\operatorname{Re} s > \mu_1$, $\operatorname{Re} y > \nu_\infty$, $\operatorname{Re} w > \mu_\infty - 1$, $\xi_D(u, s; y, w; g)$ is defined and analytic in u, s, y, w .

PROOF. By Lemma 3.5,

$$\begin{aligned} &\int_0^\infty \int_0^\infty \left| \frac{t^{s-1} x^{u-1} e^{-wt} e^{-yx}}{j_D(g, e^t)} \frac{1}{1 - (ge^t)e^{-x}} \right| dt dx \\ &\leq M \int_0^\infty \int_0^\infty t^{\operatorname{Re} s - 1 - \mu_1} x^{\operatorname{Re} u - 1 - \nu_1} e^{(\mu_\infty - 1 - \operatorname{Re} w)t} \\ &\quad \times e^{(\nu_\infty - \operatorname{Re} y)x} (t+1)^{\mu_1} (x+1)^{\nu_1} dt dx < \infty. \end{aligned}$$

The analyticity in u, s, y, w follows from the Morera theorem and the Fubini theorem. □

EXAMPLE 3.7. Let $g_\eta := \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$. We can see that $g_\eta^{-1} = g_\eta$ and $\det g_\eta = -1$ which are important properties. For $g = g_\eta$, we have $gT = 1 - T$, namely, $g1 = 0$, $g\infty = \infty$ and

$$g([1, +\infty]) \cap [1, +\infty] = \{\infty\} = V(g).$$

Hence, by Definition 3.4, we obtain $\mu_1 = 0$ and $\nu_1 = 0$. Since ∞ is not a cusp, we have $\mu_\infty, \nu_\infty \in [0, 1]$ satisfying $\mu_\infty + \nu_\infty = 1$. Therefore $\xi_D(u, s; y, w; g_\eta)$ is defined for $\operatorname{Re} u > 0, \operatorname{Re} s > 0, \operatorname{Re} y > \nu_\infty, \operatorname{Re} w > \mu_\infty - 1$, where $\mu_\infty, \nu_\infty \in [0, 1]$ with $\mu_\infty + \nu_\infty = 1$. We see that

$$\operatorname{Li}_u(g e^t) = \operatorname{Li}_u(1 - e^t), \quad j_D(g, e^t) = 1, \quad j_N(g, e^t) = 1 - e^t.$$

Hence, noting (2.1) and (3.3), we define

$$\eta(u; s) = \xi_N(u, s; 0, 0; g_\eta) = \xi_D(u, s; 1, 0; g_\eta),$$

which was already considered by Yamamoto [12].

Let $g_\xi := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. Then $\det g_\xi = 1$. For $g = g_\xi$, we have $gT = 1 - T^{-1}$, namely, $g1 = 0, g\infty = 1$ and

$$g([1, +\infty]) \cap [1, +\infty] = \{1\} = V(g).$$

Hence we obtain $\mu_1 = 0$ and $\nu_\infty = 0$. Since 1 is not a cusp, we have $\mu_\infty, \nu_1 \in [0, 1]$ satisfying $\mu_\infty + \nu_1 = 1$. Therefore $\xi_D(u, s; y, w; g_\xi)$ is defined for $\operatorname{Re} u > \nu_1, \operatorname{Re} s > 0, \operatorname{Re} y > 0, \operatorname{Re} w > \mu_\infty - 1$, where $\mu_\infty, \nu_1 \in [0, 1]$ with $\mu_\infty + \nu_1 = 1$. We have

$$\operatorname{Li}_u(g e^t) = \operatorname{Li}_u(1 - e^{-t}), \quad j_D(g, e^t) = e^t, \quad j_N(g, e^t) = e^t - 1.$$

Hence, noting (2.1) and (3.3), we define

$$\xi(u; s; w) = \xi_N(u, s; 0, w - 1; g_\xi) = \xi_D(u, s; 1, w - 1; g_\xi)$$

and, in particular,

$$\xi(u; s) = \xi_N(u, s; 0, 0; g_\xi) = \xi_D(u, s; 1, 0; g_\xi),$$

which is a generalization of (1.6).

4. Relations among Arakawa–Kaneko zeta-functions.

In this section, we give two types of functional relation formulas for ξ_D and ξ_N (see Theorems 4.1 and 4.3). We will see that these give functional relations which interpolate the well-known relations among poly-Bernoulli numbers in Section 6.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we put $j_N(g, D_u) = aD_u + b$ for the difference operator D_u and so on.

THEOREM 4.1 (Difference relations). *For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have*

$$j_N(g, D_w^{-1})\xi_N(u, s; y, w; g) = j_D(g, D_w^{-1})\xi_D(u, s; y, w; g) - y^{-u}w^{-s}, \quad (4.1)$$

namely,

$$\begin{aligned}
& a\xi_N(u, s; y, w-1; g) + b\xi_N(u, s; y, w; g) \\
& (= a\xi_D(u, s; y+1, w-1; g) + b\xi_D(u, s; y+1, w; g)) \\
& = c\xi_D(u, s; y, w-1; g) + d\xi_D(u, s; y, w; g) - y^{-u}w^{-s}.
\end{aligned}$$

PROOF. The assertion follows from the integral representations (3.2) and (3.4) with

$$\frac{aT+b}{j_N(g, T)} - \frac{cT+d}{j_D(g, T)} = 1 - 1 = 0$$

and

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-wt} y^{-u} dt = y^{-u} w^{-s}. \quad \square$$

For $g \in GL_2(\mathbb{C})$ and indeterminates X, T , we define

$$\begin{aligned}
F_D &= \frac{1}{1 - (gT)X^{-1}}, & G_D &= \frac{1}{1 - (g^{-1}X)T^{-1}}, \\
F_N &= \frac{(gT)X^{-1}}{1 - (gT)X^{-1}}, & G_N &= \frac{(g^{-1}X)T^{-1}}{1 - (g^{-1}X)T^{-1}}.
\end{aligned}$$

From Lemma 3.3, we see that these come from the integrands of the double integral representations. We have the key relations, which are the core of duality relations.

LEMMA 4.2.

$$\begin{aligned}
\frac{T}{j_D(g, T)} F_D &= -\frac{1}{\det g} \frac{X}{j_D(g^{-1}, X)} G_D, \\
\frac{1}{j_N(g, T)} F_N &= -\frac{1}{\det g} \frac{1}{j_N(g^{-1}, X)} G_N, \\
\frac{1}{j_D(g, T)} F_D &= -\frac{1}{\det g} \frac{X}{j_N(g^{-1}, X)} G_N.
\end{aligned}$$

PROOF. The first equation follows from

$$\begin{aligned}
F_D &= \frac{1}{1 - (gT)X^{-1}} \\
&= \frac{j_D(g, T)X}{(cX - a)T - (-dX + b)} \\
&= -\frac{j_D(g, T)T^{-1}}{1 - (g^{-1}X)T^{-1}} \frac{X}{-cX + a} \\
&= -\frac{1}{\det g} \frac{j_D(g, T)T^{-1}}{1 - (g^{-1}X)T^{-1}} \frac{X}{j_D(g^{-1}, X)} \\
&= -\frac{1}{\det g} \frac{j_D(g, T)T^{-1}X}{j_D(g^{-1}, X)} G_D,
\end{aligned}$$

and the third, from

$$\begin{aligned} F_D &= -\frac{1}{\det g} \frac{j_D(g, T)T^{-1}X}{j_D(g^{-1}, X)} \frac{j_D(g^{-1}, X)}{j_N(g^{-1}, X)T^{-1}} G_N \\ &= -\frac{1}{\det g} \frac{j_D(g, T)X}{j_N(g^{-1}, X)} G_N, \end{aligned}$$

and finally the second, from

$$\begin{aligned} F_N &= \frac{j_N(g, T)X^{-1}}{j_D(g, T)} F_D \\ &= -\frac{1}{\det g} \frac{j_N(g, T)X^{-1}}{j_D(g, T)} \frac{j_D(g, T)X}{j_N(g^{-1}, X)} G_N \\ &= -\frac{1}{\det g} \frac{j_N(g, T)}{j_N(g^{-1}, X)} G_N. \quad \square \end{aligned}$$

There are three types of duality relations, namely, ascending-ascending, descending-descending, and ascending-descending types.

THEOREM 4.3 (Duality relations). For $n \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} j_D(g^{-1}, D_y^{-1})^n D_w^{-n-1} \left(\prod_{k=1}^n (D_u^{-1} - y + k) \right) \xi_D(u, s; y, w; g) \\ = \left(\frac{-1}{\det g} \right)^{n+1} j_D(g, D_w^{-1})^n D_y^{-n-1} \left(\prod_{k=1}^n (D_s^{-1} - w + k) \right) \xi_D(s, u; w, y; g^{-1}), \end{aligned} \quad (4.2)$$

$$\begin{aligned} j_N(g^{-1}, D_y^{-1})^n \left(\prod_{k=1}^n (D_u^{-1} - y - k) \right) \xi_N(u, s; y, w; g) \\ = \left(\frac{-1}{\det g} \right)^{n+1} j_N(g, D_w^{-1})^n \left(\prod_{k=1}^n (D_s^{-1} - w - k) \right) \xi_N(s, u; w, y; g^{-1}) \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} j_N(g^{-1}, D_y^{-1})^n \left(\prod_{k=1}^n (D_u^{-1} - y + k) \right) \xi_D(u, s; y, w; g) \\ = \left(\frac{-1}{\det g} \right)^{n+1} j_D(g, D_w^{-1})^n D_y^{-n-1} \left(\prod_{k=1}^n (D_s^{-1} - w - k) \right) \xi_N(s, u; w, y; g^{-1}). \end{aligned} \quad (4.4)$$

PROOF. From Lemmas 2.3, 2.4 and 3.2, we have

$$\begin{aligned} \frac{1}{n!} \left(\prod_{k=1}^n (D_u^{-1} - y + k) \right) \xi_D(u, s; y, w; g) \\ = \frac{1}{\Gamma(s)\Gamma(u)} \int_0^\infty \int_0^\infty \frac{t^{s-1} x^{u-1} e^{-wt} e^{-yx}}{j_D(g, e^t)} \frac{1}{(1 - (ge^t)e^{-x})^{n+1}} dt dx, \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{n!} \left(\prod_{k=1}^n (D_u^{-1} - y - k) \right) \xi_N(u, s; y, w; g) \\ &= \frac{1}{\Gamma(s)\Gamma(u)} \int_0^\infty \int_0^\infty \frac{t^{s-1} x^{u-1} e^{-wt} e^{-yx}}{j_N(g, e^t)} \left(\frac{(ge^t)e^{-x}}{1 - (ge^t)e^{-x}} \right)^{n+1} dt dx. \end{aligned}$$

Lemma 4.2 implies

$$\begin{aligned} j_D(g^{-1}, X)^n \frac{T^{n+1}}{j_D(g, T)} F_D^{n+1} &= \left(\frac{-1}{\det g} \right)^{n+1} j_D(g, T)^n \frac{X^{n+1}}{j_D(g^{-1}, X)} G_D^{n+1}, \\ j_N(g^{-1}, X)^n \frac{1}{j_N(g, T)} F_N^{n+1} &= \left(\frac{-1}{\det g} \right)^{n+1} j_N(g, T)^n \frac{1}{j_N(g^{-1}, X)} G_N^{n+1}, \\ j_N(g^{-1}, X)^n \frac{1}{j_D(g, T)} F_D^{n+1} &= \left(\frac{-1}{\det g} \right)^{n+1} j_D(g, T)^n \frac{X^{n+1}}{j_N(g^{-1}, X)} G_N^{n+1}. \end{aligned}$$

By noting that

$$j_D(g^{-1}, D_y^{-1})^n D_w^{-n-1} (t^{s-1} x^{u-1} e^{-wt} e^{-yx}) = j_D(g^{-1}, e^x)^n e^{(n+1)t} (t^{s-1} x^{u-1} e^{-wt} e^{-yx})$$

and so on, we obtain the result. □

The $n = 0$ case reduces to the following.

COROLLARY 4.4.

$$\xi_D(u, s; y, w - 1; g) = -\frac{1}{\det g} \xi_D(s, u; w, y - 1; g^{-1}), \tag{4.5}$$

$$\xi_N(u, s; y, w; g) = -\frac{1}{\det g} \xi_N(s, u; w, y; g^{-1}), \tag{4.6}$$

$$\xi_D(u, s; y, w; g) = -\frac{1}{\det g} \xi_N(s, u; w, y - 1; g^{-1}), \tag{4.7}$$

which are essentially the same formulas.

EXAMPLE 4.5. As for $\eta(u; s) = \xi_D(u, s; 1, 0; g_\eta)$ defined in Example 3.7, noting $g_\eta^{-1} = g_\eta$, we see that (4.5) (resp. (4.6) and (4.7)) with $(y, w) = (1, 1)$ (resp. $(0, 0)$ and $(1, 0)$) implies Yamamoto’s result $\eta(u; s) = \eta(s; u)$ in (1.10), which interpolates (1.3) (for the values of $\eta(u; s)$ at nonpositive integers, see (6.4)). We will further introduce several duality relations for $\xi_D(u, s; y, w; g)$ in Section 5 (see Examples 5.8 and 5.9).

REMARK 4.6. $\xi(u, s; y, w; g)$ can be slightly generalized with two elements $g, h \in GL_2(\mathbb{C})$ and two appropriate paths I, J which start at 0 and go to $+\infty$ as

$$\xi(u, s; y, w; h, g; I, J) = \frac{1}{\Gamma(s)\Gamma(u)} \int_J dt \int_I dx \frac{t^{s-1} x^{u-1} e^{-wt} e^{-yx}}{j_D(g, e^t) j_D(h, e^x)} \frac{1}{h e^x - g e^t}.$$

Since variables are treated completely symmetrically, it is easy to see that the trivial symmetry

$$\xi(u, s; y, w; h, g; I, J) = -\xi(s, u; w, y; g, h; J, I)$$

holds. Moreover we can show that

$$\xi(u, s; y - 1, w; h, g; (0, +\infty), (0, +\infty)) = \frac{1}{\det h} \xi_D(u, s; y, w; h^{-1}g),$$

which implies (4.5).

From the above we see that the pair (g, h) does not give rise to a generalization, while two paths I, J are essential because by this modification, it is possible to avoid cusps and to define ξ_D for any element $g \in GL_2(\mathbb{C})$ without the restriction (3.1).

5. Analytic continuation.

We give integral representations with Hankel contours to enlarge the domain of $\xi_D(u, s; y, w; g)$. In the following, $H_{\epsilon, R}$ denotes the Hankel contour, which consists of a path from R to ϵ on the real axis, around the origin counter clockwise with radius ϵ , and back to R , where $R \in (0, +\infty]$ and ϵ is an arbitrarily small positive number. We abbreviate $H = H_{\epsilon, +\infty}$.

In the following proofs, since the analyticities follow from the Morera theorem and the Fubini theorem, we omit them.

LEMMA 5.1. *Let $k \in \mathbb{Z}_{\geq 0}$. Then $\Phi(z, u, y)$ has the integral representation*

$$\Phi(z, u, y) = \frac{1}{\Gamma(u)(e^{2\pi i u} - 1)} \int_H x^{u-1} e^{-(y+k)x} \frac{z^k}{1 - ze^{-x}} dx + \sum_{n=0}^{k-1} \frac{z^n}{(y+n)^u}.$$

This expression gives the analytic continuation of $\Phi(z, u, y)$ and is valid for $z \in \mathbb{C} \setminus [1, +\infty)$ or $z = 1$, $u \in \mathbb{C}$ and $y \in \mathbb{C} \setminus (-\infty, 0]$ with $\operatorname{Re} y > -k$ except for appropriate branch cuts. Therefore $\Phi(z, u, y)$ is analytically continued in $z \in \mathbb{C} \setminus [1, +\infty)$ or $z = 1$, $u \in \mathbb{C}$ and $y \in \mathbb{C} \setminus (-\infty, 0]$ except for appropriate branch cuts.

PROOF. By (2.2) and Lemma 2.1, we have

$$\Phi(z, u, y) = \frac{1}{\Gamma(u)} \int_0^\infty x^{u-1} e^{-(y+k)x} \frac{z^k}{1 - ze^{-x}} dx + \sum_{n=0}^{k-1} \frac{z^n}{(y+n)^u},$$

which gives the integral representation with the Hankel contour. □

We study integral representations of $\xi_D(u, s; y, w; g)$ with Hankel contours by considering slightly general forms given in Lemma 3.3, namely, for $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} \xi_D(u, s; y, w; g) &= \frac{1}{\Gamma(s)\Gamma(u)} \int_0^\infty \int_0^\infty \frac{t^{s-1} x^{u-1} e^{-wt} e^{-(y+k)x}}{j_D(g, e^t)} \frac{(ge^t)^k}{1 - (ge^t)e^{-x}} dt dx \\ &\quad + \frac{1}{\Gamma(s)} \sum_{n=0}^{k-1} \frac{1}{(y+n)^u} \int_0^\infty t^{s-1} e^{-wt} \frac{(ge^t)^n}{j_D(g, e^t)} dt, \end{aligned}$$

by (2.3) and (3.2). We denote the first term and the second term by $\xi_{1,k}(u, s; y, w; g)$ and $\xi_{2,k}(u, s; y, w; g)$ respectively so that

$$\xi_D(u, s; y, w; g) = \xi_{1,k}(u, s; y, w; g) + \xi_{2,k}(u, s; y, w; g).$$

First we give the explicit form of $\xi_{2,k}(u, s; y, w; g)$, which gives its analytic continuation.

LEMMA 5.2. *Let $k \in \mathbb{Z}_{\geq 0}$.*

$$\begin{aligned} \xi_{2,k}(u, s; y, w; g) &= \sum_{n=0}^{k-1} \frac{1}{(y+n)^u} j_N(g, D_w^{-1})^n \\ &\quad \times \begin{cases} \frac{1}{d^{n+1}} \frac{1}{w^s} & (g\infty = \infty), \\ \frac{1}{c^{n+1}} \frac{1}{(w+n+1)^s} & (g0 = \infty), \\ \frac{1}{c^{n+1}} \frac{1}{n!} D_w^{n+1} \left(\prod_{j=1}^n (D_s^{-1} - w + j) \right) \Phi(-d/c, s, w) & (\text{otherwise}), \end{cases} \end{aligned} \quad (5.1)$$

which gives the analytic continuation to the whole space in u, s, y, w except for appropriate branch cuts.

PROOF. If $g\infty = \infty$, then $c = 0$ and

$$\begin{aligned} \int_0^\infty t^{s-1} e^{-wt} \frac{(ge^t)^n}{j_D(g, e^t)} dt &= \frac{1}{d^{n+1}} j_N(g, D_w^{-1})^n \int_0^\infty t^{s-1} e^{-wt} dt \\ &= \frac{\Gamma(s)}{d^{n+1}} j_N(g, D_w^{-1})^n \frac{1}{w^s}, \end{aligned}$$

which implies (5.1) in this case. If $g0 = \infty$, then $d = 0$ and

$$\begin{aligned} \int_0^\infty t^{s-1} e^{-wt} \frac{(ge^t)^n}{j_D(g, e^t)} dt &= \frac{1}{c^{n+1}} j_N(g, D_w^{-1})^n \int_0^\infty t^{s-1} e^{-(w+n+1)t} dt \\ &= \frac{\Gamma(s)}{c^{n+1}} j_N(g, D_w^{-1})^n \frac{1}{(w+n+1)^s}, \end{aligned}$$

which implies (5.1) in this case. If $g0, g\infty \neq \infty$, then $c, d \neq 0$ and

$$\begin{aligned} \int_0^\infty t^{s-1} e^{-wt} \frac{(ge^t)^n}{j_D(g, e^t)} dt &= \frac{1}{c^{n+1}} j_N(g, D_w^{-1})^n D_w^{n+1} \int_0^\infty t^{s-1} e^{-wt} \frac{1}{(1 - (-d/c)e^{-t})^{n+1}} dt \\ &= \frac{\Gamma(s)}{c^{n+1}} j_N(g, D_w^{-1})^n D_w^{n+1} \frac{1}{n!} \left(\prod_{j=1}^n (D_s^{-1} - w + j) \right) \Phi(-d/c, s, w), \end{aligned}$$

by Lemma 2.4. If $-d/c = T$ with $1 < T < \infty$, then $cT + d = 0$, which implies $\{\infty\} \in g((1, +\infty))$ and contradicts to the assumption (3.1) and hence $-d/c \in \mathbb{C} \setminus (1, +\infty)$. Hence we obtain (5.1) in this case. \square

THEOREM 5.3. *Let $k \in \mathbb{Z}_{\geq 0}$. Assume $g1 \neq 1$. Then we have*

$$\begin{aligned} &\xi_D(u, s; y, w; g) \\ &= \frac{1}{\Gamma(s)\Gamma(u)(e^{2\pi is} - 1)(e^{2\pi iu} - 1)} \int_{H_{\epsilon,1}} dx \int_{H_{\epsilon,1}} dt \frac{t^{s-1} x^{u-1} e^{-wt} e^{-(y+k)x}}{j_D(g, e^t)} \frac{(ge^t)^k}{1 - (ge^t)e^{-x}} \\ &\quad + \frac{1}{\Gamma(s)\Gamma(u)(e^{2\pi is} - 1)} \int_1^\infty dx \int_{H_{\epsilon e^{-x},1}} dt \frac{t^{s-1} x^{u-1} e^{-wt} e^{-(y+k)x}}{j_D(g, e^t)} \frac{(ge^t)^k}{1 - (ge^t)e^{-x}} \\ &\quad + \frac{1}{\Gamma(s)\Gamma(u)(e^{2\pi iu} - 1)} \int_1^\infty dt \int_{H_{\epsilon e^{-t},1}} dx \frac{t^{s-1} x^{u-1} e^{-wt} e^{-(y+k)x}}{j_D(g, e^t)} \frac{(ge^t)^k}{1 - (ge^t)e^{-x}} \\ &\quad + \frac{1}{\Gamma(s)\Gamma(u)} \int_1^\infty dt \int_1^\infty dx \frac{t^{s-1} x^{u-1} e^{-wt} e^{-(y+k)x}}{j_D(g, e^t)} \frac{(ge^t)^k}{1 - (ge^t)e^{-x}} \\ &\quad + \xi_{2,k}(u, s; y, w; g), \end{aligned} \tag{5.2}$$

which, except for the branch cuts due to $\xi_{2,k}(u, s; y, w; g)$, gives the analytic continuation for $u, s \in \mathbb{C}$, $\text{Re } y > \nu_\infty - k$, $\text{Re } w > \mu_\infty - 1 + k(\delta_{g_\infty, \infty} - \delta_{g_\infty, 0})$, and the continuous extension for $\text{Re } y = \nu_\infty - k$ when $\text{Re } u < 0$ and $\text{Re } w = \mu_\infty - 1 + k(\delta_{g_\infty, \infty} - \delta_{g_\infty, 0})$ when $\text{Re } s < 0$.

PROOF. There exists $M > 0$ such that for all sufficiently large $t > R'$,

$$\begin{aligned} |ge^t| &\leq \begin{cases} Me^{-t} & (g_\infty = 0), \\ Me^t & (g_\infty = \infty), \\ M & (\text{otherwise}) \end{cases} \\ &= Me^{(\delta_{g_\infty, \infty} - \delta_{g_\infty, 0})t} \end{aligned}$$

and for all sufficiently small $|t| < \epsilon'$

$$\begin{aligned} |ge^t| &\leq \begin{cases} M|t| & (g1 = 0), \\ M|t|^{-1} & (g1 = \infty), \\ M & (\text{otherwise}) \end{cases} \\ &= M|t|^{(\delta_{g1, 0} - \delta_{g1, \infty})}. \end{aligned}$$

Assume $g1 \neq 1$. By Lemma 3.5,

$$\left| \frac{t^{s-1}x^{u-1}e^{-wt}e^{-yx}}{j_D(g, e^t)} \frac{1}{1 - (ge^t)e^{-x}} \right| \leq M' \begin{cases} e^{(\mu_\infty - 1 - \operatorname{Re} w)t} e^{(\nu_\infty - \operatorname{Re} y)x} & ((t, x) \in (\epsilon', +\infty)^2), \\ |t|^{\operatorname{Re} s - 1} |x|^{\operatorname{Re} u - 1} & ((t, x) \in W_{0, \epsilon', R'}^2). \end{cases}$$

Thus for $\operatorname{Re} u > \nu_1, \operatorname{Re} s > \mu_1 + k(\delta_{g1, \infty} - \delta_{g1, 0}), \operatorname{Re} y > \nu_\infty - k, \operatorname{Re} w > \mu_\infty - 1 + k(\delta_{g\infty, \infty} - \delta_{g\infty, 0})$, we see that

$$\int_{\epsilon'}^\infty \int_{\epsilon'}^\infty \frac{t^{s-1}x^{u-1}e^{-wt}e^{-(y+k)x}}{j_D(g, e^t)} \frac{(ge^t)^k}{1 - (ge^t)e^{-x}} dt dx$$

and

$$\begin{aligned} & \int_0^{R'} \int_0^{R'} \frac{t^{s-1}x^{u-1}e^{-wt}e^{-(y+k)x}}{j_D(g, e^t)} \frac{(ge^t)^k}{1 - (ge^t)e^{-x}} dt dx \\ &= \frac{1}{(e^{2\pi i s} - 1)(e^{2\pi i u} - 1)} \int_{H_{\epsilon, R'}} \int_{H_{\epsilon, R'}} \frac{t^{s-1}x^{u-1}e^{-wt}e^{-(y+k)x}}{j_D(g, e^t)} \frac{(ge^t)^k}{1 - (ge^t)e^{-x}} dt dx \end{aligned}$$

are integrable. Let

$$A = \int_{R'}^\infty dx \int_0^{\epsilon'} dt \frac{t^{s-1}x^{u-1}e^{-wt}e^{-(y+k)x}}{j_D(g, e^t)} \frac{(ge^t)^k}{1 - (ge^t)e^{-x}}.$$

If $g_1 = \infty$, then by (3.10), the denominator does not vanish for $\epsilon e^{-x} \geq |t|$. Hence

$$A = \frac{1}{e^{2\pi i s} - 1} \int_{R'}^\infty dx \int_{H_{\epsilon e^{-x}, \epsilon'}} dt \frac{t^{s-1}x^{u-1}e^{-wt}e^{-(y+k)x}}{j_D(g, e^t)} \frac{(ge^t)^k}{1 - (ge^t)e^{-x}}. \tag{5.3}$$

If $g_1 \neq \infty$, then it is easier to see that the denominator does not vanish in the same region as the above, and (5.3) holds.

In the region $(0, \epsilon') \times (R', \infty)$, the same argument works well and we have the assertion by rearranging the regions. □

REMARK 5.4. For $k \in \mathbb{Z}_{<0}$, we have similar results as in Lemma 5.2 and Theorem 5.3 by use of (2.3), though we omit the detail.

In the case $k = 0$, we obtain the following theorem.

THEOREM 5.5. *If $g_1 \notin \{1, \infty\}$, then we have*

$$\xi_D(u, s; y, w; g) = \frac{1}{\Gamma(s)\Gamma(u)(e^{2\pi i s} - 1)} \int_H dt \int_0^\infty dx \frac{t^{s-1}x^{u-1}e^{-wt}e^{-yx}}{j_D(g, e^t)} \frac{1}{1 - (ge^t)e^{-x}},$$

which gives the analytic continuation for $\operatorname{Re} u > \nu_1, s \in \mathbb{C}, \operatorname{Re} y > \nu_\infty, \operatorname{Re} w > \mu_\infty - 1$, and the continuous extension for $\operatorname{Re} w = \mu_\infty - 1$ when $\operatorname{Re} s < 0$.

PROOF. If $g_1 \notin \{1, \infty\}$, then in the proof of Theorem 5.3, the radius of the Hankel contours can be taken uniformly in t while $x \in (0, +\infty)$. Thus patching contours, we

have the assertion. □

When u, s or both are nonpositive integers, further analytic continuation is possible, which leads us to generalizations of the poly-Bernoulli polynomials.

THEOREM 5.6. *Assume $g_1 \neq 1$. For $s = -m \in \mathbb{Z}_{\leq 0}$, $\xi_D(u, s; y, w; g)$ is analytically continued to $u, y, w \in \mathbb{C}$ except for appropriate branch cuts and we have the integral representation*

$$\begin{aligned} & \xi_D(u, -m; y, w; g) \\ &= \frac{(-1)^m m!}{2\pi i} \frac{1}{\Gamma(u)(e^{2\pi i u} - 1)} \int_H dx \int_{|t|=\epsilon e^{-\operatorname{Re} x}} dt \frac{t^{-m-1} x^{u-1} e^{-wt} e^{-(y+k)x}}{j_D(g, e^t)} \frac{(ge^t)^k}{1 - (ge^t)e^{-x}} \\ & \quad + \xi_{2,k}(u, -m; y, w; g). \end{aligned} \tag{5.4}$$

For $u = -m \in \mathbb{Z}_{\leq 0}$, $\xi_D(u, s; y, w; g)$ is analytically continued to $s, y, w \in \mathbb{C}$ except for appropriate branch cuts.

PROOF. If $s = -m \in \mathbb{Z}_{\leq 0}$, then by Theorem 5.3, we see that the Hankel contour in the first and the second terms of (5.2) with respect to t reduces to a small circle around the origin and that the third and the fourth terms vanish. Thus we obtain the integral representation. The integral converges for any $w \in \mathbb{C}$. Since the analytic continuation is valid for $\operatorname{Re} y > \nu_\infty - k$ with arbitrary $k \in \mathbb{Z}_{\geq 0}$, we have the first assertion.

The second assertion follows from Corollary 4.4. □

EXAMPLE 5.7. For $g = g_\eta, g_\xi$ in Example 3.7, we see that $g_1 \neq 1$. Hence, by Theorem 5.3, we see that $\xi_D(u, s; y, w; g_\eta)$ is analytic for $u, s \in \mathbb{C}$, $\operatorname{Re} y > \nu_\infty$ and $\operatorname{Re} w > \mu_\infty - 1$ with $\mu_\infty + \nu_\infty = 1$. In the case when $(y, w) = (1, 0)$, $\eta(u; s) = \xi_D(u, s; 1, 0; g_\eta)$ is analytic for $u, s \in \mathbb{C}$. Furthermore, when $(y, w) = (1, -1)$, we can define

$$\tilde{\xi}(u; s) = \xi_N(u, s; 0, -1; g_\eta) = \xi_D(u, s; 1, -1; g_\eta)$$

for $u, s \in \mathbb{C}$ with $\operatorname{Re} u < 0$ and $\operatorname{Re} s < 0$. In particular when $u = -k \in \mathbb{Z}_{\leq 0}$, by Theorem 5.6, we see that $\tilde{\xi}(-k; s)$ can be analytically continued to $s \in \mathbb{C}$, which was already considered in [10, Section 4].

Also $\xi_D(u, s; y, w; g_\xi)$ is analytic for $u, s \in \mathbb{C}$, $\operatorname{Re} y > 0$ and $\operatorname{Re} w > -1$. In particular, $\xi(u; s) = \xi_D(u, s; 1, 0; g_\xi)$ is analytic for $u, s \in \mathbb{C}$.

EXAMPLE 5.8. Consider $\tilde{\xi}(u; s) = \xi_D(u, s; 1, -1; g_\eta)$. By (4.2) with $(n, y, w, g) = (1, 1, 1, g_\eta)$, we obtain

$$\tilde{\xi}(u - 1; s) = \tilde{\xi}(s - 1; u), \tag{5.5}$$

which interpolates (1.4) (see Example 6.3). Note that from (4.7) with $(y, w) = (0, 0)$, we have

$$\xi_D(u, s; 0, 0; g_\eta) = \xi_N(s, u; 0, -1; g_\eta) = \xi_D(s, u; 1, -1; g_\eta).$$

Therefore it follows from (5.5) that

$$\xi_D(u, s; 0, 0; g_\eta) = \xi_D(u - 1, s + 1; 1, -1; g_\eta) = \tilde{\xi}(u - 1; s + 1). \tag{5.6}$$

Let $g = g_\xi$ and $\xi(u; s) := \xi_D(u, s; 1, 0; g_\xi)$ in Example 3.7. Since $g_\xi^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, we have $g_\xi^{-1}T = 1/(1-T)$ which satisfies (3.1). Let $h = g_\xi^{-1}$. Then $hT = 1/(1-T)$, namely, $h1 = \infty$, $h\infty = 0$ and

$$h([1, +\infty]) \cap [1, +\infty] = \{\infty\} = V(h).$$

Hence we obtain $\mu_\infty = 0$ and $\nu_1 = 0$. Therefore, noting (4.5) with $(y, w, g) = (0, 0, g_\xi)$, we can define

$$\check{\xi}(u; s) := \xi_D(u, s; 0, -1; g_\xi^{-1}) = -\xi_D(s, u; 0, -1; g_\xi) \tag{5.7}$$

for $\text{Re } u < 0$ and $\text{Re } s < 0$. Setting $(n, y, w, g) = (1, 1, 0, g_\xi)$ in (4.4) and noting (3.3), we obtain

$$\xi_D(u - 1, s; 1, 0; g_\xi) = \xi_D(s - 1, u; 0, -1; g_\xi^{-1}).$$

Therefore we see from (5.7) that

$$\xi(u - 1; s) = \check{\xi}(s - 1; u), \tag{5.8}$$

which also interpolates (1.4) (see Example 6.6). The symbol $\check{\xi}$ is derived from this fact. From this relation, $\check{\xi}(s; u)$ is analytic for $u, s \in \mathbb{C}$.

EXAMPLE 5.9. From (4.2) with $n = 1$, we obtain

$$\begin{aligned} j_D(g^{-1}, D_y^{-1})D_w^{-2}(D_u^{-1} - y + 1)\xi_D(u, s; y, w; g) \\ = \left(-\frac{1}{\det g}\right)^2 j_D(g, D_w^{-1})D_y^{-2}(D_s^{-1} - w + 1)\xi_D(s, u; w, y; g^{-1}). \end{aligned}$$

Substituting (4.7) into the right-hand side and noting $j_D(g, D_w^{-1})w = wj_D(g, D_w^{-1}) - cD_w^{-1}$, we have

$$\begin{aligned} j_D(g^{-1}, D_y^{-1})D_w^{-2}(D_u^{-1} - y + 1)\xi_D(u, s; y, w; g) \\ = \left(-\frac{1}{\det g}\right)^2 (-\det g)j_D(g, D_w^{-1})D_y^{-2}(D_s^{-1} - w + 1)\xi_D(u, s; y + 1, w - 1; g) \\ = -\frac{1}{\det g}D_y^{-2}((D_s^{-1} - w + 1)j_D(g, D_w^{-1}) + cD_w^{-1})\xi_D(u, s; y + 1, w - 1; g) \tag{5.9} \\ = -\frac{1}{\det g}D_y^{-2}(D_s^{-1} - w + 1)j_D(g, D_w^{-1})\xi_D(u, s; y + 1, w - 1; g) \\ \quad - \frac{1}{\det g}cD_y^{-2}D_w^{-1}\xi_D(u, s; y + 1, w - 1; g). \end{aligned}$$

Moreover, substituting (4.1) into the right-hand side of (5.9), we obtain

$$\begin{aligned} & j_D(g^{-1}, D_y^{-1})D_w^{-2}(D_u^{-1} - y + 1)\xi_D(u, s; y, w; g) \\ &= -\frac{1}{\det g}D_y^{-2}(D_s^{-1} - w + 1)(j_N(g, D_w^{-1})\xi_D(u, s; y + 2, w - 1; g) + (y + 1)^{-u}(w - 1)^{-s}) \\ &\quad - \frac{1}{\det g}cD_y^{-2}D_w^{-1}\xi_D(u, s; y + 1, w - 1; g). \end{aligned}$$

In particular, setting $(y, w, g_\eta) = (1, 1, g_\eta)$, we obtain

$$\xi_D(u - 1, s; 1, -1; g_\eta) = \xi_D(u, s - 1; 1, 0; g_\eta) - \xi_D(u, s - 1; 1, -1; g_\eta),$$

namely,

$$\eta(u; s - 1) = \tilde{\xi}(u; s - 1) + \tilde{\xi}(u - 1; s). \tag{5.10}$$

This can be regarded as an interpolation formula of (1.5) (see Example 6.8).

REMARK 5.10. If $g1 = 1$, then the analytic properties of $\xi_D(u, s; y, w; g)$ in u, s drastically change because the two paths of the integral can not be replaced by the Hankel contours due to the singularities of the integrand near the origin in t, x . In this case, by use of the technique employed in the case of multiple zeta functions (see [11]), we see that $\xi_D(u, s; y, w; g)$ has possible singularities on the hyperplanes $s + u \in \mathbb{Z}$.

6. Poly-Bernoulli polynomials associated with $GL_2(\mathbb{C})$.

In this section, let $g \in GL_2(\mathbb{C})$ satisfying (3.1) and $g1 \neq 1$. We generalize the poly-Bernoulli polynomials from the result in Theorem 5.6.

DEFINITION 6.1. For $u, y, w \in \mathbb{C}$ except for appropriate branch cuts, we define the poly-Bernoulli polynomials $\{\mathbb{B}_m^{(u)}(y, w; g)\}$ associated with g by

$$\mathbb{B}_m^{(u)}(y, w; g) = \xi_D(u, -m; y, w; g) \quad (m \in \mathbb{Z}_{\geq 0}). \tag{6.1}$$

In particular when $g1 \neq \infty$, it follows from Lemma 2.1 and Theorem 5.5 that

$$\xi_D(u, s; y, w; g) = \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \int_H t^{s-1} e^{-wt} \frac{\Phi(ge^t, u, y)}{j_D(g, e^t)} dt$$

for $\text{Re } u > \nu_1, s \in \mathbb{C}, \text{Re } y > \nu_\infty$ and $\text{Re } w > \mu_\infty - 1$. Let $s \rightarrow -m \in \mathbb{Z}_{\leq 0}$. Then we obtain the following result.

THEOREM 6.2. If $g1 \neq \infty$, then

$$e^{wt} \frac{\Phi(ge^{-t}, u, y)}{j_D(g, e^{-t})} = \sum_{m=0}^{\infty} \mathbb{B}_m^{(u)}(y, w; g) \frac{t^m}{m!} \tag{6.2}$$

for $u, y, w \in \mathbb{C}$ except for appropriate branch cuts. $\mathbb{B}_m^{(u)}(y, w; g)$ is a polynomial in w .

EXAMPLE 6.3. We consider the poly-Bernoulli polynomials defined by

$$e^{-wt} \frac{\text{Li}_u(1 - e^{-t})}{1 - e^{-t}} = \sum_{m=0}^{\infty} B_m^{(u)}(w) \frac{t^m}{m!} \quad (u \in \mathbb{C}) \quad (6.3)$$

(see Coppo–Candelpergher [5] in the case $u \in \mathbb{Z}$). We define $B_m^{(u)} := B_m^{(u)}(0)$ and $C_m^{(u)} := B_m^{(u)}(1)$ which are generalizations of (1.1) and (1.2). Furthermore, we have $B_m^{(1)}(w) = B_m(1-w) = (-1)^m B_m(w)$, where $B_m(w)$ is the classical Bernoulli polynomial. From Example 3.7, for $g = g_\eta = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$, we see that the left-hand side of (6.2) is equal to that of (6.3) with replacing $-w$ by w and y by 1. Hence we have $\mathbb{B}_m^{(u)}(1, w; g_\eta) = B_m^{(u)}(-w)$. Note that $\mathbb{B}_m^{(k)}(1, w; g_\eta)$ ($k \in \mathbb{Z}$) coincides with the poly-Bernoulli polynomial defined by Bayad and Hamahata in [4]. We emphasize that

$$\begin{aligned} \eta(u; -m) &= \xi_D(u, -m; 1, 0; g_\eta) = \mathbb{B}_m^{(u)}(1, 0; g_\eta) = B_m^{(u)}, \\ \tilde{\xi}(u; -m) &= \xi_D(u, -m; 1, -1; g_\eta) = \mathbb{B}_m^{(u)}(1, -1; g_\eta) = C_m^{(u)} \end{aligned} \quad (6.4)$$

for $m \in \mathbb{Z}_{\geq 0}$. Hence, from (5.5), we obtain (1.4). Further, from (5.6), we obtain

$$\mathbb{B}_m^{(u)}(0, 0; g_\eta) = \mathbb{B}_{m-1}^{(u-1)}(1, -1; g_\eta) = C_{m-1}^{(u-1)} \quad (m \in \mathbb{Z}_{\geq 1}). \quad (6.5)$$

Therefore it follows from (6.2) with $(y, w, g) = (0, 0, g_\eta)$ that

$$\Phi(1 - e^{-t}, u, 0) = \text{Li}_u(1 - e^{-t}) = \sum_{m=1}^{\infty} C_{m-1}^{(u-1)} \frac{t^m}{m!}. \quad (6.6)$$

Combining (5.4) with $k = 0$ and (6.1), we obtain the following.

THEOREM 6.4. For $y, w \in \mathbb{C}$,

$$\frac{e^{wt} e^{yx}}{j_D(g, e^{-t})} \frac{1}{1 - (ge^{-t})e^x} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \mathbb{B}_k^{(-l)}(y, w; g) \frac{t^k x^l}{k! l!}.$$

$\mathbb{B}_k^{(-l)}(y, w; g)$ is a polynomial in y and w .

$\xi_D(u, s; y, w; g)$ or $\mathbb{B}_m^{(u)}(y, w; g)$ satisfies simple transformation formulas for $g = hf$ with a general $h \in GL_2(\mathbb{C})$ and a special $f \in GL_2(\mathbb{C})$.

THEOREM 6.5. Let $h \in GL_2(\mathbb{C})$ and $\alpha \in \mathbb{C} \setminus \{0\}$.

1. For $f = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$,

$$\xi_D(u, s; y, w; hf) = \frac{1}{\alpha} \xi_D(u, s; y, w; h).$$

2. For $f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which corresponds to the inversion $T \mapsto 1/T$,

$$\mathbb{B}_m^{(u)}(y, w; hf) = (-1)^m \mathbb{B}_m^{(u)}(y, -w - 1; h). \quad (6.7)$$

PROOF. The first statement follows directly from the definition. We show the second statement.

$$\begin{aligned}
 & \mathbb{B}_m^{(u)}(y, w; hf) \\
 &= \xi_D(u, -m; y, w; hf) \\
 &= \frac{(-1)^m m!}{2\pi i} \frac{1}{\Gamma(u)(e^{2\pi i u} - 1)} \int_H dx \int_{|t|=\epsilon e^{-\operatorname{Re} x}} dt \frac{t^{-m-1} x^{u-1} e^{-wt} e^{-yx}}{j_D(hf, e^t)} \frac{1}{1 - (hfe^t)e^{-x}} \\
 &= \frac{(-1)^m m!}{2\pi i} \frac{1}{\Gamma(u)(e^{2\pi i u} - 1)} \int_H dx \int_{|t|=\epsilon e^{-\operatorname{Re} x}} dt \frac{t^{-m-1} x^{u-1} e^{-wt} e^{-yx}}{j_D(h, e^{-t})j_D(f, e^t)} \frac{1}{1 - (he^{-t})e^{-x}} \\
 &= \frac{(-1)^m m!}{2\pi i} \frac{1}{\Gamma(u)(e^{2\pi i u} - 1)} \int_H dx \int_{|t|=\epsilon e^{-\operatorname{Re} x}} dt \frac{t^{-m-1} x^{u-1} e^{-(w+1)t} e^{-yx}}{j_D(h, e^{-t})} \frac{1}{1 - (he^{-t})e^{-x}} \\
 &= (-1)^{m+1-1} \frac{(-1)^m m!}{2\pi i} \frac{1}{\Gamma(u)(e^{2\pi i u} - 1)} \\
 &\quad \times \int_H dx \int_{|v|=\epsilon e^{-\operatorname{Re} x}} dv \frac{v^{-m-1} x^{u-1} e^{(w+1)v} e^{-yx}}{j_D(h, e^v)} \frac{1}{1 - (hev)e^{-x}} \\
 &= (-1)^m \mathbb{B}_m^{(u)}(y, -w - 1; h),
 \end{aligned}$$

where we changed variables as $v = -t$. □

EXAMPLE 6.6. Consider g_η and g_ξ in Example 3.7. Since $g_\xi = g_\eta f$ for $f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have from (6.7) that

$$\mathbb{B}_m^{(u)}(y, w; g_\xi) = (-1)^m \mathbb{B}_m^{(u)}(y, -w - 1; g_\eta).$$

Therefore, from (6.4), we obtain

$$\xi(u; -m) = \mathbb{B}_m^{(u)}(1, 0; g_\xi) = (-1)^m \mathbb{B}_m^{(u)}(1, -1; g_\eta) = (-1)^m C_m^{(u)} \tag{6.8}$$

for $m \in \mathbb{Z}_{\geq 0}$, which includes (1.7). Hence, by (5.8) and (1.4), we obtain

$$\check{\xi}(-l; -m) = \xi(-m - 1; -l + 1) = (-1)^{l-1} C_{l-1}^{(-m-1)} = (-1)^{l-1} C_m^{(-l)} \tag{6.9}$$

for $l \in \mathbb{Z}_{\geq 1}$ and $m \in \mathbb{Z}_{\geq 0}$. It follows from (6.8) and (6.9) that (5.8) is an interpolation formula of (1.4).

THEOREM 6.7 (Difference relations). For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{aligned}
 a\mathbb{B}_m^{(u)}(y + 1, w - 1; g) + b\mathbb{B}_m^{(u)}(y + 1, w; g) \\
 = c\mathbb{B}_m^{(u)}(y, w - 1; g) + d\mathbb{B}_m^{(u)}(y, w; g) - y^{-u} w^m \tag{6.10}
 \end{aligned}$$

holds for $u, y, w \in \mathbb{C}$ except for appropriate branch cuts.

PROOF. Letting $s = -m \in \mathbb{Z}_{\leq 0}$ in Theorem 4.1 and using Theorem 6.2, we obtain the assertion. □

EXAMPLE 6.8. It follows from (6.4) and (6.5) that (6.10) with $(y, w, g) = (0, 0, g_\eta)$ gives

$$B_m^{(u)} = C_m^{(u)} + C_{m-1}^{(u-1)} \quad (6.11)$$

(see [2, Section 3] with $u \in \mathbb{Z}$). It is to be noted that (5.10) with $s = -m + 1$ implies (6.11).

Next we prove the duality relations for poly-Bernoulli polynomials associated with g which include ordinary duality relations (1.3) and (1.4). Let $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]$ ($n, m \in \mathbb{Z}_{\geq 0}$) be the Stirling numbers of the first kind defined by

$$\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] = 1, \quad \left[\begin{smallmatrix} 0 \\ m \end{smallmatrix} \right] = 0 \quad (m \geq 1), \quad \prod_{j=0}^{n-1} (X + j) = \sum_{m=0}^n \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] X^m \quad (n \geq 1).$$

Note that

$$\prod_{j=0}^{n-1} (X - j) = \sum_{m=0}^n (-1)^{n+m} \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] X^m \quad (n \geq 1).$$

THEOREM 6.9 (Duality relations). *Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For $k, m, n \in \mathbb{Z}_{\geq 0}$ and $y, w \in \mathbb{C}$,*

$$\sum_{\tau=0}^n \binom{n}{\tau} (-c)^\tau a^{n-\tau} \sum_{j=0}^n \left[\begin{smallmatrix} n \\ j \end{smallmatrix} \right] \sum_{\sigma=0}^j \binom{j}{\sigma} (\tau - y + 1)^{j-\sigma} \mathbb{B}_m^{(-k-\sigma)}(y - \tau, w - n - 1; g) \quad (6.12)$$

$$= \frac{(-1)^{n+1}}{\det g} \sum_{\tau=0}^n \binom{n}{\tau} c^\tau d^{n-\tau} \sum_{j=0}^n \left[\begin{smallmatrix} n \\ j \end{smallmatrix} \right] \sum_{\sigma=0}^j \binom{j}{\sigma} (\tau - w + 1)^{j-\sigma} \mathbb{B}_k^{(-m-\sigma)}(w - \tau, y - n - 1; g^{-1}),$$

$$\sum_{\tau=0}^n \binom{n}{\tau} d^\tau (-b)^{n-\tau} \sum_{j=0}^n (-1)^j \left[\begin{smallmatrix} n \\ j \end{smallmatrix} \right] \sum_{\sigma=0}^j \binom{j}{\sigma} (\tau - y - 1)^{j-\sigma} \mathbb{B}_m^{(-k-\sigma)}(y + 1 - \tau, w; g) \quad (6.13)$$

$$= \frac{(-1)^{n+1}}{\det g} \sum_{\tau=0}^n \binom{n}{\tau} a^\tau b^{n-\tau} \sum_{j=0}^n (-1)^j \left[\begin{smallmatrix} n \\ j \end{smallmatrix} \right] \sum_{\sigma=0}^j \binom{j}{\sigma} (\tau - w - 1)^{j-\sigma} \\ \times \mathbb{B}_k^{(-m-\sigma)}(w + 1 - \tau, y; g^{-1}),$$

$$\sum_{\tau=0}^n \binom{n}{\tau} d^\tau (-b)^{n-\tau} \sum_{j=0}^n \left[\begin{smallmatrix} n \\ j \end{smallmatrix} \right] \sum_{\sigma=0}^j \binom{j}{\sigma} (\tau - y + 1)^{j-\sigma} \mathbb{B}_m^{(-k-\sigma)}(y - \tau, w; g) \quad (6.14)$$

$$= \frac{(-1)}{\det g} \sum_{\tau=0}^n \binom{n}{\tau} c^\tau d^{n-\tau} \sum_{j=0}^n (-1)^j \left[\begin{smallmatrix} n \\ j \end{smallmatrix} \right] \sum_{\sigma=0}^j \binom{j}{\sigma} (\tau - 1 - w)^{j-\sigma} \\ \times \mathbb{B}_k^{(-m-\sigma)}(w + 1 - \tau, y - n - 1; g^{-1}).$$

In particular when $n = 0$,

$$\mathbb{B}_k^{(-m)}(y, w - 1; g) = -\frac{1}{\det g} \mathbb{B}_m^{(-k)}(w, y - 1; g^{-1}). \quad (6.15)$$

PROOF. First we assume that $\operatorname{Re} w$ and $\operatorname{Re} y$ are sufficiently large. By (4.2), we obtain

$$\begin{aligned} & \sum_{\tau=0}^n \binom{n}{\tau} (-c)^\tau a^{n-\tau} \sum_{j=0}^n \binom{n}{j} \sum_{\sigma=0}^j \binom{j}{\sigma} (1-y+\tau)^{j-\sigma} \xi_D(u-\sigma, s; y-\tau, w-n-1; g) \\ &= \frac{(-1)^{n+1}}{\det g} \sum_{\tau=0}^n \binom{n}{\tau} c^\tau d^{n-\tau} \sum_{j=0}^n \binom{n}{j} \sum_{\sigma=0}^j \binom{j}{\sigma} (1-w+\tau)^{j-\sigma} \xi_D(s-\sigma, u; w-\tau, y-n-1; g^{-1}). \end{aligned}$$

It is noted that, for example, D_y^{-1} and D_u^{-1} are commutative and $D_y^{-\tau}(D_u^{-1}-y+k) = (D_u^{-1}-(y-\tau)+k)D_y^{-\tau}$. Letting $(u, s) = (-k, -m)$, we obtain from (6.1) that (6.12) holds for $y, w \in \mathbb{C}$ if $\operatorname{Re} y$ and $\operatorname{Re} w$ are sufficiently large. Since $\mathbb{B}_m^{(-k)}(y, w; g)$ is a polynomial in y, w , we see that (6.12) holds for all $y, w \in \mathbb{C}$. Similar argument works well for (6.13) and (6.14) by considering (4.3) and (4.4), respectively. When $n = 0$, each equation gives (6.15). This completes the proof. \square

EXAMPLE 6.10. Let $(y, w, g) = (1, 1, g_\eta)$ in (6.12). Then, from Example 6.3, we obtain

$$\sum_{j=0}^n \binom{n}{j} B_m^{(-k-j)}(n) = \sum_{j=0}^n \binom{n}{j} B_k^{(-m-j)}(n),$$

which was given by Kaneko, Sakurai and the second-named author (see [9]). In particular when $n = 0$ and 1, we obtain (1.3) and (1.4). Hence we can regard (4.2)–(4.4) in Theorem 4.3 as interpolation formulas of the duality relations (1.3) and (1.4) and their generalizations. Therefore we can give more general examples. For $\alpha \in \mathbb{C}$, let $g = g_\alpha = \begin{pmatrix} -1 & \alpha \\ 0 & 1 \end{pmatrix}$. Suppose $\operatorname{Re} \alpha < 2$ and let $(y, w) = (1, 1)$ in (6.2). Then $g1 = \alpha - 1 \notin \{1, \infty\}$ and

$$e^{wt} \frac{\operatorname{Li}_u(\alpha - e^t)}{\alpha - e^t} = \sum_{m=0}^{\infty} \mathbb{B}_m^{(u)}(1, w; g_\alpha) \frac{t^m}{m!}. \tag{6.16}$$

We have $\det g_\alpha = -1$ and $g_\alpha^{-1} = g_\alpha$. By (6.12) with g_α , we have

$$\sum_{j=0}^n \binom{n}{j} \mathbb{B}_m^{(-k-j)}(1, -n; g_\alpha) = \sum_{j=0}^n \binom{n}{j} \mathbb{B}_k^{(-m-j)}(1, -n; g_\alpha). \tag{6.17}$$

Note that (6.17) holds for $\alpha \in \mathbb{C} \setminus \{2\}$. In fact, $\mathbb{B}_m^{(-k)}(1, -n; g_\alpha)$ is a rational function in α and continuous for $\alpha \in \mathbb{C} \setminus \{2\}$, because the left-hand side of (6.16) is analytic around $t = 0$ when $\alpha \in \mathbb{C} \setminus \{2\}$. In particular,

$$\mathbb{B}_m^{(-k)}(1, 0; g_\alpha) = \mathbb{B}_k^{(-m)}(1, 0; g_\alpha).$$

For example, when $\alpha = 3, -2$ and $\sqrt{-1}$, then we can check that

$$\mathbb{B}_2^{(-3)}(1, 0; g_3) = \mathbb{B}_3^{(-2)}(1, 0; g_3) = 242,$$

$$\begin{aligned}\mathbb{B}_2^{(-3)}(1, 0; g_{-2}) &= \mathbb{B}_3^{(-2)}(1, 0; g_{-2}) = -\frac{1}{512}, \\ \mathbb{B}_2^{(-3)}(1, 0; g_{\sqrt{-1}}) &= \mathbb{B}_3^{(-2)}(1, 0; g_{\sqrt{-1}}) = -\frac{4}{125} - \frac{22}{125}\sqrt{-1}.\end{aligned}$$

EXAMPLE 6.11. By (6.15) with $(y, w, g) = (-l, -l, g_\eta)$ for $l \in \mathbb{Z}_{\geq 0}$, we have

$$\mathbb{B}_k^{(-m)}(-l, -l-1; g_\eta) = \mathbb{B}_m^{(-k)}(-l, -l-1; g_\eta) \quad (k, m \in \mathbb{Z}_{\geq 0}). \quad (6.18)$$

Since

$$\Phi(z, -k, -l) = \sum_{n=0}^{\infty} z^n (n-l)^k = \sum_{i=0}^{l-1} z^i (i-l)^k + z^l \text{Li}_{-k}(z),$$

we obtain from (6.6) that

$$\begin{aligned}e^{-(l+1)t} \Phi(1 - e^{-t}, -k, -l) &= \sum_{i=0}^{l-1} (i-l)^k \sum_{j=0}^i \binom{i}{j} (-1)^j e^{-(l+j+1)t} \\ &\quad + \sum_{j=0}^l (-1)^j e^{-(l+j+1)t} \sum_{n=1}^{\infty} C_{n-1}^{(-k-1)} \frac{t^n}{n!}.\end{aligned}$$

Hence, by (6.2), we have

$$\begin{aligned}\mathbb{B}_m^{(-k)}(-l, -l-1; g_\eta) &= (-1)^m \sum_{i=0}^{l-1} (i-l)^k \sum_{j=0}^i \binom{i}{j} (-1)^j (l+j+1)^m \\ &\quad + \sum_{i=1}^m \binom{m}{i} \sum_{j=0}^l (-1)^j (-l-j-1)^{m-i} C_{i-1}^{(-k-1)}.\end{aligned}$$

Therefore, for example, (6.18) in the cases $l = 0, 1$ give new duality relations

$$\sum_{i=1}^m \binom{m}{i} (-1)^{m-i} C_{i-1}^{(-k-1)} = \sum_{i=1}^k \binom{k}{i} (-1)^{k-i} C_{i-1}^{(-m-1)}, \quad (6.19)$$

$$\begin{aligned}(-1)^{k+m} 2^m + \sum_{i=1}^m \binom{m}{i} \{(-2)^{m-i} - (-3)^{m-i}\} C_{i-1}^{(-k-1)} \\ = (-1)^{k+m} 2^k + \sum_{i=1}^k \binom{k}{i} \{(-2)^{k-i} - (-3)^{k-i}\} C_{i-1}^{(-m-1)}\end{aligned} \quad (6.20)$$

for $k, m \in \mathbb{Z}_{\geq 1}$.

By (6.15) with $(y, w, g) = (-l, l+1, g_\eta)$ for $l \in \mathbb{Z}_{\geq 0}$, we obtain

$$\mathbb{B}_k^{(-m)}(-l, l; g_\eta) = \mathbb{B}_m^{(-k)}(l+1, -l-1; g_\eta).$$

Similar to the above consideration, this produces new duality relations among $C_m^{(-k)}$

different from the above formulas. For example, the case $l = 0$ implies (1.4), and the case $l = 1$ gives a new formula

$$\sum_{j=1}^{k-1} \binom{k}{j} C_{j-1}^{(-m-1)} = \sum_{j=0}^m \binom{m}{j} \frac{C_{m-j} C_{j+1}^{(-k)}}{j+1} \quad (k, m \in \mathbb{Z}_{\geq 0}). \tag{6.21}$$

Finally, we give certain explicit expressions of poly-Bernoulli polynomials.

LEMMA 6.12. *Assume $g_1 = 0$. For $m \in \mathbb{Z}_{\geq 0}$ and $u, y, w \in \mathbb{C}$ except for appropriate branch cuts,*

$$\mathbb{B}_m^{(u)}(y, w; g) = \xi_{2,m+1}(u, -m; y, w; g). \tag{6.22}$$

PROOF. Since $g_1 = 0$, we have $O(ge^t) = O(t)$ ($t \rightarrow 0$). Substitute $s = 1 - k$ ($k \in \mathbb{Z}_{\geq 1}$) into (5.4). Then the first term on the right-hand side of (5.4) vanishes, because its integrand is holomorphic in t around the origin. Hence, from Theorem 6.2, we see that (6.22) holds for $u, y, w \in \mathbb{C}$ except for appropriate branch cuts. Replacing $m = k - 1$, we have the assertion. \square

Combining Lemmas 5.2 and 6.12, we have the following.

EXAMPLE 6.13. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. First we assume $g_1 = 0$ and $g_\infty = \infty$, namely, $a + b = 0$ and $c = 0$. By Theorem 6.5, we have only to consider $g = h_d := \begin{pmatrix} -1 & 1 \\ 0 & d \end{pmatrix}$ for $d \in \mathbb{C} \setminus \{0\}$. Note that $h_1 = g_\eta$ (see Example 3.7). Combining Lemma 5.2 with $k = m + 1$, Theorem 6.2 and Lemma 6.12, we have

$$\begin{aligned} \mathbb{B}_m^{(u)}(y, w; h_d) &= \sum_{n=0}^m \frac{1}{(y+n)^u} (1 - D_w^{-1})^n \frac{w^m}{d^{n+1}} \\ &= \sum_{n=0}^m \frac{1}{(y+n)^u} \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{(w-j)^m}{d^{n+1}}. \end{aligned}$$

In particular when $(d, y, w) = (1, 1, 0)$, from Example 6.3, we obtain the well-known expression

$$B_m^{(u)} = (-1)^m \sum_{n=0}^m \frac{(-1)^n n!}{(n+1)^u} \left\{ \begin{matrix} m \\ n \end{matrix} \right\}$$

(see [7, Theorem 1]), where $\left\{ \begin{matrix} m \\ n \end{matrix} \right\}$ is the Stirling number of the second kind determined by

$$\left\{ \begin{matrix} m \\ n \end{matrix} \right\} = \frac{(-1)^n}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} j^m \quad (m, n \in \mathbb{Z}_{\geq 0}).$$

Next we assume $g_1 = 0$ and $g_0 = \infty$, namely, $a + b = 0$ and $d = 0$. Hence we consider $g = h'_c := \begin{pmatrix} 1 & -1 \\ c & 0 \end{pmatrix}$ for $c \in \mathbb{C} \setminus \{0\}$. Note that $h'_1 = g_\xi$ (see Example 3.7). Combining Lemma 5.2 with $k = m + 1$, Theorem 6.2 and Lemma 6.12, we have

$$\begin{aligned} \mathbb{B}_m^{(u)}(y, w; h'_c) &= \sum_{n=0}^m \frac{1}{(y+n)^u} (1 - D_w^{-1})^n \frac{(w+n+1)^m}{c^{n+1}} \\ &= \sum_{n=0}^m \frac{1}{(y+n)^u} \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{(w+n+1-j)^m}{c^{n+1}}. \end{aligned}$$

7. Proofs of Lemmas 2.5 and 2.6.

LEMMA 7.1. *Let N be a neighborhood of the origin in $\mathbb{R}_{\geq 0}$. Let $a(U), b(U), c(U)$ be real continuous functions in $U \in N$ such that $a(U), c(U) > 0$ and $-\sqrt{a(U)c(U)} \leq b(U) < \sqrt{a(U)c(U)}$ for all $U \in N$. Let $0 \leq q \leq 1$. Then there exists $M > 0$ such that*

$$F(U, Y) = \frac{a(U)Y^2 - 2b(U)UY + c(U)U^2}{U^{2-q}Y^q} \geq M$$

for all (U, Y) in a sufficiently small neighborhood of the origin in $\mathbb{R}_{\geq 0}^2$ unless the denominator vanishes.

PROOF. We denote $a(U_0), b(U_0), c(U_0)$ by a, b, c respectively for short. First assume $0 < q \leq 1$. Fix a sufficiently small $U_0 > 0$. Then

$$\frac{\partial F(U_0, Y)}{\partial Y} = \frac{a(2-q)Y^2 - 2b(1-q)U_0Y - cqU_0^2}{U_0^{2-q}Y^{q+1}} = 0$$

implies the unique solution

$$Y_0 = AU_0 > 0 \tag{7.1}$$

with

$$A = \frac{b(1-q) + \sqrt{b^2(1-q)^2 + ac(2-q)q}}{a(2-q)} > 0.$$

Thus we have

$$F(U_0, Y) \geq F(U_0, Y_0) = 2 \frac{ac(2-q) - b^2(1-q) - b\sqrt{b^2(1-q)^2 + ac(2-q)q}}{a(2-q)^2 A^q}.$$

Here

$$\begin{aligned} ac(2-q) - b^2(1-q) &= ac + (ac - b^2)(1-q) \geq ac > 0, \\ -b\sqrt{b^2(1-q)^2 + ac(2-q)q} &\geq -|b|\sqrt{ac(1-q)^2 + ac(2-q)q} \geq -|b|\sqrt{ac}. \end{aligned}$$

If $a(0)c(0) \neq b(0)^2$, then

$$F(U_0, Y_0) \geq 2 \frac{\sqrt{ac}(\sqrt{ac} - |b|)}{a(2-q)^2 A^q}$$

and there exists $M > 0$ such that

$$F(U, Y) \geq M$$

for all (U, Y) in a sufficiently small neighborhood of the origin in $\mathbb{R}_{>0}^2$. If $a(0)c(0) = b(0)^2$, then by the assumption we have $b(0) = -\sqrt{a(0)c(0)} < 0$ and $b(U) < 0$ for all sufficiently small $U \geq 0$. Then

$$-b\sqrt{b^2(1-q)^2 + ac(2-q)q} \geq 0$$

and

$$F(U_0, Y_0) \geq \frac{2ac}{a(2-q)^2 A^q}.$$

Hence we have the same conclusion.

Next assume $q = 0$. Fix a sufficiently small $U_0 > 0$. Then

$$\frac{\partial F(U_0, Y)}{\partial Y} = 2 \frac{aY - bU_0}{U_0^2} = 0$$

implies the unique solution

$$Y_0 = \frac{bU_0}{a} \in \mathbb{R}. \tag{7.2}$$

If $a(0)c(0) \neq b(0)^2$, then we have

$$F(U_0, Y) \geq F(U_0, Y_0) = \frac{ac - b^2}{a}$$

and there exists $M > 0$ such that

$$F(U, Y) \geq M$$

for all (U, Y) in a sufficiently small neighborhood of the origin in $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}$. If $a(0)c(0) = b(0)^2$, then by the assumption we have $b(0) = -\sqrt{a(0)c(0)} < 0$ and $b(U) < 0$ for all sufficiently small $U \geq 0$. Then

$$F(U_0, Y) \geq F(U_0, 0) = c$$

for $Y \geq 0$ and we have the same conclusion. □

LEMMA 7.2. *Let N be a neighborhood of the origin in $\mathbb{R}_{\geq 0}$. Let $a(U), b(U), c(U)$ be real continuous functions in $U \in N$ such that $a(U), c(U) > 0$, $-\sqrt{a(U)c(U)} \leq b(U) < \sqrt{a(U)c(U)}$ for all $U \in N \setminus \{0\}$, $b(0) = \sqrt{a(0)c(0)}$ and*

$$K = \lim_{U \rightarrow 0} \frac{a(U)c(U) - b(U)^2}{U^2} > 0.$$

Let $0 \leq q \leq 2$. Then there exists $M > 0$ such that

$$G(U, Y) = \frac{a(U)Y^2 - 2b(U)UY + c(U)U^2}{U^{4-q}Y^q} \geq M$$

for all (U, Y) in a sufficiently small neighborhood of the origin in $\mathbb{R}_{\geq 0}^2$ unless the denominator vanishes.

PROOF. We denote $a(U_0), b(U_0), c(U_0)$ by a, b, c respectively for short. Note that $G(U, Y) = U^{-2}F(U, Y)$, where $F(U, Y)$ is given in Lemma 7.1.

First assume $0 < q < 2$. Fix a sufficiently small $U_0 > 0$. Then $G(U_0, Y)$ attains its minimum at the same Y_0 as (7.1), which is also valid for $1 \leq q < 2$ and

$$G(U_0, Y) \geq G(U_0, Y_0) = 2 \frac{ac(2-q) - b^2(1-q) - b\sqrt{b^2(1-q)^2 + ac(2-q)q}}{a(2-q)^2 A^q U_0^2}$$

with

$$\begin{aligned} ac(2-q) - b^2(1-q) &= b^2 + (ac - b^2)(2-q) \geq 0, \\ (ac(2-q) - b^2(1-q))^2 - (b\sqrt{b^2(1-q)^2 + ac(2-q)q})^2 &= ac(ac - b^2)(2-q)^2. \end{aligned}$$

Since

$$\begin{aligned} B &= ac(2-q) - b^2(1-q) + b\sqrt{b^2(1-q)^2 + ac(2-q)q} \rightarrow 2a(0)c(0), \\ A &\rightarrow \sqrt{c(0)/a(0)} \end{aligned}$$

as $U_0 \rightarrow 0$, we have

$$G(U_0, Y) \geq G(U_0, Y_0) = 2 \frac{ac(2-q)^2}{a(2-q)^2 A^q B} \frac{ac - b^2}{U_0^2} \rightarrow \frac{K}{a(0)\sqrt{c(0)/a(0)}^q} > 0$$

as $U_0 \rightarrow 0$. Thus there exists $M > 0$ such that

$$G(U, Y) \geq M$$

for all (U, Y) in a sufficiently small neighborhood of the origin in $\mathbb{R}_{> 0}^2$.

Secondly assume $q = 0$. Fix a sufficiently small $U_0 > 0$. Then $G(U_0, Y)$ attains its minimum at the same Y_0 as (7.2) and

$$G(U_0, Y) \geq G(U_0, Y_0) = \frac{ac - b^2}{aU_0^2} \rightarrow \frac{K}{a(0)} > 0$$

as $U_0 \rightarrow 0$. Thus there exists $M > 0$ such that

$$G(U, Y) \geq M$$

for all (U, Y) in a sufficiently small neighborhood of the origin in $\mathbb{R}_{> 0} \times \mathbb{R}_{\geq 0}$.

Thirdly we assume $q = 2$. Fix a sufficiently small $U_0 > 0$. Then

$$\frac{\partial G(U, Y)}{\partial Y} = 2 \frac{bU_0Y - cU_0^2}{U_0^2Y^3} = 0$$

implies the unique solution

$$Y_0 = \frac{cU_0}{b} > 0$$

because by the assumption, $b(U) > 0$ for all sufficiently small $U \geq 0$. Thus we have

$$G(U, Y) \geq G(U, Y_0) = \frac{ac - b^2}{cU_0^2} \rightarrow \frac{K}{c(0)} > 0$$

as $U_0 \rightarrow 0$ and there exists $M > 0$ such that

$$G(U, Y) \geq M$$

for all (U, Y) in a sufficiently small neighborhood of the origin in $\mathbb{R}_{>0}^2$. □

LEMMA 7.3. *Assume that $h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathbb{C})$ satisfies $hU = Y$ for only $(U, Y) = (0, 0)$ in a neighborhood of the origin in $\mathbb{R}_{\geq 0}^2$. Then for $0 \leq q \leq 1$, there exists $M > 0$ such that*

$$\frac{1}{|\alpha U + \beta - Y(\gamma U + \delta)|} \leq \begin{cases} \frac{M}{U^{1-q}Y^q} & \text{if the origin is not a cusp,} \\ \frac{M}{U^{2(1-q)}Y^{2q}} & \text{if the origin is a cusp} \end{cases}$$

in a neighborhood of the origin in $\mathbb{R}_{>0}^2$.

PROOF. Assume that $h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathbb{C})$ satisfies $hU = Y$ for only $(U, Y) = (0, 0)$ in the neighborhood of the origin in $\mathbb{R}_{\geq 0}^2$, Then $h0 = 0$ implies $\beta = 0$ and $\det h = \alpha\delta \neq 0$. Hence $hU = Y$ is rewritten as

$$Y = \frac{\alpha U}{\gamma U + \delta} = \frac{(\alpha\bar{\gamma}U + \alpha\bar{\delta})U}{|\gamma U + \delta|^2}. \tag{7.3}$$

Assume that $\alpha\bar{\delta} \in \mathbb{R}_{>0}$ and $\alpha\bar{\gamma} \in \mathbb{R}$. Then in any neighborhood of the origin, a pair (U, Y) with a small $U > 0$ and Y given by (7.3) is a solution. Thus if the solution is only $(U, Y) = (0, 0)$ in a neighborhood of the origin in $\mathbb{R}_{\geq 0}^2$, then $\alpha\bar{\delta} \notin \mathbb{R}_{>0}$ or $\alpha\bar{\gamma} \notin \mathbb{R}$.

If the origin is a cusp, then $(d/dU)hU|_{U=0} = (\det h)/\delta^2 = \alpha\bar{\delta}/|\delta|^2 > 0$ and hence $\alpha\bar{\delta} \in \mathbb{R}_{>0}$. The converse is also true.

Assume $0 \leq q \leq 1/2$. Consider

$$|\alpha U - \delta Y - \gamma UY|^2 = |\delta + \gamma U|^2 Y^2 - 2 \operatorname{Re}(\alpha\bar{\delta} + \alpha\bar{\gamma}U)UY + |\alpha|^2 U^2$$

and let

$$a(U) = |\delta + \gamma U|^2, \quad b(U) = \operatorname{Re}(\alpha\bar{\delta} + \alpha\bar{\gamma}U), \quad c(U) = |\alpha|^2.$$

We check the assumptions in Lemmas 7.1 and 7.2. Since $\alpha, \delta \neq 0$, we see that $a(U), c(U) > 0$ for all sufficiently small $U \geq 0$. Furthermore

$$a(U)c(U) - b(U)^2 = |\alpha\bar{\delta} + \alpha\bar{\gamma}U|^2 - (\operatorname{Re}(\alpha\bar{\delta} + \alpha\bar{\gamma}U))^2 = (\operatorname{Im}(\alpha\bar{\delta} + \alpha\bar{\gamma}U))^2 \geq 0,$$

which implies $-\sqrt{a(U)c(U)} \leq b(U) \leq \sqrt{a(U)c(U)}$. Since $\alpha\bar{\delta} \notin \mathbb{R}_{>0}$ or $\alpha\bar{\gamma} \notin \mathbb{R}$,

$$\sqrt{a(U)c(U)} - b(U) = |\alpha\bar{\delta} + \alpha\bar{\gamma}U| - \operatorname{Re}(\alpha\bar{\delta} + \alpha\bar{\gamma}U) \neq 0$$

holds for all sufficiently small $U \geq 0$ if $\alpha\bar{\delta} \notin \mathbb{R}_{>0}$, and for all sufficiently small $U > 0$ if $\alpha\bar{\delta} \in \mathbb{R}_{>0}$. In the latter case,

$$b(0) = \operatorname{Re} \alpha\bar{\delta} = |\alpha\bar{\delta}| = \sqrt{a(0)c(0)}$$

and

$$\frac{a(U)c(U) - b(U)^2}{U^2} = \frac{(\operatorname{Im}(\alpha\bar{\delta} + \alpha\bar{\gamma}U))^2}{U^2} = \frac{(\operatorname{Im} \alpha\bar{\gamma}U)^2}{U^2} = (\operatorname{Im} \alpha\bar{\gamma})^2 > 0.$$

Thus we have checked the assumptions required and have the assertions in this case.

For $1/2 < q \leq 1$, exchanging the roles of U and Y , and applying Lemmas 7.1 and 7.2 with

$$|\alpha U - \delta Y - \gamma UY|^2 = |\alpha - \gamma Y|^2 U^2 - 2 \operatorname{Re}(\delta\bar{\alpha} - \delta\bar{\gamma}Y)UY + |\delta|^2 Y^2$$

and

$$a(Y) = |\alpha - \gamma Y|^2, \quad b(Y) = \operatorname{Re}(\delta\bar{\alpha} - \delta\bar{\gamma}Y), \quad c(Y) = |\delta|^2,$$

we have the assertions in this case. Here we used the fact that $\alpha\bar{\delta} \in \mathbb{R}_{>0}$ implies $\alpha\bar{\gamma} \notin \mathbb{R}$, and hence $\delta\bar{\alpha} \in \mathbb{R}_{>0}$ and $\delta\bar{\gamma} \notin \mathbb{R}$. \square

LEMMA 7.4. *Assume that $h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathbb{C})$ satisfies $hU = Y$ for only $(U, Y) = (0, 0)$ in a neighborhood of the origin in $\mathbb{R}_{\geq 0}^2$. Then there exists $\epsilon > 0$ such that*

$$\frac{1}{\epsilon}|Y| > |U| > \epsilon|Y|$$

for any pair (U, Y) satisfying $hU = Y$ in a sufficiently small neighborhood of the origin in \mathbb{C}^2 .

PROOF. From the first paragraph of the proof of Lemma 7.3, we see that $\beta = 0$ and $\alpha\delta \neq 0$. Since $hU = Y$ is rewritten as $Y = \alpha U / (\gamma U + \delta)$, we have

$$|Y| \geq \frac{|\alpha|}{|\delta| \left| 1 + \frac{\gamma}{\delta} U \right|} |U| \geq \frac{|\alpha|}{2|\delta|} |U|.$$

Similarly $U = \delta Y / (\gamma Y - \alpha)$ implies

$$|U| \geq \frac{|\delta|}{2|\alpha|}|Y|. \quad \square$$

PROOF OF LEMMA 2.5. For $Z \in \{1, \infty\}$, let

$$k_Z = \begin{pmatrix} \tilde{Z} & & \\ & -1 & \\ -(-1)^{\tilde{Z}} & & 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} & (Z = 1), \\ \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & (Z = \infty). \end{cases}$$

Note that k_Z maps a neighborhood of Z in $[1, +\infty]$ to a neighborhood of the origin in $\mathbb{R}_{\geq 0}$.

By putting $U = k_{T_0}T$ and $Y = k_{X_0}X$, we see that $h = k_{X_0}gk_{T_0}^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ satisfies the assumption in Lemma 7.3. Since

$$k_Z^{-1} = \begin{pmatrix} 0 & -(-1)^{\tilde{Z}} \\ -1 & -(-1)^{\tilde{Z}}\tilde{Z} \end{pmatrix}$$

and

$$j_D(k_{X_0}^{-1}hk_{T_0}, k_{T_0}^{-1}U) = \frac{j_D(k_{X_0}^{-1}h, U)}{j_D(k_{T_0}^{-1}, U)},$$

$$j_D(k_{X_0}^{-1}h, U) = j_D(k_{X_0}^{-1}, hU)j_D(h, U), \quad j_N(k_{X_0}^{-1}h, U) = j_N(k_{X_0}^{-1}, hU)j_D(h, U),$$

we have

$$\begin{aligned} j_D(g, T)(1 - (gT)X^{-1}) &= j_D(k_{X_0}^{-1}hk_{T_0}, k_{T_0}^{-1}U)(1 - (k_{X_0}^{-1}hU)(k_{X_0}^{-1}Y)^{-1}) \\ &= \frac{j_D(k_{X_0}^{-1}h, U)}{j_D(k_{T_0}^{-1}, U)} \left(1 - \frac{j_N(k_{X_0}^{-1}h, U) j_D(k_{X_0}^{-1}, Y)}{j_D(k_{X_0}^{-1}h, U) j_N(k_{X_0}^{-1}, Y)} \right) \\ &= \frac{j_D(k_{X_0}^{-1}h, U)j_N(k_{X_0}^{-1}, Y) - j_N(k_{X_0}^{-1}h, U)j_D(k_{X_0}^{-1}, Y)}{j_D(k_{T_0}^{-1}, U)j_N(k_{X_0}^{-1}, Y)} \\ &= \frac{j_D(k_{X_0}^{-1}, hU) - j_D(k_{X_0}^{-1}, Y)}{j_D(k_{T_0}^{-1}, U)} j_D(h, U) \\ &= \frac{j_N(h, U) - Yj_D(h, U)}{U + (-1)^{\tilde{T}_0}\tilde{T}_0} \\ &= \frac{\alpha U + \beta - Y(\gamma U + \delta)}{U + (-1)^{\tilde{T}_0}\tilde{T}_0}. \end{aligned} \quad (7.4)$$

Hence

$$\left| \frac{1}{j_D(g, T)} \frac{1}{(1 - (gT)X^{-1})} \right|$$

$$\leq M|U + (-1)^{\tilde{T}_0} \tilde{T}_0| \times \begin{cases} \frac{1}{U^{1-q} Y^q} & \text{if the vertex is not a cusp,} \\ \frac{1}{U^{2(1-q)} Y^{2q}} & \text{if the vertex is a cusp.} \end{cases}$$

Since

$$U = k_{T_0} T = \frac{\tilde{T}_0 T - 1}{-(-1)^{\tilde{T}_0} T},$$

$$Y = k_{X_0} X = \frac{\tilde{X}_0 X - 1}{-(-1)^{\tilde{X}_0} X},$$

we obtain the first result.

The second statement follows from Lemma 7.4. □

PROOF OF LEMMA 2.6. We use the same notation as in Lemma 2.5. If $V(g) \neq \emptyset$, then we fix $X_0 \in V(g)$ and $T_0 = g^{-1} X_0$, and otherwise put $X_0 = T_0 = \infty$. Further put $U = k_{T_0} T$, $Y = k_{X_0} X$, $h = k_{X_0} g k_{T_0}^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $S(g) = \{(U, Y) \in [0, 1]^2 \mid hU = Y\}$. We see that

$$S(g) = \begin{cases} \emptyset & (\#V(g) = 0), \\ \{(0, 0)\} & (\#V(g) = 1), \\ \{(0, 0), (1, 1)\} & (\#V(g) = 2), \end{cases}$$

and $S(g)$ coincides with the set of all solutions of $\alpha U + \beta = Y(\gamma U + \delta)$ in $[0, 1]^2$. Let $N_{\epsilon'} \subset (k_{T_0} \times k_{X_0})(N)$ be an open ϵ' -neighborhood of $S(g)$ in \mathbb{C}^2 and $B_{\epsilon''}$ be an ϵ'' -neighborhood of $[0, 1]$ in \mathbb{C} . Since $[0, 1]^2 \setminus N_{\epsilon'}$ is a compact set in \mathbb{C}^2 , there exists $M > 0$ and $\epsilon'' > 0$ such that

$$|\alpha U + \beta - Y(\gamma U + \delta)| > \frac{1}{M}$$

for all $(U, Y) \in B_{\epsilon''}^2 \setminus N_{\epsilon'}$. By the same calculation as (7.4), we have

$$j_D(g, T)(1 - (gT)X^{-1}) = \frac{\alpha U + \beta - Y(\gamma U + \delta)}{U + (-1)^{\tilde{T}_0} \tilde{T}_0}.$$

Hence

$$\left| \frac{1}{j_D(g, T)} \frac{1}{(1 - (gT)X^{-1})} \right| \leq \frac{M}{|T|}$$

for all $(T, X) \in (k_{T_0} \times k_{X_0})^{-1}(B_{\epsilon''}^2 \setminus N_{\epsilon'}) \cap \mathbb{C}^2$. Since $k_1^{-1}(B_{\epsilon''}) = k_\infty^{-1}(B_{\epsilon''}) \supset W_{1,\epsilon}$ for a sufficiently small $\epsilon > 0$, we have

$$(k_{T_0} \times k_{X_0})^{-1}(B_{\epsilon''}^2 \setminus N_{\epsilon'}) \cap \mathbb{C}^2 \supset ((k_{T_0}^{-1}(B_{\epsilon''}) \times k_{X_0}^{-1}(B_{\epsilon''})) \setminus N) \cap \mathbb{C}^2 \supset W_{1,\epsilon}^2 \setminus N,$$

and the assertion. □

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