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Analyticity of the Stokes semigroup in BMO-type spaces

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Abstract. We consider the Stokes semigroup in a large class of domains including bounded domains, the half-space and exterior domains. We will prove that the Stokes semigroup is analytic in a certain type of solenoidal subspaces of BMO.

1. Introduction.

We will investigate the homogeneous Stokes equations

$$\begin{aligned} u_t - \Delta u + \nabla \pi &= 0 \quad \text{in } \Omega \times (0,T) \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \times (0,T) \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T) \\ u(0) &= u_0 \end{aligned} \tag{1.1}$$

in a uniformly C^3-domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$. The L^p-theory for $1 < p < \infty$ of the Stokes equations is quite well understood if the Helmholtz projection in L^p exists. For this let $L^p_\sigma(\Omega)$ be the closure of $C^\infty_{c,\sigma}(\Omega)$, the space of smooth solenoidal vector fields with compact support, in $L^p(\Omega)$. The Helmholtz projection is then the projection operator from $L^p(\Omega)$ into $L^p_\sigma(\Omega)$ derived from the Helmholtz decomposition. In [**15**] the second author proved that the Stokes operator generates an analytic semigroup in $L^p_\sigma(\Omega)$ if Ω is a bounded domain. The same result was proved in [**13**], [**14**] for general domains under the assumption that the Helmholtz decomposition of $L^p_\sigma(\Omega)$ exists. For domains not admitting the L^p-Helmholtz decomposition this result is still unknown.

In [**1**] and [**2**] K. Abe and the second author proved similar analyticity results in solenoidal subspaces of $L^\infty(\Omega)$ for a certain class of domains called admissible. Similar analyticity results in L^∞ by resolvent estimates were obtained in [**3**].

In this work we want to generalize these analyticity results to a subspace of BMO. In order to do so we introduce a norm measuring the mean oscillation of the function inside the domain and the mean value of the function near the boundary. We define this BMO-type norm in the following way. Let for $f \in L^1_{\mathrm{loc}}(\Omega)$ and $B \subset \Omega$ the mean value

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f_B be defined as

$$f_B := \frac{1}{|B|}\int_B f(y)\,dy.$$

For the parameter $\mu \in (0,\infty]$ we define the BMO-seminorm

$$[f]_{BMO^\mu(\Omega)} := \sup_{B_r(x)\subset\Omega, r<\mu} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y) - f_{B_r(x)}|\,dy.$$

We will usually omit Ω in the notation of the seminorm if no confusion may arise. The space $BMO^\mu(\Omega)$ is then defined as

$$BMO^\mu(\Omega) := \{f \in L^1_{\mathrm{loc}}(\Omega) : [f]_{BMO^\mu} < \infty\}.$$

We define for $\nu \in (0,\infty]$ the seminorm

$$[f]_{b^\nu} := \sup\left\{ r^{-n}\int_{B_r(x_0)\cap\Omega} |f(y)|\,dy : x_0 \in \partial\Omega, 0 < r < \nu \right\}.$$

Then

$$\|f\|_{BMO_b^{\mu,\nu}} := [f]_{BMO^\mu} + [f]_{b^\nu}$$

will be called the BMO-type norm. The space $BMO_b^{\mu,\nu}(\Omega)$ is then defined as the space of all functions $f \in L^1_{\mathrm{loc}}(\Omega)$ satisfying $\|f\|_{BMO_b^{\mu,\nu}} < \infty$. Let $VMO_{b,0}^{\mu,\nu}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ and $VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ the closure of $C_{c,\sigma}^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{BMO_b^{\mu,\nu}}$. Furthermore, let $C_{0,\sigma}(\Omega)$ be the closure of $C_{c,\sigma}^\infty(\Omega)$ with respect to the L^∞-norm. It is obvious that $C_{0,\sigma}(\Omega) \hookrightarrow VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$.

Further we define for $p \in (1,\infty)$

$$\begin{aligned}
[f]_{BMO^{\mu}p} &:= \sup_{B_r(x)\subset\Omega, r<\mu} \left(\frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y) - f_{B_r(x)}|^p\,dy\right)^{1/p}, \\
[f]_{b^\nu p} &:= \sup_{x_0\in\partial\Omega, 0<r<\nu} \left(r^{-n}\int_{B_r(x_0)\cap\Omega} |f(y)|^p\,dy\right)^{1/p}, \\
\|f\|_{BMO_b^{\mu,\nu}p} &:= [f]_{BMO^\mu p} + [f]_{b^\nu p}.
\end{aligned}$$

Note that by the John–Nirenberg inequality the seminorm $[f]_{BMO^\mu p}$ is equivalent to $[f]_{BMO^\mu}$ provided that $p \in (1,\infty)$ and $\mu \in (0,\infty]$.

In [**10**], [**11**] it was proved that for the space $\tilde{L}^r := L^2 \cap L^r$ if $r \geq 2$, $\tilde{L}^r := L^2 + L^r$ otherwise, there is a bounded Helmholtz projection P_r from $\tilde{L}^r(\Omega)$ to $\tilde{L}^r_\sigma(\Omega)$ in uniformly C^2-domains. Furthermore, it was proved that the Stokes operator generates an analytic semigroup in $\tilde{L}^r_\sigma(\Omega)$. Here $\tilde{L}^r_\sigma(\Omega)$ is the closure of $C_{c,\sigma}^\infty(\Omega)$ in the $\tilde{L}^r$-norm. The Sobolev space $\tilde{W}_0^{1,r}$ is defined as the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{\tilde{W}^{1,r}} := \|\cdot\|_{\tilde{L}^r} + \|\nabla\cdot\|_{\tilde{L}^r}$. In [**10**], [**12**] it was proved that for every $u_0 \in \tilde{L}^r(\Omega)$ there is a unique solution $u(t) \in \tilde{W}_0^{1,r}(\Omega) \cap \tilde{L}^r_\sigma(\Omega)$ with $\nabla^2 u(t), u_t(t), \nabla\pi(t) \in \tilde{L}^r(\Omega)$. We call such a solution $\tilde{L}^r$-solution.

We are now ready to define the notion of an admissible domain in the sense of [**1**]. Let Ω be a uniformly C^2-domain. The domain Ω is then called admissible if there are $r > n$ and a constant $C > 0$ such that for all matrix-valued functions $f \in C^1(\overline{\Omega})$ with $\operatorname{div} f \in \tilde{L}^r(\Omega)$, $\operatorname{tr} f = 0$ and $\partial_l f_{ij} = \partial_j f_{il}$ $(1 \leq i, j, l \leq n)$

$$\sup_{x \in \Omega} \operatorname{dist}(x, \partial\Omega) |(I - P_r)(\nabla f)(x)| \leq C \|f\|_{L^\infty(\partial\Omega)}$$

holds. Examples of admissible domains are bounded domains, the half space ([**1**]) and exterior domains ([**2**]). A layer domain of dimension $n \geq 3$ is an example of a domain that is not admissible ([**6**]) but has a Helmholtz decomposition in L^p ([**21**]). Furthermore, there are also examples of admissible domains that do not have a Helmholtz decomposition in L^p as constructed in [**4**].

Having these definitions the first and the second author proved in [**8**] that for the Stokes equations the L^∞-norm of the derivatives of the solution can be estimated by the BMO_b-norm of the initial data as in the following theorem.

Theorem 1.1. *Let $n \geq 2$, $r > n$ and*

$$\tilde{N}(u, \pi)(x, t) := t^{1/2} |\nabla u(x, t)| + t |\nabla^2 u(x, t)| + t |u_t(x, t)| + t |\nabla \pi(x, t)|.$$

Let Ω be an admissible, uniformly C^3-domain in $\mathbb{R}^n$, $\mu, \nu \in (0, \infty]$. Then there exist a solution operator S to (1.1) *and constants $C, T_0 > 0$ depending only on μ, ν, n and Ω such that*

$$\sup_{0 < t < T_0} \|\tilde{N}(u, \pi)(\cdot, t)\|_\infty \leq C \|u_0\|_{BMO_b^{\mu,\nu}}$$

holds for every $\tilde{L}^r$-solution $(u, \nabla\pi)$ with $u_0 \in C_{c,\sigma}^\infty(\Omega)$. By density the estimate holds also for each $u_0 \in VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ with $S(t)u_0 = u$ and a suitable choice of π. The solution operator S is taken so that it agrees with the L^2-Stokes semigroup on $C_{c,\sigma}^\infty(\Omega)$.

The estimate $t\|u_t(t)\|_{BMO_b^{\mu,\nu}} \leq C \|u_0\|_{BMO_b^{\mu,\nu}}$ for $t < T_0$ which is a consequence of the theorem is the estimate needed for proving the analyticity of a semigroup. Nevertheless, in our case we have the required estimate but this is not enough to conclude that the Stokes operator actually generates a semigroup on $VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ since the theorem does not give us sufficient control about the solution u itself. It is the aim of this paper to close this gap and to show that the Stokes semigroup is analytic in $VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$.

For this we will need to assume some regularity at the boundary and will make use of the following property.

Lemma 1.2. *Let Ω be a uniformly C^2-domain. Then there exists a constant R such that for all $x \in \Omega$ with* $\operatorname{dist}(x, \partial\Omega) < R$ *there is a unique projection to a boundary point $x_c \in \partial\Omega$ such that the line between x_c and x is normal to $\partial\Omega$ in x_c.*

Proof. For a proof see [**16**, appendix] and [**19**, Section 4.4]. $\square$

We define then for a uniformly C^2-domain the number $R^* > 0$ to be the supremum of all R satisfying the above for Ω and its complement. This R^* is often called the reach of $\partial\Omega$ ([**19**]).

Our main result then states that in an admissible domain the Stokes operator generates an analytic semigroup in $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ for suitable choices of μ and ν. The constant $C_{n,L}$ denotes here a constant depending on the regularity of the domain which will be defined in section 4.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be an admissible, uniformly C^3-domain. Let $0 < \nu \leq R^*$ and $\mu \in (R^*, \infty]$. Then the Stokes operator generates a C_0-analytic semigroup in $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$.*

The main idea of the proof is deriving estimates for

$$\int_{B_r(x)} \left|u(y,t) - u_{B_r(x)}(t)\right|^2 dy \quad \text{and} \quad \int_{B_r(x_0)\cap\Omega} |u(y,t)|^2 \, dy$$

for $B_r(x) \subset \Omega$ and $x_0 \in \partial\Omega$. This can be done by using the fundamental theorem of calculus $u(t) = \int_0^t u_s(s)\, ds - u_0$, the equality $u_t = \Delta u - \nabla\pi$ and integration by parts such that we only need to estimate π and the gradient of u. Via an estimate on harmonic functions the pressure in this calculation is also controlled by the gradient of u. By the estimate

$$\sup_{0<t<T_0} t^{1/2}\|\nabla u(t)\|_\infty \leq C\|u_0\|_{BMO_b^{\mu,\nu}}$$

of Theorem 1.1 we then obtain for $t < T_0$ the inequality

$$\|u(t)\|_{BMO_b^{\mu,2\nu}2} \leq C\|u_0\|_{BMO_b^{\mu,\nu}}.$$

Finally, we will need equivalence results between different BMO_b-norms to compare these two norms and get the boundedness in $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$. Together with the time derivative estimate of Theorem 1.1 this yields the analyticity of the Stokes operator in $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$.

This paper is organized as follows. In section 2 we will prove estimates that will be needed to get control of the pressure terms that will appear in our calculations. In section 3 we will prove that we can estimate the BMO-type norm of the solution by another BMO-type norm of the initial data and that the solution is in $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$. In section 4 we will prove the required equivalence results of different BMO-type norms. In section 5 we will consider the Stokes semigroup in the half-space and prove the global boundedness of the semigroup and its derivatives.

2. Boundary estimate for the pressure.

In this section we will prove estimates for harmonic functions in order to estimate the pressure terms in Section 3 in a suitable way.

Theorem 2.1. *Let Ω be a bounded C^2-domain and consider the equation*

$$\begin{aligned} \Delta\pi &= 0 \qquad \text{in } \Omega \\ \partial\pi/\partial\boldsymbol{n} &= \operatorname{div}_{\partial\Omega} g \text{ on } \partial\Omega \\ \textstyle\int_\Omega \pi\, dx &= 0. \end{aligned}$$

Then there is a constant $C > 0$ depending only on C^2-regularity of Ω and the second eigenvalue of the Neumann Laplacian in Ω such that

$$\|\pi\|_{L^2(\partial\Omega)} \leq C\|g\|_{L^2(\partial\Omega)} \tag{2.1}$$

holds for all $g \in L^2(\partial\Omega)$ with $g \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. The constant C is additionally invariant under scaling transformations of the domain Ω.

We shall prove this theorem in several steps. For Lipschitz domains we consider the Sobolev space on the boundary $\partial\Omega$. Let $H^1(\partial\Omega)$ denote the space of all $f \in L^2(\partial\Omega)$ whose weak tangential derivative $\nabla_{\partial\Omega} f$ is also in $L^2(\partial\Omega)$. We equip this space with an inner product in the same way as in the definition of $H^1(\Omega)$. The space $H^s(\partial\Omega)$ $(0 \leq s \leq 1)$ is given as the complex interpolation space $\left[L^2(\partial\Omega), H^1(\partial\Omega)\right]_s$ based on fractional powers of the self-adjoint operator associated with the inner product of H^1 ([**20**]). It is well-known that the trace space $H^{1/2}(\partial\Omega)$ of $H^1(\Omega)$ agrees with this characterization of the interpolation ([**20**]). Let $H^{-s}(\partial\Omega)$ be the dual space of $H^s(\partial\Omega)$.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then*

$$\|\nabla_{\partial\Omega} f\|_{H^{-1}(\partial\Omega)} \leq \|f\|_{L^2(\partial\Omega)} \tag{2.2}$$

for all $f \in L^2(\partial\Omega)$, where $\nabla_{\partial\Omega}$ denotes the weak tangential gradient.

Proof. This can be seen immediately from the definition of $H^{-1}(\partial\Omega)$. □

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then*

$$\|\nabla_{\partial\Omega} f\|_{H^{-s}(\partial\Omega)} \leq \|f\|_{H^{1-s}(\partial\Omega)} \tag{2.3}$$

for all $f \in H^{1-s}(\partial\Omega)$ and $s \in [0,1]$. In particular,

$$\|\nabla_{\partial\Omega} f\|_{H^{-1/2}(\partial\Omega)} \leq \|f\|_{H^{1/2}(\partial\Omega)} \tag{2.4}$$

for all $f \in H^{1/2}(\partial\Omega)$.

Proof. We interpolate (2.2) with

$$\|\nabla_{\partial\Omega} f\|_{L^2(\partial\Omega)} \leq \|f\|_{H^1(\partial\Omega)}$$

to get (2.3) by complex interpolation theory ([**20**]). Note that $H^{-s}(\partial\Omega) = \left[L^2(\partial\Omega), H^{-1}(\partial\Omega)\right]_s$. □

We next recall the solvability of the Neumann problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in} \quad \Omega \\ \partial u/\partial \boldsymbol{n} &= h \quad \text{on} \quad \partial\Omega \end{aligned} \tag{2.5}$$

under the compatibility condition $\int_{\partial\Omega} h \, d\mathcal{H}^{n-1} = 0$. The Lax–Milgram theorem or even the Riesz representation theorem for a Hilbert space guarantees the existence of a solution $u \in H^1(\Omega)$ for $h \in H^{-1/2}(\partial\Omega)$. If h is regular, say $h \in H^{1/2}(\partial\Omega)$ and if $\partial\Omega$ is C^2, then u is H^2. This is also standard. We just summarize these results which are for example found in [**7**, Theorem III, 4.3] including the case when the Laplace equation (2.5) is replaced by the Poisson equation $\Delta u = f$.

Lemma 2.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. For a given $h \in H^{-1/2}(\partial\Omega)$ with $\int_{\partial\Omega} h \, d\mathcal{H}^{n-1} = 0$, there is a unique weak solution $u \in H^1(\Omega)$ of* (2.5) *satisfying $\int_\Omega u \, dx = 0$. This linear operator $h \mapsto u$ fulfills the estimate*

$$\|u\|_{H^1(\Omega)} \le C\|h\|_{H^{-1/2}(\partial\Omega)} \tag{2.6}$$

with C depending only on Ω through its Lipschitz regularity of Ω as well as the second eigenvalue of the Laplacian with Neumann boundary conditions.

Moreover, if Ω is C^2 and $h \in H^{1/2}(\partial\Omega)$, then $u \in H^2(\Omega)$. The linear operator $h \mapsto u$ fulfills the estimate

$$\|u\|_{H^2(\Omega)} \le C\|h\|_{H^{1/2}(\partial\Omega)}. \tag{2.7}$$

Here, the constant C depends in addition on C^2-regularity of Ω.

The dependence of C with respect to the second eigenvalue of the Laplacian with Neumann boundary condition appears when one uses the Poincaré type inequality to control the L^2-norm of u by the L^2-norm of ∇u. If the boundary regularity is fixed, then the constant decreases as the second eigenvalue increases.

The estimate (2.6) together with the well-known trace theorem [**7**, Theorem III, 2.2] and (2.4) yield key estimates for the boundary value of the solution π in Theorem 2.1.

Lemma 2.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $g \in H^{1/2}(\partial\Omega)$ satisfy $g \cdot n = 0$ on $\partial\Omega$ and let $\pi \in H^1(\Omega)$ with $\int_\Omega \pi \, dx = 0$ be the unique solution of* (2.5) *with $h = \operatorname{div}_{\partial\Omega} g$. Then*

$$\|\gamma\pi\|_{H^{1/2}(\partial\Omega)} \le C\|g\|_{H^{1/2}(\partial\Omega)} \tag{2.8}$$

with C depending only on the Lipschitz regularity of Ω and the second eigenvalue of the Laplacian with Neumann boundary condition, where γ denotes the trace on $\partial\Omega$.

Proof. We first notice that $\int_{\partial\Omega} h \, d\mathcal{H}^{n-1} = 0$ because g is tangential. By the inequality (2.4) we observe that $\operatorname{div} g \in H^{-1/2}(\partial\Omega)$, which guarantees the existence of an H^1-solution π (Lemma 2.4). We now observe by the trace theorem, (2.6) and (2.4) that

$$\|\gamma\pi\|_{H^{1/2}(\partial\Omega)} \le C_1\|\pi\|_{H^1(\Omega)}$$

$$\leq C_2 \| \operatorname{div}_{\partial\Omega} g \|_{H^{-1/2}(\partial\Omega)}$$
$$\leq C_3 \| g \|_{H^{1/2}(\partial\Omega)}$$

which yields (2.8) where C_j denotes a constant depending only on Ω. Here we only used Lipschitz regularity of the boundary. □

We finally apply a duality argument.

Lemma 2.6. *Assume that Ω is a bounded C^2-domain in $\mathbb{R}^n$. Let g and π be as in Lemma 2.5. Then*

$$\|\gamma\pi\|_{H^{-1/2}(\partial\Omega)} \leq C \|g\|_{H^{-1/2}(\partial\Omega)} \tag{2.9}$$

with C depending only on C^2-regularity of Ω as well as the second eigenvalue of the Laplacian with Neumann boundary condition in Ω.

Proof. Let u_h be the H^2-solution (satisfying $\int_\Omega u_h \, dx = 0$) of (2.5) with $h \in H^{1/2}(\partial\Omega)$ satisfying $\int_{\partial\Omega} h \, d\mathcal{H}^{n-1} = 0$. By the Green formula we have

$$\int_{\partial\Omega} (\gamma\pi) h \, d\mathcal{H}^{n-1} - \int_{\partial\Omega} \frac{\partial\pi}{\partial n} u_h \, d\mathcal{H}^{n-1} = \int_\Omega (\pi \Delta u_h - u_h \Delta \pi) \, dx = 0,$$

where γu_h is denoted simply by u_h. Thus

$$\int_{\partial\Omega} (\gamma\pi) h \, d\mathcal{H}^{n-1} = \int_{\partial\Omega} (\operatorname{div}_{\partial\Omega} g) u_h \, d\mathcal{H}^{n-1} = -\int_{\partial\Omega} g \cdot \nabla_{\partial\Omega} u_h \, d\mathcal{H}^{n-1}.$$

This representation yields

$$\left| \int_{\partial\Omega} (\gamma\pi) h \, d\mathcal{H}^{n-1} \right| \leq \|g\|_{H^{-1/2}(\partial\Omega)} \|\gamma \nabla_{\partial\Omega} u_h\|_{H^{1/2}(\partial\Omega)}.$$

By the trace theorem and (2.7) we get

$$\left| \int_{\partial\Omega} (\gamma\pi) h \, d\mathcal{H}^{n-1} \right| \leq \|g\|_{H^{-1/2}(\partial\Omega)} \|h\|_{H^{1/2}(\partial\Omega)}$$

which yields (2.9). □

Proof of Theorem 2.1. Since the estimate (2.9) guarantees that $g \mapsto \gamma\pi$ is extendable from tangential $H^{-1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$, interpolating (2.8) with (2.9) yields (2.1), where we suppress the trace symbol γ. Here we invoke the property that $\left[H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\right]_{1/2} = L^2(\partial\Omega)$ ([**20**]). The scaling invariance follows directly from scaling Ω, π and g. □

Remark 2.7. In the application of Theorem 2.1 in Section 3 we will need to consider this estimate for domains of the form $\Omega \cap B_r(x_0)$ $(x_0 \in \partial\Omega, r < 2c_0R^*)$ with smoothed corners and for balls. For the balls we can invoke that the constant C is scaling invariant and thus depends only on dimension. For the other domains we will need some uniform control of the constant in this estimate. We can achieve this through control on the C^2-regularity of the domains and an estimate from below on the second eigenvalue λ_2 of the Neumann Laplacian. For convex domains $\tilde{\Omega}$ such an estimate is explicitly known by [**22**], where $\lambda_2 \geq \pi^2/\operatorname{diam}(\tilde{\Omega})^2$ was shown. But not all of the domains we consider are convex. For these remaining domains one can obtain an estimate from below directly from [**9**], where this was proved for manifolds satisfying an "interior rolling ε-ball condition" which is satisfied for all C^2-domains. Another possibility is to use the result of [**5**] where it was proved that small perturbations of the domain result in small changes of the eigenvalues. Thus, if we choose $0 < c_0 < 1/2$ sufficiently small we get from the estimate $\lambda_2 \geq \pi^2/r^2$ for the upper half of a ball with radius r the estimate $\lambda_2 \geq \pi^2/2r^2$ for all $B_r(x_0) \cap \Omega$ with $x_0 \in \partial\Omega$, $r < 2c_0R^*$ and smoothed corners. From this we can conclude that the constant C of (2.1) is bounded from above in all applications of Theorem 2.1 if we choose $\nu < c_0R^*$ and have control on the C^2-regularity.

3. Boundedness in BMO-type spaces.

In this section we will prove that the solution operator maps $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ to $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ under suitable choices of μ and ν and finally conclude the analyticity of the Stokes semigroup in these BMO-type spaces. We will distinguish between small and large balls and use the derivative estimate of Theorem 1.1 in order to prove this boundedness. It will be easier to do most of the calculations with the BMO_b2-norms since in this case we do not have to take care of the absolute value in the definition and it enables us to integrate by parts in a way that fits to our needs.

Since we will also need some control over the mean values we will start with an estimate on mean values of the solution.

Lemma 3.1. *Let $\mu, \nu \in (0,\infty]$ and Ω an admissible uniformly C^3-domain. Then there are constants $C, T_0 > 0$ which are independent of r, u_0 and t such that*

$$\left|u_{B_r(x)}(t) - u_{0B_r(x)}\right| \leq C\frac{t^{1/2}}{r}\|u_0\|_{BMO^{\mu,\nu}_b}$$

holds for all solutions $u := S(t)u_0$ of (1.1) *with $u_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$, $t \in (0,T_0)$ and $B_r(x) \subset \Omega$.*

Proof. By the fundamental theorem of calculus, equation $(1.1)_1$ and integration by parts we get

$$\begin{aligned}
&\frac{1}{|B_r(x)|}\int_{B_r(x)}(u(y,t) - u_0(y))\,dy \\
&= \frac{1}{|B_r(x)|}\int_{B_r(x)}\int_0^t \frac{\partial u}{\partial s}(y,s)\,ds\,dy \\
&= \frac{1}{|B_r(x)|}\int_0^t\int_{B_r(x)}(\Delta u(y,s) - \nabla\pi(y,s))\,dy\,ds \\
&= \frac{1}{|B_r(x)|}\int_0^t\int_{\partial B_r(x)}\left(\frac{\partial u}{\partial \boldsymbol{n}}(y,s) - \pi(y,s)\boldsymbol{n}\right)d\mathcal{H}^{n-1}(y)\,ds.
\end{aligned}$$

Then we can estimate this in the following way by using the Hölder inequality $\|f\|_{L^1(\partial B_r)} \le \|1\|_{L^2(\partial B_r)}\|f\|_{L^2(\partial B_r)}$, where $\|1\|_{L^2(\partial B_r)} = Cr^{(n-1)/2}$

$$\begin{aligned}
&\left|\frac{1}{|B_r(x)|}\int_{B_r(x)}(u(y,t)-u_0(y))\,dy\right| \\
&\le \frac{\omega_n^{-1}}{r}\int_0^t \frac{1}{r^{n-1}}\int_{\partial B_r(x)}\left(\left|\frac{\partial u}{\partial \boldsymbol{n}}(y,s)\right| + |\pi(y,s)\boldsymbol{n}|\right)ds\,d\mathcal{H}^{n-1}(y),\ (\omega_n = |B_1(0)|) \\
&\le \frac{\omega_n^{-1}}{r}\int_0^t \frac{1}{r^{n-1}}\left(\int_{\partial B_r(x)}\|\nabla u(s)\|_{L^\infty(\Omega)}\,d\mathcal{H}^{n-1}(y) + \|\pi(s)\|_{L^1(\partial B_r(x))}\right)ds \\
&\le \frac{C}{r}\int_0^t (\|\nabla u(s)\|_{L^\infty(\Omega)} + r^{-(n-1)/2}\|\pi(s)\|_{L^2(\partial B_r(x))})\,ds.
\end{aligned}$$

Here, we used that $\nabla u(t)$ is in $W^{1,\infty}(\Omega)$ by Theorem 1.1 and thus $\nabla u \in C(\overline{\Omega})$ such that we can estimate $\|\partial u/\partial \boldsymbol{n}\|_{L^\infty(\partial\Omega)}$ by $\|\nabla u\|_{L^\infty(\Omega)}$.

We get then by Theorem 1.1, Theorem 2.1 with choosing π such that $\int_{B_r(x)}\pi = 0$, $(1.1)_1$ and the Hölder inequality $\|f\|_{L^2(\partial B_r)} \le \|1\|_{L^2(\partial B_r)}\|f\|_{L^\infty(\partial B_r)}$

$$\begin{aligned}
&\left|\frac{1}{|B_r(x)|}\int_{B_r(x)}(u(y,t)-u_0(y))\,dy\right| \\
&\le \frac{C}{r}\int_0^t (\|\nabla u(s)\|_\infty + r^{-(n-1)/2}\|\operatorname{curl} u(s)\times \boldsymbol{n}\|_{L^2(\partial B_r(x))})\,ds \\
&\le \frac{C}{r}\int_0^t \|\nabla u(s)\|_\infty\,ds \\
&\le \frac{C}{r}\int_0^t s^{-1/2}\|u_0\|_{BMO_b^{\mu,\nu}}\,ds \\
&\le C\frac{t^{1/2}}{r}\|u_0\|_{BMO_b^{\mu,\nu}}. \qquad \square
\end{aligned}$$

In the next theorem we obtain bounds for the mean oscillation of the solution in large balls.

THEOREM 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an admissible, uniformly C^3-domain, $\mu,\nu \in (0,\infty]$. Then there are constants $C, T_0 > 0$ depending only on Ω, n, μ and ν such that for all $0 < r < \mu$ and $x \in \Omega$ with $B_r(x) \subset \Omega$, $t \in (0,T_0)$ and all $u_0 \in VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ with $u(t) = S(t)u_0$*

$$\frac{1}{|B_r(x)|}\int_{B_r(x)}|u(y,t)-u_{B_r(x)}(t)|^2\,dy \le C\left(1+\frac{t}{r^2}\right)\|u_0\|^2_{BMO_b^{\mu,\nu}}.$$

PROOF. By the fundamental theorem of calculus, $(1.1)_1$ and integration by parts we get

$$\int_{B_r}|u(y,t)-u_{B_r}(t)|^2\,dy$$

$$
\begin{aligned}
&= \int_{B_r} (u(y,t) - u_{B_r}(t)) \left(\int_0^t \frac{\partial (u - u_{B_r})}{\partial s}(y,s)\, ds - (u_0(y) - u_{0B_r}) \right) dy \\
&\leq \left| \int_{B_r} (u(y,t) - u_{B_r}(t)) \int_0^t (\Delta u(y,s) - \nabla \pi(y,s))\, ds\, dy \right| \\
&\quad + \left| \int_{B_r} (u(y,t) - u_{B_r}(t)) \int_0^t \frac{\partial u_{B_r}}{\partial s}(s)\, ds\, dy \right| \\
&\quad + \|u(y,t) - u_{B_r}(t)\|_{L^2(B_r)} \|u_0 - u_{0B_r}\|_{L^2(B_r)} \\
&\leq \left| \int_0^t \int_{B_r} \nabla u(y,t) \nabla u(y,s)\, dy\, ds \right| + \left| \int_0^t \int_{\partial B_r} (u(y,t) - u_{B_r}(t)) \frac{\partial u}{\partial \boldsymbol{n}}(s)\, d\mathcal{H}^{n-1}(y)\, ds \right| \\
&\quad + \left| \int_0^t \int_{\partial B_r} (u(y,t) - u_{B_r}(t)) \pi(y,s) \boldsymbol{n}\, d\mathcal{H}^{n-1}(y)\, ds \right| \\
&\quad + \left| \int_{B_r} (u(y,t) - u_{B_r}(t))\, dy (u_{B_r}(t) - u_{0B_r}) \right| \\
&\quad + \frac{1}{2} \int_{B_r} |u(y,t) - u_{B_r}(t)|^2\, dy + \frac{1}{2} \int_{B_r} |u_0(y,t) - u_{0B_r}|^2\, dy,
\end{aligned}
$$

where we can take the second to last term to the left hand side and the last term can be estimated by $r^n [u_0]^2_{BMO^{\mu}2}$ which by the John-Nirenberg inequality is equivalent to $r^n [u_0]^2_{BMO^{\mu}}$. We will use the derivative estimate of Theorem 1.1 for estimating the other parts. The first term can be estimated by

$$
\begin{aligned}
\left| \int_0^t \int_{B_r} \nabla u(y,t) \nabla u(y,s)\, dy\, ds \right| &\leq \int_{B_r} \|\nabla u(t)\|_\infty \int_0^t \|\nabla u(s)\|_\infty\, ds\, dy \\
&\leq C r^n t^{-1/2} \|u_0\|_{BMO_b^{\mu,\nu}} \int_0^t s^{-1/2} \|u_0\|_{BMO_b^{\mu,\nu}}\, ds \\
&\leq C r^n \|u_0\|^2_{BMO_b^{\mu,\nu}}.
\end{aligned}
$$

For the second summand we get

$$
\begin{aligned}
&\left| \int_0^t \int_{\partial B_r} (u(y,t) - u_{B_r}(t)) \frac{\partial u}{\partial \boldsymbol{n}}(s)\, d\mathcal{H}^{n-1}(y)\, ds \right| \\
&\leq \int_{\partial B_r} |u(y,t) - u_{B_r}(t)| \int_0^t \|\nabla u(s)\|_\infty\, ds\, d\mathcal{H}^{n-1}(y) \\
&\leq C \int_{\partial B_r} |u(y,t) - u_{B_r}(t)|\, d\mathcal{H}^{n-1}(y) \int_0^t s^{-1/2} \|u_0\|_{BMO_b^{\mu,\nu}}\, ds \\
&\leq C t^{1/2} \|u_0\|_{BMO_b^{\mu,\nu}} \int_{\partial B_r} |u(y,t) - u_{B_r}(t)|\, d\mathcal{H}^{n-1}(y).
\end{aligned}
$$

In order to estimate the third term we estimate the pressure part by using Theorem 2.1, $(1.1)_1$ and Hölder's inequality.

$$
\left| \int_0^t \int_{\partial B_r} (u(y,t) - u_{B_r}(t)) \pi(y,s) \boldsymbol{n}\, d\mathcal{H}^{n-1}(y)\, ds \right|
$$

$$
\begin{aligned}
&\le \|u(t) - u_{B_r}(t)\|_{L^2(\partial B_r)} \int_0^t \|\pi(s)\|_{L^2(\partial B_r)}\,ds \\
&\le Cr^{(n-1)/2}\|u(t) - u_{B_r}(t)\|_{L^\infty(\partial B_r)} \int_0^t r^{(n-1)/2}\|\operatorname{curl} u(s) \times \boldsymbol{n}\|_{L^\infty(\partial B_r)}\,ds \\
&\le Cr^n \|\nabla u(t)\|_\infty \int_0^t \|\nabla u(s)\|_\infty\,ds \\
&\le Cr^n \|u_0\|^2_{BMO_b^{\mu,\nu}},
\end{aligned}
$$

where we used Poincaré's inequality with constant Cr in B_r in the second to last line. For the fourth term we use Lemma 3.1

$$
\begin{aligned}
&\left|\int_{B_r} (u(y,t) - u_{B_r}(t))\,dy(u_{B_r}(t) - u_{0B_r})\right| \\
&\le C\left(\int_{B_r} |u(y,t) - u_{B_r}(t)|^2\,dy\right)^{1/2} r^{n/2}\frac{t^{1/2}}{r}\|u_0\|_{BMO_b^{\mu,\nu}} \\
&\le \varepsilon \int_{B_r} |u(y,t) - u_{B_r}(t)|^2\,dy + C_\varepsilon \frac{t}{r^2} r^n \|u_0\|^2_{BMO_b^{\mu,\nu}}.
\end{aligned}
$$

Thus we have the estimate

$$
\begin{aligned}
\int_{B_r} |u(y,t) - u_{B_r}(t)|^2\,dy \le C_\varepsilon r^n\left(1 + \frac{t}{r^2}\right)\|u_0\|^2_{BMO_b^{\mu,\nu}} + \varepsilon \int_{B_r} |u(y,t) - u_{B_r}(t)|^2\,dy \\
+ Ct^{1/2}\|u_0\|_{BMO_b^{\mu,\nu}} \int_{\partial B_r} |u(y,t) - u_{B_r}(t)|\,d\mathcal{H}^{n-1}(y).
\end{aligned}
$$

After taking the term containing ε to the left hand side it is left to estimate $\int_{\partial B_r} |u(y,t) - u_{B_r}(t)|\,d\mathcal{H}^{n-1}(y)$. By the trace theorem and Poincaré's inequality we obtain

$$
\begin{aligned}
\int_{\partial B_r} |u(y,t) - u_{B_r}(t)|\,d\mathcal{H}^{n-1}(y) &\le C_r\left(\int_{B_r} |\nabla u(y,t)|^2 + |u(y,t) - u_{B_r}(t)|^2\,dy\right)^{1/2} \\
&\le C_r\left(\int_{B_r} |\nabla u(y,t)|^2\,dy\right)^{1/2}.
\end{aligned}
$$

We see by a scaling argument that $C_r = Cr^{n/2}$. Then

$$
\begin{aligned}
Ct^{1/2}\int_{\partial B_r} |u(y,t) - u_{B_r}(t)|\,d\mathcal{H}^{n-1}(y) &\le Ct^{1/2}r^{n/2}\|\nabla u(t)\|_{L^2(B_r)} \\
&\le Ct^{1/2}r^n\|\nabla u(t)\|_\infty \\
&\le Cr^n\|u_0\|_{BMO_b^{\mu,\nu}}
\end{aligned}
$$

such that we finally obtain

$$
\int_{B_r(x)} |u(y,t) - u_{B_r(x)}(t)|^2\,dy \le Cr^n\left(1 + \frac{t}{r^2}\right)\|u_0\|^2_{BMO_b^{\mu,\nu}}. \qquad \square
$$

For boundedness we will need similar estimates for small r. These estimates can be proved in a much simpler way by using Poincaré's inequality.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an admissible, uniformly C^3-domain, $\mu, \nu \in (0, \infty]$. There are constants $C, T_0 > 0$ depending only on Ω, n, μ and ν such that for all $r > 0$ and $x \in \Omega$ with $B_r(x) \subset \Omega$, $t \in (0, T_0)$ and all $u_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ with $u(t) = S(t)u_0$*

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |u(y,t) - u_{B_r(x)}(t)|^2 \, dy \leq C \frac{r^2}{t} \|u_0\|^2_{BMO^{\mu,\nu}_b}.$$

Proof. By Poincaré's inequality in B_r with constant Cr and Theorem 1.1 we can estimate

$$\begin{aligned}\int_{B_r(x)} |u(y,t) - u_{B_r(x)}(t)|^2 \, dy &\leq \int_{B_r(x)} \|u(t) - u_{B_r(x)}(t)\|^2_\infty \, dy \\ &\leq C \int_{B_r(x)} r^2 \|\nabla u(t)\|^2_\infty \, dy \\ &\leq C r^n \frac{r^2}{t} \|u_0\|^2_{BMO^{\mu,\nu}_b}. \end{aligned}$$

□

We can now estimate the BMO-part of the BMO_b-norm in a suitable way. In a similar way we will get estimates for the boundary part of the norm. Since $B_r(x_0) \cap \Omega$ for $x_0 \in \partial\Omega$ is not a C^2-domain which we will need for the estimate of the pressure, we need to change the parameter ν in a certain way.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be an admissible, uniformly C^3-domain, $\mu \in (0, \infty]$, $0 < \nu \leq c_0 R^*$, where c_0 is the constant of Remark 2.7. There are constants $C, T_0 > 0$ depending only on Ω, n, μ and ν such that for all $x_0 \in \partial\Omega$, $r < \nu$, $t \in (0, T_0)$ and all $u_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ with $u(t) = S(t)u_0$*

$$\frac{1}{r^n} \int_{B_r(x_0) \cap \Omega} |u(y,t)|^2 \, dy \leq C(\|u_0\|^2_{BMO^{\mu,\nu}_b} + [u_0]^2_{b^{2\nu}2}).$$

Proof. Let $B_r(x_0) \cap \Omega \subset \tilde{B} \subset B_{2r}(x_0) \cap \Omega$ be a domain with C^2-regularity, where the C^2-regularity of $\tilde{B}$ depends only on ν and the C^3-regularity of Ω. Again by the fundamental theorem of calculus and integration by parts we obtain

$$\begin{aligned}&\int_{\tilde{B}} |u(y,t)|^2 \, dy \\ &= \int_{\tilde{B}} u(y,t) \int_0^t \frac{\partial u}{\partial s}(y,s) \, ds \, dy - \int_{\tilde{B}} u(y,t) u_0(y) \, dy \\ &\leq \left| \int_{\tilde{B}} u(y,t) \int_0^t (\Delta u(y,s) - \nabla \pi(y,s)) \, ds \, dy \right| + \|u(t)\|_{L^2(\tilde{B})} \|u_0\|_{L^2(\tilde{B})} \\ &\leq \left| \int_{\tilde{B}} u(y,t) \int_0^t (\Delta u(y,s) - \nabla \pi(y,s)) \, ds \, dy \right| + \frac{1}{2} \int_{\tilde{B}} |u(y,t)|^2 \, dy \end{aligned}$$

$$+\frac{1}{2}\int_{B_{2r}(x_0)\cap\Omega}|u_0|^2\,dy,$$

where we take the second summand to the left hand side. The last summand can be estimated by $r^n[u_0]^2_{b^{2\nu}2}$. For the first summand we obtain by using the estimate $\|u\|_{L^\infty(B_{2r}(x_0)\cap\Omega)}\le Cr\|\nabla u\|_\infty$, which follows from the homogeneous boundary condition, estimating the part with pressure π in the same way as in Theorem 3.2 and integrating by parts

$$\begin{aligned}
&\frac{C}{r^n}\left|\int_{\tilde B}u(y,t)\int_0^t(\Delta u(y,s)-\nabla\pi(y,s))\,ds\,dy\right|\\
&\le\frac{C}{r^n}\left(\left|\int_{\tilde B}\nabla u(y,t)\int_0^t\nabla u(y,s)\,ds\,dy\right|+\left|\int_{\partial\tilde B}u(y,t)\int_0^t\frac{\partial u}{\partial\boldsymbol{n}}(y,s)\,ds\,d\mathcal{H}^{n-1}(y)\right|\right.\\
&\qquad\left.+\left|\int_{\partial\tilde B}u(y,t)\int_0^t\pi(y,s)\boldsymbol{n}\,ds\,d\mathcal{H}^{n-1}(y)\right|\right)\\
&\le\frac{C}{r^n}\left(\int_{\tilde B}\|\nabla u(t)\|_\infty\int_0^t\|\nabla u(s)\|_\infty\,ds\,dy\right.\\
&\qquad+\int_{\partial\tilde B}r\|\nabla u(t)\|_\infty\int_0^t\|\nabla u(s)\|_\infty\,ds\,d\mathcal{H}^{n-1}(y)\\
&\qquad\left.+\int_{\partial\tilde B}|u(y,t)|\int_0^t|\pi(y,s)\boldsymbol{n}|\,ds\,d\mathcal{H}^{n-1}(y)\right)\\
&\le C\|u_0\|^2_{BMO_b^{\mu,\nu}}+\frac{C}{r^n}\|u(t)\|_{L^2(\partial\tilde B)}\int_0^t\|\pi(s)\|_{L^2(\partial\tilde B)}\,ds\\
&\le C\|u_0\|^2_{BMO_b^{\mu,\nu}}.
\end{aligned}$$

Finally we obtain

$$\int_{\tilde B}|u(y,t)|^2\,dy\le C\|u_0\|^2_{BMO_b^{\mu,\nu}}+C[u_0]^2_{b^{2\nu}2}.\qquad\square$$

Let $C_{n,L}$ be a constant depending on Ω which will be defined in section 4. Roughly speaking, $C_{n,L}$ measures the degree of shrinkage of transforms from neighborhoods near the boundary to $\mathbb{R}^n_+$.

Theorem 3.5. *Let $\Omega\subset\mathbb{R}^n$ be an admissible, uniformly C^3-domain, $0<\nu\le R^*$, $\mu\in(R^*,\infty]$. Then there are constants $C,T_0>0$ depending only on Ω, n, μ and ν such that for all $t\in(0,T_0)$ and all $u_0\in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ with $u(t)=S(t)u_0$*

$$\|u(t)\|_{BMO_b^{\mu,\nu}}\le C\|u_0\|_{BMO_b^{\mu,\nu}}$$

holds.

Proof. Since the norms with different such ν are equivalent by Theorem 4.3 that will be proved later, we can assume that $\nu<\min\{R^*/(4C_{n,L}),c_0R^*\}$. By Theorem 3.2, Lemma 3.3 and Theorem 3.4 we obtain for some T_0 and C depending only on Ω,n,μ,ν

$$\|u(t)\|_{BMO_b^{\mu,\nu}2} \le C\|u_0\|_{BMO_b^{\mu,\nu}} + C[u_0]_{b^{2\nu}2} \quad (t \in (0,T_0)). \tag{3.1}$$

We will now use two different equivalence results on the BMO_b-norms. At first note that it is immediate from the definition and Hölder's inequality that

$$\|u(t)\|_{BMO_b^{\mu,\nu}} \le C\|u(t)\|_{BMO_b^{\mu,\nu}2}.$$

Since $2\nu < R^*/(2C_{n,L})$ we can use the equivalence between $\|\cdot\|_{BMO_b^{\mu,2\nu}2}$ and $\|\cdot\|_{BMO_b^{\mu,2\nu}}$ that will be proved in the next section (Theorem 4.7) to estimate $C[u_0]_{b^{2\nu}2}$ such that we get

$$\|u(t)\|_{BMO_b^{\mu,\nu}} \le C\|u_0\|_{BMO_b^{\mu,\nu}} + C\|u_0\|_{BMO_b^{\mu,2\nu}} \quad (t \in (0,T_0)).$$

Now we will use the equivalence between $\|\cdot\|_{BMO_b^{\mu,\nu}}$ and $\|\cdot\|_{BMO_b^{\mu,2\nu}}$ (Theorem 4.3) which yields

$$\|u(t)\|_{BMO_b^{\mu,\nu}} \le C\|u_0\|_{BMO_b^{\mu,\nu}} \quad (t \in (0,T_0)). \qquad \square$$

Now we have all estimates that are necessary to obtain a semigroup. However, as in the L^∞-case $C_{c,\sigma}^\infty(\Omega)$ is not dense in the largest solenoidal subspace of $BMO_b^{\mu,\nu}(\Omega)$. Thus, in order to get a semigroup on $VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ we have to ensure that the solutions $u(t) \in VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ for $u_0 \in VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$. This will be done in the appendix.

We are now able to show our main result, the analyticity of the $VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$-Stokes semigroup.

Proof of Theorem 1.3. By Theorem 1.1 and the embedding $L^\infty(\Omega) \hookrightarrow BMO_b^{\mu,\nu}(\Omega)$ we know that the solution operator $S(t)$ satisfies the estimate

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}S(t)u_0\right\|_{BMO_b^{\mu,\nu}} \le \frac{C}{t}\|u_0\|_{BMO_b^{\mu,\nu}} \quad (t \in (0,T_0)).$$

Furthermore, we know by the previous theorem that

$$\|S(t)u_0\|_{BMO_b^{\mu,\nu}} \le C_0\|u_0\|_{BMO_b^{\mu,\nu}} \quad (t \in (0,T_0)). \tag{3.2}$$

By the appendix we obtain that $S(t)u_0 \in VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ for every $u_0 \in VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ and $t \in (0,T_0)$. From this we can conclude that $S(t)u_0 \in VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ for every $t > 0$. This together with the above estimates yields that S is an analytic semigroup. It is left to show that S is a C_0-semigroup. It was proved in Proposition 5.3 of [**1**] that for all $u_0 \in C_{c,\sigma}^\infty(\Omega)$

$$\lim_{t\to 0}\|S(t)u_0 - u_0\|_\infty = 0.$$

If we now take $u_0 \in VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$, then there exists by definition of $VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ a sequence $u_0^m \in C_{c,\sigma}^\infty(\Omega)$ such that u_0^m converges to u_0 with respect to the $BMO_b^{\mu,\nu}$-norm. Then we have by (3.2) for $t < T_0$

$$
\begin{aligned}
&\|S(t)u_0 - u_0\|_{BMO_b^{\mu,\nu}} \\
&\le \|S(t)(u_0 - u_0^m)\|_{BMO_b^{\mu,\nu}} + \|S(t)u_0^m - u_0^m\|_{BMO_b^{\mu,\nu}} + \|u_0^m - u_0\|_{BMO_b^{\mu,\nu}} \\
&\le (C_0+1)\|u_0^m - u_0\|_{BMO_b^{\mu,\nu}} + (2+\omega_n)\|S(t)u_0^m - u_0^m\|_\infty.
\end{aligned}
$$

For given $\varepsilon > 0$ we choose then $m \in \mathbb{N}$ such that $\|u_0^m - u_0\|_{BMO_b^{\mu,\nu}} < \varepsilon/2(C_0+1)$ and then $t_0 > 0$ sufficiently small such that $\|S(t)u_0^m - u_0^m\|_\infty < \varepsilon/2(2+\omega_n)$ for all $0 < t < t_0$. Then

$$\|S(t)u_0 - u_0\|_{BMO_b^{\mu,\nu}} < \varepsilon$$

for $t < t_0$ which proves that S is a C_0-semigroup. $\square$

4. Remark on equivalences of BMO_b-norms.

In this section we will prove the equivalence results for different BMO_b-norms that were used in the proof of Theorem 3.5.

For these equivalence results we will need a fundamental theorem on BMO-functions that states that the L^1-norm of a function in a large area can be controlled by the L^1-norm of the function in a small area and the BMO-seminorm of f.

THEOREM 4.1. *Let $\mu \in (0,\infty]$ and $\Omega \subset \mathbb{R}^n$ be a domain. Then for all $f \in BMO^\mu(\Omega)$, $a > 1$, $r > 0$, $x_1, x_2 \in \Omega$ with $B_r(x_1) \subset B_{ar}(x_2) \subset \Omega$ and $ar \le \mu$ holds the inequality*

$$\|f\|_{L^1(B_{ar}(x_2))} \le |B_{ar}(x_2)|(1+a^n)[f]_{BMO^\mu(\Omega)} + a^n\|f\|_{L^1(B_r(x_1))}. \tag{4.1}$$

PROOF. Let $B_1 := B_r(x_1)$, $B_2 := B_{ar}(x_2)$ and $\tilde f := f - f_{B_1}$. By $\int_{B_1}(\tilde f - \tilde f_{B_2})\,dy = -|B_1|\tilde f_{B_2}$ we obtain

$$|B_1||\tilde f_{B_2}| \le \int_{B_1} |\tilde f - \tilde f_{B_2}|\,dy$$

and thus

$$
\begin{aligned}
|B_2|[\tilde f]_{BMO^\mu} &\ge \int_{B_2} |\tilde f - \tilde f_{B_2}|\,dy \\
&\ge \int_{B_1} |\tilde f - \tilde f_{B_2}|\,dy \\
&\ge |B_1||\tilde f_{B_2}|.
\end{aligned}
$$

From this we can estimate the mean value of $\tilde f$ in B_2 by

$$|\tilde f_{B_2}| \le a^n[\tilde f]_{BMO^\mu}.$$

Then we can estimate the L^1-norm of f by using estimates on the mean values together with the L^1-norm of f on a small ball.

$$\begin{aligned}\|f\|_{L^1(B_2)} &\le \|f - f_{B_1}\|_{L^1(B_2)} + |B_2||f_{B_1}| \\ &= \|\tilde{f}\|_{L^1(B_2)} + |B_2||f_{B_1}| \\ &\le \|\tilde{f} - \tilde{f}_{B_2}\|_{L^1(B_2)} + |B_2||\tilde{f}_{B_2}| + \frac{|B_2|}{|B_1|}\|f\|_{L^1(B_1)} \\ &\le |B_2|[\tilde{f}]_{BMO^\mu} + |B_2|a^n[\tilde{f}]_{BMO^\mu} + a^n\|f\|_{L^1(B_1)} \\ &= |B_2|(1+a^n)[f]_{BMO^\mu} + a^n\|f\|_{L^1(B_r(x_1))}. \qquad \square\end{aligned}$$

Since we consider BMO-functions on domains it will be useful to extend those functions to the more classical BMO-functions on $\mathbb{R}^n$. P. W. Jones proved in [**18**] the exact condition when this is possible. This condition is in particular satisfied if the domain is a bounded Lipschitz domain.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there is a constant C depending only on Lipschitz regularity of $\partial\Omega$ such that for each $f \in BMO^\infty(\Omega)$ there is an extension $\bar{f} \in BMO^\infty(\mathbb{R}^n)$ such that*

$$[\bar{f}]_{BMO^\infty(\mathbb{R}^n)} \le C[f]_{BMO^\infty(\Omega)}.$$

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a uniformly C^2-domain, $\nu_1 < \nu_2 \le R^*$ and $\mu \in [\nu_2, \infty]$. The norm $\|\cdot\|_{BMO_b^{\mu,\nu_1}}$ is then equivalent to $\|\cdot\|_{BMO_b^{\mu,\nu_2}}$.*

Proof. It follows immediately from the definition that for $\nu_1 < \nu_2$, $\|f\|_{BMO_b^{\mu,\nu_1}} \le \|f\|_{BMO_b^{\mu,\nu_2}}$. Thus it is left to show that

$$\frac{1}{r^n}\int_{\Omega\cap B_r(x_0)} |f(y)|\,dy \le C\|f\|_{BMO_b^{\mu,\nu_1}}$$

with a constant $C > 0$ independent of $x_0 \in \partial\Omega$ and $\nu_1 \le r < \nu_2$. Since $\nu_1 \le r < R^*$, every $B_{\nu_1/2}(x_0) \cap \Omega \subset B_r(x_0) \cap \Omega$ contains a closed ball B_0 of radius $\nu_1/4$ and the Lipschitz regularity of $\Omega \cap B_r(x_0)$ is uniform. Thus by Theorem 4.2 there is a uniform constant $C > 0$ such that for all $x_0 \in \partial\Omega$ and all $\nu_1 \le r < \nu_2$ there is an extension of $f|_{\Omega\cap B_r(x_0)}$ to $\bar{f} \in BMO^\infty(\mathbb{R}^n)$ with

$$[\bar{f}]_{BMO^\infty(\mathbb{R}^n)} \le C[f]_{BMO^\infty(\Omega\cap B_r(x_0))} \le C[f]_{BMO^\mu(\Omega)}.$$

Since $\int_{B_0} |f(y)|\,dy \le \nu_1^n[f]_{b^{\nu_1}}$ we obtain by Theorem 4.1 for $\nu_1 \le r < \nu_2$ that

$$\begin{aligned}\frac{1}{r^n}\int_{\Omega\cap B_r(x_0)} |f(y)|\,dy &\le \frac{1}{r^n}\int_{B_r(x_0)} |\bar{f}(y)|\,dy \\ &\le \omega_n\left(1 + \left(\frac{4\nu_2}{\nu_1}\right)^n\right)[\bar{f}]_{BMO^\infty(\mathbb{R}^n)} + \frac{(4r/\nu_1)^n}{r^n}\|f\|_{L^1(B_0)} \\ &\le C[f]_{BMO^\mu(\Omega)} + C[f]_{b^{\nu_1}(\Omega)}\end{aligned}$$

with a constant independent of r and x_0. $\square$

We now want to prove the equivalence between $BMO_b^{\mu,\nu}p$ and $BMO_b^{\mu,\nu}$. Our proof is divided into two parts. One concerning Hölder type estimates and one concerning reverse Hölder type estimates which will be the crucial part.

Lemma 4.4 (Hölder type estimates). *Let $\Omega \subset \mathbb{R}^n$ be a domain, $p \in [1, \infty)$, $\mu, \nu \in (0, \infty]$ and $f \in BMO_b^{\mu,\nu}p(\Omega)$. Then f satisfies the following estimate*

$$\|f\|_{BMO_b^{\mu,\nu}} \leq C\|f\|_{BMO_b^{\mu,\nu}p}$$

for some constant $C = C(n, p) > 0$.

Proof. This Lemma is easily obtained by the use of Hölder's inequality. □

For the reverse Hölder type inequality we need the John–Nirenberg inequality.

Theorem 4.5 (John–Nirenberg inequality). *Let $\Omega \subset \mathbb{R}^n$ be a domain, $p \in [1, \infty)$, $f \in BMO^{\mu}(\Omega)$. Then, there exists $C = C(n, p) > 0$ such that*

$$[f]_{BMO^{\mu}p} \leq C[f]_{BMO^{\mu}}.$$

Proof. This inequality is rather different from the original John–Nirenberg inequality ([**17**]), but it can be obtained from this inequality. □

Let $\Omega \subset \mathbb{R}^n$ be a uniformly C^2-domain with Lipschitz constant L and let $x_0 \in \partial\Omega$. We define $\Phi_{x_0} : \Omega \cap \bar{B}_{R^*}(x_0) \to \mathbb{R}^n_+$ by $\Phi_{x_0}(x) = (x', x_n - \phi_{x_0}(x'))$ where ϕ_{x_0} is a Lipschitz function with Lipschitz constant L which is a local coordinate of $\partial\Omega$ at x_0. Let $d(A)$ denote the diameter of A. Then we define the degree of shrinkage of Ω (denoted by $C_{n,L}$) by

$$\sup\left\{\frac{d(\Phi_{x_0}(B_r(x) \cap \Omega))}{d(B_r(x) \cap \Omega)}, \frac{d(\Phi_{x_0}^{-1}(B_r(x) \cap \Omega))}{d(B_r(x) \cap \Omega)} : x \in \Omega, B_r(x) \subset B_{R^*}(x_0), x_0 \in \partial\Omega\right\}.$$

We remark that this degree depends only on n and L because Ω is uniformly Lipschitz. Now we want to state the reverse Hölder type estimates up to the boundary.

Lemma 4.6 (Reverse Hölder type estimates up to the boundary). *Let $\Omega \subset \mathbb{R}^n$ be a uniformly C^2-domain with Lipschitz constant L. Let $C_{n,L}$ denote the degree of shrinkage of Ω. Let $\nu \in (0, R^*/(2C_{n,L}^2)]$, $\mu \in [R^*, \infty]$, $p \in [1, \infty)$, $f \in BMO_b^{\mu,\nu}(\Omega)$. Then there exists a constant $C = C(n, p, \Omega, \nu) > 0$ such that*

$$[f]_{b^{\nu}p} \leq C\|f\|_{BMO_b^{\mu,\nu}(\Omega)}.$$

Proof. Let $x_0 \in \partial\Omega$ and $r < \nu$ be given. We will then write Φ for Φ_{x_0}. Then, by changing variables

$$\left(r^{-n}\int_{\Omega \cap B_r(x_0)} |f(y)|^p \, dy\right)^{1/p}$$

$$= \left(r^{-n} \int_{\Phi(\Omega \cap B_r(x_0))} |(f \circ \Phi^{-1})(z)|^p |J_{\Phi^{-1}}| \, dz \right)^{1/p}$$
$$\leq (1+L) \left(\frac{|\Phi(B_r)|}{r^n} \right)^{1/p} \left(|\Phi(B_r)|^{-1} \int_{\Phi(\Omega \cap B_r(x_0))} |(f \circ \Phi^{-1})(z)|^p \, dz \right)^{1/p}$$
$$\leq (1+L)(\omega_n C_{n,L})^{1/p} \left(|\Phi(B_r)|^{-1} \int_{\Phi(\Omega \cap B_r(x_0))} |(f \circ \Phi^{-1})(z)|^p \, dz \right)^{1/p},$$

where $J_{\Phi^{-1}}$ denotes the Jacobian of Φ^{-1}. Let $E_{\mathbb{R}^n_+}$ be the x_n-odd extension from $\mathbb{R}^n_+$ to $\mathbb{R}^n$. We define the function g by $g = E_{\mathbb{R}^n_+}(f \circ \Phi^{-1})$ and set

$$Q_R = \Phi(\Omega \cap B_R(x_0)) \cup (-\Phi(\Omega \cap B_R(x_0))) \text{ for } R = r, R^*.$$

Then, $\int_{Q_R} g dx = 0$ for $R = r, R^*$. We want to apply Theorem 4.5, so we check that g satisfies the assumption of Theorem 4.5, i.e., $g \in BMO^{C_{n,L}\nu}(Q_{R^*})$. For this we will show that

$$[g]_{BMO^{C_{n,L}\nu}(Q_{R^*})} \leq C \|f\|_{BMO_b^{\mu,\nu}(\Omega)}. \tag{4.2}$$

Take $B_s(x) \subset Q_{R^*}$ with $s < C_{n,L}\nu < \mu/C_{n,L}$. There are two cases we have to consider.

(1) $B_s(x) \cap \partial \mathbb{R}^n_+ = \emptyset$,

(2) $B_s(x) \cap \partial \mathbb{R}^n_+ \neq \emptyset$.

In the case (1), we may assume $B_s(x) \subset \mathbb{R}^n_+$. We remark that $g = f \circ \Phi^{-1}$ in this case. We will show that

$$\frac{1}{|B_s(x)|} \int_{B_s(x)} |g(y) - g_{B_s(x)}| \, dy \leq C \|f\|_{BMO_b^{\mu,\nu}(\Omega)}.$$

Take arbitrary $c \in \mathbb{R}$. Then, by changing variables

$$\int_{B_s(x)} |g(z) - g_{B_s(x)}| \, dz$$
$$\leq \int_{B_s(x)} |f \circ \Phi^{-1}(z) - c| \, dz + |B_s(x)||c - (f \circ \Phi^{-1})_{B_s(x)}|$$
$$\leq 2 \int_{B_s(x)} |f \circ \Phi^{-1}(z) - c| \, dz$$
$$= 2 \int_{\Phi^{-1}(B_s(x))} |f(y) - c||J_\Phi| \, dy$$
$$\leq 2(1+L) \int_{\Phi^{-1}(B_s(x))} |f(y) - c| \, dy.$$

Let $d > 0$ be the distance from $\Phi^{-1}(B_s(x))$ to the boundary of $B_{R^*} \cap \Omega$. If the diameter of $\Phi^{-1}(B_s(x))$ is smaller than d, we can take the smallest ball $B_{s'}(z')$ with $s' < d < R^*$

and $z' \in \Omega$ so that $\Phi^{-1}(B_s(x)) \subset B_{s'}(z') \subset B_{R^*}(x_0) \cap \Omega$. Then $s' \leq C_{n,L}s < \mu$ and we obtain

$$\int_{\Phi^{-1}(B_s(x))} |f(y) - c| \, dy \leq \int_{B_{s'}(z')} |f(y) - c| \, dy.$$

Since c is arbitrary, this implies

$$\frac{1}{|B_s(x)|} \int_{B_s(x)} |g(y) - g_{B_s(x)}| \, dy \leq C[f]_{BMO^\mu} < +\infty.$$

If the diameter of $\Phi^{-1}(B_s(x))$ is bigger than d, then we take a perpendicular from $\Phi^{-1}(x)$ to $\partial\Omega$, and let x' denote a point at which the perpendicular intersects with $\partial\Omega$. Take the smallest ball $B_{s'}(x') \subset B_{R^*}(x_0)$ which contains $\Phi^{-1}(B_s(x))$. Then,

$$\frac{1}{|B_s(x)|} \int_{B_s(x)} |g(y) - g_{B_s(x)}| \, dy \leq C \frac{s'^n}{|B_s|} \frac{1}{s'^n} \int_{B_{s'}(x') \cap \Omega} |f(y) - c| \, dy.$$

By taking $c = 0$ in the integral,

$$\begin{aligned} &\frac{1}{|B_s(x)|} \int_{B_s(x)} |g(y) - g_{B_s(x)}| \, dy \\ &\leq C \frac{s'^n}{|B_s|} \frac{1}{s'^n} \int_{B_{s'}(x') \cap \Omega} |f(y)| \, dy \leq C_{n,L,d} \frac{s'^n}{|B_s|} [f]_{b^{R^*}}. \end{aligned}$$

We remark that $[f]_{b^{R^*}}$ is estimated by $C\|f\|_{BMO_b^{\mu,\nu}(\Omega)}$ because $f \in BMO_b^{\mu,\nu}(\Omega)$ and $BMO_b^{\mu,\nu}(\Omega)$ is equivalent to $BMO_b^{\mu,R^*}(\Omega)$ by Theorem 4.3. We also remark that $s'^n/|B_s|$ is finite because $d(\Phi^{-1}(B_s(x))) \leq C_{n,L}s$. In the case (2), $B_s(x)$ can be decomposed up to a null set as

$$B_s(x) = (B_s(x) \cap \mathbb{R}^n_+) \cup (B_s(x) \cap (-\mathbb{R}^n_+)) = B^1 \cup B^2.$$

Then, $\int_{B_s(x)} |g(z) - g_{B_s(x)}| \, dz \leq 2 \int_{B^1} |g(z)| \, dz + 2 \int_{B^2} |g(z)| \, dz$. Since the second term can be estimated in the same way as the first term, we only need to estimate the first term. By change of variables,

$$\int_{B^1} |g(z)| \, dz = \int_{\Phi^{-1}(B^1)} |f(z)||J_\Phi| \, dz \leq (1 + L) \int_{\Phi^{-1}(B^1)} |f(z)| \, dz.$$

Let us take a perpendicular from $\Phi^{-1}(x)$ to $\partial\Omega$, and let x' denote the point at which the perpendicular intersects with $\partial\Omega$. Take the smallest ball $B_{s'}(x') \subset B_{R^*}(x_0)$ which contains $\Phi^{-1}(B^1)$. Then,

$$\begin{aligned} &\int_{\Phi^{-1}(B^1)} |f(z)| \, dz \\ &\leq C s'^n \frac{1}{s'^n} \int_{B_{s'}(x')} |f(z)| \, dz \\ &\leq C s'^n [f]_{b^{R^*}} < +\infty. \end{aligned}$$

We have thus proved (4.2).

As a consequence, we can apply Theorem 4.5 to g and get for the largest ball $B_{\tilde{r}}(\tilde{x})$ satisfying $B_{\tilde{r}}(\tilde{x}) \subset Q_r$ and the smallest ball $B_{r'}(x')$ satisfying $Q_r \subset B_{r'}(x')$

$$\begin{aligned}
&\left(r^{-n} \int_{\Omega\cap B_r(x_0)} |f(y)|^p \, dy\right)^{1/p} \\
&\le C \left(|\Phi(B_r)|^{-1} \int_{\Phi(\Omega\cap B_r(x_0))} |(f\circ\Phi^{-1})(z)|^p \, dz\right)^{1/p} \\
&= C \left(2|Q_r|^{-1} \frac{1}{2} \int_{Q_r} |g(z) - g_{Q_r}|^p \, dz\right)^{1/p} \\
&\le \left(C|B_{\tilde{r}}(\tilde{x})|^{-1} \int_{Q_r} |g(z) - g_{B_{r'}(x')}|^p \, dz\right)^{1/p} \\
&\le \left(C|B_{r'}(x')|^{-1} \int_{B_{r'}(x')} |g(z) - g_{B_{r'}(x')}|^p \, dz\right)^{1/p} \\
&\le C[g]_{BMO^{C_{n,L}\nu}(Q_{R^*})}.
\end{aligned}$$

Here, we used $r \le C_{n,L}\tilde{r}$ and $r' \le C_{n,L} r \le C_{n,L}\nu$. By (4.2) we obtain as a consequence for arbitrarily given $x_0 \in \partial\Omega$ and $r < \nu$,

$$\left(r^{-n} \int_{\Omega\cap B_r(x_0)} |f(y)|^p \, dy\right)^{1/p} \le C\|f\|_{BMO_b^{\mu,\nu}(\Omega)}.$$

Therefore, we obtain the reverse Hölder type estimates up to the boundary. □

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^n$ be a uniformly C^2-domain with Lipschitz constant L. Let $C_{n,L}$ denote the degree of shrinkage of Ω. Let $\nu \in (0, R^*/(2C_{n,L}^2)]$, $\mu \in [R^*, \infty]$, $p \in [1,\infty)$, $f \in BMO_b^{\mu,\nu}(\Omega)$. Then, $\|\cdot\|_{BMO_b^{\mu,\nu}p}$ is equivalent to $\|\cdot\|_{BMO_b^{\mu,\nu}}$.*

Proof. Lemma 4.4 and Theorem 4.6 imply the equivalence. □

5. Bounded analyticity in the half-space.

In this section we will prove that the Stokes semigroup is a bounded analytic semigroup in a solenoidal subspace of $BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$. Furthermore, we will obtain global derivative estimates of the solution.

Theorem 5.1. *Let $\Omega = \mathbb{R}^n_+$ be the half-space. Then there is a constant C which only depends on the dimension n such that for all $u_0 \in VMO_{b,0,\sigma}^{\infty,\infty}(\mathbb{R}^n_+)$*

$$\sup_{t>0} \|u(t)\|_{BMO_b^{\infty,\infty}} \le C\|u_0\|_{BMO_b^{\infty,\infty}}, \tag{5.1}$$

$$\sup_{t>0} t^{1/2}\|\nabla u(t)\|_\infty \le C\|u_0\|_{BMO_b^{\infty,\infty}}, \tag{5.2}$$

$$\sup_{t>0} t\|\nabla^2 u(t)\|_\infty \leq C\|u_0\|_{BMO_b^{\infty,\infty}}, \tag{5.3}$$

$$\sup_{t>0} t\|u_t(t)\|_\infty \leq C\|u_0\|_{BMO_b^{\infty,\infty}}, \tag{5.4}$$

$$\sup_{t>0} t\|\nabla\pi(t)\|_\infty \leq C\|u_0\|_{BMO_b^{\infty,\infty}}, \tag{5.5}$$

where $(u, \nabla\pi)$ *is the solution of the Stokes equations with* $S(t)u_0 = u(t)$*. In particular,* S *is a bounded analytic semigroup on* $VMO_{b,0,\sigma}^{\infty,\infty}(\mathbb{R}_+^n)$*.*

PROOF. We will use that the spaces $BMO_b^{\infty,\infty}(\mathbb{R}_+^n)$ and $L^\infty(\mathbb{R}_+^n)$ are scaling-invariant. By Theorem 1.1 and Theorem 3.5 we obtain the existence of some $T_0 > 0$ such that for all $u_0 \in VMO_{b,0,\sigma}^{\infty,\infty}(\mathbb{R}_+^n)$ the estimate

$$\sup_{0<t<T_0} (\|u(t)\|_{BMO_b^{\infty,\infty}} + \|\tilde{N}(u,\pi)(\cdot,t)\|_\infty) \leq C_{T_0}\|u_0\|_{BMO_b^{\infty,\infty}}$$

holds. By taking $u_0^\lambda(x) := u_0(\lambda x)$ as initial data for $\lambda > 0$ we obtain the same estimate for $u^\lambda(x,t) = u(\lambda x, \lambda^2 t)$ and $\pi^\lambda = \lambda\pi(\lambda x, \lambda^2 t)$ with the right hand side $C_{T_0}\|u_0^\lambda\|_{BMO_b^{\infty,\infty}}$ which is equal to $C_{T_0}\|u_0\|_{BMO_b^{\infty,\infty}}$. By the scaling-invariance of the spaces we can conclude from the estimate for (u^λ, π^λ) that

$$\sup_{0<t<\lambda^2 T_0} (\|u(t)\|_{BMO_b^{\infty,\infty}} + \|\tilde{N}(u,\pi)(\cdot,t)\|_\infty) \leq C_{T_0}\|u_0\|_{BMO_b^{\infty,\infty}}$$

with C_{T_0} independent of $\lambda > 0$. Since λ was arbitrary we can replace $\sup_{0<t<\lambda^2 T_0}$ by $\sup_{t>0}$ in the above inequality and get the desired estimates. The bounded analyticity follows then from the time derivative estimate. □

A. Appendix.

Our goal in this section is to prove a density result. Let $\tilde{A}_r$ be the Stokes operator in the space $\tilde{L}_\sigma^r$ which is constructed in [**10**], [**11**].

THEOREM A.1. *Let* Ω *be a uniformly* C^2*-domain in* $\mathbb{R}^n$ $(n \geq 2)$*. For* $f \in D(\tilde{A}_{r_0})$*,* $r_0 > 2$*, there exists a sequence* $\{f_m\} \subset C_{c,\sigma}^\infty(\Omega)$ *such that* $\|f - f_m\|_{\tilde{W}^{1,r}(\Omega)} \to 0$ *as* $m \to \infty$ *for all* $r \in [2, r_0)$*.*

This density result yields the following property for the Stokes semigroup $S(t)$. Let $\tilde{W}_{\sigma,0}^{1,r}(\Omega)$ denote the $\tilde{W}^{1,r}$-closure of $C_{c,\sigma}^\infty(\Omega)$.

COROLLARY A.2. *Let* Ω *be a uniformly* C^2*-domain and* $u_0 \in C_{c,\sigma}^\infty(\Omega)$*. Then* $S(t)u_0 \in \tilde{W}_{\sigma,0}^{1,r}(\Omega)$ *for all* $r \geq 2$ *and* $t > 0$*. In particular,* $S(t)u_0 \in C_{0,\sigma}(\Omega) \subset VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ *with* $\mu, \nu \in (0, \infty]$*.*

This follows from Theorem A.1. Indeed, since S is an analytic semigroup in $\tilde{L}_\sigma^r(\Omega)$ we observe that $S(t)u_0 \in D(\tilde{A}_r)$ for $t > 0$ and $u_0 \in \tilde{L}_\sigma^r(\Omega)$. If $u_0 \in C_{c,\sigma}^\infty(\Omega)$ so that $u_0 \in \tilde{L}_\sigma^{r_0}(\Omega)$ for any $r_0 \geq 2$, then we get $S(t)u_0 \in D(\tilde{A}_{r_0})$. Thus, applying Theorem A.1 implies that $S(t)u_0 \in \tilde{W}_{\sigma,0}^{1,r}(\Omega)$ for any $r \geq 2$. The remaining assertion follows from the

Sobolev embedding for $r > n$ and $L^\infty(\Omega) \hookrightarrow BMO_b^{\mu,\nu}(\Omega)$.

The rest of this section is devoted to the proof of Theorem A.1. For this purpose we need an approximation of the domain Ω.

For a uniformly C^2-domain Ω of type (α, β, K) in the sense of [**10**] one can easily construct a sequence of uniformly C^2-domains Ω_m of type (α, β, K) such that $\Omega_m \subset \Omega$, $\mathrm{dist}(\Omega_m, \partial\Omega) \geq 1/m$ and

$$\Omega \subset \left\{ x \in \mathbb{R}^n : \mathrm{dist}(x, \Omega_m) \leq \frac{2}{m} \right\}$$

for $m \in \mathbb{N}$.

Lemma A.3. *For $f \in D(\tilde{A}_{r_0})$ with $r_0 > 2$ and $r \in [2, r_0)$ there exists a sequence $\{f_m\} \subset \tilde{W}_0^{1,r}(\Omega_m) \cap \tilde{L}_\sigma^r(\Omega_m)$ such that $\|f_m - f\|_{\tilde{W}^{1,r}(\Omega)} \to 0$ as $m \to \infty$. Here we interpret f_m as a function defined on Ω by extending via $f_m = 0$ in $\Omega \setminus \Omega_m$.*

Proof. Let $\tilde{A}_{r_0,m}$ be the Stokes operator in $\tilde{L}_\sigma^{r_0}(\Omega_m)$. By the construction of the operator there exists λ_0 such that if $\lambda \geq \lambda_0$, then $\lambda + \tilde{A}_{r_0,m}$ is invertible in $\tilde{L}_\sigma^{r_0}(\Omega_m)$, where λ_0 is independent of Ω_m since this property only depends on (α, β, K). We fix λ_0. For $f \in D(\tilde{A}_{r_0})$ we define $g \in \tilde{L}_\sigma^{r_0}(\Omega)$ by

$$g = (\lambda_0 + \tilde{A}_{r_0})f.$$

We approximate g by $g_m \in C_{c,\sigma}^\infty(\Omega)$ such that $\|g - g_m\|_{\tilde{L}^r(\Omega)} \to 0$ as $m \to \infty$. We may assume that $\operatorname{supp} g_m \subset \Omega_m$ by taking a subsequence. We set

$$f_m = (\lambda_0 + \tilde{A}_{r_0,m})^{-1}(g_m|_{\Omega_m}).$$

Since $f_m \in D(\tilde{A}_{r_0,m})$, it is clear that $f_m \in \tilde{W}_0^{1,r}(\Omega_m) \cap \tilde{L}_\sigma^r(\Omega_m)$ for all $r \in [2, r_0]$. We extend f_m by 0 and obtain a sequence of functions f_m defined on Ω. By the a priori estimate of [**10**], [**11**] we see that

$$\|f_m\|_{\tilde{W}^{2,r}(\Omega_m)} \leq C\|g_m\|_{\tilde{L}^r(\Omega_m)} \quad (r \in [2, r_0))$$

with C depending only on (α, β, K). It is not difficult to show that $f_m \to f$ in the sense of distributions in Ω. Since $\|g_m\|_{\tilde{L}^r(\Omega)}$ is bounded by a constant multiple of $\|g\|_{\tilde{L}^r(\Omega)}$, this implies that $\|f_m\|_{\tilde{W}^{1,r_0}(\Omega)}$ is bounded. By

$$\|\nabla f - \nabla f_m\|_{L^r(\Omega)} \leq \|\nabla f - \nabla f_m\|_{L^2(\Omega)}^\theta \|\nabla f - \nabla f_m\|_{L^{r_0}(\Omega)}^{1-\theta}$$

with $1/r = (\theta/r_0) + ((1-\theta)/2)$ and the same estimate for $f - f_m$ it suffices to prove that $f_m \to f$ strongly in $H^1(\Omega)$. We consider $H^1(\Omega)$ equipped with the scalar product $(f, g) = \int_\Omega (\lambda_0 + A_{r_0}) f \cdot g$ which is equivalent to the standard scalar product in $H^1(\Omega)$.

Since we already know that $f_m \to f$ in the sense of distributions and since $\|f_m\|_{H^1(\Omega)}$ is bounded, we can conclude that $f_m \to f$ weakly in $H^1(\Omega)$. To obtain strong convergence it remains to prove that $\|f_m\|_{H^1} \to \|f\|_{H^1}$. For this purpose we observe that

$$\|f_m\|_{H^1(\Omega)}^2 = \int_{\Omega_m} (\lambda_0 + \tilde{A}_{r_0,m}) f_m \cdot f_m \, dx = \int_{\Omega_m} g_m \cdot f_m \, dx.$$

Since $f_m \to f$ weakly in $L^2(\Omega)$ and $g_m \to g$ strongly in L^2 we conclude that

$$\|f_m\|_{H^1}^2 \to \int_\Omega f \cdot g \, dx \quad (m \to \infty).$$

The limit equals to

$$\|f\|_{H^1}^2 = \int_\Omega (\lambda_0 + A_{r_0}) f \cdot f \, dx.$$

Thus $f_m \to f$ in H^1. The proof is now complete. □

Lemma A.4. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $1 < r < \infty$. Let $f \in \tilde{W}_0^{1,r}(\Omega) \cap \tilde{L}_\sigma^r(\Omega)$ with $c_0 := \operatorname{dist}(\operatorname{supp} f, \partial\Omega) > 0$. Then there exists a sequence $f_m \in C_{c,\sigma}^\infty(\Omega)$ such that $\|f_m - f\|_{\tilde{W}^{1,r}} \to 0$ as $m \to \infty$.*

Proof. Let $\varepsilon > 0$ and take some $\delta < \min\{\varepsilon, c_0/2\}$. Let Ω' be defined by

$$\Omega' = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > c_0/2\}.$$

Since f is regarded as an element of $\tilde{L}_\sigma^r(\Omega')$, there exists a sequence $f_k \in C_{c,\sigma}^\infty(\Omega')$ such that $f_k \to f$ in $\tilde{L}_\sigma^r(\Omega')$. Let ϱ_δ be the standard mollifier whose support is contained in a ball of radius δ centered at zero. We define $f_\delta = f * \varrho_\delta$. We construct a sequence $f_{k,\delta} \in C_{c,\sigma}^\infty(\Omega)$ by $f_{k,\delta} = f_k * \varrho_\delta$ such that $f_{k,\delta}$ converges to f_δ in $\tilde{W}^{1,r}(\Omega)$. Note that the support of $f_{k,\delta}$ is contained in Ω by the choice of Ω' and ϱ. We observe that

$$\begin{aligned}\|f - f_{k,\delta}\|_{\tilde{W}^{1,r}} &\le \|f - f_\delta\|_{\tilde{W}^{1,r}} + \|f_\delta - f_{k,\delta}\|_{\tilde{W}^{1,r}} \\ &\le \|f - f_\delta\|_{\tilde{W}^{1,r}} + C_\delta \|f - f_k\|_{\tilde{L}^r}.\end{aligned}$$

For $\varepsilon > 0$ we take δ sufficiently small such that $\|f - f_\delta\|_{\tilde{W}^{1,r}} \le \varepsilon/2$ and then choose k_0 large enough to obtain for all $k \ge k_0$ that $C_\delta \|f - f_k\|_{\tilde{L}^r} \le \varepsilon/2$. □

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Note added in proof.

(i) As an application of our main result (Theorem 1.3), we are able to prove in [BGMST] that for any $p > 2$ there exists a domain Ω not admitting the L^p-Helmholtz decomposition but the Stokes operator generates an analytic semigroup in $L^p_\sigma(\Omega)$.

(ii) There are several ways to define BMO like spaces. It turns out the analyticity results (Theorem 1.3) can be extended to some other spaces closely related to the present BMO spaces [BGST]. [BGMST] M. Bolkart, Y. Giga, T.-H. Miura, T. Suzuki and Y. Tsutsui, On analyticity of the L^p-Stokes semigroup for some non-Helmholtz domains, Math. Nachr., 290 (2017), 2524-2546. [BGST] M. Bolkart, Y. Giga, T. Suzuki and Y. Tsutsui, Equivalence of BMO-type norms with applications to the heat and Stokes semigroup, Potential Analysis, Online. DOI:10.1007/s11118-017-9650-x