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Foundation of symbol theory for analytic pseudodifferential operators, I

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Abstract. A new symbol theory for pseudodifferential operators in the complex analytic category is given. Here the pseudodifferential operators mean integral operators with real holomorphic microfunction kernels. The notion of real holomorphic microfunctions had been introduced by Sato, Kawai and Kashiwara by using sheaf cohomology theory. Symbol theory for those operators was partly developed by Kataoka and by the first author and it has been effectively used in the analysis of operators of infinite order. However, there was a missing part that links the symbol theory and the cohomological definition of operators, that is, the consistency of the Leibniz–Hörmander rule and the cohomological definition of composition for operators. This link has not been established completely in the existing symbol theory. This paper supplies the link and provides a cohomological foundation of the symbolic calculus of pseudodifferential operators.

Introduction.

The aim of this series of papers is to establish a complete symbol theory for the sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudodifferential operators in the complex analytic category. Here we distinguish a little difference between the usage of hyphenation in the words "pseudodifferential" and "pseudo-differential." The latter might be more familiar than the former for most of readers. To clarify this distinction, we have to mention some of history.

The notion of the pseudo-differential operators in the analytic category was introduced by Boutet de Monvel and Kreé [**10**] and by Boutet de Monvel [**9**] for the real domain and by Sato, Kawai and Kashiwara [**24**] for the complex domain about forty years ago. Note that [**10**] introduced the notion in the category of ultradifferentiable functions of Gevrey class which contains the analytic category for a special case and treated operators of finite order. On the other hand, [**9**] and [**24**] considered operators of infinite order and these operators play an essential role in [**24**] and in Kashiwara and Kawai [**16**].

The definition of the pseudo-differential operators given in [**9**] used oscillatory integrals and analytic symbols, while [**24**] employed the cohomology theory. One of the advantages of the latter theory is invariance which comes from the cohomology theory.

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Symbol theory for pseudo-differential operators was also developed in [**24**], where the sheaf of them was denoted by $\mathscr{P}_X$. This sheaf is recently denoted by $\mathscr{E}_X^{\infty}$ after the work of Kashiwara and Schapira [**17**] and called the sheaf of microdifferential operators (of finite or infinite order). Note that it is a subsheaf of $\mathscr{E}_X^{\mathbb{R}}$. The notion of symbols defined in [**24**] is different from that of [**9**]. A symbol of a microdifferential operator (or a pseudo-differential operator in the sense of [**24**]) is a formal sum

$$\sum_{i \in \mathbb{Z}} P_i(x, \xi)$$

of holomorphic functions satisfying some conditions. On the other hand, a pseudo-differential operator in the sense of [**9**] is defined by a total symbol $p(x, \xi)$ which is real analytic in (x, ξ) satisfying a growth condition in ξ variables. The relation between those two theories was clarified by Kataoka [**20**]. He defined symbols of operators in $\mathscr{E}_X^{\mathbb{R}}$ by using the Radon transform and through his theory, we knew that pseudo-differential operators of [**9**] is obtained by restriction of $\mathscr{E}_X^{\mathbb{R}}$ to the real domain.

The essential idea of the definition of $\mathscr{E}_X^{\mathbb{R}}$ was given in [**24**] but the definition itself was not given there explicitly. The definition first appeared in the work of Kashiwara and Kawai [**15**], where the notation $\mathscr{P}_X^{\mathbb{R}}$ was used, although the name of the sections of the sheaf was not given. As well as the case of $\mathscr{E}_X^{\infty}$, we use the notation $\mathscr{E}_X^{\mathbb{R}}$ instead of $\mathscr{P}_X^{\mathbb{R}}$ after [**17**] and we call the sections of $\mathscr{E}_X^{\mathbb{R}}$ pseudodifferential operators after [**2**].

Since the symbol theory developed in [**20**] was not published, some parts of it were supplied by the first author [**2**] and the theory played a role in the analysis of operators of infinite order (cf. Aoki [**1**], [**3**], Aoki, Kawai, Koike and Takei [**7**], Aoki, Kawai and Takei [**8**], Kajitani and Wakabayashi [**12**], Kataoka [**21**], Uchikoshi [**25**]). Some systematic description of the theory has been included in the book of Aoki, Kataoka and Yamazaki [**6**]. The foundation of the symbol theory of $\mathscr{E}_X^{\mathbb{R}}$ at the present stage is, however, quite unsatisfactory. There are two issues: first one is that, as Kamimoto and Kataoka have pointed out in their work [**13**, Example 1.1], the space of the kernel functions which comes from standard Čech representation of cohomology groups is not closed under composition of kernel functions defined by naive integration employed in [**2**], [**6**]. Regarding this issue, [**13**] gives a possible solution by introducing the notion of formal kernels. Second issue is that the consistency of the action of operators by integration of kernel functions and canonical action through cohomological definition was not proved. We note that the notion of formal kernels given in [**13**] has not yet given a solution to this issue. Thus we think we have to provide a complete symbol theory of $\mathscr{E}_X^{\mathbb{R}}$ which solves these issues.

We mention that some modifications of the symbol theory are given by Uchikoshi [**26**] and by Ishimura [**11**] for microlocal operators and non-local operators in the analytic category, respectively. But there are analogous issues in these theories.

In this series of papers, we establish a new symbol theory of $\mathscr{E}_X^{\mathbb{R}}$ which completely fits in the cohomological definition of the sheaf. In the first part, we present a foundation of symbol theory for $\mathscr{E}_X^{\mathbb{R}}$. Our main idea is to introduce a redundant parameter, which we call an apparent parameter, in the definitions of (real) holomorphic microfunctions and symbols. By introducing this parameter, cohomological definition of operation such as composition, formal adjoint, coordinate transformation, etc. are directly related to those of symbols (see Kashiwara–Schapira [**18**], [**19**]). To clarify the relation between Čech cohomology classes and symbols, we fully use the theory of the action of microdifferential operators on microfunctions established by Kashiwara and Kawai ([**15**], [**16**]). We also develop a theory of formal symbols which was firstly introduced for operators of infinite order by [**9**] and generalized by [**1**], [**2**] and by Laurent [**22**]. The formal symbol theory established in this article is exactly based on the cohomological definition of $\mathscr{E}_X^{\mathbb{R}}$. To develop this theory, we employ an idea introduced by [**22**]. Our forthcoming second paper will be devoted to the symbol theory for operators with Gevrey growth and the cohomology theory for Whitney holomorphic functions. It will be useful for applications.

The plan of this paper, the first part, is as follows. In Section 1, we prepare a proposition of the local cohomology group theory on a vector space which we shall use in this article. Section 2 gives a new formulation of the sheaf of (real) holomorphic microfunctions utilizing an apparent parameter. Applying this formulation, we give a cohomological representation of pseudodifferential operators in Section 3. In Section 4, we define symbol spaces with an apparent parameter. The relation between symbols and cohomological representation of pseudodifferential operators is clarified in Section 5. Sections 6 and 7 are devoted to establishing a theory of formal symbols with an apparent parameter for pseudodifferential operators. We can express basic operations of pseudodifferential operators such as composition and coordinate transformation algebraically in terms of the formal symbols with cohomology theoretical foundation in these sections.

In Appendix A, we confirm the compatibility of actions of pseudodifferential operators on the sheaf of holomorphic microfunctions. Appendix B gives a general construction of the sheaf of microfunctions which can manage the symbol mapping on the space of kernel functions with respect to arbitrary coverings.

This work, especially, the idea of introducing redundant parameter, is inspired by [**13**]. The authors would like to express their sincere thanks to Professor K. Kataoka and Dr. S. Kamimoto. They also thank Professor T. Kawai and Professor Y. Okada for encouragement to them. In addition, they appreciate the constructive comments and suggestions from the reviewers.

1. Local cohomology groups on a vector space.

We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of integers, of real numbers and of complex numbers respectively. Further, set $\mathbb{N} := \{m \in \mathbb{Z};\, m > 0\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{C}^{\times} := \{c \in \mathbb{C};\, c \neq 0\}$.

Let X be a finite dimensional $\mathbb{R}$-vector space, and define an open proper sector $S \subset \mathbb{C}$ by

$$S := \{\eta \in \mathbb{C};\, a < \arg \eta < b,\, 0 < |\eta| < r\}$$

for some $0 < b - a < \pi$ and $r > 0$. We set $\widehat{X} := X \times \mathbb{C}_\eta$ with coordinates (x, η), and let $\pi_\eta \colon \widehat{X} \ni (x, \eta) \mapsto x \in X$ be the canonical projection. Let $G \subset X$ be a closed subset (not necessarily convex) and $U \subset X$ an open neighborhood of the origin. In this section we give another representation of local cohomology groups $H^k_{G\cap U}(U; \mathcal{F})$ for a sheaf $\mathcal{F}$ on X. For this purpose, we need some preparations. Let Z be a closed subset in X

and $\varphi\colon X \times [0,1] \to X$ a continuous deformation mapping which satisfies the following conditions:

(i) $\varphi(x,1) = x$ for any $x \in X$ and $\varphi(z,s) = z$ for any $z \in Z$.

(ii) $\varphi(\varphi(x,s),0) = \varphi(x,0)$ for any $s \in [0,1]$ and $x \in X$.

(iii) We set

$$\rho_\varphi(x,s) := |\varphi(x,s) - \varphi(x,0)|.$$

Then $\rho_\varphi(x,s)$ is a strictly increasing function of s outside Z, i.e. if $s_1 < s_2$, we have $\rho_\varphi(x,s_1) < \rho_\varphi(x,s_2)$ for any $x \in X \setminus Z$.

Note that $\varphi(x,0) \in Z$ for any $x \in X$ follows from the above conditions. Further, we set, for short

$$\rho_\varphi(x) := \rho_\varphi(x,1) = |\varphi(x,1) - \varphi(x,0)| = |x - \varphi(x,0)|.$$

Here we remark

$$\rho_\varphi(\varphi(x,s)) = |\varphi(x,s) - \varphi(\varphi(x,s),0)| = |\varphi(x,s) - \varphi(x,0)| = \rho_\varphi(x,s).$$

Let us see a typical example of such a deformation mapping.

Example 1.1. Let ζ be a unit vector in $X := \mathbb{C}^n$ and $Z = \{x \in X;\ \langle x,\zeta\rangle = 0\}$ with $\langle x,\zeta\rangle := \sum_{i=1}^n x_i\zeta_i$. Define the deformation mapping $\varphi\colon X \times [0,1] \to X$ by

$$\varphi(x,s) := x + (s-1)\langle x,\zeta\rangle\overline{\zeta}.$$

Here $\overline{\zeta}$ denotes the complex conjugate of ζ. Note that $\varphi(x,1) = x$ and $\varphi(x,0)$ gives the orthogonal projection to the complex hyperplane Z with respect to the standard Hermitian metric $|x| = \langle x,\overline{x}\rangle^{1/2}$.

Let $\varrho > 0$ a positive constant. We define the subsets in $\widehat{X}$ by

$$\begin{aligned} \widehat{G} &:= \{(\varphi(x,s),\eta) \in \widehat{X};\ \rho_\varphi(x) \le \varrho|\eta|,\ 0 \le s \le 1,\ x \in G\}, \\ \widehat{U} &:= \{(x,\eta) \in U \times S;\ \rho_\varphi(x) < \varrho|\eta|\}. \end{aligned} \tag{1.1}$$

Then we have the following fundamental lemma for these subsets:

Lemma 1.2. *Under the above situation, we have the followings.*

(1) *$\widehat{U}$ is an open subset in $\widehat{X}$ and $\pi_\eta(\widehat{U}) = U$ holds if U satisfies the condition $\sup_{x\in U}\rho_\varphi(x) < \varrho r$.*

(2) *$\widehat{G} \cap \widehat{U}$ is a closed subset in $\widehat{U}$.*

Proof. The claim (1) clearly holds. Let us show the claim (2). Let $\{(w_k,\eta_k,s_k)\}_{k=1}^\infty$ be a sequence in the subset

$$T := \{(w,\, \eta,\, s) \in \widehat{X} \times [0,1];\ \rho_\varphi(w) \le \varrho|\eta|,\ w \in G\}$$

such that the sequence $\{(\varphi(w_k, s_k),\, \eta_k)\}_{k=1}^\infty$ in $\widehat{G}$ converges to some point $(x_\infty,\, \eta_\infty)$ in $\widehat{U}$. We will see $(x_\infty,\, \eta_\infty) \in \widehat{G}$, which comes from boundness of $\{w_k\}_{k=1}^\infty$. Since $\{\rho_\varphi(w_k)\}_{k=1}^\infty$ is bounded because of $(w_k, \eta_k, s_k) \in T$ and $\eta_k \to \eta_\infty$, and since we have

$$\varphi(w_k,\, 0) = \varphi(\varphi(w_k, s_k),\, 0) \to \varphi(x_\infty,\, 0) \quad (k \to \infty),$$

the boundness of $\{w_k\}_{k=1}^\infty$ follows from $\rho_\varphi(w_k) = |w_k - \varphi(w_k,\, 0)|$. This completes the proof. □

We have the following proposition.

Proposition 1.3. *Let $\mathcal{F}$ be a complex of Abelian sheaves on X. Assume that U satisfies*

$$\sup_{x \in U} \rho_\varphi(x) < \varrho r. \tag{1.2}$$

Then there exists the following isomorphism:

$$\boldsymbol{R}\varGamma_{G \cap U}(U; \mathcal{F}) \xrightarrow{\sim} \boldsymbol{R}\varGamma_{\widehat{G} \cap \widehat{U}}(\widehat{U}; \pi_\eta^{-1}\mathcal{F}).$$

Proof. Since $\pi_\eta^{-1}(G) \cap \widehat{U}$ is closed in $\widehat{G} \cap \widehat{U}$ and $\widehat{U}$ is open in $\pi_\eta^{-1}(U)$, we obtain

$$\mathbb{Z}_{\widehat{G} \cap \widehat{U}} \to \mathbb{Z}_{\pi_\eta^{-1}(G) \cap \widehat{U}} \to \mathbb{Z}_{\pi_\eta^{-1}(G) \cap \pi_\eta^{-1}(U)} = \pi_\eta^{-1}\mathbb{Z}_{G \cap U} = \pi_\eta^{!}\mathbb{Z}_{G \cap U}[-2],$$

and this induces the canonical morphism

$$\boldsymbol{R}\pi_{\eta!}\mathbb{Z}_{\widehat{G} \cap \widehat{U}} \to \mathbb{Z}_{G \cap U}[-2]. \tag{1.3}$$

Hence, we have

$$\begin{aligned} \boldsymbol{R}\varGamma_{G \cap U}(U; \mathcal{F}) &\simeq \boldsymbol{R}\mathrm{Hom}_{\mathbb{Z}_X}(\mathbb{Z}_{G \cap U}, \mathcal{F}) \to \boldsymbol{R}\mathrm{Hom}_{\mathbb{Z}_X}(\boldsymbol{R}\pi_{\eta!}\mathbb{Z}_{\widehat{G} \cap \widehat{U}}, \mathcal{F})[-2] \\ &\simeq \boldsymbol{R}\mathrm{Hom}_{\mathbb{Z}_{\widehat{X}}}(\mathbb{Z}_{\widehat{G} \cap \widehat{U}}, \pi_\eta^{!}\mathcal{F})[-2] \simeq \boldsymbol{R}\varGamma_{\widehat{G} \cap \widehat{U}}(\widehat{U}; \pi_\eta^{-1}\mathcal{F}), \end{aligned}$$

and to show Proposition 1.3, it suffices to prove that (1.3) is an isomorphism. We first give some properties of φ and ρ_φ.

(1) If $\varphi(x, s) = \varphi(x', s')$ holds, we have $\varphi(x, 0) = \varphi(\varphi(x, s), 0) = \varphi(\varphi(x', s'), 0) = \varphi(x', 0)$. In particular, $\rho_\varphi(x, s) = \rho_\varphi(x', s')$.

(2) For any $x^* \in X$ we set

$$G(x^*) := \{(g, t) \in G \times [0,1];\ \varphi(g, t) = x^*\}.$$

If $G(x^*) \neq \emptyset$, there exists $(x, s) \in G(x^*)$ such that $\rho_\varphi(x)$ attains the value

$$a(x^*) := \inf\{\rho_\varphi(g);\ (g, t) \in G(x^*)\}.$$

Let us compute $\boldsymbol{R}\pi_{\eta!}\mathbb{Z}_{\widehat{G}\cap\widehat{U}}$. If $x^* \notin U$, clearly we have $(\boldsymbol{R}\pi_{\eta!}\mathbb{Z}_{\widehat{G}\cap\widehat{U}})_{x^*} = 0$. Hence in what follows, we assume $x^* \in U$, in particular, $\pi_\eta^{-1}(x^*) \cap \widehat{U} \neq \emptyset$ holds thanks to the assumption. Now we can calculate the stalk $(\boldsymbol{R}\pi_{\eta!}\mathbb{Z}_{\widehat{G}\cap\widehat{U}})_{x^*}$ for $x^* \in U$ as follows. If $x^* \in G$, we get

$$\pi_\eta^{-1}(x^*) \cap \widehat{G} = \{\eta \in \mathbb{C};\ \rho_\varphi(x^*) \leq \varrho|\eta|\}$$

because of $(x^*, 1) \in G(x^*)$ and

$$\rho_\varphi(x^*) = \rho_\varphi(x^*, 1) = \rho_\varphi(x, s) \leq \rho_\varphi(x, 1) = \rho_\varphi(x)$$

for any $(x, s) \in G(x^*)$. Hence we have

$$\pi_\eta^{-1}(x^*) \cap \widehat{G} \cap \widehat{U} = \{\eta \in S;\ \rho_\varphi(x^*) < \varrho|\eta|\},$$

which implies

$$(\boldsymbol{R}\pi_{\eta!}\mathbb{Z}_{\widehat{G}\cap\widehat{U}})_{x^*} = \boldsymbol{R}\varGamma_{\mathrm{c}}(\pi_\eta^{-1}(x^*) \cap \widehat{G} \cap \widehat{U}; \mathbb{Z}_{\widehat{X}}) = \mathbb{Z}[-2].$$

On the other hand, if $x^* \notin G$, we obtain

$$(\boldsymbol{R}\pi_{\eta!}\mathbb{Z}_{\widehat{G}\cap\widehat{U}})_{x^*} = \boldsymbol{R}\varGamma_{\mathrm{c}}(\pi_\eta^{-1}(x^*) \cap \widehat{G} \cap \widehat{U}; \mathbb{Z}_{\widehat{X}}) = 0.$$

As a matter of fact, if $\pi_\eta^{-1}(x^*) \cap \widehat{G} = \emptyset$, the claim clearly holds. Otherwise, we have $\rho_\varphi(x^*) < a(x^*)$ which can be shown by the following argument. Let (x, s) be a point in $G(x^*)$ with $\rho_\varphi(x) = a(x^*)$. Since $x^* \notin G$, $x \in G$ and $x^* = \varphi(x, s)$, we have $x \notin Z$ and $s < 1$. From these facts, it follows that

$$\rho_\varphi(x^*) = \rho_\varphi(x^*, 1) = \rho_\varphi(x, s) < \rho_\varphi(x, 1) = \rho_\varphi(x) = a(x^*).$$

Hence we have

$$\pi_\eta^{-1}(x^*) \cap \widehat{G} \cap \widehat{U} = \{\eta \in S;\ \rho_\varphi(x^*) < \varrho|\eta|,\, a(x^*) \leq \varrho|\eta|\} = \{\eta \in S;\ a(x^*) \leq \varrho|\eta|\},$$

which implies the claim.

Summing up, we have obtained

$$(\boldsymbol{R}\pi_{\eta!}\mathbb{Z}_{\widehat{G}\cap\widehat{U}})_{x^*} = \begin{cases} \mathbb{Z}[-2] & (x^* \in G \cap U), \\ 0 & (\text{otherwise}), \end{cases}$$

and hence (1.3) is an isomorphism. This completes the proof. □

Remark 1.4. Without (1.2), we have the following claim by the same argument as that in the proof above: Set $U' := \{x \in U;\ \rho_\varphi(x) < \varrho r\}$. Then there exists the canonical isomorphism

$$\boldsymbol{R}\varGamma_{G\cap U'}(U'; \mathcal{F}) \simeq \boldsymbol{R}\varGamma_{\widehat{G}\cap\widehat{U}}(\widehat{U}; \pi_\eta^{-1}\mathcal{F}).$$

2. Holomorphic microfunctions with an apparent parameter.

Let X be an n-dimensional $\mathbb{C}$-vector space with the coordinates $z = (z_1, \dots, z_n)$, and Y the closed complex submanifold of X defined by $\{z' = 0\}$ where $z = (z', z'')$ with $z' := (z_1, \dots, z_d)$ for some $1 \leq d \leq n$. Set $\widehat{X} := X \times \mathbb{C}$, and let $\pi_\eta \colon \widehat{X} \ni (z, \eta) \mapsto z \in X$ be the canonical projection as in Section 1. In what follows, we denote an object defined on the space $\widehat{X}$ by a symbol with $\widehat{\cdot}$ like $\widehat{U}_{\boldsymbol{\kappa}}$ etc. For any $z \in \mathbb{C}^n$, we set $\|z\| := \max_{1 \leq i \leq n}\{|z_i|\}$. Let $\mathscr{O}_X$ be the sheaf of *holomorphic functions* on X, and $\mathscr{C}^{\mathbb{R}}_{Y|X}$ the sheaf of *real holomorphic microfunctions* along Y on the conormal bundle $T^*_Y X$ to Y. Let $z_0 = (0, z''_0) \in Y$ and $z^*_0 = (z''_0; \zeta'_0) \in T^*_Y X$ with $|\zeta'_0| = 1$. Set

$$f_1(z) := \langle z', \zeta'_0 \rangle, \quad f'(z) := z' - \langle z', \zeta'_0 \rangle \overline{\zeta}'_0 \,.$$

Remark 2.1. The subsequent arguments can be applied to a general family of a function $f_1(z)$ and a mapping $f'(z)$, which enables us to develop the theory not only on a vector space but also on a complex manifold. It is, however, rather technical. Hence we put such a generalization in Appendix B.

By the definition of $\mathscr{C}^{\mathbb{R}}_{Y|X}$, we have

$$\mathscr{C}^{\mathbb{R}}_{Y|X, z^*_0} = \varinjlim_{\varrho, L, U} H^d_{G_{\varrho,L} \cap U}(U; \mathscr{O}_X).$$

Here $U \subset X$ ranges through open neighborhoods of z_0, and $G_{\varrho,L}$ denotes the closed set

$$G_{\varrho,L} := \{z \in X;\, \varrho^2 |f'(z)| \leq |f_1(z)|,\, f_1(z) \in L\},$$

where $L \subset \mathbb{C}$ ranges through closed convex cones with $L \subset \{\tau \in \mathbb{C}; \operatorname{Re}\tau > 0\} \cup \{0\}$. Now we apply the result in the previous section to the case above. We take the open sector $S_{r,\theta}$ defined by

$$S_{r,\theta} := \{\eta \in \mathbb{C};\, |\arg \eta| < \theta,\, 0 < |\eta| < r\}$$

for $0 < \theta < \pi/2$ and $r > 0$ as an S in the previous section. Set

$$\zeta := (0, \zeta'_0) \in \mathbb{C} \times \mathbb{C}^{n-1}.$$

We adopt the deformation mapping given in Example 1.1 and assume that U is sufficiently small so that the assumption of Proportion 1.3 is satisfied. Therefore there exists the canonical isomorphism

$$\boldsymbol{R}\varGamma_{G_{\varrho,L} \cap U}(U; \mathscr{O}_X) \simeq \boldsymbol{R}\varGamma_{\widehat{G}_{\varrho,L} \cap \widehat{U}_{\varrho,r,\theta}}(\widehat{U}_{\varrho,r,\theta}; \pi_\eta^{-1}\mathscr{O}_X),$$

where $\widehat{G}_{\varrho,L}$ and $\widehat{U}_{\varrho,r,\theta}$ are defined by (1.1) with respect to $G = G_{\varrho,L}$ and U. By easy computations, these sets are given by

$$\widehat{G}_{\varrho,L} = \{(z,\eta) \in \widehat{X};\ \varrho|f'(z)| \le |\eta|,\ f_1(z) \in L\},$$
$$\widehat{U}_{\varrho,r,\theta} = \{(z,\eta) \in U \times S_{r,\theta};\ |f_1(z)| < \varrho|\eta|\}, \tag{2.1}$$

respectively. Thus, from the exact sequence

$$0 \to \pi_\eta^{-1}\mathscr{O}_X \to \mathscr{O}_{\widehat{X}} \xrightarrow{\partial_\eta} \mathscr{O}_{\widehat{X}} \to 0,$$

we obtain the following distinguished triangle:

$$\boldsymbol{R}\varGamma_{G_{\varrho,L}\cap U}(U;\mathscr{O}_X) \to \boldsymbol{R}\varGamma_{\widehat{G}_{\varrho,L}\cap\widehat{U}_{\varrho,r,\theta}}(\widehat{U}_{\varrho,r,\theta};\mathscr{O}_{\widehat{X}}) \xrightarrow{\partial_\eta} \boldsymbol{R}\varGamma_{\widehat{G}_{\varrho,L}\cap\widehat{U}_{\varrho,r,\theta}}(\widehat{U}_{\varrho,r,\theta};\mathscr{O}_{\widehat{X}}) \xrightarrow{+1} .$$

We will see later the fact

$$\varinjlim_{\varrho,r,\theta,L,U} H^k_{\widehat{G}_{\varrho,L}\cap\widehat{U}_{\varrho,r,\theta}}(\widehat{U}_{\varrho,r,\theta};\mathscr{O}_{\widehat{X}}) = 0 \quad (k \ne d). \tag{2.2}$$

Hence we have reached the following definition and theorem.

Definition 2.2. We define

$$\widehat{C}^{\mathbb{R}}_{Y|X,z_0^*} := \varinjlim_{\varrho,r,\theta,L,U} H^d_{\widehat{G}_{\varrho,L}\cap\widehat{U}_{\varrho,r,\theta}}(\widehat{U}_{\varrho,r,\theta};\mathscr{O}_{\widehat{X}}),$$

where $U \subset X$ and $L \subset \mathbb{C}$ range through open neighborhoods of z_0 and closed convex cones in $\mathbb{C}$ with $L \subset \{\tau \in \mathbb{C};\ \mathrm{Re}\,\tau > 0\} \cup \{0\}$ respectively, and the subsets $\widehat{U}_{\varrho,r,\theta}$ and $\widehat{G}_{\varrho,L}$ are given in (2.1). Further we define

$$C^{\mathbb{R}}_{Y|X,z_0^*} := \mathrm{Ker}(\partial_\eta\colon \widehat{C}^{\mathbb{R}}_{Y|X,z_0^*} \to \widehat{C}^{\mathbb{R}}_{Y|X,z_0^*}).$$

Therefore, we obtain:

Theorem 2.3. *There exists the following canonical isomorphism*

$$\mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*} \simeq C^{\mathbb{R}}_{Y|X,z_0^*}.$$

Let us show (2.2). We may assume $z_0^* = (z_0'';\zeta_0') = (0;1,0,\dots,0)$. Let $\boldsymbol{\kappa} := (r,r',\varrho,\theta) \in \mathbb{R}^4$ be a 4-tuple of positive constants with

$$0 < \theta < \frac{\pi}{2}, \quad 0 < \varrho < 1, \quad 0 < r < \varrho r'. \tag{2.3}$$

Then we set

$$S_{\boldsymbol{\kappa}} := S_{r,\theta/4} = \left\{\eta \in \mathbb{C};\ 0 < |\eta| < r,\ |\arg\eta| < \frac{\theta}{4}\right\} \tag{2.4}$$

and define the open subset

$$\widehat{U}_{\boldsymbol{\kappa}} := \bigcap_{i=2}^{n} \{(z,\eta) \in X \times S_{\boldsymbol{\kappa}};\ |z_1| < \varrho|\eta|,\ |z_i| < r'\}.$$

We also define the closed cone

$$\widehat{G}_{\boldsymbol{\kappa}} := \bigcap_{i=2}^{d} \left\{(z,\eta) \in \widehat{X};\ |\arg z_1| \le \frac{\pi}{2} - \theta,\ \varrho|z_i| \le |\eta|\right\}.$$

By using these subsets, we introduce objects corresponding to $\widehat{C}^{\mathbb{R}}_{Y|X,z_0^*}$ and $C^{\mathbb{R}}_{Y|X,z_0^*}$ at z_0^*, which are easily manipulated by Čech cohomology groups.

Definition 2.4. We define

$$\widehat{C}^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) := H^d_{\widehat{G}_{\boldsymbol{\kappa}} \cap \widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}}),$$
$$C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) := \operatorname{Ker}(\partial_\eta \colon \widehat{C}^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) \to \widehat{C}^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa})).$$

Clearly we have

$$\widehat{C}^{\mathbb{R}}_{Y|X,z_0^*} = \varinjlim_{\boldsymbol{\kappa}} \widehat{C}^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) \quad\text{and}\quad C^{\mathbb{R}}_{Y|X,z_0^*} = \varinjlim_{\boldsymbol{\kappa}} C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}),$$

since families of closed cones and open subsets appearing in inductive limits of the both sides are equivalent with respect to inclusion of sets.

Proposition 2.5. *If $k \neq d$, then*

$$H^k_{\widehat{G}_{\boldsymbol{\kappa}} \cap \widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}}) = 0.$$

In particular, (2.2) *holds.*

Proof. We set

$$\begin{aligned} \widehat{V}^{(1)}_{\boldsymbol{\kappa}} &:= \left\{(z,\eta) \in \widehat{U}_{\boldsymbol{\kappa}};\ \frac{\pi}{2} - \theta < \arg z_1 < \frac{3\pi}{2} + \theta\right\}, \\ \widehat{V}^{(i)}_{\boldsymbol{\kappa}} &:= \left\{(z,\eta) \in \widehat{U}_{\boldsymbol{\kappa}};\ \varrho|z_i| > |\eta|\right\} \qquad (2 \le i \le d). \end{aligned} \tag{2.5}$$

Since each $\widehat{V}^{(i)}_{\boldsymbol{\kappa}}$ is pseudoconvex and $\widehat{U}_{\boldsymbol{\kappa}} \setminus \widehat{G}_{\boldsymbol{\kappa}} = \bigcup_{i=1}^{d} \widehat{V}^{(i)}_{\boldsymbol{\kappa}}$, we have $H^k_{\widehat{G}_{\boldsymbol{\kappa}} \cap \widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}}) = 0$ for $k > d$. Let us show the assertion for $k < d$. As $\varrho < 1$ and $r < \varrho r'$ hold, we have

$$\boldsymbol{R}\Gamma_{\widehat{G}_{\boldsymbol{\kappa}} \cap \widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}}) \simeq \boldsymbol{R}\Gamma_{\widehat{G}_{\boldsymbol{\kappa}} \cap \widehat{D}}(\widehat{D}; \mathscr{O}_{\widehat{X}}),$$

where

$$\widehat{D} := \bigcap_{i=d+1}^{n} \{(z,\eta) \in X \times S_{\boldsymbol{\kappa}};\ |z_1| < \varrho|\eta|,\ |z_i| < r'\}.$$

Let us consider the holomorphic mapping on $\widehat{X}$ defined by

$$\varphi(z,\eta) := (z_1, \eta z_2, \dots, \eta z_d, z'', \eta).$$

Since φ is bi-holomorphic on $X \times \mathbb{C}^{\times}$, we have

$$\boldsymbol{R}\Gamma_{\widehat{G}_{\boldsymbol{\kappa}} \cap \widehat{D}}(\widehat{D}; \mathscr{O}_{\widehat{X}}) \simeq \boldsymbol{R}\Gamma_{\widehat{K} \cap \widehat{D}}(\widehat{D}; \mathscr{O}_{\widehat{X}}).$$

Here we set $\widehat{K} := \widehat{K}_1 \cap \widehat{K}_2$ with

$$\widehat{K}_1 := \left\{(z,\eta) \in \widehat{X};\ |\arg z_1| \le \frac{\pi}{2} - \theta\right\}, \qquad \widehat{K}_2 := \bigcap_{i=2}^{d} \{(z,\eta) \in \widehat{X};\ \varrho |z_i| \le 1\}.$$

Then we have the distinguished triangle

$$\boldsymbol{R}\Gamma_{\widehat{K} \cap \widehat{D}}(\widehat{D}; \mathscr{O}_{\widehat{X}}) \to \boldsymbol{R}\Gamma_{\widehat{K}_2 \cap \widehat{D}}(\widehat{D}; \mathscr{O}_{\widehat{X}}) \to \boldsymbol{R}\Gamma_{(\widehat{K}_2 \setminus \widehat{K}_1) \cap \widehat{D}}(\widehat{D} \setminus \widehat{K}_1; \mathscr{O}_{\widehat{X}}) \xrightarrow{+1} .$$

Hence the claim of the proposition follows from the following well-known lemma. □

Lemma 2.6 ([**14**, Theorem 4.1.6]). *Let $\mathbb{D}$ be a closed disk with positive radius in $\mathbb{C}$ and U a pseudoconvex open subset in $\mathbb{C}^m$. Then*

$$H^{\nu}_{\mathbb{D}^k \times U}(\mathbb{C}^k \times U; \mathscr{O}_{\mathbb{C}^{k+m}}) = 0 \qquad (\nu \neq k).$$

Furthermore, for any pseudoconvex open subsets $U_1 \subset U_2$ in $\mathbb{C}^m$ which are non-empty and connected, the following canonical morphism is injective:

$$H^{k}_{\mathbb{D}^k \times U_2}(\mathbb{C}^k \times U_2; \mathscr{O}_{\mathbb{C}^{k+m}}) \to H^{k}_{\mathbb{D}^k \times U_1}(\mathbb{C}^k \times U_1; \mathscr{O}_{\mathbb{C}^{k+m}}).$$

Next, we set

$$U_{\boldsymbol{\kappa}} := \bigcap_{i=2}^{n} \{z \in X;\ |z_1| < \varrho r,\ |z_i| < r'\},$$
$$G_{\boldsymbol{\kappa}} := \bigcap_{i=2}^{d} \left\{z \in X;\ |\arg z_1| \le \frac{\pi}{2} - \theta,\ \varrho^2 |z_i| \le |z_1|\right\}.$$

Corollary 2.7. *If $k \neq d$, then*

$$H^{k}_{G_{\boldsymbol{\kappa}} \cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X) = 0,$$

and there exists the following exact sequence:

$$0 \to H^{d}_{G_{\boldsymbol{\kappa}} \cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X) \to \widehat{C}^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) \xrightarrow{\partial_\eta} \widehat{C}^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) \to 0. \tag{2.6}$$

Proof. We set

$$\begin{aligned} V_{\boldsymbol{\kappa}}^{(1)} &:= \left\{ z \in U_{\boldsymbol{\kappa}};\ \frac{\pi}{2} - \theta < \arg z_1 < \frac{3\pi}{2} + \theta \right\}, \\ V_{\boldsymbol{\kappa}}^{(i)} &:= \{ z \in U_{\boldsymbol{\kappa}};\ \varrho^2 |z_i| > |z_1| \} \qquad (2 \le i \le d). \end{aligned} \tag{2.7}$$

Since each $V_{\boldsymbol{\kappa}}^{(i)}$ is pseudoconvex and $U_{\boldsymbol{\kappa}} \setminus G_{\boldsymbol{\kappa}} = \bigcup_{i=1}^{d} V_{\boldsymbol{\kappa}}^{(i)}$, we have $H^k_{G_{\boldsymbol{\kappa}} \cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X) = 0$ for $k > d$. By Proposition 1.3 and Remark 1.4, we have the following distinguished triangle

$$\boldsymbol{R}\varGamma_{G_{\boldsymbol{\kappa}} \cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X) \to \boldsymbol{R}\varGamma_{\widehat{G}_{\boldsymbol{\kappa}} \cap \widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}}) \xrightarrow{\partial_\eta} \boldsymbol{R}\varGamma_{\widehat{G}_{\boldsymbol{\kappa}} \cap \widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}}) \xrightarrow{+1} .$$

By Definition 2.4 and Proposition 2.5, we have (2.6) and $H^k_{G_{\boldsymbol{\kappa}} \cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X) = 0$ for $k < d$. □

Note that, since

$$\widehat{U}_{\boldsymbol{\kappa}} \subset \pi_\eta^{-1}(U_{\boldsymbol{\kappa}}), \quad \pi_\eta^{-1}(G_{\boldsymbol{\kappa}}) \cap \widehat{U}_{\boldsymbol{\kappa}} \subset \widehat{G}_{\boldsymbol{\kappa}} \cap \widehat{U}_{\boldsymbol{\kappa}},$$

the morphism $H^d_{G_{\boldsymbol{\kappa}} \cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X) \to \widehat{C}^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa})$ is defined by a natural way associated with inclusion of sets. By Proposition 2.5 and (2.6), we obtain the following corollary.

Corollary 2.8. *Let $z_0^* = (0; 1, 0, \dots, 0)$. Then there exist isomorphisms*

$$\begin{array}{ccc} H^d_{G_{\boldsymbol{\kappa}} \cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X) & \xrightarrow{\sim} & C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) \\ \downarrow & & \downarrow \\ \mathscr{C}^{\mathbb{R}}_{Y|X, z_0^*} & \xrightarrow{\sim} & \varinjlim_{\boldsymbol{\kappa}} C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}). \end{array} \tag{2.8}$$

We now consider a Čech representation of $C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa})$. Recall $\widehat{V}_{\boldsymbol{\kappa}}^{(i)} \subset \widehat{X}$ of (2.5) and $V_{\boldsymbol{\kappa}}^{(i)} \subset X$ of (2.7) for $1 \le i \le d$. Let $\mathcal{P}_d$ be the set of all the subsets of $\{1, \dots, d\}$ and $\mathcal{P}_d^\vee \subset \mathcal{P}_d$ consisting of $\alpha \in \mathcal{P}_d$ with $\#\alpha = d - 1$ ($\#\alpha$ denotes the number of elements in α). For $\alpha \in \mathcal{P}_d$, we define

$$\widehat{V}_{\boldsymbol{\kappa}}^{(\alpha)} := \bigcap_{i \in \alpha} \widehat{V}_{\boldsymbol{\kappa}}^{(i)}, \qquad V_{\boldsymbol{\kappa}}^{(\alpha)} := \bigcap_{i \in \alpha} V_{\boldsymbol{\kappa}}^{(i)}. \tag{2.9}$$

In what follows, the symbol $*$ denotes the set $\{1, \dots, d\}$ by convention, for example,

$$\widehat{V}_{\boldsymbol{\kappa}}^{(*)} := \widehat{V}_{\boldsymbol{\kappa}}^{(\{1,\dots,d\})} = \bigcap_{i=1}^{d} \widehat{V}_{\boldsymbol{\kappa}}^{(i)}.$$

As each $\widehat{V}_{\boldsymbol{\kappa}}^{(\alpha)}$ (resp. $V_{\boldsymbol{\kappa}}^{(\alpha)}$) is pseudoconvex, we have

$$\widehat{C}^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) = \varGamma(\widehat{V}_{\boldsymbol{\kappa}}^{(*)}; \mathscr{O}_{\widehat{X}}) \Big/ \sum_{\alpha \in \mathcal{P}_d^\vee} \varGamma(\widehat{V}_{\boldsymbol{\kappa}}^{(\alpha)}; \mathscr{O}_{\widehat{X}}),$$

$$C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) = \{u \in \widehat{C}^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa});\, \partial_\eta u = 0\},$$
$$H^d_{G_{\boldsymbol{\kappa}}\cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}};\mathscr{O}_X) = \Gamma(V^{(*)}_{\boldsymbol{\kappa}};\mathscr{O}_X)\Big/\sum_{\alpha\in\mathcal{P}^\vee_d}\Gamma(V^{(\alpha)}_{\boldsymbol{\kappa}};\mathscr{O}_X).$$

Since $\widehat{V}^{(\alpha)}_{\boldsymbol{\kappa}} \subset \pi_\eta^{-1}(V^{(\alpha)}_{\boldsymbol{\kappa}})$ holds, we can regard a holomorphic function φ on $V^{(\alpha)}_{\boldsymbol{\kappa}}$ as that on $\widehat{V}^{(\alpha)}_{\boldsymbol{\kappa}}$, and thus, we have the natural morphism $\Gamma(V^{(\alpha)}_{\boldsymbol{\kappa}};\mathscr{O}_X) \to \Gamma(\widehat{V}^{(\alpha)}_{\boldsymbol{\kappa}};\mathscr{O}_{\widehat{X}})$. This induces the canonical morphism between the Čech cohomology groups

$$\begin{aligned} H^d_{G_{\boldsymbol{\kappa}}\cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}};\mathscr{O}_X) &= \frac{\Gamma(V^{(*)}_{\boldsymbol{\kappa}};\mathscr{O}_X)}{\sum_{\alpha\in\mathcal{P}^\vee_d}\Gamma(V^{(\alpha)}_{\boldsymbol{\kappa}};\mathscr{O}_X)} \\ &\to \left\{u \in \frac{\Gamma(\widehat{V}^{(*)}_{\boldsymbol{\kappa}};\mathscr{O}_{\widehat{X}})}{\sum_{\alpha\in\mathcal{P}^\vee_d}\Gamma(\widehat{V}^{(\alpha)}_{\boldsymbol{\kappa}};\mathscr{O}_{\widehat{X}})};\, \partial_\eta u = 0\right\} = C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}). \end{aligned}$$

Clearly this morphism coincides with (2.8), and hence it gives an isomorphism by Corollary 2.8.

3. Cohomological representation of $\mathscr{E}^{\mathbb{R}}_X$ with an apparent parameter.

We inherit the same notation from the previous section. Set $X^2 := X \times X$ with the coordinates (z, w), and let (z, w, η) be coordinates of $\widehat{X}^2 := X^2 \times \mathbb{C}$. Let $\Delta \subset X^2$ be the diagonal set. We identify X with Δ, and

$$T^*X = \{(z;\zeta)\} \simeq \{(z, z;\zeta, -\zeta)\} = T^*_\Delta X^2.$$

Let $\mathscr{E}^{\mathbb{R}}_X$ denote the sheaf of pseudodifferential operators on the cotangent bundle T^*X of X, and $z^*_0 = (z_0;\zeta_0) \in T^*X$ with $|\zeta_0| = 1$. Set

$$f_{\Delta,1}(z, w) := \langle z - w, \zeta_0\rangle, \qquad f'_\Delta(z, w) := z - w - \langle z - w, \zeta_0\rangle\overline{\zeta}_0.$$

See also Appendix B for a generalization of the mappings above and the following arguments on a complex manifold. For a closed convex cone $L \subset \mathbb{C}$, set

$$G_{\Delta,\varrho,L} := \{(z, w) \in X^2;\, \varrho^2|f'_\Delta(z, w)| \le |f_{\Delta,1}(z, w)|,\, f_{\Delta,1}(z, w) \in L\}.$$

Then it follows from the definition of $\mathscr{E}^{\mathbb{R}}_X$ that we have

$$\mathscr{E}^{\mathbb{R}}_{X,z^*_0} = \varinjlim_{\varrho,L,U} H^n_{G_{\Delta,\varrho,L}\cap U}(U;\mathscr{O}^{(0,n)}_{X^2}).$$

Here $\mathscr{O}^{(0,n)}_{X^2}$ is the sheaf of holomorphic n-forms with respect to $dw_1, \dots, dw_n$, $U \subset X^2$ and $L \subset \mathbb{C}$ range through open neighborhoods of (z_0, z_0) and closed convex cones in $\mathbb{C}$ with $L \subset \{\tau \in \mathbb{C};\, \operatorname{Re}\tau > 0\} \cup \{0\}$ respectively.

Now we introduce the corresponding cohomology group with an apparent parameter. Set, for an open subset $U \subset X^2$ and a closed convex cone $L \subset \mathbb{C}$,

$$\begin{aligned}\widehat{U}_{\Delta,\varrho,r,\theta} &:= \{(z,w,\eta) \in U \times S_{r,\theta};\ |f_{\Delta,1}(z,w)| < \varrho|\eta|\},\\ \widehat{G}_{\Delta,\varrho,L} &:= \{(z,w,\eta) \in \widehat{X}^2;\ \varrho|f'_{\Delta}(z,w)| \le |\eta|,\ f_{\Delta,1}(z,w) \in L\}.\end{aligned}$$

Definition 3.1. We set

$$\widehat{E}^{\mathbb{R}}_{X,z_0^*} := \varinjlim_{\varrho,r,\theta,L,U} H^n_{\widehat{G}_{\Delta,\varrho,L}\cap \widehat{U}_{\Delta,\varrho,r,\theta}}(\widehat{U}_{\Delta,\varrho,r,\theta};\mathscr{O}^{(0,n,0)}_{\widehat{X}^2}).$$

Here $\mathscr{O}^{(0,n,0)}_{\widehat{X}^2}$ is the sheaf of holomorphic n-forms with respect to $dw_1,\dots,dw_n$, $U \subset X^2$ and $L \subset \mathbb{C}$ range through open neighborhoods of (z_0,z_0) and closed convex cones in $\mathbb{C}$ with $L \subset \{\tau \in \mathbb{C};\ \operatorname{Re}\tau > 0\} \cup \{0\}$ respectively. Further we define

$$E^{\mathbb{R}}_{X,z_0^*} := \operatorname{Ker}(\partial_\eta\colon \widehat{E}^{\mathbb{R}}_{X,z_0^*} \to \widehat{E}^{\mathbb{R}}_{X,z_0^*}).$$

From the consequence of the previous section, the following theorem immediately follows.

Theorem 3.2. *There exists the canonical isomorphism*

$$\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \xrightarrow{\sim} E^{\mathbb{R}}_{X,z_0^*}.$$

We assume $z_0^* = (z_0;\zeta_0) = (0;1,0,\dots,0)$ in what follows and consider a Čech representation of $E^{\mathbb{R}}_{X,z_0^*}$. Let $\boldsymbol{\kappa} = (r,r',\varrho,\theta) \in \mathbb{R}^4$ be parameters satisfying the conditions (2.3). Then we define

$$\begin{aligned}\widehat{U}_{\Delta,\boldsymbol{\kappa}} &:= \bigcap_{i=2}^{n}\{(z,w,\eta) \in \widehat{X}^2;\ \|z\| < r',\ \eta \in S_{\boldsymbol{\kappa}},\ |z_1 - w_1| < \varrho|\eta|,\ |z_i - w_i| < r'\},\\ \widehat{G}_{\Delta,\boldsymbol{\kappa}} &:= \bigcap_{i=2}^{n}\Big\{(z,w,\eta) \in \widehat{X}^2;\ |\arg(z_1 - w_1)| \le \frac{\pi}{2} - \theta,\ \varrho|z_i - w_i| \le |\eta|\Big\}.\end{aligned}$$

We also set

$$\begin{aligned}U_{\Delta,\boldsymbol{\kappa}} &:= \bigcap_{i=2}^{n}\{(z,w) \in X^2;\ \|z\| < r',\ |z_1 - w_1| < \varrho r,\ |z_i - w_i| < r'\},\\ G_{\Delta,\boldsymbol{\kappa}} &:= \bigcap_{i=2}^{n}\Big\{(z,w) \in X^2;\ |\arg(z_1 - w_1)| \le \frac{\pi}{2} - \theta,\ \varrho^2|z_i - w_i| \le |z_1 - w_1|\Big\}.\end{aligned}$$

Definition 3.3. We define

$$\begin{aligned}\widehat{E}^{\mathbb{R}}_{X}(\boldsymbol{\kappa}) &:= H^n_{\widehat{G}_{\Delta,\boldsymbol{\kappa}}\cap \widehat{U}_{\Delta,\boldsymbol{\kappa}}}(\widehat{U}_{\Delta,\boldsymbol{\kappa}};\mathscr{O}^{(0,n,0)}_{\widehat{X}^2}),\\ E^{\mathbb{R}}_{X}(\boldsymbol{\kappa}) &:= \operatorname{Ker}(\partial_\eta\colon \widehat{E}^{\mathbb{R}}_{X}(\boldsymbol{\kappa}) \to \widehat{E}^{\mathbb{R}}_{X}(\boldsymbol{\kappa})).\end{aligned}$$

Note that

$$\widehat{E}^{\mathbb{R}}_{X,z_0^*} = \varinjlim_{\boldsymbol{\kappa}} \widehat{E}^{\mathbb{R}}_{X}(\boldsymbol{\kappa}) \quad\text{and}\quad E^{\mathbb{R}}_{X,z_0^*} = \varinjlim_{\boldsymbol{\kappa}} E^{\mathbb{R}}_{X}(\boldsymbol{\kappa})$$

hold. Then by employing the coordinates transformation $(z, w) \mapsto (z, z - w)$, it follows from Proposition 2.5, Corollaries 2.7 and 2.8 that the both complexes

$$\boldsymbol{R}\Gamma_{\widehat{G}_{\Delta,\boldsymbol{\kappa}}\cap\widehat{U}_{\Delta,\boldsymbol{\kappa}}}(\widehat{U}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2}),$$
$$\boldsymbol{R}\Gamma_{G_{\Delta,\boldsymbol{\kappa}}\cap U_{\Delta,\boldsymbol{\kappa}}}(U_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n)}_{X^2}) \simeq \boldsymbol{R}\Gamma_{\widehat{G}_{\Delta,\boldsymbol{\kappa}}\cap\widehat{U}_{\Delta,\boldsymbol{\kappa}}}(\widehat{U}_{\Delta,\boldsymbol{\kappa}}; \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_{\widehat{X}^2}}(\mathscr{D}_{\widehat{X}^2}/\mathscr{D}_{\widehat{X}^2}\partial_\eta, \mathscr{O}^{(0,n,0)}_{\widehat{X}^2}))$$

are concentrated in degree n, and we have the canonical isomorphism

$$H^n_{G_{\Delta,\boldsymbol{\kappa}}\cap U_{\Delta,\boldsymbol{\kappa}}}(U_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n)}_{X^2}) \xrightarrow{\sim} E^{\mathbb{R}}_X(\boldsymbol{\kappa}).$$

Furthermore we have

$$\mathscr{E}^{\mathbb{R}}_{X,z_0^*} = \varinjlim_{\boldsymbol{\kappa}} H^n_{G_{\Delta,\boldsymbol{\kappa}}\cap U_{\Delta,\boldsymbol{\kappa}}}(U_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n)}_{X^2}).$$

By these facts, we get

$$\mathscr{E}^{\mathbb{R}}_{X,z_0^*} = \varinjlim_{\boldsymbol{\kappa}} E^{\mathbb{R}}_X(\boldsymbol{\kappa}).$$

Now we give the Čech representations of these cohomology groups. Recall that the open subset $\widehat{U}_{\Delta,\boldsymbol{\kappa}} \subset \widehat{X}^2$ is defined by

$$\bigcap_{i=2}^{n}\{(z, w, \eta) \in X^2 \times S_{\boldsymbol{\kappa}};\ \|z\| < r',\ |z_1 - w_1| < \varrho|\eta|,\ |z_i - w_i| < r'\}.$$

Here the open sector $S_{\boldsymbol{\kappa}}$ was given by (2.4). Set

$$\widehat{V}^{(1)}_{\Delta,\boldsymbol{\kappa}} := \left\{(z, w, \eta) \in \widehat{U}_{\Delta,\boldsymbol{\kappa}};\ \frac{\pi}{2} - \theta < \arg(z_1 - w_1) < \frac{3\pi}{2} + \theta\right\},$$
$$\widehat{V}^{(i)}_{\Delta,\boldsymbol{\kappa}} := \{(z, w, \eta) \in \widehat{U}_{\Delta,\boldsymbol{\kappa}};\ \varrho|z_i - w_i| > |\eta|\} \quad (2 \le i \le n).$$

We also set

$$V^{(1)}_{\Delta,\boldsymbol{\kappa}} := \left\{(z, w) \in U_{\Delta,\boldsymbol{\kappa}};\ \frac{\pi}{2} - \theta < \arg(z_1 - w_1) < \frac{3\pi}{2} + \theta\right\},$$
$$V^{(i)}_{\Delta,\boldsymbol{\kappa}} := \{(z,\ w) \in U_{\Delta,\boldsymbol{\kappa}};\ \varrho^2|z_i - w_i| > |z_1 - w_1|\} \quad (2 \le i \le n).$$

For any $\alpha \in \mathcal{P}_n$, the subset $\widehat{V}^{(\alpha)}_{\Delta,\boldsymbol{\kappa}}$, $V^{(\alpha)}_{\Delta,\boldsymbol{\kappa}}$ etc. are defined in the same way as those in (2.9). Then, using these coverings, we have

$$\widehat{E}^{\mathbb{R}}_X(\boldsymbol{\kappa}) = \Gamma(\widehat{V}^{(*)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2}) \Big/ \sum_{\alpha\in\mathcal{P}_n^\vee} \Gamma(\widehat{V}^{(\alpha)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2}),$$
$$E^{\mathbb{R}}_X(\boldsymbol{\kappa}) = \{K \in \widehat{E}^{\mathbb{R}}_X(\boldsymbol{\kappa});\ \partial_\eta K = 0\},$$

$$H^n_{G_{\Delta,\boldsymbol{\kappa}} \cap U_{\Delta,\boldsymbol{\kappa}}}(U_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n)}_{X^2}) = \mathit{\Gamma}(V^{(*)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n)}_{X^2}) \Big/ \sum_{\alpha \in \mathcal{P}^\vee_n} \mathit{\Gamma}(V^{(\alpha)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n)}_{X^2}).$$

Let us take any $K(z,w)\,dw = [\psi(z,w,\eta)\,dw] \in E^{\mathbb{R}}_X(\boldsymbol{\kappa})$ and $f(z) = [u(z,\eta)] \in C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa})$ with representatives $\psi(z,w,\eta)\,dw \in \mathit{\Gamma}(\widehat{V}^{(*)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2})$ and $u(z,\eta) \in \mathit{\Gamma}(\widehat{V}^{(*)}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}})$ respectively, which were introduced in the previous section. We will define the action μ_K on $C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa})$ associated with the kernel $K(z,w)\,dw$. For that purpose, we first introduce the paths of the integration related to μ_K. Let $(z,\eta) \in \widehat{X}$. Set $\beta_0 := (\varrho/2)\,e^{-\sqrt{-1}(\pi+\theta)/2}$ and $\beta_1 := (\varrho/2)\,e^{\sqrt{-1}(\pi+\theta)/2}$, and we define, for a sufficiently small $\varepsilon > 0$, the path $\gamma_1(z,\eta;\varrho,\theta)$ in $\mathbb{C}_{w_1}$ by

$$\{w_1 = z_1 + t\beta_0\eta;\ 1 \geq t \geq \varepsilon\} \vee \Big\{w_1 = z_1 + \frac{\varepsilon\varrho\eta}{2}\,e^{-\sqrt{-1}(\pi+\theta)t/2};\ -1 \leq t \leq 1\Big\}$$
$$\vee\ \{w_1 = z_1 + t\beta_1\eta;\ \varepsilon \leq t \leq 1\}.$$

Note that $\gamma_1(z,\eta;\varrho,\theta)$ joins the two points $z_1 + \beta_0\eta$ and $z_1 + \beta_1\eta$, which depend on the variables z_1 and η holomorphically. We introduce another path $\overline{\gamma}_1(z,\eta;\varrho,\theta)$ in $\mathbb{C}_{w_1}$ by the straight segment from $z_1 + \beta_0\eta$ to $z_1 + \beta_1\eta$. We also define the path $\gamma_i(z,\eta;\varrho)$ in $\mathbb{C}_{w_i}$ $(i = 2,\dots,n)$ by the circle with center at z_i and radius $|\eta|/\varrho + \varepsilon$, i.e.

$$\gamma_i(z,\eta;\varrho) := \Big\{w_i = z_i + \Big(\frac{|\eta|}{\varrho} + \varepsilon\Big)e^{2\pi\sqrt{-1}\,t};\ 0 \leq t \leq 1\Big\}.$$

Define the real n-dimensional chain in X made from these paths by

$$\gamma(z,\eta;\varrho,\theta) := \gamma_1(z,\eta;\varrho,\theta) \times \gamma_2(z,\eta;\varrho) \times \cdots \times \gamma_n(z,\eta;\varrho) \subset X, \tag{3.1}$$
$$\overline{\gamma}(z,\eta;\varrho,\theta) := \overline{\gamma}_1(z,\eta;\varrho,\theta) \times \gamma_2(z,\eta;\varrho) \times \cdots \times \gamma_n(z,\eta;\varrho) \subset X.$$

Let $\widehat{\pi}_2\colon \widehat{X}^2 \ni (z,w,\eta) \mapsto (w,\eta) \in \widehat{X}$ be the canonical projection. For $\alpha \in \mathcal{P}_n$ and $\beta \in \mathcal{P}_d$, we set

$$\widehat{W}^{(\alpha,\,\beta)}_{\boldsymbol{\kappa}} := \widehat{V}^{(\alpha)}_{\Delta,\boldsymbol{\kappa}} \cap \widehat{\pi}_2^{-1}(\widehat{V}^{(\beta)}_{\boldsymbol{\kappa}}),$$
$$\widehat{W}^{(*,*)}_{\boldsymbol{\kappa}} := \widehat{V}^{(*)}_{\Delta,\boldsymbol{\kappa}} \cap \widehat{\pi}_2^{-1}(\widehat{V}^{(*)}_{\boldsymbol{\kappa}}).$$

We also set $\widehat{W}^{(\alpha,*)}_{\boldsymbol{\kappa}} := \widehat{W}^{(\alpha,\{1,\dots,d\})}_{\boldsymbol{\kappa}}$ and $\widehat{W}^{(*,\,\beta)}_{\boldsymbol{\kappa}} := \widehat{W}^{(\{1,\dots,n\},\,\beta)}_{\boldsymbol{\kappa}}$. Then the following lemma is easily obtained by elementary computations.

Lemma 3.4. *Let $\widetilde{\boldsymbol{\kappa}} = (\tilde{r}, \tilde{r}', \tilde{\varrho}, \tilde{\theta}) \in \mathbb{R}^4$ satisfy*

$$0 < \tilde{r} < r, \quad 0 < \tilde{r}' < \frac{r'}{2}, \quad 0 < \tilde{\theta} < \frac{\theta}{4}, \quad 0 < \tilde{\varrho} < \frac{\varrho}{2}\sin\frac{\theta}{4},$$

and the conditions corresponding to (2.3). *Then the following hold for sufficiently small* $\varepsilon > 0$:

(1) *For any $(z,\eta) \in \widehat{V}^{(*)}_{\widetilde{\boldsymbol{\kappa}}}$, in $\widehat{X}^2$*

$$\{z\} \times \gamma(z, \eta; \varrho, \theta) \times \{\eta\} \subset \widehat{W}_{\boldsymbol{\kappa}}^{(*,*)}.$$

Here $\{z\} \times \gamma(z, \eta; \varrho, \theta) \times \{\eta\}$ *denotes the product of these three subsets in* $\widehat{X}^2 = X \times X \times \mathbb{C}$.

(2) *For any* $(z, \eta) \in \widehat{V}_{\widetilde{\boldsymbol{\kappa}}}^{(\beta)}$ *with* $\beta \in \mathcal{P}_d^{\vee}$,

$$\{z\} \times \gamma(z, \eta; \varrho, \theta) \times \{\eta\} \subset \widehat{W}_{\boldsymbol{\kappa}}^{(*,\beta)}.$$

(3) *For any* $(z, \eta) \in \widehat{V}_{\widetilde{\boldsymbol{\kappa}}}^{(\{2,\ldots,d\})}$,

$$\{z\} \times \overline{\gamma}(z, \eta; \varrho, \theta) \times \{\eta\} \subset \widehat{W}_{\boldsymbol{\kappa}}^{(\{2,\ldots,n\},*)}.$$

Furthermore

$$\{z\} \times \partial\gamma(z, \eta; \varrho, \theta) \times \{\eta\} \subset \widehat{W}_{\boldsymbol{\kappa}}^{(*,*)},$$

where $\partial\gamma(z, \eta; \varrho, \theta)$ *denotes the boundary of* $\gamma(z, \eta; \varrho, \theta)$ *as a real* n*-dimensional chain.*

Now we are ready to define the action μ_K of $K(z, w)\,dw \in E_X^{\mathbb{R}}(\boldsymbol{\kappa})$ on $C_{Y|X}^{\mathbb{R}}(\boldsymbol{\kappa})$.

Theorem 3.5. *The bi-linear morphism*

$$\mu\colon E_X^{\mathbb{R}}(\boldsymbol{\kappa}) \underset{\mathbb{C}}{\otimes} C_{Y|X}^{\mathbb{R}}(\boldsymbol{\kappa}) \to C_{Y|X}^{\mathbb{R}}(\widetilde{\boldsymbol{\kappa}})$$

defined by

$$\begin{aligned} K(z, w)\,dw \otimes f(z) &= [\psi(z, w, \eta)\,dw] \otimes [u(z, \eta)] \\ \mapsto \mu(K dw \otimes f) &:= \left[\int_{\gamma(z,\eta;\varrho,\theta)} \psi(z, w, \eta)\, u(w, \eta)\, dw \right] \end{aligned}$$

is well-defined. Here $\widetilde{\boldsymbol{\kappa}}$ *is a* 4*-tuple of positive constants satisfying the conditions given in Lemma* 3.4*. In particular, there exists the following linear morphism*:

$$\mu_K\colon C_{Y|X}^{\mathbb{R}}(\boldsymbol{\kappa}) \ni f(z) \mapsto \mu(K dw \otimes f) \in C_{Y|X}^{\mathbb{R}}(\widetilde{\boldsymbol{\kappa}}).$$

Remark 3.6. The same result holds for $\psi(z, w, \tau, \eta)dw$ and $u(w, \tau, \eta)$ with additional holomorphic parameters τ.

Proof of Theorem 3.5. For any $\varphi(z, w, \eta) \in \Gamma(\widehat{W}_{\boldsymbol{\kappa}}^{(*,*)}; \mathscr{O}_{\widehat{X}^2})$, set

$$\mu(\varphi)(z, \eta) := \int_{\gamma(z,\eta;\varrho,\theta)} \varphi(z, w, \eta)\, dw.$$

Note that, by Lemma 3.4 (1) we have $\mu(\varphi)(z, \eta) \in \Gamma(\widehat{V}_{\widetilde{\boldsymbol{\kappa}}}^{(*)}; \mathscr{O}_{\widehat{X}})$. Recall that $\mathcal{P}_n^{\vee}$ denotes the subset of $\mathcal{P}_n$ consisting $\alpha \in \mathcal{P}_n$ with $|\alpha| = n - 1$.

LEMMA 3.7. *Assume that* $\varphi(z,w,\eta) \in \Gamma(\widehat{W}_{\boldsymbol{\kappa}}^{(\alpha,\,\beta)}; \mathscr{O}_{\widehat{X}^2})$ *with* $\alpha \in \mathcal{P}_n^{\vee}$ *and* $\beta = *$ *or with* $\alpha = *$ *and* $\beta \in \mathcal{P}_d^{\vee}$. *Then* $\mu(\varphi)(z,\eta) \in \Gamma(\widehat{V}_{\widetilde{\boldsymbol{\kappa}}}^{(\beta)}; \mathscr{O}_{\widehat{X}})$ *for some* $\beta \in \mathcal{P}_d^{\vee}$.

PROOF. If $\varphi(z,w,\eta) \in \Gamma(\widehat{W}_{\boldsymbol{\kappa}}^{(\alpha,*)}; \mathscr{O}_{\widehat{X}^2})$ for some $\alpha \in \mathcal{P}_n^{\vee}$, we have

$$\begin{cases} \mu(\varphi)(z,\eta) \in \Gamma(\widehat{V}_{\widetilde{\boldsymbol{\kappa}}}^{(\{2,\dots,d\})}; \mathscr{O}_{\widehat{X}}) & (\alpha = \{2,\dots,n\}), \\ \mu(\varphi)(z,\eta) = 0 & (\text{otherwise}). \end{cases}$$

Here we remark that the first fact comes from Lemma 3.4 (3) by deforming the path of integration to $\overline{\gamma}(z,\eta;\varrho,\theta)$. In the same way, by Lemma 3.4 (2), it follows that if $\varphi(z,w,\eta) \in \Gamma(\widehat{W}_{\boldsymbol{\kappa}}^{(*,\,\beta)}; \mathscr{O}_{\widehat{X}^2})$ for some $\beta \in \mathcal{P}_d^{\vee}$, then

$$\mu(\varphi)(z,\eta) \in \Gamma(\widehat{V}_{\widetilde{\boldsymbol{\kappa}}}^{(\beta)}; \mathscr{O}_{\widehat{X}}). \qquad \square$$

It follows from Lemma 3.7 that μ induces the canonical morphism

$$\mu\colon \frac{\Gamma(\widehat{W}_{\boldsymbol{\kappa}}^{(*,*)}; \mathscr{O}_{\widehat{X}^2}^{(0,n,0)})}{\displaystyle\sum_{(\alpha,\,\beta)\in\Lambda} \Gamma(\widehat{W}_{\boldsymbol{\kappa}}^{(\alpha,\,\beta)}; \mathscr{O}_{\widehat{X}^2}^{(0,n,0)})} \to \frac{\Gamma(\widehat{V}_{\widetilde{\boldsymbol{\kappa}}}^{(*)}; \mathscr{O}_{\widehat{X}})}{\displaystyle\sum_{\beta\in\mathcal{P}_d^{\vee}} \Gamma(\widehat{V}_{\widetilde{\boldsymbol{\kappa}}}^{(\beta)}; \mathscr{O}_{\widehat{X}})} = \widehat{C}_{Y|X}^{\mathbb{R}}(\widetilde{\boldsymbol{\kappa}})$$

where $\Lambda := \{(\alpha,*);\, \alpha \in \mathcal{P}_n^{\vee}\} \sqcup \{(*,\,\beta);\, \beta \in \mathcal{P}_d^{\vee}\}$. Furthermore, we have the canonical morphism

$$E_X^{\mathbb{R}}(\boldsymbol{\kappa}) \underset{\mathbb{C}}{\otimes} C_{Y|X}^{\mathbb{R}}(\boldsymbol{\kappa}) \to \frac{\Gamma(\widehat{W}_{\boldsymbol{\kappa}}^{(*,*)}; \mathscr{O}_{\widehat{X}^2}^{(0,n,0)})}{\displaystyle\sum_{(\alpha,\,\beta)\in\Lambda} \Gamma(\widehat{W}_{\boldsymbol{\kappa}}^{(\alpha,\,\beta)}; \mathscr{O}_{\widehat{X}^2}^{(0,n,0)})}$$

by $[\psi(z,w,\eta)\,dw] \otimes [u(z,\eta)] \mapsto [\psi(z,w,\eta)\,u(w,\eta)\,dw]$. Hence we have obtained the morphism

$$\begin{aligned} \mu\colon E_X^{\mathbb{R}}(\boldsymbol{\kappa}) \underset{\mathbb{C}}{\otimes} C_{Y|X}^{\mathbb{R}}(\boldsymbol{\kappa}) &\ni [\psi(z,w,\eta)\,dw] \otimes [u(z,\eta)] \\ &\mapsto \left[\int\limits_{\gamma(z,\eta;\varrho,\theta)} \psi(z,w,\eta)\,u(w,\eta)\,dw\right] \in \widehat{C}_{Y|X}^{\mathbb{R}}(\widetilde{\boldsymbol{\kappa}}). \end{aligned}$$

Thus to complete the proof, it suffices to show the image of μ is contained in $C_{Y|X}^{\mathbb{R}}(\widetilde{\boldsymbol{\kappa}})$. We have

$$\begin{aligned} \partial_\eta \int\limits_{\gamma(z,\eta;\varrho,\theta)} \psi(z,w,\eta)\,u(w,\eta)\,dw &= \int\limits_{\gamma_2(z,\eta;\varrho)\times\cdots\times\gamma_n(z,\eta;\varrho)} \big[\tau\psi(z,z_1+\tau\eta,w',\eta)\,u(z_1+\tau\eta,w',\eta)\big]_{\tau=\beta_0}^{\beta_1} dw_2\cdots dw_n \\ &\quad + \int\limits_{\gamma(z,\eta;\varrho,\theta)} \partial_\eta\psi(z,w,\eta)\,u(w,\eta)\,dw + \int\limits_{\gamma(z,\eta;\varrho,\theta)} \psi(z,w,\eta)\,\partial_\eta u(w,\eta)\,dw. \end{aligned}$$

By Lemma 3.4 (3), the first term belongs to $\Gamma(\widehat{V}_{\widetilde{\boldsymbol{\kappa}}}^{(2,\dots,d)}; \mathscr{O}_{\widehat{X}})$. For the second and third terms, as each integrand belongs to $\sum_{(\alpha,\beta)\in\Lambda}\Gamma(\widehat{W}^{(\alpha,\beta)}_{\boldsymbol{\kappa}};\mathscr{O}_{\widehat{X}^2})$, the corresponding integral also belongs to $\sum_{\beta\in\mathcal{P}^\vee_d}\Gamma(\widehat{V}^{(\beta)}_{\widetilde{\boldsymbol{\kappa}}};\mathscr{O}_{\widehat{X}})$. Hence we have obtained $(\partial/\partial\eta)\,\mu([\psi dw]\otimes[u]) = 0 \in \widehat{C}^{\mathbb{R}}_{Y|X}(\widetilde{\boldsymbol{\kappa}})$, which implies $\mu([\psi dw]\otimes[u]) \in C^{\mathbb{R}}_{Y|X}(\widetilde{\boldsymbol{\kappa}})$. The proof is complete. □

As a corollary of the theorem, we have the result on the composition on $E^{\mathbb{R}}_X(\boldsymbol{\kappa})$.

Corollary 3.8. *Let* $\widetilde{\boldsymbol{\kappa}} = (\tilde{r},\tilde{r}',\tilde{\varrho},\tilde{\theta})\in\mathbb{R}^4$ *satisfy*

$$0<\tilde{r}<r,\quad 0<\tilde{r}'<\frac{r'}{8},\quad 0<\tilde{\theta}<\frac{\theta}{4},\quad 0<\tilde{\varrho}<\frac{\varrho}{2}\sin\frac{\theta}{4},$$

and the conditions corresponding to (2.3)*. Then there exists the bi-linear morphism*

$$\mu\colon E^{\mathbb{R}}_X(\boldsymbol{\kappa})\underset{\mathbb{C}}{\otimes}E^{\mathbb{R}}_X(\boldsymbol{\kappa})\to E^{\mathbb{R}}_X(\widetilde{\boldsymbol{\kappa}})$$

defined by

$$[\psi_1(z,w,\eta)\,dw]\otimes[\psi_2(z,w,\eta)\,dw]\mapsto[\mu(\psi_1\otimes\psi_2)(z,w,\eta)\,dw],$$

where

$$\mu(\psi_1\otimes\psi_2)(z,w,\eta):=\int_{\gamma(z,\eta;\varrho,\theta)}\psi_1(z,\widetilde{w},\eta)\,\psi_2(\widetilde{w},w,\eta)\,d\widetilde{w}.$$

Proof. By employing the coordinates transformation $z=z'+w$, $\widetilde{w}=\widetilde{w}'+w$ and $w=w$, the integration above becomes

$$\int_{\gamma(z',\eta;\varrho,\theta)}\psi_1(z'+w,\widetilde{w}'+w,\eta)\,\psi_2(\widetilde{w}'+w,w,\eta)\,d\widetilde{w}'.$$

Then, under the new coordinates $(z',\widetilde{w}',w)$, the ψ_2 (resp. the result of the integration) can be regarded as a holomorphic microfunction along $\{\widetilde{w}'=0\}$ (resp. $\{z'=0\}$). Hence, by noticing the simple fact that $|w_i|<r'/2$ and $|\widetilde{w}'_i|<r'/2$ imply $|\widetilde{w}_i|<r'$, we can easily obtain the result by the theorem. □

The following theorem can be shown by the same arguments as in Kashiwara–Kawai [**16**]. We give the detailed proof in Appendix A for the reader's convenience. See also Theorem B.8 for the corresponding claim at an arbitrary point in T^*X.

Theorem 3.9. *The action*

$$\mathscr{E}^{\mathbb{R}}_{X,z_0^*}\underset{\mathbb{C}}{\otimes}\mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*}=\varinjlim_{\boldsymbol{\kappa}}(E^{\mathbb{R}}_X(\boldsymbol{\kappa})\underset{\mathbb{C}}{\otimes}C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}))\xrightarrow{\mu}\varinjlim_{\boldsymbol{\kappa}}C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa})=\mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*}$$

coincides with the cohomological action of $\mathscr{E}^{\mathbb{R}}_{X,z_0^*}$ *on* $\mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*}$.

As an immediate corollary, we have:

Corollary 3.10. *The multiplication of the ring $\mathscr{E}^{\mathbb{R}}_{X,z_0^*}$ coincides with the composition defined by*

$$\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \underset{\mathbb{C}}{\otimes} \mathscr{E}^{\mathbb{R}}_{X,z_0^*} = \varinjlim_{\boldsymbol{\kappa}} (E^{\mathbb{R}}_X(\boldsymbol{\kappa}) \underset{\mathbb{C}}{\otimes} E^{\mathbb{R}}_X(\boldsymbol{\kappa})) \xrightarrow{\mu} \varinjlim_{\boldsymbol{\kappa}} E^{\mathbb{R}}_X(\boldsymbol{\kappa}) = \mathscr{E}^{\mathbb{R}}_{X,z_0^*}.$$

4. Symbols with an apparent parameter.

Let $X := \mathbb{C}^n$ and consider $T^*X \simeq X \times \mathbb{C}^n = \{(z;\zeta)\}$. Let $\pi\colon T^*X \to X$ be the canonical projection. If $V \subset T^*X$ is a conic set and $d > 0$, we set

$$V[d] := \{(z;\zeta) \in V;\ \|\zeta\| \geq d\}.$$

For any open conic subset $\Omega \subset T^*X$ and $\rho \geq 0$, we set

$$\Omega_\rho := \mathrm{Cl}\Big[\bigcup_{(z,\zeta)\in\Omega} \{(z+z';\zeta+\zeta') \in \mathbb{C}^{2n};\ \|z'\| \leq \rho,\ \|\zeta'\| \leq \rho\|\zeta\|\}\Big].$$

Here Cl means the closure. In particular, $\Omega_0 = \mathrm{Cl}\,\Omega$. For any $d > 0$ and $\rho \in [0,1[$, we set for short:

$$d_\rho := d(1-\rho).$$

Let U, V be conic subsets of T^*X. Then we write $V \underset{\text{conic}}{\Subset} U$ if V is generated by a compact subset of U. We recall the definition of symbols of $\mathscr{E}^{\mathbb{R}}_X$:

Definition 4.1 (see [**2**], [**6**]). Let $\Omega \underset{\text{conic}}{\Subset} T^*X$ be an open conic subset.

(1) We call $P(z,\zeta)$ a *symbol* on Ω if there exist $d > 0$ and $\rho \in]0,1[$ such that $P(z,\zeta) \in \Gamma(\Omega_\rho[d_\rho];\mathscr{O}_{T^*X})$, and for any $h > 0$ there exists $C_h > 0$ such that

$$|P(z,\zeta)| \leq C_h e^{h\|\zeta\|} \quad ((z;\zeta) \in \Omega_\rho[d_\rho]).$$

We denote by $\mathscr{S}(\Omega)$ the set of symbols on Ω.

(2) We call $P(z,\zeta)$ a *null-symbol* on Ω if there exist $d > 0$ and $\rho \in]0,1[$ such that $P(z,\zeta) \in \Gamma(\Omega_\rho[d_\rho];\mathscr{O}_{T^*X})$, and there exist C, $\delta > 0$ such that

$$|P(z,\zeta)| \leq Ce^{-\delta\|\zeta\|} \quad ((z;\zeta) \in \Omega_\rho[d_\rho]).$$

We denote by $\mathscr{N}(\Omega)$ the set of null-symbols on Ω.

(3) For any $z_0^* \in T^*X$, we set

$$\mathscr{S}_{z_0^*} := \varinjlim_{\Omega\ni z_0^*} \mathscr{S}(\Omega) \supset \mathscr{N}_{z_0^*} := \varinjlim_{\Omega\ni z_0^*} \mathscr{N}(\Omega)$$

where $\Omega \underset{\text{conic}}{\Subset} T^*X$ ranges through open conic neighborhoods of z_0^*.

Next, set for short

$$S := S_{\boldsymbol{\kappa}} \tag{4.1}$$

for some r, $\theta \in \left]0, 1/2\right[$ (recall (2.4)). In particular we always assume that $|\eta| < 1/2$ for any $\eta \in S$. For $Z \Subset S$, we set $m_Z := \min_{\eta\in Z} |\eta| > 0$.

DEFINITION 4.2. We define a set $\mathfrak{N}(\Omega; S)$ as follows: $P(z, \zeta, \eta) \in \mathfrak{N}(\Omega; S)$ if

(i) $P(z, \zeta, \eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$ for some $d > 0$ and $\rho \in \left]0, 1\right[$,

(ii) there exists $\delta > 0$ so that for any $Z \Subset S$, there exists a constant $C_Z > 0$ satisfying

$$|P(z, \zeta, \eta)| \le C_Z e^{-\delta\|\eta\zeta\|} \quad ((z; \zeta, \eta) \in \Omega_\rho[d_\rho] \times Z). \tag{4.2}$$

LEMMA 4.3. *If $P(z, \zeta, \eta) \in \mathfrak{N}(\Omega; S)$, it follows that $\partial_\eta P(z, \zeta, \eta) \in \mathfrak{N}(\Omega; S)$.*

PROOF. We assume that $P(z, \zeta, \eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$. For any $Z \Subset S$, we take $\delta' \in \left]0, \delta/2\right[$ as

$$Z' := \bigcup_{\eta\in Z} \{\eta' \in \mathbb{C};\ |\eta - \eta'| \le \delta'|\eta|\} \Subset S. \tag{4.3}$$

Then by the Cauchy inequality, there exists a constant $C_{Z'} > 0$ such that

$$|\partial_\eta P(z, \zeta, \eta)| \le \frac{1}{\delta'|\eta|} \sup_{|\eta-\eta'|=\delta'|\eta|} |P(z, \zeta, \eta')| \le \frac{C_{Z'} e^{-\delta\|\eta\zeta\|/2}}{\delta' m_Z} \quad ((z; \zeta, \eta) \in \Omega_\rho[d_\rho] \times Z). \quad \square$$

PROPOSITION 4.4. *Let $P(z, \zeta, \eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$. Assume that $\partial_\eta P(z, \zeta, \eta) \in \mathfrak{N}(\Omega; S)$.*

(1) *The following conditions are equivalent*:

(i) *there exists a constant $\upsilon > 0$ satisfying the following*: *for any $Z \Subset S$ there exists a constant $C_Z > 0$ such that*

$$|P(z, \zeta, \eta)| \le C_Z e^{\upsilon|\eta|\|\zeta\|} \quad ((z; \zeta, \eta) \in \Omega_\rho[d_\rho] \times Z). \tag{4.4}$$

(ii) *for any $h > 0$ and $Z \Subset S$ there exists constant $C_{h,Z} > 0$ such that*

$$|P(z, \zeta, \eta)| \le C_{h,Z} e^{h\|\zeta\|} \quad ((z; \zeta, \eta) \in \Omega_\rho[d_\rho] \times Z).$$

(2) *Assume that $P(z, \zeta, \eta)$ satisfies the equivalent conditions of* (1) (*resp. $P(z, \zeta, \eta) \in \mathfrak{N}(\Omega; S)$*). *Then for any $\eta_0 \in S$, it follows that $P(z, \zeta, \eta_0) \in \mathscr{S}(\Omega)$* (*resp. $P(z, \zeta, \eta_0) \in \mathscr{N}(\Omega)$*) *and further $P(z, \zeta, \eta) - P(z, \zeta, \eta_0) \in \mathfrak{N}(\Omega; S)$.*

PROOF. (1) (i) $\Longrightarrow$ (ii). For any $h > 0$, we choose $\eta_0 \in S \cap \mathbb{R}$ as $\upsilon\eta_0 < h$. Then there exists a constant $C_{\eta_0} > 0$ such that

$$|P(z, \zeta, \eta_0)| \le C_{\eta_0} e^{\upsilon\eta_0\|\zeta\|} \le C_{\eta_0} e^{h\|\zeta\|} \quad ((z; \zeta) \in \Omega_\rho[d_\rho]).$$

For any $Z \Subset S$, let $Z' \Subset S$ be the convex hull of $Z \cup \{\eta_0\}$. Since $\partial_\eta P(z,\zeta,\eta) \in \mathfrak{N}(\Omega;S)$, we can find $\delta > 0$ and a constant $C_{Z'} > 0$ such that for any $(z;\zeta,\eta) \in \Omega_\rho[d_\rho] \times Z \subset \Omega_\rho[d_\rho] \times Z'$ the following holds:

$$\begin{aligned}|P(z,\zeta,\eta)| &= \left|P(z,\zeta,\eta_0) + \int_{\eta_0}^{\eta} \partial_\eta P(z,\zeta,\tau)\,d\tau\right| \le C_{\eta_0} e^{h\|\zeta\|} + |\eta-\eta_0| C_{Z'} e^{-\delta m_{Z'}\|\zeta\|}\\ &\le (C_{\eta_0} + rC_{Z'}) e^{h\|\zeta\|}.\end{aligned}$$

(ii) $\Longrightarrow$ (i). For any $Z \Subset S$, we take $0 < h \le m_Z$. Then there exists $C_{h,Z} > 0$ such that

$$|P(z,\zeta,\eta)| \le C_{h,Z} e^{h\|\zeta\|} \le C_{h,Z} e^{|\eta|\|\zeta\|} \quad ((z;\zeta,\eta) \in \Omega_\rho[d_\rho] \times Z).$$

(2) Taking $Z = \{\eta_0\}$, we see that $P(z,\zeta,\eta_0) \in \mathscr{S}(\Omega)$ by (1). Set $\delta_0 := \delta|\eta_0|$. As in the proof of (i) $\Longrightarrow$ (ii) in (1), we see that for any $(z;\zeta,\eta) \in \Omega_\rho[d_\rho] \times Z \subset \Omega_\rho[d_\rho] \times Z'$ the following holds: if $|\eta| \ge |\eta_0|$

$$|P(z,\zeta,\eta) - P(z,\zeta,\eta_0)| = \left|\int_{\eta_0}^{\eta} \partial_\eta P_\nu(z,\zeta,\tau)\,d\tau\right| \le |\eta-\eta_0| C_{h,Z'} e^{-\delta\|\eta_0\zeta\|} \le rC_{h,Z'} e^{-\delta_0\|\eta\zeta\|},$$

and if $|\eta| \le |\eta_0|$

$$|P(z,\zeta,\eta) - P(z,\zeta,\eta_0)| \le rC_{h,Z'} e^{-\delta\|\eta\zeta\|} \le rC_{h,Z'} e^{-\delta_0\|\eta\zeta\|}.$$

Hence $P(z,\zeta,\eta) - P(z,\zeta,\eta_0) \in \mathfrak{N}(\Omega;S)$. If $P(z,\zeta,\eta) \in \mathfrak{N}(\Omega;S)$, the proof is same. □

Definition 4.5. (1) We define a set $\mathfrak{S}(\Omega;S)$ as follows: $P(z,\zeta,\eta) \in \mathfrak{S}(\Omega;S)$ if

(i) $P(z,\zeta,\eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$ for some $d > 0$ and $\rho \in \left]0,1\right[$,

(ii) $\partial_\eta P(z,\zeta,\eta) \in \mathfrak{N}(\Omega;S)$,

(iii) $P(z,\zeta,\eta)$ satisfies the equivalent conditions of Proposition 4.4.

Note that $\mathfrak{N}(\Omega;S) \subset \mathfrak{S}(\Omega;S)$ holds by Lemma 4.3.

(2) For $z_0^* \in \dot{T}^*X$, we set

$$\mathfrak{S}_{z_0^*} := \varinjlim_{\Omega,S} \mathfrak{S}(\Omega;S) \supset \mathfrak{N}_{z_0^*} := \varinjlim_{\Omega,S} \mathfrak{N}(\Omega;S).$$

Here $\Omega \underset{\text{conic}}{\Subset} T^*X$ ranges through open conic neighborhoods of z_0^*, and the inductive limits with respect to S are taken by r, $\theta \to 0$ in (4.1).

We call each element of $\mathfrak{S}(\Omega;S)$ (resp. $\mathfrak{N}(\Omega;S)$) a *symbol* (resp. *null-symbol*) *on* Ω *with an apparent parameter in* S. It is easy to see that $\mathfrak{S}(\Omega;S)$ is a $\mathbb{C}$-algebra under the ordinary operations of functions, and $\mathfrak{N}(\Omega;S)$ is a subalgebra. By definition, we can regard that

$$\mathscr{S}(\Omega) = \{P(z,\zeta,\eta) \in \mathfrak{S}(\Omega;S);\ \partial_\eta P(z,\zeta,\eta) = 0\} \subset \mathfrak{S}(\Omega;S),$$

$$\mathscr{N}(\Omega) = \mathscr{S}(\Omega) \cap \mathfrak{N}(\Omega; S) \subset \mathfrak{N}(\Omega; S).$$

Hence we have an injective mapping $\mathscr{S}(\Omega)/\mathscr{N}(\Omega) \hookrightarrow \mathfrak{S}(\Omega; S)/\mathfrak{N}(\Omega; S)$. Moreover

PROPOSITION 4.6. *There exists the following isomorphism*:

$$\mathscr{S}(\Omega)/\mathscr{N}(\Omega) \xrightarrow{\sim} \mathfrak{S}(\Omega; S)/\mathfrak{N}(\Omega; S).$$

PROOF. Let $P(z,\zeta,\eta) \in \mathfrak{S}(\Omega; S)$. We fix $\eta_0 \in S$. Then by Proposition 4.4, we have $P(z,\zeta,\eta_0) \in \mathscr{S}(\Omega)$ and $[P(z,\zeta,\eta)] = [P(z,\zeta,\eta_0)] \in \mathfrak{S}(\Omega; S)/\mathfrak{N}(\Omega; S)$. □

DEFINITION 4.7. We set

$$:\!P(z,\zeta,\eta)\!: := P(z,\zeta,\eta) \bmod \mathfrak{N}(\Omega; S) \in \mathfrak{S}(\Omega; S)/\mathfrak{N}(\Omega; S)$$

which is called the *normal product* or the *Wick product* of $P(z,\zeta,\eta)$.

5. Kernel functions and symbols.

In this section, we shall establish the correspondence of kernel functions and symbols. For this purpose, first we define two mappings that give the correspondence above. Set $z_0^* = (z_0;\zeta_0) := (0;1,0,\dots,0)$. Take any element $K(z,w)\,dw = [\psi(z,w,\eta)\,dw] \in \varinjlim_{\boldsymbol{\kappa}} E_X^{\mathbb{R}}(\boldsymbol{\kappa})$. Then a representative $\psi(z,z+w,\eta)$ of $K(z,z+w)$ is holomorphic on

$$\bigcap_{i=2}^{n}\left\{(z,w,\eta) \in \mathbb{C}^{2n} \times S;\, \|z\| < r',\, \frac{1}{\varrho}|\eta| < |w_i| < r',\, |w_1| < \varrho|\eta|,\, |\arg w_1| < \frac{\pi}{2} + \theta\right\}.$$

DEFINITION 5.1. We set

$$\sigma(\psi)(z,\zeta,\eta) := \int_{\gamma(0,\eta;\varrho,\theta)} \psi(z,z+w,\eta)\, e^{\langle w,\zeta\rangle} dw.$$

Here the path $\gamma(0,\eta;\varrho,\theta)$ is given in (3.1) with $z=0$. In Proposition 5.4 below, we show that σ induces a mapping $\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \to \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*}$.

In order to construct the inverse of σ, we make full use of the following family of functions (see Laurent [**22**, p. 39]):

DEFINITION 5.2. We set

$$\Gamma_\nu(\tau,\eta) := \begin{cases} 1 & (\nu = 0), \\ \dfrac{1}{(\nu-1)!}\displaystyle\int_0^\eta e^{-s\tau} s^{\nu-1} ds & (\nu \in \mathbb{N}). \end{cases}$$

Let $z_0^* = (0;1,0,\dots,0) \in \dot{T}^*X$, and $P(z,\zeta,\eta) \in \mathfrak{S}_{z_0^*}$. By Proposition 4.4, for any sufficiently small $\eta_0 > 0$ we have $P(z,\zeta,\eta) - P(z,\zeta,\eta_0) \in \mathfrak{N}_{z_0^*}$. We may assume that $\|\zeta\| = |\zeta_1|$ on a neighborhood of z_0^*. We develop $P(z,\zeta,\eta_0)$ into the Taylor series with respect to $\zeta'/\zeta_1 = (\zeta_2/\zeta_1,\dots,\zeta_n/\zeta_1)$:

$$P(z,\zeta,\eta_0) = \sum_{\alpha\in\mathbb{N}_0^{n-1}} P_\alpha(z,\zeta_1,\eta_0)\Big(\frac{\zeta'}{\zeta_1}\Big)^{\alpha}. \tag{5.1}$$

Then we set $P^{\mathcal{B}}_\alpha(z,\zeta_1,\eta) := P_\alpha(z,\zeta_1,\eta_0)\,\zeta_1^{|\alpha|}\varGamma_{|\alpha|}(\zeta_1,\eta)$ and

$$P^{\mathcal{B}}(z,\zeta,\eta) := \sum_{\alpha\in\mathbb{N}_0^{n-1}} P^{\mathcal{B}}_\alpha(z,\zeta_1,\eta)\Big(\frac{\zeta'}{\zeta_1}\Big)^{\alpha}. \tag{5.2}$$

Definition 5.3. Under the preceding notation, we set

$$\begin{aligned}\varpi_\alpha(P)(z,w_1,\eta) &:= \int_d^\infty P^{\mathcal{B}}_\alpha(z,\zeta_1,\eta)\,\frac{e^{-w_1\zeta_1}}{\zeta_1^{|\alpha|}}\,d\zeta_1\\ &= \int_d^\infty P_\alpha(z,\zeta_1,\eta_0)\,\varGamma_{|\alpha|}(\zeta_1,\eta)\,e^{-w_1\zeta_1}\,d\zeta_1,\end{aligned} \tag{5.3}$$

and further define

$$\varpi(P)(z,z+w,\eta) := \sum_{\alpha\in\mathbb{N}_0^{n-1}} \frac{\alpha!\,\varpi_\alpha(P)(z,w_1,\eta)}{(2\pi\sqrt{-1}\,)^n\,(w')^{\alpha+\mathbf{1}_{n-1}}}\,. \tag{5.4}$$

Here we set $w' := (w_2,\dots,w_n)$ and $\mathbf{1}_{n-1} := (1,\dots,1)$. In Proposition 5.5 below, we show that ϖ induces a mapping $\mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*} \to \mathscr{E}^{\mathbb{R}}_{X,z_0^*}$.

Proposition 5.4. *The* σ *in Definition* 5.1 *induces the linear mapping*

$$\begin{array}{ccccc}\sigma\colon \mathscr{E}^{\mathbb{R}}_{X,z_0^*} & = & \varinjlim_{\boldsymbol{\kappa}} E^{\mathbb{R}}_X(\boldsymbol{\kappa}) & \longrightarrow & \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*}\\ \Psi & & \Psi & & \Psi\\ K(z,w)\,dw & = & [\psi(z,w,\eta)\,dw] & \mapsto & \sigma(K)(z,\zeta) = [\sigma(\psi)(z,\zeta,\eta)].\end{array}$$

The σ *does not depend on the choice of the path of the integration.*

We call σ the *symbol mapping*, and $\sigma(K)$ the *symbol* of $K(z,w)\,dw \in \mathscr{E}^{\mathbb{R}}_{X,z_0^*}$.

Proof. We expand

$$\psi(z,z+w,\eta) = \sum_{\alpha\in\mathbb{Z}^{n-1}} \frac{\psi_\alpha(z,w_1,\eta)}{(2\pi\sqrt{-1}\,)^{n-1}\,(w')^{\alpha+\mathbf{1}_{n-1}}}.$$

If $\alpha_i+1\le 0$ for some $2\le i\le n$, this term is zero in $\varinjlim_{\boldsymbol{\kappa}} E^{\mathbb{R}}_X(\boldsymbol{\kappa})$, and hence we may assume from the beginning that

$$\psi(z,z+w,\eta) = \sum_{\alpha\in\mathbb{N}_0^{n-1}} \frac{\psi_\alpha(z,w_1,\eta)}{(2\pi\sqrt{-1}\,)^{n-1}\,(w')^{\alpha+\mathbf{1}_{n-1}}}\,. \tag{5.5}$$

Here

$$\psi_\alpha(z, w_1, \eta) := \oint\limits_{|\widetilde{w}_2|=c|\eta|,\dots,|\widetilde{w}_n|=c|\eta|} \psi(z, z_1 + w_1, z' + \widetilde{w}', \eta)(\widetilde{w}')^\alpha \, d\widetilde{w}' \tag{5.6}$$

for $c > 1/\varrho$. Hence we may assume that $\psi(z, z + w, \eta)$ is holomorphic on

$$\bigcap_{i=2}^{n} \left\{ (z, w, \eta) \in \mathbb{C}^{2n} \times S;\ \|z\| < r',\ |w_1| < \varrho|\eta|,\ |\arg w_1| < \frac{\pi}{2} + \theta,\ \frac{|\eta|}{|w_i|} < \varrho \right\}.$$

Take $c' > 0$, an open conic neighborhood $\Omega \underset{\text{conic}}{\Subset} T^*X$ and $0 < \rho \ll 1$ as

$$\Omega_\rho \underset{\text{conic}}{\Subset} \{(z, \zeta) \in \mathbb{C}^{2n};\ \|z\| \le r',\ \|\zeta'\| \le c'|\zeta_1|,\ |\arg \zeta_1| \le r''\}.$$

Taking c' small enough, we can assume that $\|\zeta\| = |\zeta_1|$ on Ω_ρ. We have

$$\begin{aligned}
\sigma(\psi)(z, \zeta, \eta) &= \int\limits_{\gamma_1(0,\eta;\varrho,\theta)} dw_1 \oint\limits_{\gamma_2(0,\eta;\varrho)} \cdots \oint\limits_{\gamma_n(0,\eta;\varrho)} \sum_{\alpha \in \mathbb{N}_0^{n-1}} \frac{\psi_\alpha(z, w_1, \eta)}{(2\pi\sqrt{-1}\,)^{n-1}\,(w')^{\alpha + \mathbf{1}_{n-1}}} e^{\langle w, \zeta \rangle} dw' \\
&= \sum_{\alpha \in \mathbb{N}_0^{n-1}} \int\limits_{\gamma_1(0,\eta;\varrho,\theta)} \psi_\alpha(z, w_1, \eta) \frac{e^{w_1\zeta_1}}{\alpha!} \partial_{w'}^{\alpha} e^{\langle w', \zeta' \rangle} \Big|_{w'=0} dw_1 \\
&= \sum_{\alpha \in \mathbb{N}_0^{n-1}} \frac{(\zeta')^\alpha}{\alpha!} \int\limits_{\gamma_1(0,\eta;\varrho,\theta)} \psi_\alpha(z, w_1, \eta)\, e^{w_1\zeta_1} dw_1 \,.
\end{aligned}$$

We can change $\gamma_i(0, \eta; \varrho) = \{w_i = |\eta| s' e^{2\pi\sqrt{-1}\,t};\ 0 \le t \le 1\}$ with $0 < \varrho^{-1} < s'$. Deforming $\gamma_1(0, \eta; \varrho, \theta)$, we can see that for any $h > 0$ we have $e^{\mathrm{Re}\langle w_1, \zeta_1 \rangle} \le e^{h|\eta\zeta_1|}$ holds if $|\arg \zeta_1| < r''$ and $w_1 \in \gamma_1(0, \eta; \varrho, \theta)$. Thus we have

$$\begin{aligned}
\left| e^{\langle w, \zeta \rangle} \right| = e^{\mathrm{Re}\langle w, \zeta \rangle} &\le \exp\left(\mathrm{Re}\langle w_1, \zeta_1 \rangle + \sum_{i=2}^{n} |w_i\, \zeta_i| \right) \\
&\le e^{h|\eta\zeta_1| + (n-1)c's'|\eta\zeta_1|} = e^{(h + (n-1)c's')\|\eta\zeta\|}.
\end{aligned}$$

Fix $d > 0$. Take $Z \Subset S$. Then there exists a constant $C > 0$ such that

$$\left| \int\limits_{\gamma(0,\eta;\varrho,\theta)} \psi(z, z + w, \eta)\, e^{\langle w, \zeta \rangle} dw \right| \le C e^{(h + (n-1)c's')\|\eta\zeta\|} \quad ((z; \zeta, \eta) \in \Omega_\rho[d_\rho] \times Z),$$

that is, we can see that $\sigma(\psi)(z, \zeta, \eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X \times \mathbb{C}})$ and satisfies (4.4). If $[\psi(z, w, \eta)\, dw] = 0 \in \varinjlim_{\boldsymbol{\kappa}} E_X^{\mathbb{R}}(\boldsymbol{\kappa})$, we may assume that there is $\delta' > 0$ such that $|e^{\langle w_1, \zeta_1 \rangle}| \le e^{-\delta'|\eta\zeta_1|}$ holds if $|\arg \zeta_1| < r''$ and $w_1 \in \overline{\gamma}_1(0, \eta; \varrho, \theta)$. We choose c' so small that $\delta := \delta' - (n-1)c's' > 0$. Then there exists a constant $C > 0$ such that

$$\left| \int\limits_{\gamma(0,\eta;\varrho,\theta)} \psi(z, z + w, \eta)\, e^{\langle w, \zeta \rangle} dw \right| \le C e^{((n-1)c's' - \delta')|\eta\zeta_1|} \le C e^{-\delta\|\eta\zeta\|} \quad ((z; \zeta, \eta) \in \Omega_\rho[d_\rho] \times Z),$$

that is, $\sigma(\psi)(z,\zeta,\eta) \in \mathfrak{N}_{z_0^*}$. Further we can prove that

$$\int\limits_{\gamma(0,\eta;\varrho,\theta)} \psi(z,z+w,\eta)\, e^{\langle w,\zeta\rangle} dw - \int\limits_{\gamma(0,\eta;\varrho_1,\theta_1)} \psi(z,z+w,\eta)\, e^{\langle w,\zeta\rangle} dw \in \mathfrak{N}_{z_0^*}.$$

Note that

$$\begin{aligned}\partial_\eta \sigma(\psi)(z,\zeta,\eta) = &\int\limits_{\gamma_2(0,\eta;\varrho)\times\cdots\times\gamma_n(0,\eta;\varrho)} \left[\tau\psi(z,z_1+\tau\eta,z'+w',\eta)\, e^{\tau\eta\zeta_1+\langle w',\zeta'\rangle}\right]_{\tau=\beta_0}^{\beta_1} dw_2\cdots dw_n \\ &+ \int\limits_{\gamma(0,\eta;\varrho,\theta)} \partial_\eta\psi(z,z+w,\eta)\, e^{\langle w,\zeta\rangle}\, dw.\end{aligned}$$

Since $\partial_\eta\psi(z,w,\eta)$ is a zero class, $\partial_\eta\sigma(\psi)(z,\zeta,\eta) \in \mathfrak{N}_{z_0^*}$. Thus we see that $\sigma(\psi)(z,\zeta,w) \in \mathfrak{S}_{z_0^*}$ and σ is well-defined. □

Proposition 5.5. *The ϖ in Definition 5.3 induces the linear mapping*

$$\varpi\colon \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*} \ni {:}P(z,\zeta,\eta){:} \mapsto \varpi({:}P{:}) := [\varpi(P)(z,w,\eta)\,dw] \in \mathscr{E}^{\mathbb{R}}_{X,z_0^*}. \tag{5.7}$$

This mapping is independent of the choice of either η_0 or the path of the integration.

We call $\varpi({:}P{:})$ the *kernel* of ${:}P{:} \in \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*}$.

Proof. We need the following estimate to prove that ϖ is well-defined:

Lemma 5.6. *Assume that $\mathrm{Re}(\eta\tau) \geq 2\delta_0|\eta\tau| > 0$ for some $\delta_0 \in \left]0,1/2\right[$. Then for any $\nu \in \mathbb{N}$,*

$$|\varGamma_\nu(\tau,\eta)| \leq \frac{|\eta|^\nu}{\nu!}, \tag{5.8}$$

$$|1-\tau^\nu \varGamma_\nu(\tau,\eta)| \leq \frac{e^{-\delta_0|\eta\tau|}}{\delta_0^{\nu-1}}. \tag{5.9}$$

Proof. We have (5.8) as follows:

$$|\varGamma_\nu(\tau,\eta)| \leq \frac{1}{(\nu-1)!}\int_0^{|\eta|} |e^{-s\tau}s^{\nu-1}|\,ds \leq \frac{1}{(\nu-1)!}\int_0^{|\eta|} s^{\nu-1}\,ds = \frac{|\eta|^\nu}{\nu!}.$$

By the definition of $\varGamma$-function and induction on ν, we have

$$1-\tau^\nu\varGamma_\nu(\tau,\eta) = \frac{\tau^\nu}{(\nu-1)!}\int_\eta^\infty e^{-s\tau}s^{\nu-1}\,ds = \sum_{k=0}^{\nu-1}\frac{(\eta\tau)^k}{k!}\,e^{-\eta\tau}.$$

Therefore, we have

$$|1-\tau^\nu\varGamma_\nu(\tau,\eta)| = \left|\sum_{k=0}^{\nu-1}\frac{(\delta_0\eta\tau)^k}{k!\delta_0^k}\,e^{-\eta\tau}\right| \leq \sum_{k=0}^{\nu-1}\frac{(\delta_0|\eta\tau|)^k}{k!}\,\frac{e^{-2\delta_0|\eta\tau|}}{\delta_0^{\nu-1}}$$

$$\leq e^{\delta_0|\eta\tau|}\frac{e^{-2\delta_0|\eta\tau|}}{\delta_0^{\nu-1}}=\frac{e^{-\delta_0|\eta\tau|}}{\delta_0^{\nu-1}}. \qquad \square$$

Recall (5.1) and (5.2). There exist sufficiently small r_0, $\theta'>0$ and sufficiently large $d>0$ such that $P_\alpha(z,\zeta_1,\eta_0)$ is holomorphic on a common neighborhood of D for each $\alpha\in\mathbb{N}_0^{n-1}$, where

$$D:=\{(z,\zeta_1)\in\mathbb{C}^{n+1};\ \|z\|\leq r_0,\ |\arg\zeta_1|\leq\theta',\ |\zeta_1|\geq d\}.$$

It follows from the Cauchy inequality that we can take a constant $K>0$ so that for each $h>0$ there exists $C_h>0$ such that for every $\alpha\in\mathbb{N}_0^{n-1}$,

$$|P_\alpha(z,\zeta_1,\eta_0)|\leq C_hK^{|\alpha|}e^{h|\zeta_1|}\quad((z,\zeta_1)\in D). \tag{5.10}$$

We take $\delta_0\in\,]0,1/2[$ as $\mathrm{Re}(\eta\zeta_1)\geq 2\delta_0|\eta\zeta_1|>0$ if $\eta\in S$ and $|\arg\zeta_1|\leq\theta'$. Take $\varepsilon>0$ as $0<K\varepsilon/\delta_0<1/2$. For any $Z\Subset S$, we chose $h=\delta_0 m_Z/2$. Then by (5.9), for $(z,\zeta_1)\in D\times Z$ and $|\zeta_i|\leq\varepsilon|\zeta_1|$ $(2\leq i\leq n)$ we have

$$\begin{aligned}|P(z,\zeta,\eta_0)-P^{\mathcal{B}}(z,\zeta,\eta)|&=\left|\sum_{|\alpha|=1}^{\infty}P_\alpha(z,\zeta_1,\eta_0)\,(1-\zeta_1^{|\alpha|}\mathit{\Gamma}_{|\alpha|}(\zeta_1,\eta))\left(\frac{\zeta'}{\zeta_1}\right)^{\alpha}\right|\\&\leq\delta_0C_he^{-\delta_0|\eta\zeta_1|/2}\sum_{|\alpha|=1}^{\infty}\left(\frac{K\varepsilon}{\delta_0}\right)^{|\alpha|}\leq 2^{n-1}\delta_0C_he^{-\delta_0|\eta\zeta_1|/2},\end{aligned} \tag{5.11}$$

where we remark that $\#\{\alpha\in\mathbb{N}_0^{n-1};\ |\alpha|=i\}=\binom{n+i-2}{i}\leq 2^{n+i-2}$. Therefore we see that $P(z,\zeta,\eta)-P^{\mathcal{B}}(z,\zeta,\eta)=P(z,\zeta,\eta)-P(z,\zeta,\eta_0)+P(z,\zeta,\eta_0)-P^{\mathcal{B}}(z,\zeta,\eta)\in\mathfrak{N}_{z_0^*}$. Further by (5.8) and (5.10), there exists a constant $K>0$ so that for each $h>0$ there exists $C_h>0$ such that for every $\alpha\in\mathbb{N}_0^{n-1}$ and $(z,\zeta_1,\eta)\in D\times S$, we have

$$\frac{|P^{\mathcal{B}}_\alpha(z,\zeta_1,\eta)|}{|\zeta_1|^{|\alpha|}}\leq\frac{C_h(K|\eta|)^{|\alpha|}e^{h|\zeta_1|}}{|\alpha|!}. \tag{5.12}$$

We can take a sufficiently small δ_1, $\delta'>0$ such that

$$\left\{w_1\in\mathbb{C};\ |\arg w_1|<\delta'+\frac{\pi}{2}\right\}\subset\bigcup_{|\arg\zeta_1|\leq\theta'}\{w_1\in\mathbb{C};\ \mathrm{Re}(w_1\zeta_1)\geq\delta_1|w_1\zeta_1|\},$$

and we set

$$L:=\left\{(z,w_1)\in\mathbb{C}^{n+1};\ \|z\|<r_0,\ |\arg w_1|<\delta'+\frac{\pi}{2}\right\}. \tag{5.13}$$

By (5.12), for any $k\in\mathbb{N}$ there exists $C_k>0$ such that for every $\alpha\in\mathbb{N}_0^{n-1}$,

$$\frac{|P^{\mathcal{B}}_\alpha(z,\zeta_1,\eta)|}{|\zeta_1|^{|\alpha|}}\leq\frac{C_k(K|\eta|)^{|\alpha|}e^{\delta_1|\zeta_1|/k}}{|\alpha|!}\quad((z,\zeta_1,\eta)\in D\times S). \tag{5.14}$$

By changing the direction of the integration in the complex τ-plane, $\varpi_\alpha(P)(z,w_1,\eta)$

extends analytically to the domain $L \times S$. Set

$$L_k := \left\{ (z, w_1) \in \mathbb{C}^{n+1};\ \|z\| < r_0,\ |\arg w_1| < \delta' + \frac{\pi}{2},\ \frac{2}{k} < |w_1| \right\}. \tag{5.15}$$

Then by (5.14) and (5.8) for any $\eta \in S$ we have

$$\sup\{|\varpi_\alpha(P)(z, w_1, \eta)|;\ (z, w_1) \in L_k\} \leq \frac{2kC_k}{\delta_1 |\alpha|!} (K|\eta|)^{|\alpha|}. \tag{5.16}$$

Therefore the right-hand side of (5.4) converges locally uniformly in

$$V_k := \bigcap_{2 \leq i \leq n} \{(z, w, \eta) \in \mathbb{C}^{2n} \times S;\ (z, w_1) \in L_k,\ K|\eta| < |w_i|\}. \tag{5.17}$$

Hence $\varpi(P)(z, z + w, \eta)$ is a holomorphic function defined on the set

$$V := \bigcup_{k=1}^{\infty} V_k = \bigcap_{2 \leq i \leq n} \left\{ (z, w, \eta) \in \mathbb{C}^{2n} \times S;\ \|z\| < r_0,\ |\arg w_1| < \delta' + \frac{\pi}{2},\ K|\eta| < |w_i| \right\}. \tag{5.18}$$

Next, we have

$$\partial_\eta \varpi_\alpha(P)(z, w_1, \eta) = \frac{1}{(|\alpha| - 1)!} \int_d^\infty P_\alpha(z, \zeta_1, \eta_0)\, e^{-(\eta + w_1)\zeta_1} \eta^{|\alpha| - 1} d\zeta_1 \,.$$

Let $Z \Subset S$. Then choosing $h = \delta_0 m_Z$ in (5.10), for $\|z\| < r_0$, $\eta \in Z$ and $|w_1| < \delta_0|\eta|/2$, we have

$$\begin{aligned} |\partial_\eta \varpi_\alpha(P)(z, w_1, \eta)| &\leq \frac{C_{\delta_0 m_Z} (K|\eta|)^{|\alpha|}}{(|\alpha| - 1)!} \int_d^\infty e^{\delta_0 m_Z |\zeta_1| - \mathrm{Re}(\eta\zeta_1) + |w_1 \zeta_1|} d|\zeta_1| \\ &\leq \frac{C_{\delta_0 m_Z} (K|\eta|)^{|\alpha|}}{(|\alpha| - 1)!} \int_d^\infty e^{-(\delta_0|\eta| - |w_1|)|\zeta_1|} d|\zeta_1| \leq \frac{2C_{\delta_0 m_Z} (K|\eta|)^{|\alpha|}}{\delta_0 |\eta|\, (|\alpha| - 1)!}. \end{aligned}$$

Hence $\partial_\eta \varpi(P)(z, z + w, \eta)$ is holomorphic on

$$\begin{aligned} &\bigcup_{Z \Subset S} \bigcap_{2 \leq i \leq n} \left\{ (z, w, \eta) \in \mathbb{C}^{2n} \times Z;\ \|z\| < r_0,\ |w_1| < \frac{\delta_0|\eta|}{2},\ K|\eta| < |w_i| \right\} \\ &\quad = \bigcap_{2 \leq i \leq n} \left\{ (z, w, \eta) \in \mathbb{C}^{2n} \times S;\ \|z\| < r_0,\ |w_1| < \frac{\delta_0|\eta|}{2},\ K|\eta| < |w_i| \right\}. \end{aligned}$$

This entails that $[\partial_\eta \varpi(P)(z, w, \eta)\, dw] = 0 \in \mathscr{E}^{\mathbb{R}}_{X, z_0^*}$. If $P(z, \zeta, \eta) \in \mathfrak{N}_{z_0^*}$, there exist constants δ, C, $K > 0$ so that for every $\alpha \in \mathbb{N}_0^{n-1}$,

$$|P_\alpha(z, \zeta_1, \eta_0)| \leq CK^{|\alpha|} e^{-\delta|\zeta_1|} \quad ((z, \zeta_1) \in D).$$

Thus if $|w_1| < \delta/2$, we have

$$|\varpi_\alpha(P)(z,w_1,\eta)| \le \frac{C(K|\eta|)^{|\alpha|}}{|\alpha|!}\int_d^\infty e^{-(\delta-|w_1|)|\zeta_1|}d|\zeta_1| \le \frac{2C(K|\eta|)^{|\alpha|}}{\delta|\alpha|!}\,.$$

Thus $\varpi(P)(z,z+w,\eta)$ is holomorphic on

$$\bigcap_{2\le i\le n}\left\{(z,w,\eta)\in\mathbb{C}^{2n}\times S;\ \|z\|<r_0,\ |w_1|<\frac{\delta}{2},\ K|\eta|<|w_i|\right\},$$

and hence $[\varpi(P)(z,w,\eta)\,dw]=0$. If we change η_0 or d in (5.3), for the same reasoning as above, we see that $[\varpi(P)(z,w,\eta)\,dw]=0$. Therefore we obtain a well-defined linear mapping (5.7). □

Now we shall prove our fundamental theorem for the symbol theory:

THEOREM 5.7. *The mappings σ and ϖ are inverse to each other. In particular*

$$\sigma\colon \mathscr{E}^{\mathbb{R}}_{X,z_0^*} \xrightarrow{\sim} \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*}\,.$$

PROOF. We may assume that $z_0^*=(0;1,0,\dots,0)$, and we may also assume $\|\zeta\|=|\zeta_1|$ on a neighborhood of z_0^* in the course of proof.

Step 1. We shall show $\sigma\cdot\varpi=\mathrm{id}\colon\mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*}\to\mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*}$. Let $P(z,\zeta,\eta)\in\mathfrak{S}_{z_0^*}$. Assume that $P^{\mathcal{B}}(z,\zeta,\eta)\in\mathfrak{S}_{z_0^*}$ is holomorphic on a neighborhood of

$$\widetilde{V}:=\bigcap_{2\le i\le n}\{(z,\zeta,\eta)\in\mathbb{C}^{2n}\times S;\ \|z\|<r_0,\ |\zeta_1|\ge d,\ d|\arg\zeta_1|\le 1,\ d|\zeta_i|\le|\zeta_1|\}.$$

By the definition of ϖ, we have

$$\begin{aligned}\sigma\cdot\varpi(P)(z,\zeta,\eta) &= \int_{\gamma(0,\eta;\varrho,\theta)}\varpi(P)(z,z+w,\eta)\,e^{\langle w,\zeta\rangle}dw\\ &= \int_{\gamma(0,\eta;\varrho,\theta)}dw\sum_{\alpha\in\mathbb{N}_0^{n-1}}\frac{\alpha!\,e^{\langle w,\zeta\rangle}}{(2\pi\sqrt{-1}\,)^n\,(w')^{\alpha+\mathbf{1}_{n-1}}}\int_d^\infty P^{\mathcal{B}}_\alpha(z,\xi_1,\eta)\,\frac{e^{-w_1\xi_1}}{\xi_1^{|\alpha|}}\,d\xi_1\\ &= \sum_{\alpha\in\mathbb{N}_0^{n-1}}(\zeta')^\alpha\int_{\gamma_1(0,\eta;\varrho,\theta)}dw_1\,\frac{e^{w_1\zeta_1}}{2\pi\sqrt{-1}}\int_d^\infty P^{\mathcal{B}}_\alpha(z,\xi_1,\eta)\,\frac{e^{-w_1\xi_1}}{\xi_1^{|\alpha|}}\,d\xi_1\,.\end{aligned}$$

We set

$$\widetilde{V}_\varepsilon:=\bigcap_{i=2}^n\{(z,\zeta)\in\mathbb{C}^{2n};\ \|z\|<r_0,\ |\zeta_1|\ge\frac{d}{\varepsilon},\ |\arg\zeta_1|\le\varepsilon,\ |\zeta_i|\le\varepsilon|\zeta_1|\}.$$

We deform the path of integration $\int_d^\infty d\xi_1$ in two ways as follows: Let $\delta>0$ be a sufficiently small constant and $d^\pm$ intersection points of the circle $|\tau|=d$ and $\{\xi_1\in\mathbb{C};\ \pm\operatorname{Im}\xi_1=\delta\operatorname{Re}\xi_1>0\}$. Let $\varSigma_\pm$ be paths starting from d, first going to $d^\pm$ along the circle and next going to the infinity along the half lines $\{\xi_1\in\mathbb{C};\ \pm\operatorname{Im}\xi_1=\delta\operatorname{Re}\xi_1>0\}$ respectively (see Figure 1). According to these deformations, we divide the path

$\gamma_1(0,\eta;\varrho,\theta)$ into two parts:

$$\gamma_1^{\pm}(0,\eta;\varrho,\theta) := \gamma_1(0,\eta;\varrho,\theta) \cap \{w_1 \in \mathbb{C};\, \pm \operatorname{Im} w_1 > 0\}.$$

We take $a \in \gamma_1(0,\eta;\varrho,\theta) \cap \mathbb{R}$. Now we can change the order of integration in I (cf. Figure 1) and obtain:

$$\begin{aligned}\sigma \cdot \varpi(P)(z,\zeta,\eta) &= \sum_{\alpha\in\mathbb{N}_0^{n-1}} (\zeta')^\alpha \Big(\int_{\Sigma_-} d\xi_1 \, \frac{P_\alpha^{\mathcal{B}}(z,\xi_1,\eta)}{2\pi\sqrt{-1}\,\xi_1^{|\alpha|}} \int_{\gamma_1^+(0,\eta;\varrho,\theta)} e^{w_1(\zeta_1-\xi_1)}\,dw_1 \\ &\qquad + \int_{\Sigma_+} d\xi_1 \, \frac{P_\alpha^{\mathcal{B}}(z,\xi_1,\eta)}{2\pi\sqrt{-1}\,\xi_1^{|\alpha|}} \int_{\gamma_1^-(0,\eta;\varrho,\theta)} e^{w_1(\zeta_1-\xi_1)}\,dw_1 \Big) \\ &= \sum_{\alpha\in\mathbb{N}_0^{n-1}} (\zeta')^\alpha \int_{\Sigma_-} \frac{P_\alpha^{\mathcal{B}}(z,\xi_1,\eta)\,(e^{a(\zeta_1-\xi_1)} - e^{\beta_1\eta(\zeta_1-\xi_1)})}{2\pi\sqrt{-1}\,\xi_1^{|\alpha|}(\xi_1-\zeta_1)}\,d\xi_1 \\ &\quad + \sum_{\alpha\in\mathbb{N}_0^{n-1}} (\zeta')^\alpha \int_{\Sigma_+} \frac{P_\alpha^{\mathcal{B}}(z,\xi_1,\eta)\,(e^{\beta_0\eta(\zeta_1-\xi_1)} - e^{a(\zeta_1-\xi_1)})}{2\pi\sqrt{-1}\,\xi_1^{|\alpha|}(\xi_1-\zeta_1)}\,d\xi_1 \,.\end{aligned}$$

Here we remark that $a > 0$ can be taken as sufficiently small. Further we set

$$\begin{aligned} I &:= \sum_{\alpha\in\mathbb{N}_0^{n-1}} (\zeta')^\alpha \int_{\Sigma_- - \Sigma_+} \frac{P_\alpha^{\mathcal{B}}(z,\xi_1,\eta)\,e^{a(\zeta_1-\xi_1)}}{2\pi\sqrt{-1}\,\xi_1^{|\alpha|}(\xi_1-\zeta_1)}\,d\xi_1, \\ I^- &:= -\sum_{\alpha\in\mathbb{N}_0^{n-1}} (\zeta')^\alpha \int_{\Sigma_-} \frac{P_\alpha^{\mathcal{B}}(z,\xi_1,\eta)\,e^{\beta_1\eta(\zeta_1-\xi_1)}}{2\pi\sqrt{-1}\,\xi_1^{|\alpha|}(\xi_1-\zeta_1)}\,d\xi_1, \\ I^+ &:= \sum_{\alpha\in\mathbb{N}_0^{n-1}} (\zeta')^\alpha \int_{\Sigma_+} \frac{P_\alpha^{\mathcal{B}}(z,\xi_1,\eta)\,e^{\beta_0\eta(\zeta_1-\xi_1)}}{2\pi\sqrt{-1}\,\xi_1^{|\alpha|}(\xi_1-\zeta_1)}\,d\xi_1. \end{aligned}$$

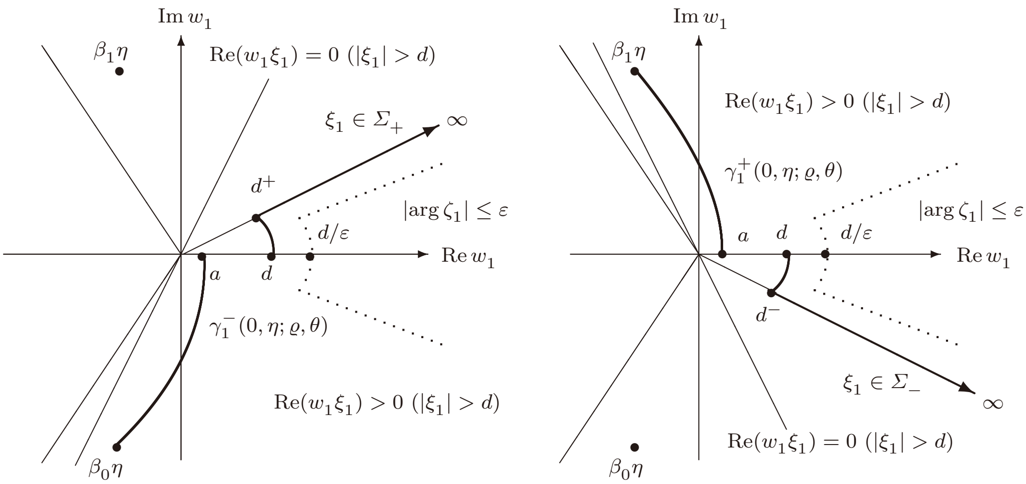

Figure 1.

Then $\sigma\cdot\varpi(P)(z,\zeta,\eta) = I + I^- + I^+$. Let us recall that we have discussed in $|\arg\zeta_1| \le \varepsilon$, $|\zeta_1| \ge d/\varepsilon > d$ and $|\zeta_i| \le \varepsilon|\zeta_1|$ $(2 \le i \le n)$. We can find ε_0, $c > 0$ such that $|\zeta_1 - \xi_1| \ge c|\xi_1| \ge cd$ and $\mathrm{Re}(\zeta_1\beta_1\eta) \le -2c|\eta\zeta_1|$ hold for any $\varepsilon \in \,]0,\varepsilon_0[$ and $(\xi_1,\eta) \in \Sigma_- \times S$. Further there exists a constant $h_0 > 0$ such that $\mathrm{Re}(\beta_1\eta\xi_1) \ge 2h_0|\eta\xi_1|$ holds for any $\xi_1 \in \Sigma_- \smallsetminus \{|\xi_1| = d\}$ and $\eta \in S$. For any $Z \Subset S$, choose $h = h_Z > 0$ as $h_Z < h_0 m_Z$ in (5.12). Hence replacing $\varepsilon > 0$ as $2K\varepsilon \le c$ on $\widetilde{V}_\varepsilon \times Z$ we have

$$\begin{aligned}|I^-| &\le \sum_{|\alpha|=0}^{\infty} \frac{C_{h_Z}(K\varepsilon|\eta\zeta_1|)^{|\alpha|}\, e^{-2c|\eta\zeta_1|}}{2\pi cd\,|\alpha|!} \\ &\quad\times \left(e^{(h_Z+|\mathrm{Re}(\beta_1\eta)|)d}\int_{|\xi_1|=d}|d\xi_1| + \int_d^\infty e^{-(2h_0|\eta|-h_Z)|\xi_1|}\,d|\xi_1|\right) \\ &\le \frac{2^{n-2}C_{h_Z}e^{-c|\eta\zeta_1|}}{c}\left(e^{(h_Z+|\beta_1|r)d} + \frac{e^{-h_0dr}}{2\pi h_0 d m_Z}\right). \end{aligned} \tag{5.19}$$

Hence we see that $I^- \in \mathfrak{N}_{z_0^*}$. Similarly, we have

$$|I^+| \le \frac{2^{n-2}C_{h_Z}e^{-c|\eta\zeta_1|}}{c}\left(e^{(h_Z+|\beta_0|r)d} + \frac{e^{-h_0dr}}{2\pi h_0 d m_Z}\right), \tag{5.20}$$

and hence $I^+ \in \mathfrak{N}_{z_0^*}$. Now we consider I. For any $K \Subset \{\zeta_1 \in \mathbb{C};\, |\arg\zeta_1| \le \varepsilon\}$, we see that the integral operator

$$\int_{\Sigma_- - \Sigma_+} d\xi_1\, \frac{e^{a(\zeta_1-\xi_1)}}{2\pi\sqrt{-1}(\xi_1-\zeta_1)}$$

has the Cauchy kernel with a damping factor since $-\mathrm{Re}(a\xi_1) < 0$. Hence,

$$I = \sum_{\alpha\in\mathbb{N}_0^{n-1}} P_\alpha^{\mathcal{B}}(z,\zeta_1,\eta)\left(\frac{\zeta'}{\zeta_1}\right)^\alpha = P^{\mathcal{B}}(z,\zeta,\eta)$$

holds if ζ_1 is located in the domain surrounded by $\Sigma_- - \Sigma_+$. Thus we have

$$\sigma\cdot\varpi(P)(z,\zeta,\eta) - P(z,\zeta,\eta) = \sigma\cdot\varpi(P)(z,\zeta,\eta) - P^{\mathcal{B}}(z,\zeta,\eta) + P^{\mathcal{B}}(z,\zeta,\eta) - P(z,\zeta,\eta) \in \mathfrak{N}_{z_0^*},$$

that is, $\sigma\cdot\varpi = \mathrm{id}\colon \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*} \xrightarrow{\sim} \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*}$.

Step 2. Let $P = [\psi(z,w,\eta)\,dw] \in \mathscr{E}^{\mathbb{R}}_{X,z_0^*}$. Then we can assume that a representative $\psi(z, z+w,\eta)$ has the form as in (5.5). By Proposition 5.4, each coefficient $P_\alpha(z,\zeta_1,\eta)$ in (5.1) is written as

$$P_\alpha(z,\zeta_1,\eta) = \frac{\zeta_1^{|\alpha|}}{\alpha!}\int_{\gamma_1(0,\eta;\varrho,\theta)} \psi_\alpha(z,p,\eta)\,e^{p\zeta_1}\,dp.$$

We assume that each $\psi_\alpha(z,p,\eta)$ is holomorphic on

$$\Big\{(z,p,\eta)\in\mathbb{C}^{n+1}\times S;\ \|z\|<2r_0,\ |w_1|<\varrho|\eta|,\ |\arg p|<\frac{\pi}{2}+\theta\Big\}.$$

Fix $\eta_0\in S\cap\mathbb{R}$, and we take $\varrho'<\varrho$ as $\varrho'|\eta|<|\eta_0|$ for any $\eta\in S$. By (5.6), there exists $c>0$ and for any $Z\Subset S$ there exists $C_Z>0$ such that for any $\eta\in Z$,

$$\sup\{|\psi_\alpha(z,p,\eta)|;\ \|z\|\le r_0,\ p\in\gamma_1(0,\eta_0;\varrho,\theta)\}\le C_Z(c|\eta|)^{|\alpha|+n-1}. \tag{5.21}$$

By the definition, we have

$$\begin{aligned}&\varpi\cdot\sigma(\psi)(z,z+w,\eta)\\ &\quad=\sum_{|\alpha|=0}^{\infty}\frac{1}{2\pi\sqrt{-1}\,(w')^{\alpha+\mathbf{1}_{n-1}}}\int_d^\infty d\zeta_1\,\zeta_1^{|\alpha|}\varGamma_{|\alpha|}(\zeta_1,\eta)\,e^{-w_1\zeta_1}\int_{\gamma_1(0,\eta_0;\varrho,\theta)}\psi_\alpha(z,p,\eta_0)\,e^{p\zeta_1}dp.\end{aligned}$$

We set

$$\varpi'\cdot\sigma(\psi)(z,z+w,\eta):=\sum_{|\alpha|=0}^{\infty}\frac{1}{2\pi\sqrt{-1}\,(w')^{\alpha+\mathbf{1}_{n-1}}}\int_d^\infty d\zeta_1\,e^{-w_1\zeta_1}\int_{\gamma_1(0,\eta_0;\varrho,\theta)}\psi_\alpha(z,p,\eta)\,e^{p\zeta_1}dp.$$

We assume that $\mathrm{Re}(\eta\zeta_1)\ge 2\delta_0|\eta\zeta_1|>0$ for some $\delta_0\in\,]0,1/2[$. We deform the path $\gamma_1(0,\eta_0;\varrho,\theta)$ as $|e^{p\zeta_1}|\le e^{-\delta|\eta\zeta_1|/2}$ for $|\arg\zeta_1|\le\varepsilon$. Then by (5.9) and (5.21), for any $Z\Subset S$, there exists $C_Z>0$ such that if $2|w_1|<\delta_0|\eta|$ and $c|\eta|<\delta_0|w_i|$ $(2\le i\le n)$, we have

$$\begin{aligned}&\Bigg|\sum_{|\alpha|=0}^{\infty}\frac{1}{2\pi\sqrt{-1}\,(w')^{\alpha+\mathbf{1}_{n-1}}}\int_d^\infty d\zeta_1\,(1-\zeta_1^{|\alpha|}\varGamma_{|\alpha|}(\zeta_1,\eta))\,e^{-w_1\zeta_1}\int_{\gamma_1(0,\eta_0;\varrho,\theta)}\psi_\alpha(z,p,\eta)\,e^{p\zeta_1}dp\Bigg|\\ &\quad\le\sum_{|\alpha|=1}^{\infty}\frac{C_Z|\gamma_1(0,\eta_0;\varrho,\theta)|(c|\eta|)^{|\alpha|+n-1}}{2\pi(w')^{\alpha+\mathbf{1}_{n-1}}}\int_d^\infty\frac{e^{-(\delta_0|\eta|/2-|w_1|)|\zeta_1|}}{\delta_0^{|\alpha|-1}}\,d\zeta_1\\ &\quad\le\sum_{|\alpha|=1}^{\infty}\frac{C_Z\delta_0^n|\gamma_1(0,\eta_0;\varrho,\theta)|}{\pi(\delta_0|\eta|-2|w_1|)}\Big|\Big(\frac{c\eta}{\delta_0w'}\Big)^{\alpha+\mathbf{1}_{n-1}}\Big|<\infty.\end{aligned}$$

Here $|\gamma_1(0,\eta_0;\varrho,\theta)|$ denotes the length of $\gamma_1(0,\eta_0;\varrho,\theta)$. Next we consider

$$\int_{\gamma_1(0,\eta_0;\varrho,\theta)}\psi_\alpha(z,p,\eta)\,e^{p\zeta_1}dp-\int_{\gamma_1(0,\eta_0;\varrho,\theta)}\psi_\alpha(z,p,\eta_0)\,e^{p\zeta_1}dp=\int_{\gamma_1(0,\eta_0;\varrho,\theta)}dp\,e^{p\zeta_1}\int_{\eta_0}^{\eta}\partial_\eta\psi_\alpha(z,p,\tau)\,d\tau\,. \tag{5.22}$$

Since $\partial_\eta\psi(z,w,\eta)$ is holomorphic on $|w_1|<\varrho|\eta|$, as in (5.21) there exists c and for any $S\Subset Z$ there exists $C_Z>0$ such that for any $\eta\in Z$,

$$\sup\{|\psi_\alpha(z,p,\eta)-\psi_\alpha(z,p,\eta_0)|;\ \|z\|\le r_0,\ |p|\le\varrho'|\eta|\}\le C_Z(c|\eta|)^{|\alpha|+n-1}.$$

Thus we can change the path of the integration in (5.22) as $|e^{p\zeta_1}|\le e^{-c'|\eta\zeta|}$ for $|\arg\zeta_1|\le\varepsilon$. Hence if $\eta\in Z$, $|w_1|<c'|\eta|$ and $c|\eta|<\delta_0|w_i|$ $(2\le i\le n)$, we have

$$\begin{aligned}&\left|\sum_{|\alpha|=0}^{\infty}\frac{1}{2\pi\sqrt{-1}(w')^{\alpha+\mathbf{1}_{n-1}}}\int_d^{\infty}d\zeta_1\, e^{-w_1\zeta_1}\int\limits_{\gamma_1(0,\eta_0;\varrho,\theta)}(\psi_\alpha(z,p,\eta)-\psi_\alpha(z,p,\eta_0))\,e^{p\zeta_1}dp\right|\\&\quad\le\sum_{|\alpha|=0}^{\infty}\frac{C_Z|\gamma_1(0,\eta_0;\varrho,\theta)|(c|\eta|)^{|\alpha|+n-1}}{2\pi(w')^{\alpha+\mathbf{1}_{n-1}}}\int_d^{\infty}e^{-(c'|\eta|-|w_1|)|\zeta_1|}\,d\zeta_1\\&\quad\le\sum_{|\alpha|=0}^{\infty}\frac{C_Z|\gamma_1(0,\eta_0;\varrho,\theta)|}{2\pi(c'|\eta|-|w_1|)}\left|\left(\frac{c\eta}{\delta_0 w'}\right)^{\alpha+\mathbf{1}_{n-1}}\right|<\infty.\end{aligned}$$

Summing up we can prove that $[\varpi'\cdot\sigma(\psi)(z,w,\eta)\,dw]=[\varpi\cdot\sigma(\psi)(z,w,\eta)\,dw]\in\varinjlim_{\boldsymbol{\kappa}}\widehat{E}_X^{\mathbb{R}}(\boldsymbol{\kappa})$.

We may assume that $\varpi'\cdot\sigma(\psi)(z,z+w,\eta)$ is holomorphic on

$$\bigcap_{2\le i\le n}\left\{(z,w,\eta)\in\mathbb{C}^{2n}\times S;\ \|z\|<r_0,\ 0<|w_1|,\ |\arg w_1|<\delta'+\frac{\pi}{2},\ c|\eta|<|w_i|\right\}$$

with some constants r_0, δ', $c>0$. Let γ_1' be a path starting from $\beta_0\eta_0$, ending at $\beta_1\eta_0$ and detouring w_1 clockwise as in Figure 2.

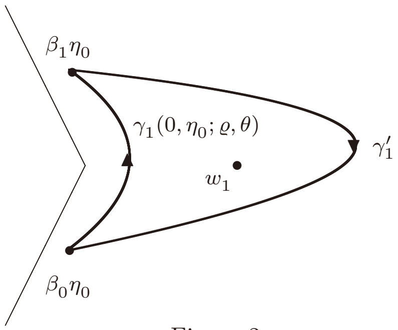

Figure 2.

If $\mathrm{Re}\,(p-w_1)<0$, we have

$$\int_d^{\infty}e^{(p-w_1)\zeta_1}d\zeta_1=-\frac{e^{(p-w_1)d}}{p-w_1},$$

and the right-hand side extends analytically. Thus on the common domain of definition we have

$$\begin{aligned}\varpi'\cdot\sigma(\psi)(z,z+w,\eta)&=\sum_{\alpha\in\mathbb{N}_0^{n-1}}\frac{-1}{(w')^{\alpha+\mathbf{1}_{n-1}}}\int\limits_{\gamma_1(0,\eta_0;\varrho,\theta)}\frac{\psi_\alpha(z,p,\eta)\,e^{(p-w_1)d}}{2\pi\sqrt{-1}\,(p-w_1)}\,dp\\&=\sum_{\alpha\in\mathbb{N}_0^{n-1}}\frac{-1}{(w')^{\alpha+\mathbf{1}_{n-1}}}\oint\limits_{\gamma_1(0,\eta_0;\varrho,\theta)\vee\gamma_1'}\frac{\psi_\alpha(z,p,\eta)\,e^{(p-w_1)d}}{2\pi\sqrt{-1}\,(p-w_1)}\,dp\\&\qquad+\sum_{\alpha\in\mathbb{N}_0^{n-1}}\frac{1}{(w')^{\alpha+\mathbf{1}_{n-1}}}\int_{\gamma_1'}\frac{\psi_\alpha(z,p,\eta)\,e^{(p-w_1)d}}{2\pi\sqrt{-1}\,(p-w_1)}\,dp\end{aligned}$$

$$= \psi(z, z+w, \eta) + \mathit{\Pi},$$

where

$$\mathit{\Pi} := \sum_{\alpha \in \mathbb{N}_0^{n-1}} \frac{1}{(w')^{\alpha + \mathbf{1}_{n-1}}} \int_{\gamma_1'} \frac{\psi_\alpha(z, p, \eta)\, e^{(p-w_1)d}}{2\pi\sqrt{-1}\,(p - w_1)}\, dp.$$

As in (5.21), there exist c, $c_1 > 0$ and for any $Z \Subset S$, there exists $C_Z > 0$ such that

$$\Big|\frac{e^{(p-w_1)d}}{p - w_1}\Big| \le C_Z, \qquad |\psi_\alpha(z, p, \eta)| \le C_Z (c|\eta|)^{|\alpha|+n-1},$$

hold on $\{\|z\| < c_1,\ |w_1| < c_1|\eta|,\ p \in \gamma_1',\ \eta \in Z\}$. Thus, on $\{\|z\| < c_1,\ |w_1| < c_1|\eta|,\ \eta \in Z\}$ we have

$$|\mathit{\Pi}| \le \sum_{\alpha \in \mathbb{N}_0^{n-1}} \frac{C_Z^2 |\gamma_1'|}{2\pi} \Big|\Big(\frac{c|\eta|}{w'}\Big)^{\alpha + \mathbf{1}_{n-1}}\Big|.$$

By taking $\delta > 0$ as $c\delta < 1$ and $\delta < c_1$, we see that $\mathit{\Pi}$ is holomorphic on

$$\begin{aligned} &\bigcup_{Z \Subset S} \bigcap_{i=2}^{n} \{(z, w, \eta) \in \mathbb{C}^{2n} \times Z;\ \|z\| < \delta,\ |w_1| < \delta|\eta| < \delta^2 |w_i|\} \\ &= \bigcap_{i=2}^{n} \{(z, w, \eta) \in \mathbb{C}^{2n} \times S;\ \|z\| < \delta,\ |w_1| < \delta|\eta| < \delta^2 |w_i|\}. \end{aligned}$$

Thus $\varpi \cdot \sigma = \mathrm{id}\colon \mathscr{E}^{\mathbb{R}}_{X,z_0^*} \xrightarrow{\sim} \mathscr{E}^{\mathbb{R}}_{X,z_0^*}$.

By Steps 1 and 2, we see that $\sigma^{-1} = \varpi$, and hence $\sigma\colon \mathscr{E}^{\mathbb{R}}_{X,z_0^*} \xrightarrow{\sim} \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*}$. □

Let $P \in \mathfrak{S}_{z_0^*}$, and consider $[\varpi(P)(z, w, \eta)\, dw] \in \varinjlim_{\boldsymbol{\kappa}} E^{\mathbb{R}}_X(\boldsymbol{\kappa})$. Here we can assume that $\varpi(P)(z, z+w, \eta)$ is holomorphic on V in (5.18). Take $c_0 > 1$ such that $c_0 \operatorname{Re}\zeta_1 \ge |\zeta_1|$ for $|\arg\zeta_1| \le \theta'$. In (5.10), we take $\{\varepsilon_\nu\}_{\nu=1}^\infty \subset \mathbb{R}_{>0}$ and $C > 0$ as

$$1 \gg \varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_\nu \underset{\nu}{\to} 0, \qquad \frac{2C_{\varepsilon_\nu/2}}{\varepsilon_\nu} \le 2^\nu C.$$

Set $\varepsilon_\alpha := \varepsilon_{|\alpha|}$ for short, and we define

$$\begin{aligned} \varpi_{0,\alpha}(P)(z, w_1) &:= \int_d^\infty P_\alpha(z, \zeta_1, \eta_0)\, \mathit{\Gamma}_{|\alpha|}(\zeta_1, c_0\varepsilon_\alpha - w_1)\, e^{-w_1\zeta_1}\, d\zeta_1\,, \\ \varpi_0(P)(z, z+w) &:= \sum_{\alpha \in \mathbb{N}_0^{n-1}} \frac{\alpha!\, \varpi_{0,\alpha}(P)(z, w_1)}{(2\pi\sqrt{-1}\,)^n\, (w')^{\alpha + \mathbf{1}_{n-1}}}\,. \end{aligned}$$

Theorem 5.8. (1) *The* ϖ_0 *induces the mapping* $\varpi_0\colon \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*} \to \mathscr{E}^{\mathbb{R}}_{X,z_0^*}$.

(2) *It follows that* $[\varpi(P)(z, w, \eta)\, dw] = [\varpi_0(P)(z, w)\, dw] \in \varinjlim_{\boldsymbol{\kappa}} E^{\mathbb{R}}_X(\boldsymbol{\kappa})$*, and the fol-*

lowing diagram is commutative:

$$\begin{array}{ccc} \mathscr{S}_{z_0^*}/\mathscr{N}_{z_0^*} & \xrightarrow{\sim} & \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*} \\ \wr & \overset{\varpi_0}{\swarrow} & \sigma\uparrow\downarrow\varpi \\ \mathscr{E}^{\mathbb{R}}_{X,z_0^*} & \xrightarrow{\sim} & \varinjlim_{\boldsymbol{\kappa}} E^{\mathbb{R}}_X(\boldsymbol{\kappa}). \end{array} \tag{5.23}$$

Here, the isomorphism $\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \simeq \mathscr{S}_{z_0^*}/\mathscr{N}_{z_0^*}$ *is induced by*

$$\psi(z,w)\,dw \mapsto \sigma(\psi)(z,\zeta,\eta_0) = \int_{\gamma(0,\eta_0;\varrho,\theta)} \psi(z,z+w)\,e^{\langle w,\zeta\rangle}dw$$

for any fixed $\eta_0 \in S$.

Remark 5.9. (1) The isomorphism $\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \simeq \mathscr{S}_{z_0^*}/\mathscr{N}_{z_0^*}$ is established in [**2**], [**5**] and [**6**].

(2) From the diagram (5.23), we obtain an explicit description of the isomorphism $\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \simeq \varinjlim_{\boldsymbol{\kappa}} E^{\mathbb{R}}_X(\boldsymbol{\kappa})$.

Proof. If $\varepsilon_\nu < \delta_1|w_1|$, $\mathrm{Re}(w_1\zeta_1) \geq \delta_1|w_1\zeta_1|$ and $0 \leq t \leq 1$, we have

$$|e^{-(t(c_0\varepsilon_\nu - w_1)+w_1)\zeta_1}| = e^{-tc_0\varepsilon_\nu \operatorname{Re}\zeta_1 - (1-t)\operatorname{Re}(w_1\zeta_1)} \leq e^{-t\varepsilon_\nu|\zeta_1| - (1-t)\varepsilon_\nu|\zeta_1|} = e^{-\varepsilon_\nu|\zeta_1|}.$$

Thus

$$\begin{aligned} |\Gamma_\nu(\zeta_1, c_0\varepsilon_\nu - w_1)\,e^{-w_1\zeta_1}| &= \Big|\frac{1}{(\nu-1)!}\int_0^{c_0\varepsilon_\nu - w_1} e^{-(s+w_1)\zeta_1}s^{\nu-1}ds\Big| \\ &= \Big|\frac{(c_0\varepsilon_\nu - w_1)^\nu}{(\nu-1)!}\int_0^1 e^{-(t(c_0\varepsilon_\nu - w_1)+w_1)\zeta_1}t^{\nu-1}dt\Big| \\ &\leq \frac{(c_0\varepsilon_\nu + |w_1|)^\nu e^{-\varepsilon_\nu|\zeta_1|}}{(\nu-1)!}\int_0^1 t^{\nu-1}dt = \frac{(c_0\varepsilon_\nu + |w_1|)^\nu e^{-\varepsilon_\nu|\zeta_1|}}{\nu!}. \end{aligned}$$

Set

$$L'_\alpha := \Big\{(z,w_1) \in \mathbb{C}^{n+1};\ \|z\| < r_0,\ |\arg w_1| < \delta' + \frac{\pi}{2},\ \varepsilon_\alpha < \delta_1|w_1|\Big\}.$$

Then taking $h = \varepsilon_\alpha/2$ in (5.10) we have

$$\begin{aligned} \sup_{L'_\alpha}|\varpi_{0,\alpha}(P)(z,w_1)| &\leq \frac{C_{\varepsilon_\alpha/2}(K(c_0\varepsilon_\alpha + |w_1|))^{|\alpha|}}{|\alpha|!}\int_d^\infty e^{-\varepsilon_\alpha|\zeta_1|/2}\,d|\zeta_1| \\ &\leq \frac{2C_{\varepsilon_\alpha/2}(K(c_0\varepsilon_\alpha + |w_1|))^{|\alpha|}}{\varepsilon_\alpha\,|\alpha|!} \leq \frac{C(2K(c_0\varepsilon_\alpha + |w_1|))^{|\alpha|}}{|\alpha|!}. \end{aligned}$$

If $(z,w_1) \in L'_\nu$ and $2K(c_0\varepsilon_\nu + |w_1|) < |w_i|$, we have

$$|\varpi_0(P)(z,z+w)| \le \sum_{\alpha\in\mathbb{N}_0^{n-1}} \frac{C}{(2\pi)^n|(w')^{\mathbf{1}_{n-1}}|}\Big|\Big(\frac{2K(c_0\varepsilon_\alpha+|w_1|)}{w'}\Big)^\alpha\Big| < \infty,$$

and hence $\varpi_0(P)(z,z+w)$ is holomorphic on

$$\begin{aligned}&\bigcup_{\nu=1}^{\infty}\bigcap_{2\le i\le n}\{(z,w)\in\mathbb{C}^{2n};\,(z,w_1)\in L'_\nu\,,\,2K(c_0\varepsilon_\nu+|w_1|)<|w_i|\}\\&\quad=\bigcap_{2\le i\le n}\Big\{(z,w)\in\mathbb{C}^{2n};\,\|z\|<r_0,\,2K|w_1|<|w_i|,\,|\arg w_1|<\delta'+\frac{\pi}{2}\Big\}.\end{aligned}$$

Therefore $\varpi_0(P)(z,w)\,dw$ defines a germ of $\mathscr{E}^{\mathbb{R}}_{X,z_0^*}$. There exists $\delta>0$ such that $\mathrm{Re}(\eta\zeta_1)\ge 2\delta|\eta\zeta_1|$ for any $\eta\in S$ and $|\arg\zeta_1|\le\theta'$. Suppose that $|w_1|<\delta|\eta|$. For any $Z\Subset S$, there exists $N_Z\in\mathbb{N}$ such that $\delta m_Z\ge\varepsilon_\nu$ for any $\nu\ge N_Z$. Thus if $0\le t\le 1$, we have

$$\begin{aligned}|e^{-((1-t)c_0\varepsilon_\nu+t(\eta+w_1))\zeta_1}| &= e^{-(1-t)c_0\varepsilon_\nu\,\mathrm{Re}\,\zeta_1-t\,\mathrm{Re}((\eta+w_1)\zeta_1)} \le e^{-(1-t)\varepsilon_\nu|\zeta_1|-t(2\delta|\eta|-|w_1|)|\zeta_1|}\\&\le e^{-(1-t)\varepsilon_\nu|\zeta_1|-t\delta m_Z|\zeta_1|}\le e^{-\varepsilon_\nu|\zeta_1|}.\end{aligned}$$

Thus we have

$$\begin{aligned}|(\varGamma_\nu(\zeta_1,\eta)-\varGamma_\nu(\zeta_1,c_0\varepsilon_\nu-w_1))\,e^{-w_1\zeta_1}| &= \Big|\frac{1}{(\nu-1)!}\int_{c_0\varepsilon_\nu-w_1}^{\eta}e^{-(s+w_1)\zeta_1}s^{\nu-1}ds\Big|\\&=\Big|\frac{(\eta+w_1-c_0\varepsilon_\nu)^\nu}{(\nu-1)!}\int_0^1 e^{-((1-t)c_0\varepsilon_\nu+t(\eta+w_1))\zeta_1}t^{\nu-1}dt\Big|\\&\le\frac{(|\eta|+\delta|\eta|+c_0\delta m_Z)^\nu e^{-\varepsilon_\nu|\zeta_1|}}{(\nu-1)!}\int_0^1 t^{\nu-1}dt\le\frac{((1+\delta+c_0\delta)|\eta|)^\nu e^{-\varepsilon_\nu|\zeta_1|}}{\nu!}\,.\end{aligned}$$

Set $K_1:=K(1+\delta+c_0\delta)$. Choosing $h=\varepsilon_\alpha/2$ in (5.10) ($|\alpha|\ge N_Z$) we have

$$\begin{aligned}|\varpi_\alpha(P)(z,w_1,\eta)-\varpi_{0,\alpha}(P)(z,w_1)| &\le \frac{C_{\varepsilon_\alpha/2}(K_1|\eta|)^\nu e^{-\varepsilon_\nu|\zeta_1|}}{|\alpha|!}\int_d^\infty e^{-\varepsilon_\alpha|\zeta_1|/2}\,d|\zeta_1|\\&\le\frac{2C_{\varepsilon_\alpha/2}(K_1|\eta|)^{|\alpha|}}{\varepsilon_\alpha\,|\alpha|!}\le\frac{C(2K_1|\eta|)^{|\alpha|}}{|\alpha|!}\,.\end{aligned}$$

Thus if $2K_1|\eta|<|w_i|$, we have

$$\begin{aligned}&|\varpi(P)(z,z+w,\eta)-\varpi_0(P)(z,z+w)|\\&\le\Big|\sum_{|\alpha|<N_Z}\frac{\alpha!(\varpi_\alpha(P)(z,w_1,\eta)-\varpi'_\alpha(P)(z,w_1))}{(2\pi\sqrt{-1}\,)^n\,(w')^{\alpha+\mathbf{1}_{n-1}}}\Big|+\sum_{|\alpha|\ge N_Z}\frac{C}{(2\pi)^n|(w')^{\mathbf{1}_{n-1}}|}\Big|\Big(\frac{2K_1|\eta|}{w'}\Big)^\alpha\Big|\\&<\infty,\end{aligned}$$

and hence $\varpi(P)(z,z+w,\eta)-\varpi_0(P)(z,z+w)$ is holomorphic on

$$\bigcup_{Z\Subset S}\bigcap_{2\le i\le n}\{(z,w,\eta)\in\mathbb{C}^{2n}\times Z;\,\|z\|<r_0,\,|w_1|<\delta|\eta|,\,2K_1|\eta|<|w_i|\}$$

$$= \bigcap_{2\le i\le n} \left\{(z,w,\eta)\in\mathbb{C}^{2n}\times S;\ \|z\|<r_0,\ |w_1|<\delta|\eta|,\ 2K_1|\eta|<|w_i|\right\}.$$

Therefore we have $[\varpi(P)(z,w,\eta)\,dw]=[\varpi_0(P)(z,w)\,dw]\in\varinjlim_{\boldsymbol{\kappa}} E^{\mathbb{R}}_X(\boldsymbol{\kappa})$. □

6. Classical formal symbols with an spparent parameter.

Definition 6.1 (see [**6**], [**9**]). Let t be an indeterminate.

(1) $P(t;z,\zeta)=\sum_{\nu=0}^{\infty}t^\nu P_\nu(z,\zeta)$ is an element of $\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)$ if

(i) $P(t;z,\zeta)\in\Gamma(\Omega_\rho[d_\rho];\mathscr{O}_{T^*X})[\![t]\!]$ for some $d>0$ and $\rho\in\,]0,1[$,

(ii) there exists a constant $A>0$ satisfying the following: for any $h>0$ there exists a constant $C_h>0$ such that

$$|P_\nu(z,\zeta)|\le\frac{C_h\nu!A^\nu e^{h\|\zeta\|}}{\|\zeta\|^\nu}\quad(\nu\in\mathbb{N}_0,\ (z;\zeta)\in\Omega_\rho[d_\rho]).$$

(2) $P(t;z,\zeta)=\sum_{\nu=0}^{\infty}t^\nu P_\nu(z,\zeta,\eta)\in\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)$ is an element of $\widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega)$ if there exists a constant $A>0$ satisfying the following: for any $h>0$ there exists a constant $C_h>0$ such that

$$\Bigl|\sum_{\nu=0}^{m-1}P_\nu(z,\zeta)\Bigr|\le\frac{C_h m!A^m e^{h\|\zeta\|}}{\|\zeta\|^m}\quad(m\in\mathbb{N},\ (z;\zeta)\in\Omega_\rho[d_\rho]).$$

(3) We set

$$\widehat{\mathscr{S}}_{\mathrm{cl},z_0^*}:=\varinjlim_{\Omega}\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)\supset\widehat{\mathscr{N}}_{\mathrm{cl},z_0^*}:=\varinjlim_{\Omega}\widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega).$$

We call each element of $\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)$ (resp. $\widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega)$) a *classical formal symbol* (resp. *classical formal null-symbol*) *on* Ω.

Definition 6.2. Let t be an indeterminate. Then we define a set $\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$ as follows: $P(t;z,\zeta,\eta)=\sum_{\nu=0}^{\infty}t^\nu P_\nu(z,\zeta,\eta)\in\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$ if

(i) $P(t;z,\zeta,\eta)\in\Gamma(\Omega_\rho[d_\rho]\times S;\mathscr{O}_{T^*X\times\mathbb{C}})[\![t]\!]$ for some $d>0$ and $\rho\in\,]0,1[$,

(ii) there exists a constant $A>0$, and for any $Z\Subset S$, $h>0$, there exists $C_{h,Z}>0$ such that

$$\Bigl|\sum_{\nu=0}^{m-1}P_\nu(z,\zeta,\eta)\Bigr|\le\frac{C_{h,Z}\,m!A^m e^{h\|\zeta\|}}{\|\eta\zeta\|^m}\quad(m\in\mathbb{N},\ (z;\zeta,\eta)\in\Omega_\rho[d_\rho]\times Z).\tag{6.1}$$

Definition 6.3. We define a set $\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$ as follows: $P(t;z,\zeta,\eta)=\sum_{\nu=0}^{\infty}t^\nu P_\nu(z,\zeta,\eta)\in\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$ if

(i) $P(t;z,\zeta,\eta)\in\Gamma(\Omega_\rho[d_\rho]\times S;\mathscr{O}_{T^*X\times\mathbb{C}})[\![t]\!]$ for some $d>0$ and $\rho\in\,]0,1[$,

(ii) there exists a constant $A > 0$, and for any $Z \Subset S$, $h > 0$ there exists $C_{h,Z} > 0$ such that

$$|P_\nu(z,\zeta,\eta)| \le \frac{C_{h,Z}\,\nu! A^\nu e^{h\|\zeta\|}}{\|\eta\zeta\|^\nu} \quad (\nu \in \mathbb{N}_0,\ (z;\zeta,\eta) \in \Omega_\rho[d_\rho] \times Z). \tag{6.2}$$

(iii) $\partial_\eta P(t;z,\zeta,\eta) \in \widehat{\mathfrak{N}}_{\rm cl}(\Omega;S)$.

We call each element of $\widehat{\mathfrak{S}}_{\rm cl}(\Omega;S)$ (resp. $\widehat{\mathfrak{N}}_{\rm cl}(\Omega;S)$) a *classical formal symbol* (resp. *classical formal null-symbol*) *on Ω with an apparent parameter in S*.

Lemma 6.4. $\widehat{\mathfrak{N}}_{\rm cl}(\Omega;S) \subset \widehat{\mathfrak{S}}_{\rm cl}(\Omega;S)$.

Proof. We assume (6.1). Take C', $B > 0$ as $2(\nu+1)A^{\nu+1} \le d_\rho C' B^\nu$ and $2A^\nu \le C'B^\nu$ for any $\nu \in \mathbb{N}_0$. Then for any $\nu \in \mathbb{N}$ and $(z;\zeta,\eta) \in \Omega_\rho[d_\rho] \times Z$ we have

$$\begin{aligned} |P_\nu(z,\zeta,\eta)| &= \Big|\sum_{i=0}^{\nu} P_i(z,\zeta,\eta) - \sum_{i=0}^{\nu-1} P_i(z,\zeta,\eta)\Big| \le \Big|\sum_{i=0}^{\nu} P_i(z,\zeta,\eta)\Big| + \Big|\sum_{i=0}^{\nu-1} P_i(z,\zeta,\eta)\Big| \\ &\le \frac{C_{h,Z}\,(\nu+1)! A^{\nu+1} e^{h\|\zeta\|}}{\|\eta\zeta\|^{\nu+1}} + \frac{C_{h,Z}\,\nu! A^\nu e^{h\|\zeta\|}}{\|\eta\zeta\|^\nu} \le \frac{C' C_{h,Z}\,\nu! B^\nu e^{h\|\zeta\|}}{|\eta|\|\eta\zeta\|^\nu}. \end{aligned}$$

Next, for any $Z \Subset S$, take δ' and Z' as in (4.3). Then by the Cauchy inequality, for any $h > 0$ there exists a constant $C_{h,Z'} > 0$ such that for any $m \in \mathbb{N}$ and $(z;\zeta,\eta) \in \Omega_\rho[d_\rho] \times Z$,

$$\Big|\sum_{\nu=0}^{m-1} \partial_\eta P_\nu(z,\zeta,\eta)\Big| \le \frac{1}{\delta'|\eta|} \sup_{|\eta-\eta'|=\delta'|\eta|} \Big|\sum_{\nu=0}^{m-1} P_\nu(z,\zeta,\eta')\Big| \le \frac{C_{h,Z'}\, m!(2A)^m e^{h\|\zeta\|}}{\delta' m_Z \|\eta\zeta\|^m}. \qquad \square$$

For any $z_0^* \in \dot{T}^*X$, we set

$$\widehat{\mathfrak{S}}_{{\rm cl},z_0^*} := \varinjlim_{\Omega,S} \widehat{\mathfrak{S}}_{\rm cl}(\Omega;S) \supset \widehat{\mathfrak{N}}_{{\rm cl},z_0^*} := \varinjlim_{\Omega,S} \widehat{\mathfrak{N}}_{\rm cl}(\Omega;S).$$

Proposition 6.5. *Let $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}_{\rm cl}(\Omega;S)$. Then for any $\eta_0 \in S$, it follows that $P(t;z,\zeta,\eta_0) \in \widehat{\mathscr{S}_{\rm cl}}(\Omega)$ and $P(t;z,\zeta,\eta) - P(t;z,\zeta,\eta_0) \in \widehat{\mathfrak{N}}_{\rm cl}(\Omega;S)$.*

Proof. Set $A_0 := A/|\eta_0| > A$. For any $h > 0$, there exists a constant $C_{h,\eta_0} > 0$ such that for any $(z;\zeta) \in \Omega_\rho[d_\rho]$ the following holds:

$$|P_\nu(z,\zeta,\eta_0)| \le \frac{C_{h,\eta_0}\,\nu! A_0^\nu e^{h\|\zeta\|}}{\|\zeta\|^\nu};$$

that is, $P(t;z,\zeta,\eta_0) \in \widehat{\mathscr{S}_{\rm cl}}(\Omega)$. For any $Z \Subset S$, let $Z' \Subset S$ be the convex hull of $Z \cup \{\eta_0\}$. Since

$$P_\nu(z,\zeta,\eta) = P_\nu(z,\zeta,\eta_0) + \int_{\eta_0}^{\eta} \partial_\eta P_\nu(z,\zeta,\tau)\,d\tau,$$

and $\partial_\eta P(t;z,\zeta,\eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$, there exists $A>0$ so that for any $h>0$ we can find a constant $C_{h,Z'}>0$ such that for any $m \in \mathbb{N}$ and $(z;\zeta,\eta) \in \Omega_\rho[d_\rho] \times Z \subset \Omega_\rho[d_\rho] \times Z'$ the following holds: if $|\eta| \geq |\eta_0|$

$$\Big|\sum_{\nu=0}^{m-1}(P_\nu(z,\zeta,\eta) - P_\nu(z,\zeta,\eta_0))\Big| = \Big|\sum_{\nu=0}^{m-1}\int_{\eta_0}^{\eta} \partial_\eta P_\nu(z,\zeta,\tau)\,d\tau\Big| = \Big|\int_{\eta_0}^{\eta}\sum_{\nu=0}^{m-1}\partial_\eta P_\nu(z,\zeta,\tau)\,d\tau\Big|$$
$$\leq |\eta-\eta_0|\frac{C_{h,Z'}\,m!A^m e^{h\|\zeta\|}}{\|\eta_0\zeta\|^m} \leq \frac{rC_{h,Z'}\,m!A_0^m e^{h\|\zeta\|}}{\|\eta\zeta\|^m},$$

and if $|\eta| \leq |\eta_0|$,

$$\Big|\sum_{\nu=0}^{m-1}(P_\nu(z,\zeta,\eta) - P_\nu(z,\zeta,\eta_0))\Big| \leq \frac{rC_{h,Z'}\,m!A^m e^{h\|\zeta\|}}{\|\eta\zeta\|^m} \leq \frac{rC_{h,Z'}\,m!A_0^m e^{h\|\zeta\|}}{\|\eta\zeta\|^m}.$$

Hence $P(t;z,\zeta,\eta) - P(t;z,\zeta,\eta_0) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$. □

Since $\|\eta\zeta\| < \|\zeta\|$ for any $\eta \in S$, we can regard that

$$\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega) = \{P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S);\ \partial_\eta P(t;z,\zeta,\eta) = 0\} \subset \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S),$$
$$\widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega) = \widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega) \cap \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S) \subset \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S).$$

Hence we have an injective mapping $\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)/\widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega) \hookrightarrow \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$. Moreover

Proposition 6.6. $\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)/\widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega) \simeq \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$.

Proof. Let us take any $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$. We fix $\eta_0 \in S$. Then by Proposition 6.5, we have $P(t;z,\zeta,\eta_0) \in \widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)$ and $[P(t;z,\zeta,\eta)] = [P(t;z,\zeta,\eta_0)] \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$. □

Note that an element of $\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S) \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$ is $\sum_{\nu=0}^{\infty} t^\nu P_\nu(z,\zeta,\eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$ such that $P_\nu(z,\zeta,\eta) = 0$ for $\nu \geq 1$ and $P_0(z,\zeta,\eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$. The space $\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S) \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$ is similar.

Proposition 6.7. *The following hold*:

$$\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S) \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}}) = \mathfrak{N}(\Omega;S),$$
$$\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S) \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}}) = \mathfrak{S}(\Omega;S).$$

Proof. Let $P(z,\zeta,\eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S) \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$. Then there exists $A>0$ and for any $Z \Subset S$, $h>0$ there exists $C_{h,Z}>0$ such that

$$|P(z,\zeta,\eta)| \leq \frac{C_{h,Z}\,\nu!A^\nu e^{h\|\zeta\|}}{\|\eta\zeta\|^\nu} \quad (\nu \in \mathbb{N}_0,\ (z;\zeta,\eta) \in \Omega_\rho[d_\rho] \times Z).$$

Then for any ζ with $\|\zeta\| \geq d_\rho$, taking ν as the integer part of $\|\eta\zeta\|/A$, we have

$$|P(z,\zeta,\eta)| \leq C_{h,Z}\Big(\frac{2\pi\|\eta\zeta\|}{A}\Big)^{1/2} e^{(h-|\eta|/A)\|\zeta\|-1}.$$

We choose h as $2hA = m_Z$. Hence $e^{(h-|\eta|/A)\|\zeta\|} \leq e^{-\|\eta\zeta\|/(2A)}$. Then we can find δ, $C'_Z > 0$ such that

$$C_{h,Z}\Big(\frac{2\pi\|\eta\zeta\|}{A}\Big)^{1/2} e^{-\|\eta\zeta\|/(2A)} \leq C'_Z e^{-\delta\|\eta\zeta\|}.$$

Here δ does not depend on Z. Thus (4.2) holds. Conversely, assume (4.2). Set $A := 1/\delta$. Then for any $m \in \mathbb{N}_0$ and $h > 0$, we have

$$|P(z,\zeta,\eta)| \leq C_Z e^{-\delta\|\eta\zeta\|} \leq \frac{C_Z m! A^m}{\|\eta\zeta\|^m} \leq \frac{C_Z m! A^m e^{h\|\zeta\|}}{\|\eta\zeta\|^m}.$$

Hence $P(z,\zeta,\eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S) \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$.

Next, let $P(z,\zeta,\eta) \in \mathfrak{S}(\Omega;S)$. Then $P(z,\zeta,\eta) + t\cdot 0 + t^2\cdot 0 + \cdots$ satisfies (6.2), and we have $\partial_\eta P(z,\zeta,\eta) \in \mathfrak{N}(\Omega;S) = \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S) \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$. Therefore we have $\mathfrak{S}(\Omega;S) \subset \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S) \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$. The converse inclusion is proved in the same way. □

Moreover

THEOREM 6.8. *Let $\Omega \underset{\mathrm{conic}}{\Subset} T^*X$ be any sufficiently small neighborhood of $z_0^* = (z_0;\zeta_0) \in \dot{T}^*X$. Then for any $P(t;z,\zeta,\eta) = \sum_{\nu=0}^\infty t^\nu P_\nu(z,\zeta,\eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$, there exists $P(z,\zeta) \in \mathscr{S}(\Omega)$ such that*

$$P(t;z,\zeta,\eta) - P(z,\zeta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S).$$

PROOF. We may assume that $\zeta_0 = (1,0,\dots,0)$, $\Omega \underset{\mathrm{conic}}{\Subset} \{(z;\zeta);\, \mathrm{Re}\,\zeta_1 \geq 2\delta_0|\zeta_1|\}$ for some $0 < \delta_0 < 1/2$, $\|\zeta\| = |\zeta_1|$ on $\Omega_\rho[d_\rho]$, and $P_\nu(z,\zeta,\eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$. Fix $\eta_0 \in S$. Thus $P(t;z,\zeta,\eta_0) \in \widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)$ by Proposition 6.5. Set $A_0 := A/|\eta_0|$. We take a constant a as

$$0 < a < \min\Big\{1, \frac{1}{2A_0}\Big\} \tag{6.3}$$

and set $B := \max\{1/\delta_0 a, A_0/2\delta_0\}$. Using the function $\Gamma_\nu(\tau,a)$ in Definition 5.2, we set

$$P(z,\zeta) := \sum_{\nu=0}^\infty P_\nu(z,\zeta,\eta_0)\, \zeta_1^\nu\, \Gamma_\nu(\zeta_1,a).$$

By (6.3) and (5.8) for any $h > 0$ on $\Omega_\rho[d_\rho]$ we have

$$|P(z,\zeta)| \le \sum_{\nu=0}^{\infty} \frac{C_h \nu! A_0^{\nu} e^{h\|\zeta\|} \, |\zeta_1|^{\nu} \, a^{\nu}}{\|\zeta\|^{\nu} \, \nu!} = C_h e^{h\|\zeta\|} \sum_{\nu=0}^{\infty} (aA_0)^{\nu} \le 2C_h e^{h\|\zeta\|}.$$

Therefore $P(z,\zeta) \in \mathscr{S}(\Omega)$. On the other hand, for any $m \in \mathbb{N}$, on $\Omega_\rho[d_\rho]$ we have

$$\begin{aligned} &\left| P(z,\zeta) - \sum_{\nu=0}^{m-1} P_\nu(z,\zeta,\eta_0) \right| \\ &\quad \le \sum_{\nu=1}^{m-1} |P_\nu(z,\zeta,\eta_0)| \, |1 - \zeta_1^{\nu} \varGamma_\nu(\zeta_1,a)| + \sum_{\nu=m}^{\infty} |P_\nu(z,\zeta,\eta_0) \, \zeta_1^{\nu} \varGamma_\nu(\zeta_1,a)|. \end{aligned}$$

For any $1 \le \nu \le m-1$, we have $e^{-a\delta_0|\zeta_1|} = e^{-a\delta_0\|\zeta\|} \le (m-\nu)!/(a\delta_0\|\zeta\|)^{m-\nu}$. Thus by (5.9) and (6.3) we have

$$\begin{aligned} &\sum_{\nu=1}^{m-1} |P_\nu(z,\zeta,\eta_0)| \, |1 - \zeta_1^{\nu} \, \varGamma_\nu(\zeta_1,a)| \le \sum_{\nu=1}^{m-1} \frac{\delta_0 C_h \nu! A_0^{\nu} e^{h\|\zeta\|} e^{-a\delta_0|\zeta_1|}}{(\delta_0\|\zeta\|)^{\nu}} \\ &\quad \le \sum_{\nu=1}^{m-1} \frac{\delta_0 C_h \nu! \, (m-\nu)! A_0^{\nu} e^{h\|\zeta\|}}{(\delta_0\|\zeta\|)^{\nu} \, (a\delta_0\|\zeta\|)^{m-\nu}} \le \frac{\delta_0 C_h m! B^m e^{h\|\zeta\|}}{\|\zeta\|^m} \sum_{\nu=1}^{m-1} (aA_0)^{\nu} \le \frac{\delta_0 C_h m! B^m e^{h\|\zeta\|}}{\|\zeta\|^m}. \end{aligned}$$

Next, since

$$\int_0^a e^{-2\delta_0 s\|\zeta\|} s^{k+m-1} \, ds < a^k \int_0^{\infty} e^{-2\delta_0 s\|\zeta\|} s^{m-1} \, ds = \frac{a^k (m-1)!}{(2\delta_0\|\zeta\|)^m},$$

we have

$$\begin{aligned} \sum_{\nu=m}^{\infty} &|P_\nu(z,\zeta,\eta_0) \, \zeta_1^{\nu} \, \varGamma_\nu(\zeta_1,a)| \le C_h e^{h\|\zeta\|} \sum_{\nu=m}^{\infty} \frac{\nu A_0^{\nu} |\zeta_1|^{\nu}}{\|\zeta\|^{\nu}} \int_0^a e^{-2\delta_0 s\|\zeta\|} \, s^{\nu-1} \, ds \\ &= C_h e^{h\|\zeta\|} \sum_{k=0}^{\infty} (k+m) A_0^{k+m} \int_0^a e^{-2\delta_0 s\|\zeta\|} s^{k+m-1} \, ds \\ &\le \frac{C_h (m-1)! A_0^m e^{h\|\zeta\|}}{(2\delta_0\|\zeta\|)^m} \sum_{k=0}^{\infty} (k+m)(aA_0)^k \\ &\le \frac{C_h (m-1)! B^m e^{h\|\zeta\|}}{\|\zeta\|^m} \, 2(m+1) \le \frac{4C_h \, m! B^m e^{h\|\zeta\|}}{\|\zeta\|^m}. \end{aligned}$$

Thus by Proposition 6.5, we have

$$\begin{aligned} P(t;z,\zeta,\eta) - P(z,\zeta) &= (P(t;z,\zeta,\eta) - P(t;z,\zeta,\eta_0)) + (P(t;z,\zeta,\eta_0) - P(z,\zeta)) \\ &\in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S) + \widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega) \subset \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S). \end{aligned}$$ □

Corollary 6.9. *Let $\Omega \underset{\text{conic}}{\Subset} T^*X$ be any sufficiently small neighborhood of $z_0^* = (z_0;\zeta_0) \in \dot{T}^*X$. Then $\mathfrak{S}(\Omega;S)/\mathfrak{N}(\Omega;S) \xrightarrow{\sim} \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$.*

Definition 6.10. As in the case of $\mathfrak{S}(\Omega; S)$, for any $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$ we set

$$:P(t; z, \zeta, \eta): := P(t; z, \zeta, \eta) \bmod \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$$

which is also called the *normal product* or the *Wick product* of $P(t; z, \zeta, \eta)$.

Take $\Omega \underset{\text{conic}}{\Subset} T^*\mathbb{C}^n$. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be local coordinates on a neighborhood of $\mathrm{Cl}\,\pi(\Omega) \subset X$, and $(z; \zeta)$, $(w; \lambda)$ corresponding local coordinates on a neighborhood of $\mathrm{Cl}\,\Omega$. Let $z = \Phi(w)$ be the coordinate transformation. We define $J^*_\Phi(z', z)$ by the relation $\Phi^{-1}(z') - \Phi^{-1}(z) = J^*_\Phi(z', z)(z' - z)$. Then ${}^{\mathsf{t}}J^*_\Phi(z, z)\lambda = {}^{\mathsf{t}}\bigl[(\partial w/\partial z)(z)\bigr]\lambda = \zeta$. Here $\partial w/\partial z$ stands for the Jacobian matrix of Φ^{-1}. Let $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$ with respect to $(z; \zeta)$. Then we set $\Phi^* P(t; w, \lambda, \eta) = \sum_{\nu=0}^{\infty} t^\nu \Phi^* P_\nu(w, \lambda, \eta)$ by

$$\Phi^* P(t; w, \lambda, \eta) := e^{t\langle \partial_{\zeta'}, \partial_{z'}\rangle} P(t; \Phi(w), \zeta' + {}^{\mathsf{t}}J^*_\Phi(\Phi(w) + z', \Phi(w))\lambda, \eta)\Big|_{\substack{z'=0\\ \zeta'=0}},$$

i.e.

$$(\Phi^* P)_\nu(w, \lambda, \eta) = \sum_{k+|\alpha|=\nu} \frac{1}{\alpha!} \partial_{\zeta'}^{\alpha} \partial_{z'}^{\alpha} P_k(\Phi(w), \zeta' + {}^{\mathsf{t}}J^*_\Phi(\Phi(w) + z', \Phi(w))\lambda, \eta)\Bigg|_{\substack{z'=0\\ \zeta'=0}}.$$

Theorem 6.11. *Under the notation above, the following hold.*

(1) $\Phi^* P(t; w, \lambda, \eta)$ *defines an element of* $\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$ *with respect to* $(w; \lambda)$*. Moreover if* $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$*, it follows that* $\Phi^* P(t; w, \lambda, \eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$.

(2) *If* $\Phi = \mathrm{id}$ (*the identity*), id^* *is the identity, and if* $z = \Phi(w)$ *and* $w = \Psi(v)$ *are complex coordinate transformations,* $\Psi^*\Phi^* = (\Phi\Psi)^*$ *holds.*

Proof. (1) Assume that $P_\nu(z, \zeta, \eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$. Note that $\partial_\eta \Phi^* P(t; w, \lambda, \eta) = \Phi^*(\partial_\eta P)(t; w, \lambda, \eta)$. We may assume that on a neighborhood of Ω_ρ, there exist $c > 1$ and $0 < c' < 1$ such that

$$c'\|\lambda\| < \|\zeta\| = \left\| {}^{\mathsf{t}}\left[\frac{\partial w}{\partial z}(z)\right]\lambda \right\| < c\|\lambda\|. \tag{6.4}$$

We can choose $0 < \varepsilon < 1$ such that $c' - c\varepsilon > 0$, and that there exists $\delta > 0$ such that if $\|z'\| \le \delta$, it follows that

$$\begin{cases} \|{}^{\mathsf{t}}J^*_\Phi(z + z', z)\lambda - \zeta\| \le \varepsilon\|\zeta\|, \\ c'\|\lambda\| \le \|{}^{\mathsf{t}}J^*_\Phi(z + z', z)\lambda\| \le c\|\lambda\|. \end{cases} \tag{6.5}$$

Moreover take $\rho' \in \left]0, \rho\right[$ and replacing ε, $\delta > 0$ if necessary, setting

$$\Omega'_{\rho'} := \bigcup_{(z;\zeta)\in\Omega_{\rho'}} \{(z; \zeta' + {}^{\mathsf{t}}J^*_\Phi(z + z', z)\lambda);\ \|z'\| \le \delta,\ \|\zeta'\| \le \varepsilon\|\zeta\|\},$$

we have

$$\Omega'_{\rho'}[d_{\rho'}] \underset{\text{conic}}{\Subset} \Omega_\rho[d_\rho]. \tag{6.6}$$

Thus for any $h > 0$, $Z \Subset S$ and $(z; \zeta' + {}^{\mathsf{t}}J^*_\Phi(z+z', z)\lambda, \eta) \in \Omega'_{\rho'}[d_{\rho'}] \times Z$, we have

$$|P_k(z, \zeta' + {}^{\mathsf{t}}J^*_\Phi(z+z', z)\lambda, \eta)| \le \frac{C_{h,Z}\, k! A^k e^{h\|\zeta' + {}^{\mathsf{t}}J^*_\Phi(z+z',z)\lambda\|}}{\|\eta(\zeta' + {}^{\mathsf{t}}J^*_\Phi(z+z', z)\lambda)\|^k}.$$

Set $\Phi(w) := z$. If $(z; \zeta, \eta) \in \Omega_{\rho'}[d_{\rho'}] \times Z$, we have

$$\begin{aligned}
&\frac{1}{\alpha!}\Big|\partial^{\alpha}_{\zeta'}\partial^{\alpha}_{z'} P_k(z, \zeta' + {}^{\mathsf{t}}J^*_\Phi(z+z', z)\lambda, \eta)\big|_{\substack{z'=0\\ \zeta'=0}}\Big| \\
&\quad = \left|\frac{\alpha!}{(2\pi\sqrt{-1}\,)^{2n}} \oint\limits_{\substack{|z'_i|=\delta \\ |\zeta'_i|=\varepsilon\|\zeta\|}} \frac{P_k(z, \zeta' + {}^{\mathsf{t}}J^*_\Phi(z+z', z)\lambda, \eta)\, dz' d\zeta'}{(\zeta')^{\alpha+\mathbf{1}_n}\,(z')^{\alpha+\mathbf{1}_n}}\right| \\
&\quad \le \frac{\alpha!}{(\varepsilon\delta\|\zeta\|)^{|\alpha|}} \sup_{\substack{|z'_i|=\delta \\ |\zeta'_i|=\varepsilon\|\zeta\|}} |P_k(z, \zeta' + {}^{\mathsf{t}}J^*_\Phi(z+z', z)\lambda, \eta)| \\
&\quad \le \frac{C_{h,Z}\,\alpha!\, k! A^k}{(\varepsilon\delta c'\|\lambda\|)^{|\alpha|}} \sup_{\substack{|z'_i|=\delta \\ |\zeta'_i|=\varepsilon\|\zeta\|}} \frac{e^{h\|\zeta' + {}^{\mathsf{t}}J^*_\Phi(z+z',z)\lambda\|}}{\|\eta(\zeta' + {}^{\mathsf{t}}J^*_\Phi(z+z', z)\lambda)\|^k} \\
&\quad \le \frac{C_{h,Z}\,\alpha!\, k! A^k e^{hc(1+\varepsilon)\|\lambda\|}}{(\varepsilon\delta c'\|\lambda\|)^{|\alpha|}((c'-c\varepsilon)\|\eta\lambda\|)^k} \le \frac{C_{h,Z}\,\alpha!\, k! A^k e^{2hc\|\lambda\|}}{(\varepsilon\delta c'\|\lambda\|)^{|\alpha|}((c'-c\varepsilon)\|\eta\lambda\|)^k}.
\end{aligned} \tag{6.7}$$

Set $B := 2/\varepsilon\delta c'$ and replacing ε, $\delta > 0$ as $C := \varepsilon\delta c' A/2(c' - c\varepsilon) < 1$. Then if $\|\zeta\| \ge c'\|\lambda\| \ge (\nu+1)d_{\rho'}$, we have

$$\begin{aligned}
|(\Phi^*P)_\nu(w, \lambda, \eta)| &\le C_{h,Z}\, e^{2hc\|\lambda\|} \sum_{k=0}^{\nu} \frac{k! A^k}{((c'-c\varepsilon)\|\eta\lambda\|)^k} \sum_{|\alpha|=\nu-k} \frac{\alpha!}{(\varepsilon\delta c'\|\lambda\|)^{|\alpha|}} \\
&\le C_{h,Z} e^{2hc\|\lambda\|} \sum_{k=0}^{\nu} \frac{k! A^k}{((c'-c\varepsilon)\|\eta\lambda\|)^k} \frac{2^{n+\nu-k-1}\,(\nu-k)!}{(\varepsilon\delta c'\|\lambda\|)^{\nu-k}} \\
&\le \frac{2^{n-1} C_{h,Z}\, \nu! B^\nu e^{2hc\|\lambda\|}}{\|\eta\lambda\|^\nu} \sum_{k=0}^{\infty} C^k \le \frac{2^{n-1} C_{h,Z}\, \nu! B^\nu e^{2hc\|\lambda\|}}{(1-C)\,\|\eta\lambda\|^\nu}.
\end{aligned}$$

Next, if $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$, for any $m \in \mathbb{N}$

$$\frac{1}{\alpha!}\left|\sum_{k=0}^{m-1} \partial^{\alpha}_{\zeta'}\partial^{\alpha}_{z'} P_k(z, \zeta' + {}^{\mathsf{t}}J^*_\Phi(z+z', z)\lambda, \eta)\Big|_{\substack{z'=0\\ \zeta'=0}}\right| \le \frac{C_{h,Z}\,\alpha!\, m! A^m e^{2hc\|\lambda\|}}{(\varepsilon\delta c'\|\lambda\|)^{|\alpha|}((c'-c\varepsilon)\|\eta\lambda\|)^m}.$$

Hence setting $\Phi(w) = z$, we have

$$\left|\sum_{\nu=0}^{m-1} (\Phi^*P)_\nu(w, \lambda, \eta)\right| = \left|\sum_{|\alpha|=0}^{m-1} \sum_{k=0}^{m-1-|\alpha|} \frac{1}{\alpha!} \partial^{\alpha}_{\zeta'}\partial^{\alpha}_{z'} P_k(z, \zeta' + {}^{\mathsf{t}}J^*_\Phi(z+z', z)\lambda, \eta)\Big|_{\substack{z'=0\\ \zeta'=0}}\right|$$

$$\leq C_{h,Z} e^{2hc\|\lambda\|} \sum_{|\alpha|=0}^{m-1} \frac{\alpha!\,(m-|\alpha|)! A^{m-|\alpha|}}{(\varepsilon\delta c'\|\lambda\|)^{|\alpha|}((c'-c\varepsilon)\|\eta\lambda\|)^{m-|\alpha|}}$$
$$\leq \frac{C_{h,Z} e^{2hc\|\lambda\|}}{\|\eta\lambda\|^m} \sum_{\nu=0}^{m-1} \frac{2^{n+\nu-1} m! A^{m-\nu}}{(\varepsilon\delta c')^{\nu}(c'-c\varepsilon)^{m-\nu}}$$
$$= \frac{2^{n-1} C_{h,Z}\, m! B^m e^{2hc\|\lambda\|}}{\|\eta\lambda\|^m} \sum_{\nu=0}^{m-1} C^{m-\nu} \leq \frac{2^{n-1} C_{h,Z}\, m! B^m e^{2hc\|\lambda\|}}{(1-C)\|\eta\lambda\|^m}\,.$$

(2) It is trivial that id^* is the identity. In order to prove that $\Psi^*\Phi^* = (\Phi\Psi)^*$, it is enough to show that $\Psi^*\Phi^* P(t;v,\xi,\eta) = (\Phi\Psi)^* P(t;v,\xi,\eta)$ for any $P(t;z,\zeta,\eta) \in \mathscr{O}_{T^*X\times S,(z_0,\zeta_0,\eta)}[\![t]\!]$ for any fixed $(z_0;\zeta_0) \in \mathrm{Cl}\,\Omega$. Note that ${}^{\mathsf{t}}\big[(\partial w/\partial z)(z)\big]\lambda = \zeta$ and ${}^{\mathsf{t}}\big[(\partial v/\partial w)(w)\big]\xi = \lambda$.

Lemma 6.12 (see [**4**], [**23**]). *For any n-tuple $A(t;z,\zeta) = (A_1(z,\zeta),\dots,A_n(z,\zeta))$ of holomorphic functions, and holomorphic function $Q(z,\zeta)$, the following holds*:

$$e^{\langle\partial_\zeta,\partial_z\rangle} Q(z,\zeta)\, e^{\langle z, A(z,\zeta)\rangle}\big|_{z=0} = e^{\langle\partial_\zeta,\partial_z\rangle} \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}\, \partial_\zeta^\alpha \big(Q(z,\zeta)A(z,\zeta)^\alpha\big)\Big|_{z=0}.$$

Proof. We have:

$$\begin{aligned} e^{\langle\partial_\zeta,\partial_z\rangle} Q(z,\zeta)\, e^{\langle z, A(z,\zeta)\rangle}\big|_{z=0} &= \sum_{\alpha,\beta\in\mathbb{N}_0^n} \frac{1}{\alpha!\,\beta!}\, \partial_\zeta^\beta \partial_z^\beta (Q(z,\zeta)\, z^\alpha A(z,\zeta)^\alpha)\Big|_{z=0} \\ &= \sum_{\alpha,\beta}\sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \frac{1}{\alpha!\,\beta!}\, \partial_\zeta^\beta \Big(\frac{\partial^\gamma z^\alpha}{\partial z^\gamma}\, \partial_z^{\beta-\gamma}(Q(z,\zeta)\,A(z,\zeta)^\alpha)\Big)\Big|_{z=0} \\ &= \sum_{\alpha\leq\beta} \frac{1}{\alpha!\,(\beta-\alpha)!}\, \partial_\zeta^\beta \partial_z^{\beta-\alpha}(Q(z,\zeta)\,A(z,\zeta)^\alpha)\Big|_{z=0} \\ &= e^{\langle\partial_\zeta,\partial_z\rangle} \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}\, \partial_\zeta^\alpha (Q(z,\zeta)\,A(z,\zeta)^\alpha)\Big|_{z=0}. \qquad \square \end{aligned}$$

Remark 6.13. Take $\tau \in \mathbb{C}$ and $z' \in \mathbb{C}^n$ such that $|\tau z'|$ is sufficiently small, and hence $J_\Phi^*(z+\tau z', z)$ is holomorphic. Then by Lemma 6.12 for any holomorphic function $Q(\zeta)$, we have

$$\begin{aligned} & e^{\langle\partial_{\zeta'},\partial_{z'}\rangle} Q(\zeta')\, e^{\langle z', {}^{\mathsf{t}}J_\Phi^*(z+\tau z',z)\lambda-\zeta_0\rangle}\big|_{\substack{z'=0\\ \zeta'=0}} \\ &= e^{\langle\partial_{\zeta'},\partial_{z'}\rangle} \sum_\alpha \frac{1}{\alpha!}\, \partial_{\zeta'}^\alpha \big(Q(\zeta')({}^{\mathsf{t}}J_\Phi^*(z+\tau z',z)\lambda-\zeta_0)^\alpha\big)\Big|_{\substack{z'=0\\ \zeta'=0}} \\ &= e^{\langle\partial_{\zeta'},\partial_{z'}\rangle} Q(\zeta' + {}^{\mathsf{t}}J_\Phi^*(z+\tau z',z)\lambda-\zeta_0)\big|_{\substack{z'=0\\ \zeta'=0}} \\ &= \sum_\alpha \frac{1}{\alpha!}\, \partial_{z'}^\alpha \partial_{\zeta'}^\alpha Q(\zeta' + {}^{\mathsf{t}}J_\Phi^*(z+\tau z',z)\lambda-\zeta_0)\Big|_{\substack{z'=0\\ \zeta'=0}} \end{aligned}$$

$$= \sum_{\alpha} \frac{\tau^{|\alpha|}}{\alpha!}\, \partial_{z'}^{\alpha}\partial_{\zeta'}^{\alpha} Q(\zeta' + {}^{\mathsf{t}}J_{\Phi}^{*}(z+z',z)\lambda - \zeta_0)\Big|_{\substack{z'=0\\ \zeta'=0}}.$$

Hence as a *formal power series with respect to* t, we have

$$\begin{aligned} e^{\langle \partial_{\zeta'},\partial_{z'}\rangle} Q(\zeta')\, e^{\langle z', {}^{\mathsf{t}}J_{\Phi}^{*}(z+tz',z)\lambda-\zeta_0\rangle}\big|_{\substack{z'=0\\ \zeta'=0}} &= \sum_{\alpha} \frac{t^{|\alpha|}}{\alpha!}\, \partial_{z'}^{\alpha}\partial_{\zeta'}^{\alpha} Q(\zeta' + {}^{\mathsf{t}}J_{\Phi}^{*}(z+z',z)\lambda - \zeta_0)\Big|_{\substack{z'=0\\ \zeta'=0}} \\ &= e^{t\langle \partial_{\zeta'},\partial_{z'}\rangle} Q(\zeta' + {}^{\mathsf{t}}J_{\Phi}^{*}(z+z',z)\lambda - \zeta_0)\big|_{\substack{z'=0\\ \zeta'=0}}. \end{aligned}$$

In what follows, we use this type of arguments.

By Lemma 6.12 and Remark 6.13, setting $\Phi(w) = z$ we have

$$\begin{aligned} &e^{\langle \partial_{\zeta'},\partial_{z'}\rangle} P(t;z,\zeta_0+\zeta',\eta)\, e^{\langle J_{\Phi}^{*}(z+tz',z)z',\lambda\rangle} e^{-\langle z',\zeta_0\rangle}\big|_{\substack{z'=0\\ \zeta'=0}} \\ &\quad= \sum_{\nu=0}^{\infty} t^{\nu} e^{\langle \partial_{\zeta'},\partial_{z'}\rangle} P_{\nu}(z,\zeta_0+\zeta',\eta)\, e^{\langle z', {}^{\mathsf{t}}J_{\Phi}^{*}(z+tz',z)\lambda-\zeta_0\rangle}\Big|_{\substack{z'=0\\ \zeta'=0}} \\ &\quad= \sum_{\nu=0}^{\infty} t^{\nu} e^{t\langle \partial_{\zeta'},\partial_{z'}\rangle} P_{\nu}(z,\zeta' + {}^{\mathsf{t}}J_{\Phi}^{*}(z+z',z)\lambda,\eta)\Big|_{\substack{z'=0\\ \zeta'=0}} \\ &\quad= e^{t\langle \partial_{\zeta'},\partial_{z'}\rangle} P(t;z,\zeta' + {}^{\mathsf{t}}J_{\Phi}^{*}(z+z',z)\lambda,\eta)\big|_{\substack{z'=0\\ \zeta'=0}} = \Phi^{*}P(t;w,\lambda,\eta). \end{aligned}$$

On the other hand, we have

$$w + J_{\Phi}^{*}(z+z',z)z' = \Phi^{-1}(z) + \Phi^{-1}(z+z') - \Phi^{-1}(z) = \Phi^{-1}(z+z').$$

Hence we have

$$\begin{aligned} &J_{\Psi}^{*}(w + J_{\Phi}^{*}(z+z',z)z', w)\, J_{\Phi}^{*}(z+z',z)z' \\ &\quad= J_{\Psi}^{*}(\Phi^{-1}(z+z'), \Phi^{-1}(z))\, J_{\Phi}^{*}(z+z',z)(z+z'-z) \\ &\quad= J_{\Psi}^{*}(\Phi^{-1}(z+z'), \Phi^{-1}(z))\, (\Phi^{-1}(z+z') - \Phi^{-1}(z)) \\ &\quad= \Psi^{-1}\Phi^{-1}(z+z') - \Psi^{-1}\Phi^{-1}(z) = J_{\Phi\Psi}^{*}(z+z',z)z'. \end{aligned}$$

Thus as a formal power series with respect to t, we have

$$J_{\Psi}^{*}(w + J_{\Phi}^{*}(z+tz',z)tz', w)\, J_{\Phi}^{*}(z+tz',z)z' = J_{\Phi\Psi}^{*}(z+tz',z)z'.$$

Set $z_0 := \Phi(w_0)$ and ${}^{\mathsf{t}}\big[(\partial w/\partial z)(z_0)\big]\lambda_0 = \zeta_0$. Then on a neighborhood of $(w_0;\eta_0)$, setting $z = \Phi(w) = \Phi\Psi(v)$ we have

$$\begin{aligned} \Psi^{*}\Phi^{*}P(t;v,\xi,\eta) &= e^{\langle \partial_{\lambda'},\partial_{w'}\rangle} \Phi^{*}P(t;w,\lambda_0+\lambda',\eta)\, e^{\langle J_{\Psi}^{*}(w+tw',w)w',\xi\rangle} e^{-\langle w',\lambda_0\rangle}\big|_{\substack{w'=0\\ \lambda'=0}} \\ &= e^{\langle \partial_{\lambda'},\partial_{w'}\rangle} e^{\langle \partial_{\zeta'},\partial_{z'}\rangle} \big(P(t;z,\zeta_0+\zeta',\eta)\, e^{\langle J_{\Phi}^{*}(z+tz',z)z',\lambda_0+\lambda'\rangle} e^{-\langle z',\zeta_0\rangle} \\ &\qquad \cdot e^{\langle J_{\Psi}^{*}(w+tw',w)w',\xi\rangle}\, e^{-\langle w',\lambda_0\rangle}\big)\big|_{\substack{z'=w'=0\\ \zeta'=\lambda'=0}}. \end{aligned}$$

By Lemma 6.12 and Remark 6.13 we have

$$
\begin{aligned}
&e^{\langle\partial_{\lambda'},\partial_{w'}\rangle} e^{\langle J^*_{\Phi}(z+tz',z)z',\lambda_0+\lambda'\rangle} e^{\langle J^*_{\Psi}(w+tw',w)w',\xi\rangle} e^{-\langle w',\lambda_0\rangle}\Big|_{\substack{w'=0\\ \lambda'=0}}\\
&\quad= e^{\langle\partial_{\lambda'},\partial_{w'}\rangle} e^{\langle J^*_{\Phi}(z+tz',z)z',\lambda'\rangle} e^{\langle J^*_{\Psi}(w+tw',w)w',\xi\rangle} e^{\langle J^*_{\Phi}(z+tz',z)z'-w',\lambda_0\rangle}\Big|_{\substack{w'=0\\ \lambda'=0}}\\
&\quad= e^{\langle\partial_{\lambda'},\partial_{w'}\rangle}\sum_{\alpha}\frac{1}{\alpha!}\,\partial^{\alpha}_{w'}\big((J^*_{\Phi}(z+tz',z)z')^{\alpha}\, e^{\langle J^*_{\Psi}(w+tw',w)w',\xi\rangle} e^{\langle J^*_{\Phi}(z+tz',z)z'-w',\lambda_0\rangle}\big)\Big|_{\substack{w'=0\\ \lambda'=0}}\\
&\quad= \sum_{\alpha}\frac{1}{\alpha!}(J^*_{\Phi}(z+tz',z)z')^{\alpha}\,\partial^{\alpha}_{w'}\big(e^{\langle J^*_{\Psi}(w+tw',w)w',\xi\rangle}\, e^{\langle J^*_{\Phi}(z+tz',z)z'-w',\lambda_0\rangle}\big)\Big|_{w'=0}\\
&\quad= e^{\langle J^*_{\Psi}(w+J^*_{\Phi}(z+tz',z)tz',w)J^*_{\Phi}(z+tz',z)z',\xi\rangle} = e^{\langle J^*_{\Phi\Psi}(z+tz',z)z',\xi\rangle}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\Psi^*\Phi^*P(t;v,\xi,\eta) &= e^{\langle\partial_{\zeta'},\partial_{z'}\rangle}P(t;z,\zeta_0+\zeta',\eta)\, e^{\langle J^*_{\Phi\Psi}(z+tz',z)z',\xi\rangle} e^{-\langle z',\zeta_0\rangle}\Big|_{\substack{z'=0\\ \zeta'=0}}\\
&= (\Phi\Psi)^*P(t;v,\xi,\eta).
\end{aligned}
$$
□

Definition 6.14. Under the notation above, we define a coordinate transformation Φ^* associated with Φ by

$$\Phi^*(:P:)(t;w,\lambda,\eta) := :\Phi^*P(t;w,\lambda,\eta):.$$

Lemma 6.15. *Let $\Omega \subset T^*X$ be a conic open subset, $d>0$, $\rho\in\,]0,1[$. Assume that $P(z,\zeta,\eta)\in\Gamma(\Omega_{\rho}[d_{\rho}]\times S;\mathscr{O}_{T^*X\times\mathbb{C}})$ and ν, m, $N\in\mathbb{N}_0$ satisfy the following*:

For any $h>0$ and $Z\Subset S$, there exists a constant $C_{h,Z}>0$ such that for any $\rho'\in\,]0,\rho[$, on $\Omega_{\rho'}[d_{\rho'}]\times Z$

$$|P(z,\zeta,\eta)| \le \frac{C_{h,Z}e^{h\|\zeta\|}}{(\rho-\rho')^{N\nu}\|\eta\zeta\|^{m}}. \tag{6.8}$$

Then for any α, $\beta\in\mathbb{N}_0^n$ $(|\alpha|+|\beta|>0)$, $\rho'\in\,]0,\rho[$, and $(z;\zeta,\eta)\in\Omega_{\rho'}[d_{\rho'}]\times Z$, the following hold:

(1) *If $\nu\in\mathbb{N}$, set $C_{\nu} := (\nu+1)/\nu$. Then*

$$|\partial_z^{\alpha}\partial_{\zeta}^{\beta}P(z,\zeta,\eta)| \le \frac{C_{h,Z}C_{\nu}^{m}e^{N}(\nu+1)^{|\alpha+\beta|}\alpha!\,\beta!\,e^{2h\|\zeta\|}}{(\rho-\rho')^{|\alpha+\beta|+N\nu}\,\|\eta\zeta\|^{m+|\beta|}} \quad (\beta\neq 0),$$

$$|\partial_z^{\alpha}P(z,\zeta,\eta)| \le \frac{C_{h,Z}\,e^{N}(\nu+1)^{|\alpha|}\alpha!\,e^{h\|\zeta\|}}{(\rho-\rho')^{|\alpha|+N\nu}\,\|\eta\zeta\|^{m}} \quad (\beta=0).$$

(2) *If $\nu=0$, then*

$$|\partial_z^{\alpha}\partial_{\zeta}^{\beta}P(z,\zeta,\eta)| \le \frac{C_{h,Z}\,\alpha!\,\beta!\,e^{2h\|\zeta\|}}{(\rho-\rho')^{|\alpha+\beta|}(1-\rho+\rho')^{m}\|\eta\zeta\|^{m+|\beta|}} \quad (\beta\neq 0),$$

$$|\partial_z^{\alpha}P(z,\zeta,\eta)| \le \frac{C_{h,Z}\,\alpha!\,e^{h\|\zeta\|}}{(\rho-\rho')^{|\alpha|}\|\eta\zeta\|^{m}} \quad (\beta=0).$$

Proof. (1) Set $\rho'' := \rho' + (\rho-\rho')/(\nu+1)$. Note that for any $(z;\zeta) \in \Omega_{\rho'}[d_{\rho'}]$ and (z',ζ') with $\|z'\| \le (\rho-\rho')/(\nu+1)$ and $\|\zeta'\| \le (\rho-\rho')\|\eta\zeta\|/(\nu+1) < \|\zeta\|$, we have $(z+z';\zeta+\zeta') \in \Omega_{\rho''}[d_{\rho''}]$. Indeed, by the definition there exists $(z_0;\zeta_0) \in \Omega$ such that $\|z-z_0\| \le \rho'$ and $\|\zeta-\zeta_0\| \le \rho'\|\zeta_0\|$, and hence we have $\|\zeta\| \le (\rho'+1)\|\zeta_0\| \le 2\|\zeta_0\|$. Recall that we assumed that $S \subset \{\eta \in \mathbb{C};\ |\eta| < 1/2\}$. Therefore

$$\|\zeta+\zeta'-\zeta_0\| \le \rho'\|\zeta_0\| + \frac{\rho-\rho'}{\nu+1}\|\eta\zeta\| \le \rho'\|\zeta_0\| + \frac{\rho-\rho'}{\nu+1}\|\zeta_0\| = \rho''\|\zeta_0\|.$$

Further we have

$$2\|\zeta\| \ge \|\zeta+\zeta'\| \ge \Bigl(1-\frac{|\eta|(\rho-\rho')}{\nu+1}\Bigr)\|\zeta\| \ge \Bigl(1-\frac{\rho-\rho'}{\nu+1}\Bigr)\|\zeta\| \ge d(1-\rho''), \tag{6.9}$$

and hence $(z+z';\zeta+\zeta') \in \Omega_{\rho''}[(j+1)d_{\rho''}]$. Thus replacing ρ' with ρ'' in (6.8), for any $h>0$ and $(z;\zeta,\eta) \in \Omega_{\rho'}[d_{\rho'}] \times Z$ we have

$$\begin{aligned}
\sup_{\substack{|z'_i|=(\rho-\rho')/(\nu+1)\\ |\zeta'_i|=(\rho-\rho')\|\eta\zeta\|/(\nu+1)}} |P(z+z',\zeta+\zeta',\eta)| &\le \frac{C_{h,Z}e^{2h\|\zeta\|}}{\Bigl(\Bigl(1-\dfrac{1}{\nu+1}\Bigr)(\rho-\rho')\Bigr)^{N\nu}\Bigl(\Bigl(1-\dfrac{\rho-\rho'}{\nu+1}\Bigr)\|\eta\zeta\|\Bigr)^{m}}\\
&\le \frac{C_{h,Z}e^{2h\|\zeta\|}}{\Bigl(\dfrac{\nu}{\nu+1}(\rho-\rho')\Bigr)^{N\nu}\Bigl(\dfrac{\nu}{\nu+1}\|\eta\zeta\|\Bigr)^{m}}\\
&\le \frac{C_{h,Z}C_\nu^m e^N e^{2h\|\zeta\|}}{(\rho-\rho')^{N\nu}\,\|\eta\zeta\|^m}.
\end{aligned}$$

Therefore if $\beta \ne 0$, we have

$$\begin{aligned}
|\partial_z^\alpha\partial_\zeta^\beta P(z,\zeta,\eta)| &\le \frac{(\nu+1)^{|\alpha+\beta|}\,\alpha!\,\beta!}{(\rho-\rho')^{|\alpha+\beta|}\,\|\eta\zeta\|^{|\beta|}} \sup_{\substack{|z'_i|=(\rho-\rho')/(\nu+1)\\ |\zeta'_i|=(\rho-\rho')\|\eta\zeta\|/(\nu+1)}} |P(z+z',\zeta+\zeta',\eta)|\\
&\le \frac{C_{h,Z}C_\nu^m e^N(\nu+1)^{|\alpha+\beta|}\alpha!\,\beta!\,e^{2h\|\zeta\|}}{(\rho-\rho')^{|\alpha+\beta|+N\nu}\,\|\eta\zeta\|^{m+|\beta|}}.
\end{aligned}$$

If $\beta = 0$, we have

$$\begin{aligned}
|\partial_z^\alpha P(z,\zeta,\eta)| &\le \frac{(\nu+1)^{|\alpha|}\alpha!}{(\rho-\rho')^{|\alpha|}} \sup_{|z'_i|=(\rho-\rho')/(\nu+1)} |P(z+z',\zeta,\eta)|\\
&\le \frac{C_{h,Z}(\nu+1)^{|\alpha|}\alpha!\,e^{h\|\zeta\|}}{(\rho-\rho')^{|\alpha|}\Bigl((\rho-\rho')\Bigl(1-\dfrac{1}{\nu+1}\Bigr)\Bigr)^{N\nu}\|\eta\zeta\|^m} \le \frac{C_{h,Z}\,e^N(\nu+1)^{|\alpha|}\alpha!\,e^{h\|\zeta\|}}{(\rho-\rho')^{|\alpha|+N\nu}\,\|\eta\zeta\|^m}.
\end{aligned}$$

(2) We may choose $|z'_i| = \rho-\rho'$ and $|\zeta'_i| = (\rho-\rho')\|\eta\zeta\|$. □

Theorem 6.16. *For any $P(t;z,\zeta)$, $Q(t;z,\zeta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$, set*

$$Q \circ P(t; z, \zeta, \eta) := e^{t\langle \partial_{\zeta'}, \partial_{z'}\rangle} Q(t; z, \zeta', \eta)\, P(t; z', \zeta, \eta)\big|_{\substack{z'=z\\ \zeta'=\zeta}}$$
$$= e^{t\langle \partial_{\zeta'}, \partial_{z'}\rangle} Q(t; z, \zeta+\zeta', \eta)\, P(t; z+z', \zeta, \eta)\big|_{\substack{z'=0\\ \zeta'=0}}\,.$$

(1) $Q \circ P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$. *Moreover if either* $P(t; z, \zeta, \eta)$ *or* $Q(t; z, \zeta, \eta)$ *is an element of* $\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$, *it follows that* $Q \circ P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$.

(2) $R \circ (Q \circ P) = (R \circ Q) \circ P$ *holds.*

(3) *Let* $\Phi(w) = z$ *be a holomorphic coordinate transformation. Then*

$$\Phi^* Q \circ \Phi^* P(t; w, \lambda, \eta) = \Phi^*(Q \circ P)(t; w, \lambda, \eta).$$

PROOF. (1) We assume $P(t; z, \zeta, \eta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z, \zeta, \eta)$, $Q(t; z, \zeta, \eta) = \sum_{\nu=0}^{\infty} t^\nu Q_\nu(z, \zeta, \eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})[\![t]\!]$. If we set $Q \circ P(t; z, \zeta, \eta) = \sum_{\nu=0}^{\infty} t^\nu R_\nu(z, \zeta, \eta)$, we have

$$R_\nu(z, \zeta, \eta) = \sum_{|\alpha|+k+l=\nu} \frac{1}{\alpha!}\, \partial_\zeta^\alpha Q_l(z, \zeta, \eta) \cdot \partial_z^\alpha P_k(z, \zeta, \eta).$$

Therefore $R(t; z, \zeta, \eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})[\![t]\!]$. For any $\rho' \in \left]0, \rho\right[$, $k \in \mathbb{N}_0$ and $Z \Subset S$, on $\Omega_{\rho'}[d_{\rho'}] \times Z \subset \Omega_\rho[d_\rho] \times Z$ we have $|P_k(z, \zeta, \eta)|, |Q_k(z, \zeta, \eta)| \le C_{h,Z} k! A^k e^{h\|\zeta\|}/\|\eta\zeta\|^k$. Hence by Lemma 6.15, we have

$$|\partial_\zeta^\alpha Q_l(z, \zeta, \eta)| \le \frac{C_{h,Z}\, \alpha!\, l! A^l e^{2h\|\zeta\|}}{(\rho-\rho')^{|\alpha|}(1-\rho+\rho')^l \|\eta\zeta\|^{l+|\alpha|}},$$
$$|\partial_z^\alpha P_k(z, \zeta, \eta)| \le \frac{C_{h,Z}\, \alpha!\, k! A^k e^{h\|\zeta\|}}{(\rho-\rho')^{|\alpha|}\, \|\eta\zeta\|^k}. \tag{6.10}$$

We choose $\rho' \in \left]0, \rho\right[$ as $C := (\rho-\rho')^2 A/2(1-\rho+\rho') < 1$, and set $B := 1/(\rho-\rho')^2$. Then on $\Omega_{\rho'}[d_{\rho'}] \times Z$ we have

$$|R_\nu(z, \zeta, \eta)| \le \sum_{k+l=0}^{\nu} \frac{C_{h,Z}^2\, k!\, l! A^{k+l} e^{3h\|\zeta\|}}{(1-\rho+\rho')^l} \sum_{|\alpha|=\nu-k-l} \frac{\alpha!}{(\rho-\rho')^{2|\alpha|}\, \|\eta\zeta\|^{k+l+|\alpha|}}$$
$$\le \frac{2^{n-1} C_{h,Z}^2\, \nu! B^\nu e^{3\|\zeta\|}}{\|\eta\zeta\|^\nu} \sum_{k+l=0}^{\nu} C^{k+l} \le \frac{2^{n-1} C_{h,Z}^2\, \nu! B^\nu e^{3\|\zeta\|}}{(1-C)^2\, \|\eta\zeta\|^\nu}.$$

Assume that $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$. Then by (6.10) for any $m \in \mathbb{N}$ we have

$$\left|\sum_{k=0}^{m-1} \partial_z^\alpha P_k(z, \zeta, \eta)\right| \le \frac{C_{h,Z}\, \alpha!\, m! A^m e^{2\|\zeta\|}}{(\rho-\rho')^{|\alpha|}\, \|\eta\zeta\|^m}.$$

Then on $\Omega_{\rho'}[d_{\rho'}] \times Z$ we have

$$\begin{aligned}\Bigl|\sum_{\nu=0}^{m-1} R_\nu(z,\zeta,\eta)\Bigr| &= \Bigl|\sum_{l+|\alpha|=0}^{m-1} \frac{1}{\alpha!}\,\partial_\zeta^\alpha Q_l(z,\zeta,\eta) \sum_{k=0}^{m-l-1-|\alpha|} \partial_z^\alpha P_k(z,\zeta,\eta)\Bigr| \\ &\le \sum_{l+|\alpha|=0}^{m-1} \frac{C_{h,Z}^2\,\alpha!\,l!\,(m-l-|\alpha|)!A^{m-|\alpha|}e^{3h\|\zeta\|}}{\|\eta\zeta\|^m(1-\rho+\rho')^l(\rho-\rho')^{2|\alpha|}} \\ &\le \frac{2^{n-1}C_{h,Z}^2\,m!B^m e^{3h\|\zeta\|}}{\|\eta\zeta\|^m}\sum_{l=0}^{m-1} C^l \sum_{i=0}^{m-l-1}((1-\rho+\rho')C)^{m-l-i} \\ &\le \frac{2^{n-1}CC_{h,Z}^2\,m!B^m e^{3h\|\zeta\|}}{(1-C)^2\|\eta\zeta\|^m}\,.\end{aligned}$$

Therefore $Q\circ P(t;z,\zeta,\eta)\in\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$. For the same reasoning, we can show that if $Q(t;z,\zeta,\eta)\in\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$, we have $Q\circ P(t;z,\zeta,\eta)\in\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$. In particular, since

$$\partial_\eta(Q\circ P)(t;z,\zeta,\eta)=\partial_\eta Q\circ P(t;z,\zeta,\eta)+Q\circ\partial_\eta P(t;z,\zeta,\eta),$$

we see that if $P(t;z,\zeta,\eta)$, $Q(t;z,\zeta,\eta)\in\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$, we have $Q\circ P(t;z,\zeta,\eta)\in\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$.

The proof of (2) is easy.

(3) We may show the equality around each point $(\Phi^{-1}(z_0), {}^{\mathrm{t}}[(\partial z/\partial w)(w_0)]\zeta_0)=(w_0,\lambda_0)$ as a formal series. Set $z:=\Phi(w)$. We remark that

$$\partial_{\lambda'}^\alpha e^{\langle J_\Phi^*(z+z'',\,z)z'',\,\lambda+\lambda'\rangle}=e^{\langle J_\Phi^*(z+z'',\,z)z'',\,\lambda+\lambda'\rangle}\,(\Phi^{-1}(z+z'')-w)^\alpha,$$

and hence as a formal power series with respect to t, we have

$$t^{|\alpha|}\partial_{\lambda'}^\alpha e^{\langle J_\Phi^*(z+tz'',\,z)z'',\,\lambda+\lambda'\rangle}=e^{\langle J_\Phi^*(z+tz'',\,z)z',\,\lambda+\lambda'\rangle}\,(\Phi^{-1}(z+tz'')-w)^\alpha.$$

Further since

$$\begin{aligned}J_\Phi^*(z+z'',z)z''+J_\Phi^*(z+z''+z',z+z'')z'&=\Phi^{-1}(z+z''+z')-\Phi^{-1}(z)\\ &=J_\Phi^*(z+z''+z',z)(z''+z'),\end{aligned}$$

as a formal power series with respect to t, we have

$$J_\Phi^*(z+tz'',z)z''+J_\Phi^*(z+tz''+tz',z+z'')z'=J_\Phi^*(z+tz''+tz',z)(z''+z').$$

Therefore we have

$$\begin{aligned}&\Phi^*Q\circ\Phi^*P(t;w,\lambda,\eta)\\ &\quad=e^{t\langle\partial_{\lambda'},\partial_{\widetilde w}\rangle}e^{\langle\partial_{\zeta''},\partial_{z''}\rangle}Q(t;z,\zeta_0+\zeta'',\eta)\,e^{\langle J_\Phi^*(z+tz'',\,z)z'',\,\lambda+\lambda'\rangle}e^{-\langle z'',\,\zeta_0\rangle}\\ &\qquad\cdot e^{\langle\partial_{\zeta'},\partial_{z'}\rangle}P(t;\Phi(w+\widetilde w),\zeta_0+\zeta',\eta)\,e^{\langle J_\Phi^*(\Phi(w+\widetilde w)+tz',\,\Phi(w+\widetilde w))z',\,\lambda\rangle}e^{-\langle z',\,\zeta_0\rangle}\Big|_{\substack{z'=z''=\widetilde w=0\\ \zeta'=\zeta''=\lambda'=0}}\end{aligned}$$

$$= e^{\langle \partial_{\zeta''}, \partial_{z''}\rangle} \sum_{\alpha} Q(t; z, \zeta_0 + \zeta'', \eta)\, e^{\langle J^*_{\Phi}(z+tz'', z)z'', \lambda\rangle} e^{-\langle z'', \zeta_0\rangle} \frac{(\Phi^{-1}(z+tz'') - w)^{\alpha}}{\alpha!}$$
$$\cdot\, \partial_w^{\alpha} (e^{\langle \partial_{\zeta'}, \partial_{z'}\rangle} P(t; \Phi(w), \zeta_0 + \zeta', \eta)\, e^{\langle J^*_{\Phi}(\Phi(w)+tz', \Phi(w))z', \lambda\rangle} e^{-\langle z', \zeta_0\rangle}\Big|_{\substack{z'=z''=0\\ \zeta'=\zeta''=0}}$$
$$= e^{\langle \partial_{\zeta''}, \partial_{z''}\rangle} Q(t; z, \zeta_0 + \zeta'', \eta)\, e^{\langle J^*_{\Phi}(z+tz'', z)z'', \lambda\rangle} e^{-\langle z'', \zeta_0\rangle}$$
$$\cdot\, e^{\langle \partial_{\zeta'}, \partial_{z'}\rangle} P(t; z+tz'', \zeta_0 + \zeta', \eta)\, e^{\langle J^*_{\Phi}(z+tz''+tz', z+tz'')z', \lambda\rangle} e^{-\langle z', \zeta_0\rangle}\Big|_{\substack{z'=z''=0\\ \zeta'=\zeta''=0}}$$
$$= e^{\langle \partial_{\zeta'}, \partial_{z'}\rangle} e^{\langle \partial_{\zeta''}, \partial_{z''}\rangle} Q(t; z, \zeta_0 + \zeta'', \eta)\, P(t; z+tz'', \zeta_0 + \zeta', \eta)$$
$$\cdot\, e^{\langle J^*_{\Phi}(z+tz''+tz', z)(z''+z'), \lambda\rangle} e^{-\langle z''+z', \zeta_0\rangle}\Big|_{\substack{z'=z''=0\\ \zeta'=\zeta''=0}}$$
$$= \sum_{\alpha,\beta,\gamma} \frac{t^{|\alpha|}}{\alpha!\,\beta!\,\gamma!} \partial_{\zeta'}^{\alpha+\beta} Q(t; z, \zeta_0 + \zeta', \eta)\, \partial_z^{\alpha} \partial_{\zeta'}^{\gamma} P(t; z, \zeta_0 + \zeta', \eta)$$
$$\cdot\, \partial_{z'}^{\gamma} \partial_{z''}^{\beta} e^{\langle J^*_{\Phi}(z+tz''+tz', z)(z''+z'), \lambda\rangle} e^{-\langle z''+z', \zeta_0\rangle}\Big|_{\substack{z'=z''=0\\ \zeta'=0}}$$
$$= \sum_{\beta,\gamma} \frac{1}{(\beta+\gamma)!} \partial_{\zeta}^{\beta+\gamma} (Q\circ P)(t; z, \zeta_0 + \zeta', \eta)\, \partial_{z'}^{\beta+\gamma} e^{\langle J^*_{\Phi}(z+tz', z)z', \lambda\rangle} e^{-\langle z', \zeta_0\rangle}\Big|_{\substack{z'=0\\ \zeta'=0}}$$
$$= e^{\langle \partial_{\zeta'}, \partial_{z'}\rangle} (Q\circ P)(t; z, \zeta_0 + \zeta', \eta)\, e^{\langle J^*_{\Phi}(z+tz', z)z', \lambda\rangle} e^{-\langle z', \zeta_0\rangle}\Big|_{\substack{z'=0\\ \zeta'=0}}$$
$$= \Phi^*(Q\circ P)(t; w, \lambda, \eta). \qquad \square$$

Definition 6.17. For any $:P(t; z, \zeta, \eta):$, $:Q(t; z, \zeta, \eta): \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$, we define the product by:

$$:Q(t; z, \zeta, \eta):\,:P(t; z, \zeta, \eta): := \,:Q\circ P(t; z, \zeta, \eta):.$$

Theorem 6.18. *For any* $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$, *set*

$$P^*(t; z, \zeta, \eta) := e^{t\langle \partial_\zeta, \partial_z\rangle} P(t; z, -\zeta, \eta).$$

(1) $P^*(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega^a; S)$, *where* $\Omega^a := \{(z;\zeta);\, (z; -\zeta) \in \Omega\}$, *and* $P^{**} = P$. *Moreover if* $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$, *it follows that* $P^*(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega^a; S)$.

(2) $(Q\circ P)^*(t; z, \zeta, \eta) = P^*\circ Q^*(t; z, \zeta, \eta)$.

(3) *Let* $\Phi(w) = z$ *be a holomorphic coordinate transformation. Then it follows that on* $\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega^a; S)\otimes_{\mathscr{O}_X}\Omega_X$,

$$\Phi^*(P^*(t; z, \zeta, \eta)\otimes dz) = \Phi^*(P(t; z, \zeta, \eta)\otimes dz)^*. \tag{6.11}$$

Here Φ^* *also stands for the pull-back of differential forms, and* (6.11) *means that*

$$\det\frac{\partial z}{\partial w}\,\Phi^*(P^*) = \left(\det\frac{\partial z}{\partial w}\Phi^* P\right)^* = (\Phi^* P)^*\circ\det\frac{\partial z}{\partial w}\,.$$

Proof. (1) For $P(t; z, \zeta, \eta) = \sum_{i=0}^{\infty} t^i P_i(z, \zeta, \eta)$, we set $P^*(t; z, \zeta, \eta) = \sum_{i=0}^{\infty} t^j P_i^*(z, \zeta, \eta)$. Then

$$P_i^*(z,\zeta,\eta) = \sum_{|\alpha|+k=i} \frac{1}{\alpha!}\,\partial_\zeta^\alpha\,\partial_z^\alpha P_k(t;z,-\zeta,\eta).$$

As in the proof of Theorem 6.16, we can show $P^*(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega^a;S)$, and that if $P(t;z,\zeta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$, we have $P^*(t;z,\zeta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega^a;S)$. Moreover

$$P^{**}(t;z,\zeta,\eta) = e^{t\langle\partial_\zeta,\partial_z\rangle} P^*(t;z,-\zeta,\eta) = e^{t\langle\partial_\zeta,\partial_z\rangle} e^{-t\langle\partial_\zeta,\partial_z\rangle} P(t;z,\zeta,\eta) = P(t;z,\zeta,\eta).$$

(2) We have

$$\begin{aligned}
(Q\circ P)^*(t;z,\zeta,\eta) &= e^{t\langle\partial_\zeta,\partial_z\rangle}(Q\circ P(t;z,-\zeta,\eta))\\
&= e^{t\langle\partial_\zeta,\partial_z\rangle}(e^{-t\langle\partial_{\zeta''},\partial_{z'}\rangle} Q(t;z,-\zeta-\zeta'',\eta)\,P(t;z+z',-\zeta,\eta)\big|_{\substack{z'=0\\ \zeta''=0}})\\
&= e^{t\langle\partial_{\zeta'}+\partial_{\zeta''},\partial_{z'}+\partial_{z''}\rangle} e^{-t\langle\partial_{\zeta''},\partial_{z'}\rangle} Q(t;z+z'',-\zeta-\zeta'',\eta)\,P(t;z+z',-\zeta-\zeta',\eta)\big|_{\substack{z'=z''=0\\ \zeta'=\zeta''=0}}\\
&= e^{t\langle\partial_{\zeta'},\partial_{z''}\rangle} e^{t\langle\partial_{\zeta'},\partial_{z'}\rangle} P(t;z+z',-\zeta-\zeta',\eta) e^{t\langle\partial_{\zeta''},\partial_{z''}\rangle} Q(t;z+z'',-\zeta-\zeta'',\eta)\big|_{\substack{z'=z''=0\\ \zeta'=\zeta''=0}}\\
&= e^{t\langle\partial_{\zeta'},\partial_{z''}\rangle} P^*(t;z+z',\zeta+\zeta',\eta)\,Q^*(t;z+z'',\zeta+\zeta'',\eta)\big|_{\substack{z'=z''=0\\ \zeta'=\zeta''=0}}\\
&= e^{t\langle\partial_{\zeta'},\partial_{z''}\rangle} P^*(t;z,\zeta+\zeta',\eta)\,Q^*(t;z+z'',\zeta,\eta)\big|_{\substack{z''=0\\ \zeta'=0}} = P^*\circ Q^*(t;z,\zeta,\eta).
\end{aligned}$$

(3) (i) For any holomorphic function $a(z,\eta)$, we have

$$\begin{aligned}
&\Phi^*(a(z,\eta)^*\otimes dz) = \Phi^*(a(z,\eta)\otimes dz) = a(\Phi(z),\eta)\otimes \det\frac{\partial z}{\partial w}\,dw = a(\Phi(z),\eta)\det\frac{\partial z}{\partial w}\otimes dw,\\
&\Phi^*(a(z,\eta)\otimes dz)^* = \left(a(\Phi(z),\eta)\det\frac{\partial z}{\partial w}\right)^*\otimes dw = a(\Phi(z),\eta)\det\frac{\partial z}{\partial w}\otimes dw.
\end{aligned}$$

Set $J := \partial z/\partial w$ for short. Since $\Phi^*(\zeta_i) = \sum_{k=1}^n (\partial w_k/\partial z_i)\,\lambda_k$, we have

$$\begin{aligned}
\Phi^*((\zeta_i)^*\otimes dz) &= -\Phi^*(\zeta_i\otimes dz) = -\sum_{k=1}^n \frac{\partial w_k}{\partial z_i}\,\lambda_k\otimes\det\frac{\partial z}{\partial w}\,dw = -\det J\sum_{k=1}^n\frac{\partial w_k}{\partial z_i}\,\lambda_k\otimes dw,\\
\Phi^*(\zeta_i\otimes dz)^* &= \left(\det J\sum_{k=1}^n\frac{\partial w_k}{\partial z_i}\,\lambda_k\right)^*\otimes dw\\
&= -\sum_{k=1}^n\left(\det J\,\frac{\partial w_k}{\partial z_i}\,\eta_k + \frac{\partial\det J}{\partial w_k}\,\frac{\partial w_k}{\partial z_i} + \det J\frac{\partial}{\partial w_k}\Big(\frac{\partial w_k}{\partial z_i}\Big)\right)\otimes dw.
\end{aligned}$$

Here we remark

$$\frac{\partial\det J}{\partial w_k} = \det J\,\mathrm{Tr}\Big(J^{-1}\frac{\partial J}{\partial w_k}\Big),\qquad \frac{\partial J^{-1}}{\partial w_k} = -J^{-1}\frac{\partial J}{\partial w_k}J^{-1}.$$

Hence we have

$$\sum_{k=1}^n\Big(\frac{\partial\det J}{\partial w_k}\,\frac{\partial w_k}{\partial z_i} + \det J\frac{\partial}{\partial w_k}\Big(\frac{\partial w_k}{\partial z_i}\Big)\Big)$$

$$= \det J \sum_{k=1}^{n} \Big(\mathrm{Tr}\Big(J^{-1} \frac{\partial J}{\partial w_k} \Big) \frac{\partial w_k}{\partial z_i} - \sum_{\mu,\nu=1}^{n} \frac{\partial w_k}{\partial z_\mu} \frac{\partial^2 z_\mu}{\partial w_k \partial w_\nu} \frac{\partial w_\nu}{\partial z_i} \Big)$$

$$= \det J \sum_{k,\mu,\nu=1}^{n} \Big(\frac{\partial w_\nu}{\partial z_\mu} \frac{\partial^2 z_\mu}{\partial w_\nu \partial w_k} \frac{\partial w_k}{\partial z_i} - \frac{\partial w_k}{\partial z_\mu} \frac{\partial^2 z_\mu}{\partial w_k \partial w_\nu} \frac{\partial w_\nu}{\partial z_i} \Big) = 0.$$

Therefore $\Phi^*((\zeta_i)^* \otimes dz) = \Phi^*(\zeta_i \otimes dz)^*$.

(ii) If P and Q satisfy (6.11), then

$$\Phi^*((Q \circ P)^* \otimes dz) = \Phi^*(P^* \circ Q^* \otimes dz) = \Phi^*(P^*) \circ \Phi^*(Q^*) \otimes \det \frac{\partial z}{\partial w}\, dw$$
$$= \det \frac{\partial z}{\partial w}\, \Phi^*(P^*) \circ \Phi^*(Q^*) \otimes dw = (\Phi^* P)^* \circ \det \frac{\partial z}{\partial w}\, \Phi^*(Q^*) \otimes dw$$
$$= (\Phi^* P)^* \circ \Phi^*(Q^*) \circ \det \frac{\partial z}{\partial w} \otimes dw = \Phi^*(Q \circ P)^* \circ \det \frac{\partial z}{\partial w} \otimes dw$$
$$= \Big(\det \frac{\partial z}{\partial w}\, \Phi^*(Q \circ P) \Big)^* \otimes dw = \Big(\det \frac{\partial z}{\partial w}\, \Phi^*(Q \circ P) \otimes dw \Big)^* = \Phi^*(Q \circ P \otimes dz)^*.$$

(iii) Take any point $(z_0; \zeta_0) \in \Omega_\rho$ and consider the Taylor expansion

$$P(t; z, \zeta, \eta) = \sum_\alpha p_\alpha(t; z, \zeta_0, \eta)\, (\zeta - \zeta_0)^\alpha = \sum_\alpha p_\alpha(t; z, \zeta_0, \eta) \circ (\zeta - \zeta_0)^\alpha.$$

We may prove (6.11) in a formal sense. Then by induction and (i), (ii), we obtain

$$\Phi^*(P^* \otimes dz) = \sum_\alpha \Phi^*((p_\alpha(t; z, \zeta_0, \eta) \circ (\zeta - \zeta_0)^\alpha)^* \otimes dz)$$
$$= \sum_\alpha \Phi^*(p_\alpha(t; z, \zeta_0, \eta) \circ (\zeta - \zeta_0)^\alpha \otimes dz)^* = \Phi^*(P \otimes dz)^*. \qquad \square$$

Definition 6.19. For any $:P(t; z, \zeta, \eta): \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$, we define the formal adjoint by

$$(:P(t; z, \zeta, \eta):)^* := :P^*(t; z, \zeta, \eta): \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega^a; S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega^a; S).$$

Remark 6.20. We identify X with the diagonal set of $X \times X$. Then the sheaf $\mathscr{D}_X^\infty$ of holomorphic differential operators of infinite order on X is defined by

$$\mathscr{D}_X^\infty := H_X^n(\Omega_{X \times X}^{(0,n)}) = \mathscr{E}_X^{\mathbb{R}}|_X.$$

We remark that we have $\mathscr{E}_X^{\mathbb{R}}|_X = \mathscr{D}_X^\infty$. Moreover, recall that for any open subset $U \subset X$, a section $P(z, \partial_z) = \sum_\alpha a_\alpha(z)\, \partial_z^\alpha \in \Gamma(U; \mathscr{D}_X^\infty)$ is given by the following equivalent conditions:

(1) For any $W \Subset U$ and $h > 0$ there exists $C > 0$ such that

$$\sup_{z \in W} |a_\alpha(z)| \le \frac{C h^{|\alpha|}}{|\alpha|!}\,.$$

(2) Set $P(z,\zeta) := \sum_\alpha a_\alpha(z)\,\zeta^{\,\alpha}$. For any $W \Subset U$ and $h > 0$ there exists $C > 0$ such that

$$\sup_{z\in W} |P(z,\zeta)| \le Ce^{h\|\zeta\|}.$$

$\mathscr{D}_X^\infty$ is a sheaf of rings on X, and:

(1) the coordinate transform is given by

$$\Phi^* P(w,\lambda) = e^{\langle \partial_{\zeta'}, \partial_{z'}\rangle} P(\Phi(w), \zeta' + {}^{\mathsf{t}}J_\Phi^*(\Phi(w)+z', \Phi(w))\lambda)\big|_{\substack{z'=0\\ \zeta'=0}},$$

(2) the product is given by

$$Q \circ P(z,\zeta) = e^{\langle \partial_{\zeta'}, \partial_{z'}\rangle} Q(z, \zeta+\zeta')\, P(z+z', \zeta)\big|_{\substack{z'=0\\ \zeta'=0}},$$

(3) the formal adjoint is given by

$$P^*(z,\zeta) = e^{\langle \partial_\zeta, \partial_z\rangle} P(z, -\zeta).$$

Hence, the operations above are compatible with those of classical formal symbols.

Theorem 6.21. *Let* $[\psi(z,w,\eta)\,dw]$, $[\varphi(z,w,\eta)\,dw] \in \mathscr{E}^{\mathbb{R}}_{X,z_0^*}$. *Set for short*

$$\sigma(\psi) \odot \sigma(\varphi)(z,\zeta,\eta) := \sum_\alpha \frac{1}{\alpha!} \partial_\zeta^\alpha \sigma(\psi)(z,\zeta,\eta)\, \partial_z^\alpha \sigma(\varphi)(z,\zeta,\eta).$$

Then the following hold:

(1) $\sigma(\psi) \odot \sigma(\varphi)(z,\zeta,\eta) \in \mathfrak{S}_{z_0^*}$.

(2) $\sigma(\psi) \odot \sigma(\varphi)(z,\zeta,\eta) - \sigma(\psi) \circ \sigma(\varphi)(t; z,\zeta,\eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}, z_0^*}$.

(3) $\sigma(\mu(\psi\otimes\varphi))(z,\zeta,\eta) - \sigma(\psi) \odot \sigma(\varphi)(z,\zeta,\eta) \in \mathfrak{N}_{z_0^*}$.

Proof. We assume that $\sigma(\psi)(z,\zeta,\eta)$, $\sigma(\varphi)(z,\zeta,\eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$.

(1) Take $\rho' \in \,]0,\rho[$. Fix any $Z \Subset S$ and $h > 0$. In $\gamma(0,\eta;\varrho,\theta)$, we can change $\gamma_i(0,\eta;\varrho)$ as $\{w_i = |\eta| s' e^{2\pi\sqrt{-1}\,t};\ 0 \le t \le 1\}$ with $0 < \varrho^{-1} < s'$ $(2 \le i \le n)$, and $\gamma_1(0,\eta;\varrho,\theta) \subset \{|w_1| \le |\eta| s'\}$. Therefore, we have

$$\begin{aligned}|\partial_\zeta^\alpha \sigma(\psi)(z,\zeta,\eta)| &= \Big|\partial_\zeta^\alpha \int_{\gamma(0,\eta;\varrho,\theta)} \psi(z, z+\widetilde{w}, \eta)\, e^{\langle \widetilde{w}, \zeta\rangle} d\widetilde{w}\Big| = \Big|\int_{\gamma(0,\eta;\varrho,\theta)} \widetilde{w}^\alpha \psi(z, z+\widetilde{w}, \eta)\, e^{\langle \widetilde{w}, \zeta\rangle} d\widetilde{w}\Big| \\ &\le (|\eta| s')^{|\alpha|} C_{h,Z} e^{h\|\zeta\|}.\end{aligned}$$

For the same reason, we have

$$|\partial_\eta \partial_\zeta^\alpha \sigma(\psi)(z,\zeta,\eta)| = |\partial_\zeta^\alpha \partial_\eta \sigma(\psi)(z,\zeta,\eta)| \le (|\eta| s')^{|\alpha|} C e^{-\delta\|\eta\zeta\|}.$$

In the same way taking $\|z\| \le r_0' < r'$, for some $R > 0$ we have

$$|\partial_z^\alpha \sigma(\varphi)(z,\zeta,\eta)| \le \frac{C_{h,Z}\,\alpha!\,e^{h\|\zeta\|}}{R^{|\alpha|}}, \quad |\partial_\eta \partial_z^\alpha \sigma(\varphi)(z,\zeta,\eta)| \le \frac{C_Z\,\alpha!\,e^{-\delta\|\eta\zeta\|}}{R^{|\alpha|}}.$$

Hence taking r small enough as $rs' < R$, we have

$$|\sigma(\psi)\odot\sigma(\varphi)(z,\zeta,\eta)| \le \frac{C_{h,Z}^2\,e^{2h\|\zeta\|}}{(1-|\eta|s'/R)^n}\,.$$

For any $(z;\zeta,\eta) \in \Omega_{\rho'}[d_{\rho'}]\times Z$, choosing $h = \delta m_Z/2$, we have

$$\begin{aligned}
&|\partial_\eta(\sigma(\psi)\odot\sigma(\varphi))(z,\zeta,\eta)|\\
&\quad = \sum_\alpha \frac{1}{\alpha!}(\partial_\eta\partial_\zeta^\alpha\sigma(\psi)(z,\zeta,\eta)\,\partial_z^\alpha\sigma(\varphi)(z,\zeta,\eta) + \partial_\zeta^\alpha\sigma(\psi)(z,\zeta,\eta)\,\partial_\eta\partial_z^\alpha\sigma(\varphi)(z,\zeta,\eta))\\
&\quad \le \frac{4C_{h,Z}Ce^{-\delta\|\eta\zeta\|/2}}{(1-|\eta|s'/R)^n}\,.
\end{aligned}$$

(2) By Lemma 6.15, under the same notation of proof of (1), for any $\beta,\gamma\in\mathbb{N}_0^n$, on $\Omega_{\rho'}[d_{\rho'}]\times Z$ we have

$$\begin{aligned}
|\partial_\zeta^{\beta+\gamma}\sigma(\psi)(z,\zeta,\eta)| &= \Big|\partial_\zeta^{\beta+\gamma}\int_{\gamma(0,\eta;\varrho,\theta)}\psi(z,z+\widetilde{w},\eta)\,e^{\langle\widetilde{w},\,\zeta\rangle}d\widetilde{w}\Big|\\
&= \Big|\partial_\zeta^{\beta}\int_{\gamma(0,\eta;\varrho,\theta)}\widetilde{w}^\gamma\psi(z,z+\widetilde{w},\eta)\,e^{\langle\widetilde{w},\,\zeta\rangle}d\widetilde{w}\Big| \le \frac{(|\eta|s')^{|\gamma|}C_{h,Z}\,\beta!\,e^{2h\|\zeta\|}}{(\rho-\rho')^{|\beta|}\|\eta\zeta\|^{|\beta|}}.
\end{aligned}$$

Therefore, setting $B := 2/(\rho-\rho')R$ we obtain

$$\begin{aligned}
&\Big|\sigma(\psi)\odot\sigma(\varphi)(z,\zeta,\eta) - \sum_{|\alpha|=0}^{m-1}\frac{1}{\alpha!}\partial_\zeta^\alpha\sigma(\psi)(z,\zeta,\eta)\,\partial_z^\alpha\sigma(\varphi)(z,\zeta,\eta)\Big|\\
&= \Big|\sum_{|\alpha|\ge m}\frac{1}{\alpha!}\partial_\zeta^\alpha\sigma(\psi)(z,\zeta,\eta)\,\partial_z^\alpha\sigma(\varphi)(z,\zeta,\eta)\Big|\\
&= \Big|\sum_{|\beta|=m}\sum_{|\gamma|=0}^{\infty}\frac{1}{(\beta+\gamma)!}\partial_\zeta^{\beta+\gamma}\sigma(\psi)(z,\zeta,\eta)\,\partial_z^{\beta+\gamma}\sigma(\varphi)(z,\zeta,\eta)\Big|\\
&\le \frac{C_{h,Z}^2\,m!\,e^{3h\|\zeta\|}}{(\rho-\rho')^m\|\eta\zeta\|^m}\sum_{|\beta|=m}\sum_{|\gamma|=0}^{\infty}\frac{(|\eta|s')^{|\gamma|}}{R^{|\beta+\gamma|}} \le \frac{2^{n-1}C_{h,Z}^2\,m!B^m e^{3h\|\zeta\|}}{(1-|\eta|s'/R)^n\|\eta\zeta\|^m}\,.
\end{aligned}$$

(3) Take two paths $\gamma(0,\eta;\tilde{\varrho},\tilde{\theta})$ and $\gamma(0,\eta;\varrho',\theta')$. Here we take $\tilde{\varrho}$ which is sufficiently smaller than ϱ, and next we take ϱ' sufficiently smaller than $\tilde{\varrho}$. Hence we may assume

$$\mu(\psi\otimes\varphi)(z,z+w,\eta) = \int_{\gamma(0,\eta;\tilde{\varrho},\tilde{\theta})}\psi(z,z+\widetilde{w},\eta)\,\varphi(z+\widetilde{w},z+w,\eta)\,d\widetilde{w},$$

$$\begin{aligned}\sigma(\mu(\psi\otimes\varphi))(z,\zeta,\eta) &= \int_{\gamma(0,\eta;\varrho',\theta')} \mu(\psi\otimes\varphi)(z,z+w,\eta)\, e^{\langle w,\,\zeta\rangle} dw \\ &= \int_{\gamma(0,\eta;\varrho',\theta')} dw \int_{\gamma(0,\eta;\tilde{\varrho},\tilde{\theta})} \psi(z,z+\widetilde{w},\eta)\,\varphi(z+\widetilde{w},z+w,\eta)\, e^{\langle w,\,\zeta\rangle} d\widetilde{w} \\ &= \int_{\gamma(0,\eta;\tilde{\varrho},\tilde{\theta})} d\widetilde{w} \int_{\gamma(0,\eta;\varrho',\theta')} \psi(z,z+\widetilde{w},\eta)\,\varphi(z+\widetilde{w},z+w,\eta)\, e^{\langle w,\,\zeta\rangle} dw.\end{aligned}$$

Then we find

$$\begin{aligned}\sigma(\psi)\odot\sigma(\varphi)(z,\zeta,\eta) &= \sum_\alpha \frac{1}{\alpha!}\int_{\gamma(0,\eta;\varrho,\theta)} \widetilde{w}^\alpha \psi(z,z+\widetilde{w},\eta)\, e^{\langle \widetilde{w},\,\zeta\rangle} d\widetilde{w} \int_{\gamma(0,\eta;\varrho,\theta)} \partial_z^\alpha \varphi(z,z+w,\eta)\, e^{\langle w,\,\zeta\rangle} dw \\ &= \sum_\alpha \frac{1}{\alpha!}\int_{\gamma(0,\eta;\tilde{\varrho},\tilde{\theta})} \widetilde{w}^\alpha \psi(z,z+\widetilde{w},\eta)\, e^{\langle \widetilde{w},\,\zeta\rangle} d\widetilde{w} \int_{\gamma(0,\eta;\varrho,\theta)} \partial_z^\alpha \varphi(z,z+w,\eta)\, e^{\langle w,\,\zeta\rangle} dw \\ &\quad + \sum_\alpha \frac{1}{\alpha!}\int_{\gamma(0,\eta;\varrho,\theta)\vee(-\gamma(0,\eta;\tilde{\varrho},\tilde{\theta}))} \widetilde{w}^\alpha \psi(z,z+\widetilde{w},\eta)\, e^{\langle \widetilde{w},\,\zeta\rangle} d\widetilde{w} \int_{\gamma(0,\eta;\varrho,\theta)} \partial_z^\alpha \varphi(z,z+w,\eta)\, e^{\langle w,\,\zeta\rangle} dw.\end{aligned}$$

By the Cauchy integration theorem, $\gamma_1(0,\eta;\varrho,\theta)\vee(-\gamma_1(0,\eta;\tilde{\varrho},\tilde{\theta}))$ can be changed to the following two segment paths:

$$\left[\frac{\tilde{\varrho}\eta}{2}\, e^{-\sqrt{-1}(\pi+\tilde{\theta})/2}, \frac{\varrho\eta}{2}\, e^{-\sqrt{-1}(\pi+\theta)/2}\right], \quad \left[\frac{\varrho\eta}{2}\, e^{\sqrt{-1}(\pi+\theta)/2}, \frac{\tilde{\varrho}\eta}{2}\, e^{\sqrt{-1}(\pi+\tilde{\theta})/2}\right].$$

Then we can find $\delta > 0$ such that on the two paths above $\mathrm{Re}\langle w_1,\zeta_1\rangle \le -\delta|\eta\zeta_1|$ holds. Thus as in (1) we can see

$$\sum_\alpha \frac{1}{\alpha!}\int_{\gamma(0,\eta;\varrho,\theta)\vee(-\gamma(0,\eta;\tilde{\varrho},\tilde{\theta}))} \widetilde{w}^\alpha \psi(z,z+\widetilde{w},\eta)\, e^{\langle \widetilde{w},\,\zeta\rangle} d\widetilde{w} \int_{\gamma(0,\eta;\varrho,\theta)} \partial_z^\alpha \varphi(z,z+w,\eta)\, e^{\langle w,\,\zeta\rangle} dw \in \mathfrak{N}_{z_0^*}.$$

Next we find

$$\begin{aligned}&\sum_\alpha \frac{1}{\alpha!}\int_{\gamma(0,\eta;\tilde{\varrho},\tilde{\theta})} \widetilde{w}^\alpha \psi(z,z+\widetilde{w},\eta)\, e^{\langle \widetilde{w},\,\zeta\rangle} d\widetilde{w} \int_{\gamma(0,\eta;\varrho,\theta)} \partial_z^\alpha \varphi(z,z+w,\eta)\, e^{\langle w,\,\zeta\rangle} dw \\ &= \int_{\gamma(0,\eta;\tilde{\varrho},\tilde{\theta})\times\gamma(0,\eta;\varrho,\theta)} \psi(z,z+\widetilde{w},\eta)\Big(\sum_\alpha \frac{\widetilde{w}^\alpha}{\alpha!}\partial_z^\alpha\varphi(z,z+w,\eta)\Big) e^{\langle \widetilde{w}+w,\,\zeta\rangle} d\widetilde{w}\, dw \\ &= \int_{\gamma(0,\eta;\tilde{\varrho},\tilde{\theta})\times\gamma(0,\eta;\varrho,\theta)} \psi(z,z+\widetilde{w},\eta)\,\varphi(z+\widetilde{w},z+\widetilde{w}+w,\eta)\, e^{\langle \widetilde{w}+w,\,\zeta\rangle} d\widetilde{w}\, dw \\ &= \int_{\gamma(0,\eta;\tilde{\varrho},\tilde{\theta})} d\widetilde{w} \int_{\gamma(\widetilde{w},\eta;\varrho,\theta)} \psi(z,z+\widetilde{w},\eta)\,\varphi(z+\widetilde{w},z+w,\eta)\, e^{\langle w,\,\zeta\rangle} dw\end{aligned}$$

$$= \int\limits_{\gamma(0,\eta;\tilde{\varrho},\tilde{\theta})} d\widetilde{w} \int\limits_{\gamma(0,\eta;\varrho,\theta)} \psi(z, z+\widetilde{w}, \eta)\, \varphi(z+\widetilde{w}, z+w, \eta)\, e^{\langle w,\,\zeta\rangle} dw$$

$$+ \int\limits_{\gamma(0,\eta;\tilde{\varrho},\tilde{\theta})} d\widetilde{w} \int\limits_{\gamma(\widetilde{w},\eta;\varrho,\theta)\vee(-\gamma(0,\eta;\varrho,\theta))} \psi(z, z+\widetilde{w}, \eta)\, \varphi(z+\widetilde{w}, z+w, \eta)\, e^{\langle w,\,\zeta\rangle} dw.$$

We consider

$$\sigma(\mu(\psi\otimes\varphi))(z,\zeta,\eta) - \int\limits_{\gamma(0,\eta;\tilde{\varrho},\tilde{\theta})} d\widetilde{w} \int\limits_{\gamma(0,\eta;\varrho,\theta)} \psi(z, z+\widetilde{w}, \eta)\, \varphi(z+\widetilde{w}, z+w, \eta)\, e^{\langle w,\,\zeta\rangle} dw$$

$$= \int\limits_{\gamma(0,\eta;\tilde{\varrho},\tilde{\theta})} d\widetilde{w} \int\limits_{\gamma(0,\eta;\varrho',\theta')\vee(-\gamma(0,\eta;\varrho,\theta))} \psi(z, z+\widetilde{w}, \eta)\, \varphi(z+\widetilde{w}, z+w, \eta)\, e^{\langle w,\,\zeta\rangle} dw.$$

By the Cauchy integration theorem, $\gamma_1(0,\eta;\varrho',\theta') \vee (-\gamma_1(0,\eta;\varrho,\theta))$ can be changed to the following two segment paths:

$$\left[\frac{\varrho\eta}{2}\, e^{-\sqrt{-1}(\pi+\theta)/2}, \frac{\varrho'\eta}{2}\, e^{-\sqrt{-1}(\pi+\theta')/2}\right], \quad \left[\frac{\varrho'\eta}{2}\, e^{\sqrt{-1}(\pi+\theta')/2}, \frac{\varrho\eta}{2}\, e^{\sqrt{-1}(\pi+\theta)/2}\right].$$

Then we can find $\delta > 0$ such that on the two paths above $\mathrm{Re}\langle w_1, \zeta_1\rangle \le -\delta|\eta\zeta_1|$ holds. Thus we can see

$$\sigma(\mu(\psi\otimes\varphi))(z,\zeta,\eta) - \int\limits_{\gamma(0,\eta;\tilde{\varrho},\tilde{\theta})} d\widetilde{w} \int\limits_{\gamma(0,\eta;\varrho,\theta)} \psi(z, z+\widetilde{w}, \eta)\, \varphi(z+\widetilde{w}, z+w, \eta)\, e^{\langle w,\,\zeta\rangle} dw \in \mathfrak{N}_{z_0^*}.$$

Next we consider two segment paths

$$\left[\frac{\varrho\eta}{2}\, e^{-\sqrt{-1}(\pi+\theta)/2}, \frac{\varrho\eta}{2}\, e^{-\sqrt{-1}(\pi+\theta)/2} + \widetilde{w}_1\right], \quad \left[\frac{\varrho\eta}{2}\, e^{\sqrt{-1}(\pi+\theta)/2} + \widetilde{w}_1, \frac{\varrho\eta}{2}\, e^{\sqrt{-1}(\pi+\theta)/2}\right].$$

Since $\tilde{\varrho}$ is sufficiently smaller than ϱ and $|\widetilde{w}_1| \le \tilde{\varrho}|\eta|/2$, we can find $\delta > 0$ such that on the two paths above $\mathrm{Re}\langle w_1, \zeta_1\rangle \le -\delta|\eta\zeta_1|$ holds. Therefore we can conclude that

$$\int\limits_{\gamma(0,\eta;\tilde{\varrho},\tilde{\theta})} d\widetilde{w} \int\limits_{\gamma(\widetilde{w},\eta;\varrho,\theta)-\gamma(0,\eta;\varrho,\theta)} \psi(z, z+\widetilde{w}, \eta)\, \varphi(z+\widetilde{w}, z+w, \eta)\, e^{\langle w,\,\zeta\rangle} dw \in \mathfrak{N}_{z_0^*}.$$

The proof is complete. □

REMARK 6.22. Let $P = [\psi(z,w,\eta)\,dw]$, $Q = [\varphi(z,w,\eta)\,dw] \in \mathscr{E}^{\mathbb{R}}_{X,z_0^*}$. By Corollary 3.10, the product $PQ \in \mathscr{E}^{\mathbb{R}}_{X,z_0^*}$ is represented by $\mu(\psi\otimes\varphi)(z,w,\eta)\,dw$; that is, $PQ = [\mu(\psi\otimes\varphi)(z,w,\eta)\,dw]$, and by Theorem 6.21, we have

$$[\mu(\psi\otimes\varphi)(z,w,\eta)] = \Big[\sum_{\alpha} \frac{1}{\alpha!} \partial_\zeta^\alpha \sigma(\psi)(z,\zeta,\eta)\, \partial_z^\alpha \sigma(\varphi)(z,\zeta,\eta)\Big]$$

$$= \Big[e^{t\langle \partial_{\zeta'},\, \partial_{z'}\rangle} \sigma(\psi)(z,\zeta',\eta)\, \sigma(\varphi)(z',\zeta,\eta)\big|_{\substack{z'=z\\ \zeta'=\zeta}}\Big].$$

In other words, the product in $\mathscr{E}^{\mathbb{R}}_{X,z_0^*}$ coincides with that of the classical symbols given in Theorem 6.16 through the symbol mapping σ.

THEOREM 6.23. *Let* $:P(t;z,\zeta,\eta): = [\psi(z,w,\eta)\,dw] \in \mathscr{E}^{\mathbb{R}}_{X,z_0^*}$.
(1) *It follows that*

$$:P^*(t;z,\zeta,\eta): = :\int_{\gamma(0,\eta;\varrho,\theta)} \psi(z-w,z,\eta)\,e^{-\langle w,\,\zeta\rangle}\,dw:.$$

(2) *Let* $z=\Phi(w)$ *be a complex coordinate transformation. Then*

$$:\Phi^*P(t;w,\lambda,\eta): = :\int_{\gamma(z,\eta;\varrho,\theta)} \psi(z,z',\eta)\,e^{\langle \Phi^{-1}(z')-\Phi^{-1}(z),\,\lambda\rangle} dz':.$$

PROOF. We assume that

$$P(z,\zeta,\eta) := \sigma(\psi)(z,\zeta,\eta) = \int_{\gamma(0,\eta;\varrho,\theta)} \psi(z,z+w,\eta)\,e^{\langle w,\,\zeta\rangle} dw \in \Gamma(\Omega_\rho[d_\rho]\times S;\mathscr{O}_{T^*X\times\mathbb{C}}).$$

(1) We use the notation in the proof of Theorem 6.21. Then there exists $\delta>0$ and for any $Z \Subset S$ and $h>0$, there exists $C_{h,Z}$, $C_Z>0$ such that

$$|\partial_z^\alpha \partial_\zeta^\alpha P(z,\zeta,\eta)| \le \frac{(|\eta|s')^{|\alpha|} C_{h,Z}\,\alpha!\,e^{h\|\zeta\|}}{R^{|\alpha|}},$$
$$|\partial_\eta \partial_z^\alpha \partial_\zeta^\alpha P(z,\zeta,\eta)| \le \frac{(|\eta|s')^{|\alpha|} C_Z\,\alpha!\,e^{-\delta\|\eta\zeta\|}}{R^{|\alpha|}}.$$

Further as in the proof of Theorem 6.21 (2), we have

$$|\partial_z^{\alpha+\beta} \partial_\zeta^{\alpha+\beta} P(z,\zeta,\eta)| \le \frac{(|\eta|s')^{|\alpha|} C_{h,Z}\,\alpha!(\beta!)^2\,e^{2h\|\zeta\|}}{(\rho-\rho')^{2|\beta|}R^{|\alpha|}\|\eta\zeta\|^{|\beta|}}.$$

Thus we see that $\hat{P}^*(z,\zeta,\eta) := \sum_\alpha (1/\alpha!)\,\partial_\zeta^\alpha\partial_z^\alpha P(z,-\zeta,\eta) \in \mathfrak{S}_{a(z_0^*)}$, where $a(z_0^*) := (z_0;-\zeta_0)$ for $z_0^* := (z_0;\zeta_0)$. Further setting $B := 2/(\rho-\rho')^2$ we obtain

$$\Big|\hat{P}^*(z,\zeta,\eta) - \sum_{|\alpha|=0}^{m-1} \frac{1}{\alpha!}\,\partial_\zeta^\alpha\partial_z^\alpha P(z,-\zeta,\eta)\Big| = \Big|\sum_{|\beta|=m}\sum_{|\alpha|=0}^{\infty} \frac{1}{(\alpha+\beta)!}\partial_z^{\alpha+\beta}\partial_\zeta^{\alpha+\beta} P(z,-\zeta,\eta)\Big|$$
$$\le \frac{C_{h,Z}\,m!\,e^{2h\|\zeta\|}}{(\rho-\rho')^{2m}\|\eta\zeta\|^m} \sum_{|\beta|=m}\sum_{|\alpha|=0}^{\infty} \frac{(|\eta|s')^{|\alpha|}}{R^{|\alpha|}} \le \frac{2^{n-1}C_{h,Z}^2\,m!B^m e^{2h\|\zeta\|}}{(1-|\eta|s'/R)^n\|\eta\zeta\|^m},$$

and hence $P^*(t;z,\zeta,\eta) - \hat{P}^*(z,\zeta,\eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl},a(z_0^*)}$. Moreover we have

$$\hat{P}^*(z,\zeta,\eta) = \sum_\alpha \frac{1}{\alpha!}\,\partial_\zeta^\alpha\partial_z^\alpha P(z,-\zeta,\eta) = \sum_\alpha \frac{1}{\alpha!}\,\partial_\zeta^\alpha\partial_z^\alpha \int_{\gamma(0,\eta;\varrho,\theta)} \psi(z,z+w,\eta)\,e^{-\langle w,\,\zeta\rangle} dw$$

$$= \sum_{\alpha} \int\limits_{\gamma(0,\eta;\varrho,\theta)} \frac{(-w)^{\alpha}}{(\alpha-\beta)!\,\beta!} \partial_z^{\alpha-\beta} \partial_w^{\beta} \psi(z, z+w, \eta)\, e^{-\langle w,\, \zeta\rangle} dw$$

$$= \sum_{\alpha,\beta} \int\limits_{\gamma(0,\eta;\varrho,\theta)} \frac{(-w)^{\alpha+\beta}}{\alpha!\,\beta!} \partial_z^{\alpha} \partial_w^{\beta} \psi(z, z+w, \eta)\, e^{-\langle w,\, \zeta\rangle} dw = \int\limits_{\gamma(0,\eta;\varrho,\theta)} \psi(z-w, z, \eta)\, e^{-\langle w,\, \zeta\rangle} dw.$$

(2) As in (1), we see that

$$\hat{\Phi}^* P(w, \lambda, \eta) := \sum_{\alpha} \frac{1}{\alpha!} \partial_{\zeta'}^{\alpha} \partial_{z'}^{\alpha} P(\Phi(w), \zeta' + {}^{\mathrm{t}}J_{\Phi}^*(\Phi(w)+z', \Phi(w))\lambda, \eta)\Big|_{\substack{z'=0\\ \zeta'=0}} \in \widehat{\mathfrak{S}}_{z_0^*}\,,$$

$$\hat{\Phi}^* P(w, \lambda, \eta) - \Phi^* P(t; w, \lambda, \eta) \in \widehat{\mathfrak{N}}_{z_0^*}\,.$$

Further under the relation $z = \Phi(w)$ we have

$$\hat{\Phi}^* P(w, \lambda, \eta) = \sum_{\alpha} \frac{1}{\alpha!} \partial_{\zeta'}^{\alpha} \partial_{z'}^{\alpha} \int\limits_{\gamma(0,\eta;\varrho,\theta)} \psi(\Phi(w), \Phi(w)+z'', \eta)\, e^{\langle z'',\, \zeta' + {}^{\mathrm{t}}J_{\Phi}^*(\Phi(w)+z', \Phi(w))\lambda\rangle} dz''\Big|_{\substack{z'=0\\ \zeta'=0}}$$

$$= \int\limits_{\gamma(z,\eta;\varrho,\theta)} \psi(z, z'', \eta) \sum_{\alpha} \frac{1}{\alpha!} \partial_{\zeta'}^{\alpha} \partial_{z'}^{\alpha}\, e^{\langle z''-z,\, \zeta' + {}^{\mathrm{t}}J_{\Phi}^*(z', z)\lambda\rangle} dz''\Big|_{\substack{z'=z\\ \zeta'=0}}$$

$$= \int\limits_{\gamma(z,\eta;\varrho,\theta)} \psi(z, z'', \eta) \sum_{\alpha} \frac{(z''-z)^{\alpha}}{\alpha!} \partial_{z'}^{\alpha}\, e^{\langle z''-z,\, {}^{\mathrm{t}}J_{\Phi}^*(z', z)\lambda\rangle} dz''\Big|_{z'=z}$$

$$= \int\limits_{\gamma(z,\eta;\varrho,\theta)} \psi(z, z'', \eta)\, e^{\langle z''-z, {}^{\mathrm{t}}J_{\Phi}^*(z'', z)\lambda\rangle} dz'' = \int\limits_{\gamma(z,\eta;\varrho,\theta)} \psi(z, z', \eta)\, e^{\langle \Phi^{-1}(z'')-\Phi^{-1}(z),\, \lambda\rangle} dz''.$$

□

Remark 6.24. For $K(z,w)\,dw = [\psi(z,w,\eta)\,dw] \in \mathscr{E}^{\mathbb{R}}_{X,z_0^*}$.

(1) The formal adjoint $K^*(z,w)\,dw$ is defined by $[(-1)^n \psi(w,z,\eta)\,dw]$, and the symbol is given by

$$(-1)^n \int\limits_{\gamma^a(0,\eta;\varrho,\theta)} \psi(z+w, z, \eta)\, e^{\langle w,\, \zeta\rangle} dw = \int\limits_{\gamma(0,\eta;\varrho,\theta)} \psi(z-w, z, \eta)\, e^{-\langle w,\, \zeta\rangle} dw,$$

here $\gamma^a(0,\eta;\varrho,\theta)$ is the image of $\gamma(0,\eta;\varrho,\theta)$ under the mapping $w \mapsto -w$.

(2) Let $z = \Phi(w)$ be a complex coordinate transformation. Then as we see in Appendix B (in particular see (B.4)), the associated coordinate transform of $K(z,w)\,dw$ is given by

$$:\int\limits_{\gamma(z,\eta;\varrho,\theta)} \psi(z, z', \eta)\, e^{\langle \Phi^{-1}(z')-\Phi^{-1}(z),\, \lambda\rangle} dz' : .$$

Thus, Theorem 6.23 means that as in the case of product, the operations of formal adjoint and coordinate transformation for operators are also represented by the corresponding operations for formal symbols.

7. Formal symbols with an apparent parameter.

Definition 7.1 (see [**2**], [**6**]). Let t be an indeterminate.

(1) $P(t;z,\zeta)=\sum_{\nu=0}^{\infty}t^{\nu}P_{\nu}(z,\zeta)$ is an element of $\widehat{\mathscr{S}}(\Omega)$ if

(i) $P_{\nu}(z,\zeta)\in\Gamma(\Omega_{\rho}[(\nu+1)d_{\rho}];\mathscr{O}_{T^*X})$ for some $d>0$ and $\rho\in\,]0,1[$,

(ii) there exists a constant $A\in\,]0,1[$ satisfying the following: for any $h>0$ there exists a constant $C_h>0$ such that

$$|P_{\nu}(z,\zeta)|\le C_hA^{\nu}e^{h\|\zeta\|}\quad(\nu\in\mathbb{N}_0,\,(z;\zeta)\in\Omega_{\rho}[(\nu+1)d_{\rho}]).$$

(2) Let $P(t;z,\zeta)=\sum_{\nu=0}^{\infty}t^{\nu}P_{\nu}(z,\zeta)\in\widehat{\mathscr{S}}(\Omega)$. Then $P(t;z,\zeta)$ is an element of $\widehat{\mathscr{N}}(\Omega)$ if there exists a constant $A\in\,]0,1[$ satisfying the following: for any $h>0$ there exists a constant $C_h>0$ such that

$$\left|\sum_{\nu=0}^{m-1}P_{\nu}(z,\zeta)\right|\le C_hA^{m}e^{h\|\zeta\|}\quad(m\in\mathbb{N},\,(z;\zeta)\in\Omega_{\rho}[md_{\rho}]).$$

(3) For $z_0^*\in\dot{T}^*X$, we set

$$\widehat{\mathscr{S}}_{z_0^*}:=\varinjlim_{\Omega}\widehat{\mathscr{S}}(\Omega)\supset\widehat{\mathscr{N}}_{z_0^*}:=\varinjlim_{\Omega}\widehat{\mathscr{N}}(\Omega).$$

We call each element of $\widehat{\mathscr{S}}(\Omega)$ (resp. $\widehat{\mathscr{N}}(\Omega)$) a *formal symbol* (resp. *formal null-symbol*) *on* Ω.

For $U\subset S$ and $m\in\mathbb{N}$, we set

$$(\Omega_{\rho}*U)[md_{\rho}]:=\{(z;\zeta,\eta)\in\Omega_{\rho}\times U;\,\|\eta\zeta\|\ge md_{\rho}\}\subset\Omega_{\rho}[md_{\rho}]\times U.$$

Definition 7.2. Let t be an indeterminate. We say that $P(t;z,\zeta,\eta)=\sum_{\nu=0}^{\infty}t^{\nu}P_{\nu}(z,\zeta,\eta)$ is an element of $\widehat{\mathfrak{N}}(\Omega;S)$ if

(i) $P_{\nu}(z,\zeta,\eta)\in\Gamma((\Omega_{\rho}*S)[(\nu+1)d_{\rho}];\mathscr{O}_{T^*X\times\mathbb{C}})$ for some $d>0$ and $\rho\in\,]0,1[$,

(ii) there exists a constant $A\in\,]0,1[$, and for any $Z\Subset S$, $h>0$ there exists $C_{h,Z}>0$ such that

$$\left|\sum_{\nu=0}^{m-1}P_{\nu}(z,\zeta,\eta)\right|\le C_{h,Z}A^{m}e^{h\|\zeta\|}\quad(m\in\mathbb{N},\,(z;\zeta,\eta)\in(\Omega_{\rho}*Z)[md_{\rho}]).\tag{7.1}$$

Definition 7.3. (1) We say that $P(t;z,\zeta,\eta)=\sum_{\nu=0}^{\infty}t^{\nu}P_{\nu}(z,\zeta,\eta)$ is an element of $\widehat{\mathfrak{S}}(\Omega;S)$ if

(i) $P_{\nu}(z,\zeta,\eta)\in\Gamma((\Omega_{\rho}*S)[(\nu+1)d_{\rho}];\mathscr{O}_{T^*X\times\mathbb{C}})$ for some $d>0$ and $\rho\in\,]0,1[$,

(ii) there exists a constant $A \in \,]0,1[$, and for any $Z \Subset S$, $h > 0$, there exists $C_{h,Z} > 0$ such that

$$|P_\nu(z,\zeta,\eta)| \le C_{h,Z} A^\nu e^{h\|\zeta\|} \quad (\nu \in \mathbb{N}_0,\ (z;\zeta,\eta) \in (\Omega_\rho * Z)[(\nu+1)d_\rho]).$$

(iii) $\partial_\eta P(t;z,\zeta,\eta) \in \widehat{\mathfrak{N}}(\Omega;S)$.

We call each element of $\widehat{\mathfrak{S}}(\Omega;S)$ (resp. $\widehat{\mathfrak{N}}(\Omega;S)$) a *formal symbol* (resp. *formal null-symbol*) *on Ω with an apparent parameter in S.*

LEMMA 7.4. $\widehat{\mathfrak{N}}(\Omega;S) \subset \widehat{\mathfrak{S}}(\Omega;S)$.

PROOF. We assume (7.1). For any $\nu \in \mathbb{N}$ and $(z;\zeta,\eta) \in (\Omega_\rho * Z)[(\nu+1)d_\rho] \subset (\Omega_\rho * Z)[\nu d_\rho]$, we have

$$\begin{aligned}|P_\nu(z,\zeta,\eta)| &= \Big|\sum_{i=0}^{\nu} P_i(z,\zeta,\eta) - \sum_{i=0}^{\nu-1} P_i(z,\zeta,\eta)\Big| \le \Big|\sum_{i=0}^{\nu} P_i(z,\zeta,\eta)\Big| + \Big|\sum_{i=0}^{\nu-1} P_i(z,\zeta,\eta)\Big| \\ &\le C_{h,Z}A^{\nu+1}e^{h\|\zeta\|} + C_{h,Z}A^\nu e^{h\|\zeta\|} \le C_{h,Z}(A+1)A^\nu e^{h\|\zeta\|}.\end{aligned}$$

Next, for any $Z \Subset S$, take δ' and Z' as in (4.3). Then by the Cauchy inequality, for any $h > 0$ there exist constants $C_{h,Z'}$, $R > 0$ such that for any $m \in \mathbb{N}$ and $(z;\zeta,\eta) \in (\Omega_\rho * Z)[m(2d)_\rho]$, the following holds:

$$\Big|\sum_{\nu=0}^{m-1} \partial_\eta P_\nu(z,\zeta,\eta)\Big| \le \frac{1}{\delta'|\eta|} \sup_{|\eta-\eta'|=\delta'|\eta|} \Big|\sum_{\nu=0}^{m-1} P_\nu(z,\zeta,\eta')\Big| \le \frac{C_{h,Z'}A^m e^{h\|\zeta\|}}{\delta' m_Z}. \qquad \square$$

We set

$$\widehat{\mathfrak{S}}_{z_0^*} := \varinjlim_{\Omega,S} \widehat{\mathfrak{S}}(\Omega;S) \supset \widehat{\mathfrak{N}}_{z_0^*} := \varinjlim_{\Omega,S} \widehat{\mathfrak{N}}(\Omega;S).$$

PROPOSITION 7.5. *Let $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$. Then for any $\eta_0 \in S$, $P(t;z,\zeta,\eta_0) \in \widehat{\mathscr{S}}(\Omega)$ and $P(t;z,\zeta,\eta) - P(t;z,\zeta,\eta_0) \in \widehat{\mathfrak{N}}(\Omega;S)$.*

PROOF. Set $d' := d/|\eta_0| > d$. Then for any $h > 0$, there exists a constant $C_{h,\eta_0} > 0$ such that

$$|P_\nu(z,\zeta,\eta_0)| \le C_{h,\eta_0} A^\nu e^{h\|\zeta\|} \quad ((z;\zeta) \in \Omega_\rho[(\nu+1)d'_\rho]).$$

Therefore $P(t;z,\zeta,\eta_0) \in \widehat{\mathscr{S}}(\Omega)$. For any $Z \Subset S$, let $Z' \Subset S$ be the convex hull of $Z \cup \{\eta_0\}$. Since

$$P_\nu(z,\zeta,\eta) = P_\nu(z,\zeta,\eta_0) + \int_{\eta_0}^{\eta} \partial_\eta P_\nu(z,\zeta,\tau)\,d\tau$$

and $\partial_\eta P(t;z,\zeta,\eta) \in \widehat{\mathfrak{N}}(\Omega;S)$, there exists a constant $A \in \,]0,1[$, and for any $h > 0$ we can find a constant $C_{h,Z'} > 0$ such that for any $m \in \mathbb{N}$ and $(z;\zeta,\eta) \in (\Omega_\rho * Z)[md'_\rho] \subset (\Omega_\rho * Z')[md'_\rho]$, the following holds:

$$\Big|\sum_{\nu=0}^{m-1}(P_\nu(z,\zeta,\eta) - P_\nu(z,\zeta,\eta_0))\Big| = \Big|\sum_{\nu=0}^{m-1}\int_{\eta_0}^{\eta}\partial_\eta P_\nu(z,\zeta,\tau)\,d\tau\Big| = \Big|\int_{\eta_0}^{\eta}\sum_{\nu=0}^{m-1}\partial_\eta P_\nu(z,\zeta,\tau)\,d\tau\Big|$$
$$\le |\eta-\eta_0| C_{h,Z'} A^m e^{h\|\zeta\|} \le r C_{h,Z'} A^m e^{h\|\zeta\|}.$$

Hence $P(t;z,\zeta,\eta) - P(t;z,\zeta,\eta_0) \in \widehat{\mathfrak{N}}(\Omega;S)$. □

Proposition 7.6. *There exists the following isomorphism*:

$$\widehat{\mathscr{S}}(\Omega)/\widehat{\mathscr{N}}(\Omega) \simeq \widehat{\mathfrak{S}}(\Omega;S)/\widehat{\mathfrak{N}}(\Omega;S).$$

Proof. We regard that

$$\widehat{\mathscr{S}}(\Omega) = \{P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S);\, \partial_\eta P(t;z,\zeta,\eta) = 0\} \subset \widehat{\mathfrak{S}}(\Omega;S),$$
$$\widehat{\mathscr{N}}(\Omega) = \widehat{\mathscr{S}}(\Omega) \cap \widehat{\mathfrak{N}}(\Omega;S) \subset \widehat{\mathfrak{N}}(\Omega;S).$$

Hence we have an injective mapping $\widehat{\mathscr{S}}(\Omega)/\widehat{\mathscr{N}}(\Omega) \hookrightarrow \widehat{\mathfrak{S}}(\Omega;S)/\widehat{\mathfrak{N}}(\Omega;S)$.

Let $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$ and fix $\eta_0 \in S$ arbitrary. Then by Proposition 7.5, we have $P(t;z,\zeta,\eta_0) \in \widehat{\mathscr{S}}(\Omega)$ and $P(t;z,\zeta,\eta) - P(t;z,\zeta,\eta_0) \in \widehat{\mathfrak{N}}(\Omega;S)$. □

Lemma 7.7. $\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S) \subset \widehat{\mathfrak{S}}(\Omega;S)$ *and* $\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S) \subset \widehat{\mathfrak{N}}(\Omega;S)$.

Proof. Let $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$, and assume (6.2). We replace d as $B := A/d_\rho \in \,]0,1[$ if necessary. Hence on $(\Omega_\rho * Z)[(\nu+1)d_\rho]$, we have

$$|P_\nu(z,\zeta,\eta)| \le \frac{C_{h,Z}\,\nu!\,e^{h\|\zeta\|}}{(\nu+1)^\nu}\Big(\frac{A}{d_\rho}\Big)^\nu \le C_{h,Z}B^\nu e^{h\|\zeta\|}.$$

The proof of $\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S) \subset \widehat{\mathfrak{N}}(\Omega;S)$ is similar. □

Proposition 7.8. $\widehat{\mathfrak{N}}(\Omega;S) \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}}) = \mathfrak{N}(\Omega;S)$.

Proof. If $P(z,\zeta,\eta) \in \widehat{\mathfrak{N}}(\Omega;S) \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$, we set $\delta := -2\log A/d_\rho > 0$, and for any $Z \Subset S$, take $h = \delta m_Z$. For each $(z;\zeta,\eta) \in \Omega_\rho[d_\rho] \times Z$, we take m as the integral part of $\|\eta\zeta\|/d_\rho$ so that $(m+1)d_\rho > \|\eta\zeta\| \ge md_\rho$. Thus there exists $C_Z > 0$ such that

$$|P(z,\zeta,\eta)| \le C_Z A^m e^{\delta m_Z\|\zeta\|} \le C_Z A^{\|\eta\zeta\|/d_\rho - 1} e^{\delta m_Z\|\zeta\|} \le \frac{C_Z}{A} e^{\delta m_Z\|\zeta\| - 2\delta\|\eta\zeta\|} \le \frac{C_Z}{A} e^{-\delta\|\eta\zeta\|}.$$

Hence we have (4.2). Conversely, by Proposition 6.7 and Lemma 7.7 we have

$$\mathfrak{N}(\Omega;S) = \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S) \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}}) \subset \widehat{\mathfrak{N}}(\Omega;S) \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}}). \ \square$$

THEOREM 7.9. *Let $z_0^* \in \dot{T}^*X$ and $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}_{z_0^*}$. Then there exists $\widetilde{P}(z,\zeta,\eta) \in \mathfrak{S}_{z_0^*}$ such that*

$$P(t;z,\zeta,\eta) - \widetilde{P}(z,\zeta,\eta) \in \widehat{\mathfrak{N}}_{z_0^*}.$$

PROOF. We may assume that $z_0^* = (0;1,0,\dots,0)$. We fix $\eta_0 \in S \cap \mathbb{R}$. Then by Proposition 7.5 we have $P(t;z,\zeta,\eta_0) \in \widehat{\mathscr{S}}(\Omega)$ and $P(t;z,\zeta,\eta) - P(t;z,\zeta,\eta_0) \in \widehat{\mathfrak{N}}(\Omega;S)$. We use the notation of the proof in Theorem 5.7; We develop $P_\nu(z,\zeta,\eta_0)$ into the Taylor series with respect to $(\zeta_2/\zeta_1,\dots,\zeta_n/\zeta_1)$:

$$P_\nu(z,\zeta,\eta_0) = \sum_{\alpha\in\mathbb{N}_0^{n-1}} P_{\nu,\alpha}(z,\zeta_1,\eta_0)\Big(\frac{\zeta'}{\zeta_1}\Big)^{\alpha}.$$

Then there exist sufficiently small r_0, $\theta' > 0$ and sufficiently large $d > 0$ such that $P_{\nu,\alpha}(z,\zeta_1,\eta_0)$ is holomorphic on a common neighborhood of $D[(\nu+1)d]$ for each $\alpha \in \mathbb{N}_0^{n-1}$, where

$$D[(\nu+1)d] := \{(z,\zeta_1) \in \mathbb{C}^{n+1};\ \|z\| \le r_0,\ |\arg\zeta_1| \le \theta',\ |\zeta_1| \ge d(\nu+1)\}.$$

It follows from the Cauchy inequality that we can take constants $K > 0$ and $A \in \,]0,1[$ so that for each $h > 0$ there exists $C_h > 0$ such that for every $\alpha \in \mathbb{N}_0^{n-1}$,

$$|P_{\nu,\alpha}(z,\zeta_1,\eta_0)| \le C_h A^\nu K^{|\alpha|} e^{h|\zeta_1|} \quad ((z,\zeta_1) \in D[(\nu+1)d]).$$

We set $P^{\mathcal{B}}_{\nu,\alpha}(z,\zeta_1,\eta)$ and $P^{\mathcal{B}}_\nu(z,\zeta,\eta)$ as in (5.2). Then as in (5.11), for there exists $\delta_0 > 0$ and for any $Z \Subset S$, there exists C'_Z such that for any $(z,\zeta_1,\eta) \in D[(\nu+1)d] \times Z$ and $|\zeta_i| \le \varepsilon|\zeta_1|$ we have

$$|P_\nu(z,\zeta,\eta_0) - P^{\mathcal{B}}_\nu(z,\zeta,\eta)| \le 2^{n-1} C'_Z A^\nu e^{-\delta_0|\eta\zeta_1|/2}.$$

Thus shrinking Ω if necessary, setting $A_1 := e^{-\delta_0 d_\rho/2} \in \,]0,1[$, for any $m \in \mathbb{N}$, we have on $(\Omega_\rho * Z)[md_\rho]$

$$\Big|\sum_{\nu=0}^{m-1}(P_\nu(z,\zeta,\eta_0) - P^{\mathcal{B}}_\nu(z,\zeta,\eta))\Big| \le \frac{C_Z A_1^m}{1-A}.$$

Hence

$$\begin{aligned} &P(t;z,\zeta,\eta) - P^{\mathcal{B}}(t;z,\zeta,\eta) \\ &\quad = P(t;z,\zeta,\eta) - P(t;z,\zeta,\eta_0) + P(t;z,\zeta,\eta_0) - P^{\mathcal{B}}(t;z,\zeta,\eta) \in \widehat{\mathfrak{N}}(\Omega;S). \end{aligned}$$

We set

$$\varpi_\alpha(P_\nu)(x,w_1,\eta) := \int_{(\nu+1)d}^{\infty} P^{\mathcal{B}}_{\nu,\alpha}(z,\zeta_1,\eta)\,\frac{e^{-w_1\zeta_1}}{\zeta_1^{|\alpha|}}\,d\zeta_1.$$

Recall L of (5.13) and L_k of (5.15). Hence as in (5.16), $\varpi_\alpha(P_\nu)(x,w_1,\eta)$ extends analytically to the domain $L\times S$, and for any $\eta\in S$ we have

$$\sup\{|\varpi_\alpha(P_\nu)(x,w_1,\eta)|;\,(x,w_1)\in L_k\}\le \frac{2kC_{k,Z}A^\nu}{\delta_1|\alpha|!}\,(K|\eta|)^{|\alpha|}. \tag{7.2}$$

Now we define

$$\varpi(P_\nu)(z,z+w,\eta):=\sum_{\alpha\in\mathbb{N}_0^{n-1}}\frac{\alpha!\,\varpi_\alpha(P_\nu)(z,w_1,\eta)}{(2\pi\sqrt{-1}\,)^n\,(w')^{\alpha+\mathbf{1}_{n-1}}}\,.$$

The right-hand side converges locally uniformly V_k of (5.17). Hence $\varpi(P_\nu)(z,z+w,\eta)$ is a holomorphic function defined on V of (5.18). Hence we can define

$$\begin{aligned}\widetilde{P}_\nu(z,\zeta,\eta)&:=\int\limits_{\gamma(0,\eta;\varrho,\theta)}\varpi(P_\nu)(z,z+w,\eta)e^{\langle w,\,\zeta\rangle}dw\\&=\sum_{\alpha\in\mathbb{N}_0^{n-1}}(\zeta')^\alpha\int\limits_{\gamma_1(0,\eta;\varrho,\theta)}dw_1\,\frac{e^{w_1\zeta_1}}{2\pi\sqrt{-1}}\int_{(\nu+1)d}^{\infty}P^{\mathcal{B}}_{\nu,\alpha}(z,\xi_1,\eta)\,\frac{e^{-w_1\xi_1}}{\xi_1^{|\alpha|}}\,d\xi_1\,.\end{aligned}$$

By virtue of (7.2), there exist conic neighborhood Ω of z_0^*, $\rho\in\,]0,1[$ and $d>0$ such that $\widetilde{P}_\nu(z,\zeta,\eta)\in\Gamma(\Omega_\rho[d_\rho]\times S;\mathscr{O}_{T^*X\times\mathbb{C}})$ and for any $h>0$ and $Z\Subset S$ there exist constants $C_{h,Z}>0$ such that

$$|\widetilde{P}_\nu(z,\zeta,\eta)|\le C_{h,Z}A^\nu e^{h\|\zeta\|}\quad((z;\zeta,\eta)\in\Omega_\rho[d_\rho]\times Z). \tag{7.3}$$

We set

$$\widetilde{V}_\varepsilon[(\nu+1)d]:=\bigcap_{i=2}^{n}\left\{(z,\zeta)\in\mathbb{C}^{2n};\,\|z\|<r_0,\,|\zeta_1|\ge\frac{(\nu+1)d}{\varepsilon},\,|\arg\zeta_1|\le\varepsilon,\,|\zeta_i|\le\varepsilon|\zeta_1|\right\}.$$

Recall $\Sigma_\pm$ (cf. Figure 1), and we set $\Sigma_\pm^\nu:=\{(\nu+1)\xi_1\in\mathbb{C};\,\xi_1\in\Sigma_\pm\}$. Then we have $\widetilde{P}_\nu(z,\zeta,\eta)=I_\nu+I_\nu^-+I_\nu^+$, where

$$\begin{aligned}I_\nu&:=\sum_{\alpha\in\mathbb{N}_0^{n-1}}(\zeta')^\alpha\int_{\Sigma_-^\nu-\Sigma_+^\nu}\frac{P^{\mathcal{B}}_{\nu,\alpha}(z,\xi_1,\eta)\,e^{a(\zeta_1-\xi_1)}}{2\pi\sqrt{-1}\,\xi_1^{|\alpha|}(\xi_1-\zeta_1)}\,d\xi_1,\\I_\nu^-&:=-\sum_{\alpha\in\mathbb{N}_0^{n-1}}(\zeta')^\alpha\int_{\Sigma_-^\nu}\frac{P^{\mathcal{B}}_{\nu,\alpha}(z,\xi_1,\eta)\,e^{\beta_1(\eta)(\zeta_1-\xi_1)}}{2\pi\sqrt{-1}\,\xi_1^{|\alpha|}(\xi_1-\zeta_1)}\,d\xi_1,\\I_\nu^+&:=\sum_{\alpha'\in\mathbb{N}_0^{n-1}}(\zeta')^\alpha\int_{\Sigma_+^\nu}\frac{P_{\nu,\alpha}(z,\xi_1,\eta)\,e^{\beta_0(\eta)(\zeta_1-\xi_1)}}{2\pi\sqrt{-1}\,\xi_1^{|\alpha|}(\xi_1-\zeta_1)}\,d\xi_1.\end{aligned}$$

On $\widetilde{V}_\varepsilon[(\nu+1)d]\times Z$, as in (5.19) and (5.20) we have

$$|I_\nu^-| \le \frac{2^{n-2} C_{h_Z} A^\nu e^{-c|\eta\zeta_1|}}{c} \left(e^{(h_Z + |\beta_1| r)d} + \frac{e^{-h_0 dr}}{2\pi h_0 d m_Z} \right),$$
$$|I_\nu^+| \le \frac{2^{n-2} C_{h_Z} A^\nu e^{-c|\eta\zeta_1|}}{c} \left(e^{(h_Z + |\beta_0| r)d} + \frac{e^{-h_0 dr}}{2\pi h_0 d m_Z} \right). \tag{7.4}$$

Further

$$I_\nu = \sum_{\alpha \in \mathbb{N}_0^{n-1}} P^{\mathcal{B}}_{\nu,\alpha}(z, \zeta_1, \eta) \left(\frac{\zeta'}{\zeta_1} \right)^\alpha = P^{\mathcal{B}}_\nu(z, \zeta, \eta)$$

holds if ζ_1 is located in the domain surrounded by $\varSigma^\nu_- - \varSigma^\nu_+$. Therefore, shrinking $\varOmega$ and replacing d with a larger one if necessary, by (7.4), we can find $\delta > 0$ satisfying the following: for any $Z \Subset S$ there exists $C_Z > 0$ such that on $\varOmega_\rho[(\nu+1)d_\rho] \times Z$, the following holds:

$$|\widetilde{P}_\nu(z, \zeta, \eta) - P^{\mathcal{B}}_\nu(z, \zeta, \eta)| \le C_Z A^\nu e^{-\delta \|\eta\zeta\|}.$$

We set $\widetilde{P}(t; z, \zeta, \eta) := \sum_{\nu=0}^\infty t^\nu \widetilde{P}_\nu(z, \zeta, \eta)$. We set $A_2 := e^{-\delta d_\rho} \in \left]0, 1\right[$. Then for any $m \in \mathbb{N}$, on $(\varOmega_\rho * Z)[m d_\rho]$ we have

$$\Big| \sum_{\nu=0}^{m-1} (\widetilde{P}_\nu(z, \zeta, \eta) - P^{\mathcal{B}}_\nu(z, \zeta, \eta)) \Big| \le \frac{C_Z A_2^m}{1 - A}\,,$$

i.e. $\widetilde{P}(t; z, \zeta, \eta) - P^{\mathcal{B}}(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{z_0^*}$. In particular by Lemma 7.4,

$$\partial_\eta \widetilde{P}(t; z, \zeta, \eta) = \partial_\eta (\widetilde{P}(t; z, \zeta, \eta) - P^{\mathcal{B}}(t; z, \zeta, \eta)) + \partial_\eta P^{\mathcal{B}}(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{z_0^*}\,.$$

Thus $\widetilde{P}(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{z_0^*}$. By (7.3), we can define

$$\widetilde{P}(z, \zeta, \eta) := \sum_{\nu=0}^\infty \widetilde{P}_\nu(z, \zeta, \eta) \in \varGamma(\varOmega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X \times \mathbb{C}}),$$

and we have

$$|\widetilde{P}(z,\zeta,\eta)| \le \frac{C_{h,Z} e^{h\|\zeta\|}}{1-A} \qquad ((z;\zeta,\eta) \in \varOmega_\rho[d_\rho] \times Z),$$
$$\Big| \widetilde{P}(z,\zeta,\eta) - \sum_{\nu=0}^{m-1} \widetilde{P}_\nu(z,\zeta,\eta) \Big| = \Big| \sum_{\nu=m}^{\infty} \widetilde{P}_\nu(z,\zeta,\eta) \Big| \le \frac{C_{h,Z} A^m e^{h\|\zeta\|}}{1-A} \quad ((z;\zeta,\eta) \in \varOmega_\rho[m d_\rho] \times Z),$$

i.e. $\widetilde{P}(z, \zeta, \eta) - \widetilde{P}(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{z_0^*}$. Moreover by Lemma 7.4 and Proposition 7.8, we have

$$\begin{aligned} \partial_\eta \widetilde{P}(z, \zeta, \eta) &= \partial_\eta (\widetilde{P}(z, \zeta, \eta) - \widetilde{P}(t; z, \zeta, \eta)) + \partial_\eta \widetilde{P}(t; z, \zeta, \eta) \\ &\in \widehat{\mathfrak{N}}_{z_0^*} \cap \varGamma(\varOmega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X \times \mathbb{C}}) = \mathfrak{N}_{z_0^*}\,. \end{aligned}$$

Therefore $\widetilde{P}(z,\zeta,\eta) \in \mathfrak{S}_{z_0^*}$ and

$$P(t;z,\zeta,\eta) - \widetilde{P}(z,\zeta,\eta) = P(t;z,\zeta,\eta) - P^{\mathcal{B}}(t;z,\zeta,\eta) + P^{\mathcal{B}}(t;z,\zeta,\eta) - \widetilde{P}(t;z,\zeta,\eta) + \widetilde{P}(t;z,\zeta,\eta) - \widetilde{P}(z,\zeta,\eta) \in \widehat{\mathfrak{N}}_{z_0^*}. \qquad \square$$

Theorem 7.10. *For any* $z_0^* \in \dot{T}^*X$, *the inclusions* $\mathfrak{S}_{z_0^*} \subset \widehat{\mathfrak{S}}_{\mathrm{cl},z_0^*} \subset \widehat{\mathfrak{S}}_{z_0^*}$ *and* $\mathfrak{N}_{z_0^*} \subset \widehat{\mathfrak{N}}_{\mathrm{cl},z_0^*} \subset \widehat{\mathfrak{N}}_{z_0^*}$ *induce*

$$\mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*} \xrightarrow{\sim} \widehat{\mathfrak{S}}_{\mathrm{cl},z_0^*}/\widehat{\mathfrak{N}}_{\mathrm{cl},z_0^*} \xrightarrow{\sim} \widehat{\mathfrak{S}}_{z_0^*}/\widehat{\mathfrak{N}}_{z_0^*}.$$

Proof. By Proposition 7.8 and Theorem 7.9, we obtain an isomorphism $\mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*} \xrightarrow{\sim} \widehat{\mathfrak{S}}_{z_0^*}/\widehat{\mathfrak{N}}_{z_0^*}$, and we shall show that this isomorphism is compatible with $\mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*} \xrightarrow{\sim} \widehat{\mathfrak{S}}_{\mathrm{cl},z_0^*}/\widehat{\mathfrak{N}}_{\mathrm{cl},z_0^*}$ in Corollary 6.9. For any $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl},z_0^*} \subset \widehat{\mathfrak{S}}_{z_0^*}$, by Theorems 6.8 and 7.9, there exist $P'(z,\zeta,\eta)$, $P''(z,\zeta,\eta) \in \mathfrak{S}_{z_0^*}$ such that

$$\begin{cases} P(t;z,\zeta,\eta) - P'(z,\zeta,\eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl},z_0^*}, \\ P(t;z,\zeta,\eta) - P''(z,\zeta,\eta) \in \widehat{\mathfrak{N}}_{z_0^*}. \end{cases}$$

Then, by Propositions 6.7 and 7.8 we have

$$\begin{aligned} P'(z,\zeta,\eta) - P''(z,\zeta,\eta) &\in \mathfrak{S}_{z_0^*} \cap \widehat{\mathfrak{N}}_{z_0^*} \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}}) \\ &= \mathfrak{N}_{z_0^*} = \widehat{\mathfrak{N}}_{\mathrm{cl},z_0^*} \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}}). \qquad \square \end{aligned}$$

Remark 7.11. Summing up, we have the following commutative diagram:

$$\begin{array}{ccccccc} \mathscr{E}^{\mathbb{R}}_{X,z_0^*} & \xrightarrow{\sim} & \mathscr{S}_{z_0^*}/\mathscr{N}_{z_0^*} & \xrightarrow{\sim} & \widehat{\mathscr{S}}_{\mathrm{cl},z_0^*}/\widehat{\mathscr{N}}_{\mathrm{cl},z_0^*} & \xrightarrow{\sim} & \widehat{\mathscr{S}}_{z_0^*}/\widehat{\mathscr{N}}_{z_0^*} \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ \varinjlim_{\boldsymbol{\kappa}} E^{\mathbb{R}}_X(\boldsymbol{\kappa}) & \xrightarrow{\sim} & \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*} & \xrightarrow{\sim} & \widehat{\mathfrak{S}}_{\mathrm{cl},z_0^*}/\widehat{\mathfrak{N}}_{\mathrm{cl},z_0^*} & \xrightarrow{\sim} & \widehat{\mathfrak{S}}_{z_0^*}/\widehat{\mathfrak{N}}_{z_0^*}. \end{array}$$

Definition 7.12. As in the case of $\mathfrak{S}(\Omega;S)$ and $\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$, for any $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$ we set

$$:\!P(t;z,\zeta,\eta)\!: := P(t;z,\zeta,\eta) \bmod \widehat{\mathfrak{N}}(\Omega;S) \in \widehat{\mathfrak{S}}(\Omega;S)/\widehat{\mathfrak{N}}(\Omega;S)$$

which is also called the *normal product* or the *Wick product* of $P(t;z,\zeta,\eta)$.

We use the notation of Theorem 6.11. For any $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$, we also set

$$\Phi^*P(t;w,\lambda,\eta) := e^{t\langle\partial_{\zeta'},\partial_{z'}\rangle} P(t;\Phi(w),\zeta' + {}^{\mathrm{t}}J^*_{\Phi}(\Phi(w)+z',\Phi(w))\lambda,\eta)\Big|_{\substack{z'=0\\ \zeta'=0}}.$$

Theorem 7.13. *Under the notation above, the following hold.*
(1) $\Phi^*P(t;w,\lambda,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$ *with respect to coordinate system* $(w;\lambda)$. *Further if*

$P(t;z,\zeta,\eta)\in\widehat{\mathfrak{N}}(\Omega;S)$, it follows that $\Phi^*P(t;w,\lambda,\eta)\in\widehat{\mathfrak{N}}(\Omega;S)$.

(2) *If $\Phi=\mathrm{id}$ (the identity), id^* is the identity, and for complex coordinate transformations $z=\Phi(w)$ and $w=\Psi(v)$, it follow that $\Psi^*\Phi^*P(t;v,\xi,\eta)-(\Phi\Psi)^*P(t;v,\xi,\eta)\in\widehat{\mathfrak{N}}_{(v;\xi)}$.*

Proof. (1) Suppose that $P_k(z,\zeta,\eta)\in\Gamma((\Omega_\rho * S)[(k+1)d_\rho];\mathscr{O}_{T^*X\times\mathbb{C}})$. We also assume (6.4), (6.5) and (6.6), and hence for any $h>0$ there exists $C_{h,Z}>0$ such that for any $(z;\zeta'+{}^{\mathsf{t}}J_\Phi^*(z+z',z)\lambda)\in(\Omega'_{\rho'}*Z)[(k+1)d_{\rho'}]$ we have

$$|P_k(z,\zeta'+{}^{\mathsf{t}}J_\Phi^*(z+z',z)\lambda,\eta)|\le C_{h,Z}A^ke^{h\|\zeta'+{}^{\mathsf{t}}J_\Phi^*(z+z',\,z)\lambda\|}.$$

Hence if $(z;\zeta,\eta)\in(\Omega_{\rho'}*Z)[(k+1)d_{\rho'}]$, instead of (6.7) we have

$$\begin{aligned}&\frac{1}{\alpha!}\Big|\partial_{\zeta'}^{\alpha}\partial_{z'}^{\alpha}P_k(z,\zeta'+{}^{\mathsf{t}}J_\Phi^*(z+z',z)\lambda,\eta)\big|_{\substack{z'=0\\ \zeta'=0}}\Big|\\ &\quad\le\frac{C_{h,Z}\,\alpha!\,A^k}{(\varepsilon\delta c'\|\lambda\|)^{|\alpha|}}\sup_{\substack{|z'_i|=\delta\\ |\zeta'_i|=\varepsilon\|\zeta\|}}e^{h\|\zeta'+{}^{\mathsf{t}}J_\Phi^*(z+z',\,z)\lambda)\|}\le\frac{C_{h,Z}\alpha!\,A^ke^{2hc\|\lambda\|}}{(\varepsilon\delta c'\|\eta\lambda\|)^{|\alpha|}}\,.\end{aligned}$$

We may assume that $1/2<A<1$. Replacing $d>0$ if necessary, we may assume $C:=\varepsilon\delta d_{\rho'}/2>4$, and hence $CA>2$. Hence if $\|\eta\zeta\|\ge c'\|\eta\lambda\|\ge(\nu+1)d_{\rho'}$, we have

$$\begin{aligned}\Big|(\Phi^*P)_\nu(w,\lambda,\eta)\Big|&\le C_{h,Z}\sum_{k+|\alpha|=\nu}\frac{\alpha!A^ke^{2hc\|\lambda\|}}{(\varepsilon\delta c'\|\eta\lambda\|)^{|\alpha|}}\le C_{h,Z}e^{2hc\|\lambda\|}\sum_{k=0}^{\nu}\frac{2^{n+\nu-k-1}A^k(\nu-k)!}{(\varepsilon\delta d_{\rho'})^{\nu-k}\,(\nu+1)^{\nu-k}}\\ &\le\frac{2^{n-1}C_{h,Z}e^{2hc\|\lambda\|}}{C^\nu}\sum_{k=0}^{\nu}(CA)^k=\frac{2^{n-1}C_{h,Z}e^{2hc\|\lambda\|}((CA)^{\nu+1}-1)}{C^\nu(CA-1)}\le\frac{2^nC_{h,Z}\,A^\nu e^{2hc\|\lambda\|}}{CA-1}\,.\end{aligned}$$

If $P(t;z,\zeta,\eta)\in\widehat{\mathfrak{N}}(\Omega;S)$, for any $m\in\mathbb{N}$ and $(z;\zeta,\eta)\in(\Omega_{\rho'}*Z)[md_{\rho'}]$ we have

$$\frac{1}{\alpha!}\Big|\sum_{k=0}^{m-1}\partial_{\zeta'}^{\alpha}\partial_{z'}^{\alpha}P_k(z,\zeta'+{}^{\mathsf{t}}J_\Phi^*(z+z',z)\lambda,\eta)\big|_{\substack{z'=0\\ \zeta'=0}}\Big|\le\frac{C_{h,Z}\,\alpha!A^me^{2hc\|\lambda\|}}{(\varepsilon\delta c'\|\lambda\|)^{|\alpha|}}\,.$$

Hence if $\|\eta\lambda\|\ge md_{\rho'}/c'$, we have

$$\begin{aligned}\Big|\sum_{\nu=0}^{m-1}(\Phi^*P)_\nu(w,\lambda,\eta)\Big|&\le\sum_{|\alpha|=0}^{m-1}\frac{C_{h,Z}\,\alpha!A^{m-|\alpha|}e^{2hc\|\lambda\|}}{(\varepsilon\delta c'\|\lambda\|)^{|\alpha|}}\\ &\le 2^{n-1}C_{h,Z}A^me^{2hc\|\lambda\|}\sum_{k=0}^{m-1}\Big(\frac{1}{CA}\Big)^k\Big(\frac{m-1}{m}\Big)^k\le\frac{C_{h,Z}CA^{m+1}e^{2hc\|\lambda\|}}{CA-1}\,.\end{aligned}$$

(2) Set $v^*:=(v;\xi)$. By Theorem 7.9, we can find $P_0(z,\zeta,\eta)\in\mathfrak{S}_{z_0^*}\subset\widehat{\mathfrak{S}}_{\mathrm{cl},z_0^*}$ such that

$$P(t;z,\zeta,\eta)-P_0(z,\zeta,\eta)\in\widehat{\mathfrak{N}}_{z_0^*}\,.$$

By Theorem 6.11, we have $\Psi^*\Phi^*P_0(v,\xi,\eta) = (\Phi\Psi)^*P_0(v,\xi,\eta)$. Hence by (1) we obtain

$$\begin{aligned}&\Psi^*\Phi^*P(t;v,\xi,\eta) - (\Phi\Psi)^*P(t;v,\xi,\eta)\\&\quad= (\Psi^*\Phi^*P(t;v,\xi,\eta) - \Psi^*\Phi^*P_0(v,\xi,\eta)) - ((\Phi\Psi)^*P(t;v,\xi,\eta) - (\Phi\Psi)^*P_0(v,\xi,\eta)) \in \widehat{\mathfrak{N}}_{v^*}.\end{aligned}$$

□

THEOREM 7.14. *For any $P(t;z,\zeta,\eta)$, $Q(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$, set*

$$\begin{aligned}Q \circ P(t;z,\zeta,\eta) &:= e^{t\langle\partial_{\zeta'},\partial_{z'}\rangle}Q(t;z,\zeta',\eta)\,P(t;z',\zeta,\eta)\big|_{\substack{z'=z\\ \zeta'=\zeta}}\\&= e^{t\langle\partial_{\zeta'},\partial_{z'}\rangle}Q(t;z,\zeta+\zeta',\eta)\,P(t;z+z',\lambda,\eta)\big|_{\substack{z'=0\\ \zeta'=0}}.\end{aligned}$$

(1) $Q \circ P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$. *Moreover if either $P(t;z,\zeta,\eta)$ or $Q(t;z,\zeta,\eta)$ is an element of $\widehat{\mathfrak{N}}(\Omega;S)$, it follows that $Q \circ P(t;z,\zeta,\eta) \in \widehat{\mathfrak{N}}(\Omega;S)$.*

(2) *$R \circ (Q \circ P) = (R \circ Q) \circ P$ holds.*

(3) *Let $\Phi(w) = z$ be a holomorphic coordinate transformation. Then*

$$\Phi^*Q \circ \Phi^*P(t;w,\lambda,\eta) - \Phi^*(Q \circ P)(t;w,\lambda,\eta) \in \widehat{\mathfrak{N}}_{(w;\lambda)}.$$

PROOF. (1) We assume that $P_i(z,\zeta,\eta), Q_i(z,\zeta,\eta) \in \Gamma((\Omega_\rho * S)[(i+1)d_\rho]; \mathscr{O}_{T^*X\times\mathbb{C}})$ for $P(t;z,\zeta,\eta) = \sum_{i=0}^\infty t^iP_i(z,\zeta,\eta)$ and $Q(t;z,\zeta,\eta) = \sum_{i=0}^\infty t^iQ_i(z,\zeta,\eta)$. Set $Q \circ P(t;z,\zeta,\eta) = \sum_{i=0}^\infty t^iR_i(z,\zeta,\eta)$. Then

$$R_\nu(z,\zeta,\eta) = \sum_{|\alpha|+k+l=\nu}\frac{1}{\alpha!}\,\partial_\zeta^\alpha Q_l(z,\zeta,\eta)\,\partial_z^\alpha P_k(z,\zeta,\eta).$$

Hence $R_\nu(z,\zeta,\eta) \in \Gamma((\Omega_\rho * S)[(\nu+1)d_\rho]; \mathscr{O}_{T^*X\times\mathbb{C}})$. We shall prove $R(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$. Let $Z \Subset S$. Note that for any $(z,\zeta,\eta) \in (\Omega_\rho * Z)[(k+1)d_{\rho'}]$ and (z',ζ') with $\|z'\| \le \rho-\rho'$ and $\|\zeta'\| \le (\rho-\rho')\|\eta\zeta\| < \|\zeta\|$, we have $(z+z',\zeta+\zeta') \in \Omega_\rho[(k+1)d_\rho]$. Moreover as in (6.9) we have

$$\|\eta(\zeta+\zeta')\| \ge (1-\rho+\rho')\|\eta\zeta\| \ge (k+1)d(1-\rho'').$$

For any $\rho' \in \left]0,\rho\right[$ and $h > 0$, on $(\Omega_{\rho'} * Z)[(k+1)d_{\rho'}]$ we have

$$|P_k(z,\zeta,\eta)|,\ |Q_k(z,\zeta,\eta)| \le C_{h,Z}A^ke^{h\|\zeta\|}.$$

Hence in the same way as in the proof of Lemma 6.15, on $(\Omega_{\rho'} * Z)[(k+1)d_{\rho'}]$ we have

$$\begin{aligned}|\partial_\zeta^\alpha Q_k(z,\zeta,\eta)| &\le \frac{C_{h,Z}\,\alpha!A^ke^{2h\|\zeta\|}}{(\rho-\rho')^{|\alpha|}\|\eta\zeta\|^{|\alpha|}},\\ |\partial_z^\alpha P_k(z,\zeta,\eta)| &\le \frac{C_{h,Z}\,\alpha!A^ke^{h\|\zeta\|}}{(\rho-\rho')^{|\alpha|}}.\end{aligned}\tag{7.5}$$

We replace d, $\rho' > 0$ as $C := 2/Ad_{\rho'}(\rho-\rho')^2 < 1$, and choose $C' > 0$ and $B \in \left[A,1\right[$ as

$(\nu+1)A^\nu \le C'B^\nu$ for any $\nu \in \mathbb{N}_0$. Since $\#\{(k,l)\in\mathbb{N}_0^2;\, k+l=\nu-|\alpha|\} \le \nu-|\alpha|+1 \le \nu+1$, for any $(z;\zeta,\eta)\in(\Omega_{\rho'} * Z)[(\nu+1)d_{\rho'}]$, we have

$$\begin{aligned}|R_\nu(z,\zeta,\eta)| &\le \sum_{\nu=|\alpha|+k+l} \frac{C_{h,Z}^2\,\alpha! A^{k+l} e^{3h\|\zeta\|}}{\|\eta\zeta\|^{|\alpha|}\,(\rho-\rho')^{2|\alpha|}} \le \sum_{i=0}^{\nu} \frac{2^{i+n-1} C_{h,Z}^2\,(\nu+1)\,i!\,A^{\nu-i} e^{3h\|\zeta\|}}{(d_{\rho'}(\rho-\rho')^2)^i\,(\nu+1)^i} \\ &\le 2^{n-1} C_{h,Z}^2\,(\nu+1)A^\nu e^{3h\|\zeta\|} \sum_{i=0}^{\nu} C^i \le \frac{2^{n-1}C' C_{h,Z}^2\, B^\nu e^{3h\|\zeta\|}}{1-C}\,.\end{aligned}$$

Next, we assume $P(t;z,\zeta,\eta)\in\widehat{\mathfrak{N}}(\Omega;S)$. Then, instead of (7.5), we have that for any $m\in\mathbb{N}$, on $(\Omega_{\rho'} * Z)[md_{\rho'}]$ we have

$$\Big|\sum_{k=0}^{m-1} \partial_z^\alpha P_k(z,\zeta,\eta)\Big| \le \frac{C_{h,Z}\,\alpha! A^m e^{h\|\zeta\|}}{(\rho-\rho')^{|\alpha|}}\,,$$

and thus we have

$$\begin{aligned}\Big|\sum_{\nu=0}^{m-1} R_\nu(z,\zeta,\eta)\Big| &= \Big|\sum_{i+|\alpha|=0}^{m-1} \frac{1}{\alpha!}\,\partial_\zeta^\alpha Q_i(z,\zeta,\eta) \sum_{k=0}^{m-i-1-|\alpha|} \partial_z^\alpha P_k(z,\zeta,\eta)\Big| \\ &\le \sum_{i+|\alpha|=0}^{m-1} \frac{C_{h,Z}^2\,\alpha! A^{m-|\alpha|} e^{3h\|\zeta\|}}{\|\zeta\|^{|\alpha|}\,(\rho-\rho')^{2|\alpha|}} \le 2^{n-1} C_{h,Z}^2\, A^m e^{3h\|\zeta\|} \sum_{i+\nu=0}^{m-1} C^\nu \\ &\le \frac{2^{n-1} C_{h,Z}^2\, m A^m e^{3h\|\zeta\|}}{1-C} \le \frac{2^{n-1} C' C_{h,Z}^2 B^m e^{3h\|\zeta\|}}{1-C}\,.\end{aligned}$$

The proof in the case of $Q(t;z,\zeta,\eta)\in\widehat{\mathfrak{N}}(\Omega;S)$ is similar. In particular, since

$$\partial_\eta(Q\circ P)(t;z,\zeta,\eta) = \partial_\eta Q\circ P(t;z,\zeta,\eta) + Q\circ\partial_\eta P(t;z,\zeta,\eta),$$

we see that if $P(t;z,\zeta,\eta)$, $Q(t;z,\zeta,\eta)\in\widehat{\mathfrak{S}}(\Omega;S)$, we have $Q\circ P(t;z,\zeta,\eta)\in\widehat{\mathfrak{S}}(\Omega;S)$.

(2) is easily obtained.

(3) Set $v^* := (v;\xi)$. By Theorem 7.9, we can find $P_0(z,\zeta,\eta)$, $Q_0(z,\zeta,\eta)\in\widehat{\mathfrak{S}}_{z_0^*}\subset\widehat{\mathfrak{S}}_{\mathrm{cl},z_0^*}$ such that

$$P(t;z,\zeta,\eta)-P_0(z,\zeta,\eta),\quad Q(t;z,\zeta,\eta)-Q_0(z,\zeta,\eta)\in\widehat{\mathfrak{N}}_{z_0^*}\,.$$

By Theorem 6.16, we have

$$\varPhi^* Q_0\circ\varPhi^* P_0(w,\lambda,\eta) = \varPhi^*(Q_0\circ P_0)(w,\lambda,\eta).$$

Hence by (1) we obtain

$$\begin{aligned}\varPhi^* Q\circ\varPhi^* P(t;w,\lambda,\eta) &= \varPhi^* Q_0\circ\varPhi^* P_0(t;w,\lambda,\eta) + \varPhi^*(Q-Q_0)\circ\varPhi^* P(t;w,\lambda,\eta) \\ &\quad + \varPhi^* Q_0\circ\varPhi^*(P-P_0)(t;w,\lambda,\eta)\end{aligned}$$

$$\begin{aligned}
&= \Phi^*(Q_0 \circ P_0)(t; w, \lambda, \eta) + \Phi^*(Q - Q_0) \circ \Phi^* P(t; w, \lambda, \eta) \\
&\quad + \Phi^* Q_0 \circ \Phi^*(P - P_0)(t; w, \lambda, \eta) \\
&\equiv \Phi^*(Q_0 \circ P_0)(t; w, \lambda, \eta) \\
&= \Phi^*(Q \circ P)(t; w, \lambda, \eta) + \Phi^*((Q_0 - Q) \circ P_0)(t; w, \lambda, \eta) \\
&\quad + \Phi^*(Q \circ (P_0 - P))(t; w, \lambda, \eta) \\
&\equiv \Phi^*(Q \circ P)(t; w, \lambda, \eta). \qquad \square
\end{aligned}$$

We can also prove

THEOREM 7.15. *For any* $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}(\Omega; S)$ *set*

$$P^*(t; z, \zeta, \eta) := e^{t\langle \partial_\zeta, \partial_z \rangle} P(t; z, -\zeta, \eta).$$

(1) $P^*(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}(\Omega^a; S)$ *and* $P^{**} = P$. *Moreover if* $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}(\Omega; S)$, *it follows that* $P^*(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}(\Omega^a; S)$.

(2) $(Q \circ P)^*(t; z, \zeta, \eta) = P^* \circ Q^*(t; z, \zeta, \eta)$.

(3) *Let* $\Phi(w) = z$ *be a holomorphic coordinate transformation. Then on* $\widehat{\mathfrak{S}}(\Omega^a; S) \otimes_{\mathscr{O}_X} \Omega_X$

$$\Phi^*(P^*(t; z, \zeta, \eta) \otimes dz) = \Phi^*(P(t; z, \zeta, \eta) \otimes dz)^*.$$

PROOF. (1) For $P(t; z, \zeta, \eta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z, \zeta, \eta)$, we set $P^*(t; z, \zeta, \eta) = \sum_{\nu=0}^{\infty} t^\nu P^*_\nu(z, \zeta, \eta)$. Then

$$P^*_\nu(z, \zeta, \eta) = \sum_{|\alpha|+k=\nu} \frac{1}{\alpha!} \partial_\zeta^\alpha \partial_z^\alpha P_k(t; z, -\zeta, \eta).$$

Let $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}(\Omega; S)$. As in the proof of Theorem 7.14, we can show $P^*(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}(\Omega^a; S)$, and that if $P(t; z, \zeta) \in \widehat{\mathfrak{N}}(\Omega; S)$, we have $P^*(t; z, \zeta) \in \widehat{\mathfrak{N}}(\Omega^a; S)$. Moreover for any $i \in \mathbb{N}_0$, we set $P_{(i)}(t; z, \zeta, \eta) := \sum_{\nu=0}^{i} t^\nu P_\nu(z, \zeta, \eta)$. Then we can see $P^{**}_{(i)}(t; z, \zeta, \eta) = P_{(i)}(t; z, \zeta, \eta)$ for any i, and hence $P^{**}(t; z, \zeta, \eta) = P(t; z, \zeta, \eta)$.

(2), (3). For any $i, j \in \mathbb{N}_0$ we have $(Q_{(i)} \circ P_{(j)})^*(t; z, \zeta, \eta) = P^*_{(j)} \circ Q^*_{(i)}(t; z, \zeta, \eta)$ and $\Phi^*(P^*_{(j)}(t; z, \zeta, \eta) \otimes dz) = \Phi^*(P_{(j)}(t; z, \zeta, \eta) \otimes dz)^*$, and hence the results follow. $\square$

A. The compatibility of actions.

The purpose of this appendix is to show Theorem 3.9. We follow the same notation as those in Section 3. Set $\widehat{\pi}_1 \colon \widehat{X}^2 \ni (z, w, \eta) \mapsto (z, \eta) \in \widehat{X}$ and $\widehat{\pi}_2 \colon \widehat{X}^2 \ni (z, w, \eta) \mapsto (w, \eta) \in \widehat{X}$. We also define the canonical projections $\pi_1 \colon X^2 \to X$ and $\pi_2 \colon X^2 \to X$ in the same way. Note that we consider the problem at $z_0^* = (0; 1, 0, \ldots, 0)$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \underset{\mathbb{C}}{\otimes} \mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*} & \longrightarrow & \mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*} \\
\uparrow & & \uparrow \\
H^n_{G_{\Delta,\boldsymbol{\kappa}} \cap U_{\Delta,\boldsymbol{\kappa}}}(U_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n)}_{X^2}) \underset{\mathbb{C}}{\otimes} H^d_{G_{\boldsymbol{\kappa}} \cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X) & \xrightarrow{\mu^c} & H^d_{G_{\widetilde{\boldsymbol{\kappa}}} \cap U_{\widetilde{\boldsymbol{\kappa}}}}(U_{\widetilde{\boldsymbol{\kappa}}}; \mathscr{O}_X) \\
\downarrow & & \downarrow \\
H^n_{\widehat{G}_{\Delta,\boldsymbol{\kappa}} \cap \widehat{U}_{\Delta,\boldsymbol{\kappa}}}(\widehat{U}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2}) \underset{\mathbb{C}}{\otimes} H^d_{\widehat{G}_{\boldsymbol{\kappa}} \cap \widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}}) & \xrightarrow{\mu^c} & H^d_{\widehat{G}_{\widetilde{\boldsymbol{\kappa}}} \cap \widehat{U}_{\widetilde{\boldsymbol{\kappa}}}}(\widehat{U}_{\widetilde{\boldsymbol{\kappa}}}; \mathscr{O}_{\widehat{X}}),
\end{array} \tag{A.1}$$

where the down injective morphisms are described in Section 2. We will explain the other morphisms appearing in the diagram above. The top horizontal arrow in (A.1) is associated with the cohomological action of $\mathscr{E}^{\mathbb{R}}_X$ to $\mathscr{C}^{\mathbb{R}}_{Y|X}$. The second horizontal arrow μ^c in (A.1) is given by the chain of morphisms

$$\begin{aligned}
H^n_{G_{\Delta,\boldsymbol{\kappa}} \cap U_{\Delta,\boldsymbol{\kappa}}}(U_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n)}_{X^2}) \underset{\mathbb{C}}{\otimes} H^d_{G_{\boldsymbol{\kappa}} \cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X) &\to H^{n+d}_{G \cap U}(U; \mathscr{O}^{(0,n)}_{X^2}) \\
&\to H^d_{G_{\widetilde{\boldsymbol{\kappa}}} \cap U_{\widetilde{\boldsymbol{\kappa}}}}(U_{\widetilde{\boldsymbol{\kappa}}}; \mathscr{O}_X).
\end{aligned} \tag{A.2}$$

Here we set

$$G := G_{\Delta,\boldsymbol{\kappa}} \cap \pi_2^{-1}(G_{\boldsymbol{\kappa}}), \quad U := U_{\Delta,\boldsymbol{\kappa}} \cap \pi_2^{-1}(U_{\boldsymbol{\kappa}}).$$

The first morphism in (A.2) is the usual cohomological cup product and the second morphism in (A.2) is the cohomological residue morphism. Note that since $G \subset \pi_1^{-1}(G_{\widetilde{\boldsymbol{\kappa}}})$ and $G \cap \pi_1^{-1}(K) \Subset U$ for any compact subset $K \Subset U_{\widetilde{\boldsymbol{\kappa}}}$, the second morphism in (A.2) is well-defined. The third horizontal arrow μ^c in (A.1) is defined by the cohomological cup product and residue mapping in the same way as that for the second horizontal arrow. Therefore, to show the theorem, it suffices to prove that the third horizontal arrow μ^c and μ defined in Theorem 3.5 coincide. Furthermore clearly the following diagram with respect to the cup product

$$\begin{array}{ccc}
H^n_{\widehat{G}_{\Delta,\boldsymbol{\kappa}} \cap \widehat{U}_{\Delta,\boldsymbol{\kappa}}}(\widehat{U}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2}) \underset{\mathbb{C}}{\otimes} H^d_{\widehat{G}_{\boldsymbol{\kappa}} \cap \widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}}) & \longrightarrow & H^{n+d}_{\widehat{G} \cap \widehat{U}}(\widehat{U}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2}) \\
\| & & \| \\
\dfrac{\Gamma(\widehat{V}^{(*)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2})}{\sum\limits_{\alpha \in \mathcal{P}_n^\vee} \Gamma(\widehat{V}^{(\alpha)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2})} \underset{\mathbb{C}}{\otimes} \dfrac{\Gamma(\widehat{V}^{(*)}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}})}{\sum\limits_{\beta \in \mathcal{P}_d^\vee} \Gamma(\widehat{V}^{(\beta)}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}})} & \longrightarrow & \dfrac{\Gamma(\widehat{W}^{(*,*)}_{\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2})}{\sum\limits_{(\alpha,\beta) \in \Lambda} \Gamma(\widehat{W}^{(\alpha,\beta)}_{\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2})}
\end{array}$$

commutes. Here we set

$$\widehat{G} := \widehat{G}_{\Delta,\boldsymbol{\kappa}} \cap \widehat{\pi}_2^{-1}(\widehat{G}_{\boldsymbol{\kappa}}), \quad \widehat{U} := \widehat{U}_{\Delta,\boldsymbol{\kappa}} \cap \widehat{\pi}_2^{-1}(\widehat{U}_{\boldsymbol{\kappa}}).$$

Hence the problem is reduced to the following proposition.

PROPOSITION A.1. *The diagram below commutes*:

$$\begin{array}{ccc} H^{n+d}_{\widehat{G}\cap\widehat{U}}(\widehat{U};\mathscr{O}^{(0,n,0)}_{\widehat{X}^2}) & \xrightarrow{\mu^c} & H^d_{\widehat{G}_{\widetilde{\boldsymbol{\kappa}}}\cap\widehat{U}_{\widetilde{\boldsymbol{\kappa}}}}(\widehat{U}_{\widetilde{\boldsymbol{\kappa}}};\mathscr{O}_{\widehat{X}}) \\ \| & & \| \\ \dfrac{\Gamma(\widehat{W}^{(*,*)}_{\boldsymbol{\kappa}};\mathscr{O}^{(0,n,0)}_{\widehat{X}^2})}{\sum_{(\alpha,\beta)\in\Lambda}\Gamma(\widehat{W}^{(\alpha,\beta)}_{\boldsymbol{\kappa}};\mathscr{O}^{(0,n,0)}_{\widehat{X}^2})} & \xrightarrow{\mu} & \dfrac{\Gamma(\widehat{V}^{(*)}_{\widetilde{\boldsymbol{\kappa}}};\mathscr{O}_{\widehat{X}})}{\sum_{\beta\in\mathcal{P}_d^\vee}\Gamma(\widehat{V}^{(\beta)}_{\widetilde{\boldsymbol{\kappa}}};\mathscr{O}_{\widehat{X}})} \end{array} \tag{A.3}$$

Here μ^c is given by the cohomological residue morphism and μ is given by

$$u(z,w,\eta)\,dw \mapsto \int_{\gamma(z,\eta;\varrho,\theta)} u(z,w,\eta)\,dw.$$

Proof. We first define the closed subsets in $T := \{(w_1,\eta)\in\mathbb{C}^2;\ |\arg\eta| < \theta/4\}$ by

$$L_{\varrho,\theta} := \left\{(w_1,\eta)\in T;\ |\arg w_1| \le \frac{\pi}{2} - \frac{3\theta}{4} + \arg\eta\right\},$$
$$L'_{\varrho,\theta} := \left\{(w_1,\eta)\in L_{\varrho,\theta};\ |w_1| \le \frac{\varrho}{4}|\eta|\right\}.$$

Note that $T\smallsetminus L_{\varrho,\theta}$ and $T\smallsetminus L'_{\varrho,\theta}$ are pseudoconvex open subsets. Then the top horizontal morphism μ^c in (A.3) can be decomposed to the chain of morphisms:

$$\begin{aligned} H^{n+d}_{\widehat{G}\cap\widehat{U}}(\widehat{U};\mathscr{O}^{(0,n,0)}_{\widehat{X}^2}) &\xrightarrow{\psi_1^c} H^{n+d}_{\widehat{G}_1\cap\widehat{U}'}(\widehat{U}';\mathscr{O}^{(0,n,0)}_{\widehat{X}^2}) \xleftarrow{\psi_2^c} H^{n+d}_{\widehat{G}_2\cap\widehat{U}'}(\widehat{U}';\mathscr{O}^{(0,n,0)}_{\widehat{X}^2}) \\ &\xleftarrow{\psi_3^c} H^{n+d}_{\widehat{G}_3\cap\widehat{U}'}(\widehat{U}';\mathscr{O}^{(0,n,0)}_{\widehat{X}^2}) \xrightarrow{\psi_4^c} H^{n+d}_{\widehat{G}_4\cap\widehat{U}'}(\widehat{U}';\mathscr{O}^{(0,n,0)}_{\widehat{X}^2}) \\ &\xrightarrow{\psi_5^c} H^d_{\widehat{G}_{\widetilde{\boldsymbol{\kappa}}}\cap\widehat{U}_{\widetilde{\boldsymbol{\kappa}}}}(\widehat{U}_{\widetilde{\boldsymbol{\kappa}}};\mathscr{O}_{\widehat{X}}). \end{aligned}$$

Here we will explain all the subsets appearing in the chain above. Set

$$\widehat{U}' := \widehat{U}\cap\widehat{\pi}_1^{-1}(\widehat{U}_{\widetilde{\boldsymbol{\kappa}}}) = \widehat{U}_{\Delta,\boldsymbol{\kappa}}\cap\widehat{\pi}_1^{-1}(\widehat{U}_{\widetilde{\boldsymbol{\kappa}}})\cap\widehat{\pi}_2^{-1}(\widehat{U}_{\boldsymbol{\kappa}}),$$
$$\widehat{K} := \bigcap_{i=2}^n \{(z,w,\eta);\ (z_1-w_1,\eta)\in L_{\varrho,\theta},\ \varrho|z_i-w_i|\le|\eta|\},$$
$$\widehat{K}' := \bigcap_{i=2}^n \{(z,w,\eta);\ (z_1-w_1,\eta)\in L'_{\varrho,\theta},\ \varrho|z_i-w_i|\le|\eta|\}.$$

Note that $\widehat{G}_{\Delta,\boldsymbol{\kappa}}\cap\widehat{U}'\subset\widehat{K}\cap\widehat{U}'$ holds. Then $\widehat{G}_k$ $(1\le k\le 4)$ are defined by

$$\widehat{G}_1 := \widehat{K}\cap\widehat{\pi}_2^{-1}(\widehat{G}_{\boldsymbol{\kappa}}),\quad \widehat{G}_2 := \widehat{K}'\cap\widehat{\pi}_2^{-1}(\widehat{G}_{\boldsymbol{\kappa}}),$$
$$\widehat{G}_3 := \widehat{G}_2\cap\widehat{\pi}_1^{-1}(\widehat{G}_{\widetilde{\boldsymbol{\kappa}}}),\quad \widehat{G}_4 := \widehat{K}'\cap\widehat{\pi}_1^{-1}(\widehat{G}_{\widetilde{\boldsymbol{\kappa}}}).$$

The morphism ψ_5^c is nothing but the residue morphism. The other morphisms are canonical ones associated with the inclusion of sets. If we take $\tilde{\varrho}$ of $\widetilde{\boldsymbol{\kappa}}$ sufficiently small, we have $\widehat{U}'\cap\widehat{G}_1=\widehat{U}'\cap\widehat{G}_2$. Therefore the canonical morphism ψ_2^c becomes an isomorphism. Furthermore, as $\widehat{G}_2\subset\widehat{\pi}_1^{-1}(\widehat{G}_{\widetilde{\boldsymbol{\kappa}}})$ holds, we get $\widehat{U}'\cap\widehat{G}_2=\widehat{U}'\cap\widehat{G}_3$, thus the canonical morphism ψ_3^c is also an isomorphism. The corresponding morphisms of Čech cohomology groups are given by the following chain:

$$\frac{\Gamma(\widehat{W}_{\boldsymbol{\kappa}}^{(*,*)};\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})}{\sum\limits_{(\alpha,\beta)\in\Lambda}\Gamma(\widehat{W}_{\boldsymbol{\kappa}}^{(\alpha,\beta)};\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})}\xrightarrow[\psi_1]{}\frac{\Gamma(\widehat{W}_1^{(*,*)};\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})}{\sum\limits_{(\alpha,\beta)\in\Lambda}\Gamma(\widehat{W}_1^{(\alpha,\beta)};\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})}\xleftarrow[\psi_2]{\sim}\frac{\Gamma(\widehat{W}_2^{(*,*)};\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})}{\sum\limits_{(\alpha,\beta)\in\Lambda}\Gamma(\widehat{W}_2^{(\alpha,\beta)};\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})}$$
$$\xleftarrow[\psi_3]{\sim}\frac{Z^{n+d}(\widehat{\mathfrak{W}}_3;\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})}{B^{n+d}(\widehat{\mathfrak{W}}_3;\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})}\xrightarrow[\psi_4]{}\frac{\Gamma(\widehat{W}_4^{(*,*)};\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})}{\sum\limits_{(\alpha,\beta)\in\Lambda}\Gamma(\widehat{W}_4^{(\alpha,\beta)};\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})}\xrightarrow[\psi_5]{}\frac{\Gamma(\widehat{V}_{\widetilde{\boldsymbol{\kappa}}}^{(*)};\mathscr{O}_{\widehat{X}})}{\sum\limits_{\beta\in\mathcal{P}_d^\vee}\Gamma(\widehat{V}_{\widetilde{\boldsymbol{\kappa}}}^{(\beta)};\mathscr{O}_{\widehat{X}})}. \quad (A.4)$$

We also explain all the subsets appearing in (A.4). Set

$$\begin{aligned}
\widehat{O}^{(1)}&:=\{(z,w,\eta)\in\widehat{U}';\,(z_1-w_1,\eta)\notin L_{\varrho,\theta}\},\\
\widehat{O}^{(i)}&:=\{(z,w,\eta)\in\widehat{U}';\,\varrho|z_i-w_i|>|\eta|\}\quad(i=2,\dots,n),\\
\widehat{O}'^{(1)}&:=\{(z,w,\eta)\in\widehat{U}';\,(z_1-w_1,\eta)\notin L'_{\varrho,\theta}\},\\
\widehat{O}'^{(i)}&:=\{(z,w,\eta)\in\widehat{U}';\,\varrho|z_i-w_i|>|\eta|\}\quad(i=2,\dots,n).
\end{aligned}$$

Note that these open subsets are pseudoconvex. Then the coverings $\{W_1^{(\alpha,\beta)}\}$ etc. appearing in (A.4) are given by

$$\begin{aligned}
\widehat{W}_1^{(\alpha,\beta)}&:=\widehat{O}^{(\alpha)}\cap\widehat{\pi}_2^{-1}(\widehat{V}_{\boldsymbol{\kappa}}^{(\beta)}), & \widehat{W}_2^{(\alpha,\beta)}&:=\widehat{O}'^{(\alpha)}\cap\widehat{\pi}_2^{-1}(\widehat{V}_{\boldsymbol{\kappa}}^{(\beta)}),\\
\widehat{W}_3^{(\alpha,\beta,\beta')}&:=\widehat{O}'^{(\alpha)}\cap\widehat{\pi}_2^{-1}(\widehat{V}_{\boldsymbol{\kappa}}^{(\beta)})\cap\widehat{\pi}_1^{-1}(\widehat{V}_{\widetilde{\boldsymbol{\kappa}}}^{(\beta')}), & \widehat{W}_4^{(\alpha,\beta')}&:=\widehat{O}'^{(\alpha)}\cap\widehat{\pi}_1^{-1}(\widehat{V}_{\widetilde{\boldsymbol{\kappa}}}^{(\beta')}),
\end{aligned}$$

and $Z^{n+d}(\widehat{\mathfrak{W}}_3;\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})$ (resp. $B^{n+d}(\widehat{\mathfrak{W}}_3;\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})$) stands for the $n+d$ cocycle group (resp. the $n+d$ coboundary group) of Čech complex $C^\bullet(\widehat{\mathfrak{W}}_3;\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})$ with respect to the covering $\widehat{\mathfrak{W}}_3:=\{\widehat{W}_3^{(i,j,k)}\}_{\substack{1\le i\le n\\1\le j,k\le d}}$. Let $Pdw\in H^{n+d}_{\widehat{G}\cap\widehat{U}}(\widehat{U};\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})$, and $u\,dw=u(z,w,\eta)\,dw\in\Gamma(\widehat{W}_{\boldsymbol{\kappa}}^{(*,*)};\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})$ the corresponding representative of the Čech cohomology group. Let us trace the images of P and u by the chain of morphisms.

Step 1. Set $P_1dw:=\psi_1^c(Pdw)$ and $u_1\,dw:=\psi_1(u\,dw)$. Then clearly $u_1\,dw$ is a representative of $P_1\,dw$ and we have

$$\mu(u\,dw)=\mu(u_1\,dw)\mod\sum_{\beta\in\mathcal{P}_d^\vee}\Gamma(\widehat{V}_{\widetilde{\boldsymbol{\kappa}}}^{(\beta)};\mathscr{O}_{\widehat{X}}),$$

where μ was defined in the statement of the proposition.

Step 2. As ψ_2^c is an isomorphism, there exists P_2dw with $P_1dw=\psi_2^c(P_2dw)$. Then we can find a representative $u_2\,dw\in\Gamma(\widehat{W}_2^{(*,*)};\mathscr{O}_{\widehat{X}^2}^{(0,n,0)})$ of P_2dw such that

$$u_1\,dw - \psi_2(u_2\,dw) \in \sum_{(\alpha,\,\beta)\in\Lambda} \Gamma(\widehat{W}_1^{(\alpha,\,\beta)}; \mathscr{O}_{\widehat{X}^2}^{(0,n,0)}).$$

Since we have similar claims as in Lemma 3.4, we have $\mu(u_2\,dw) \in \Gamma(\widehat{V}_{\tilde{\boldsymbol{\kappa}}}^{(*)}; \mathscr{O}_{\widehat{X}})$ and

$$\mu(u_1\,dw) = \mu(u_2\,dw) \mod \sum_{\beta\in\mathcal{P}_d^\vee} \Gamma(\widehat{V}_{\tilde{\boldsymbol{\kappa}}}^{(\beta)}; \mathscr{O}_{\widehat{X}}).$$

Furthermore, we set

$$\begin{aligned}\widetilde{\gamma}(z,\eta;\varrho,\theta) &:= \gamma(z,\eta;\varrho,\theta) \vee (-\overline{\gamma}(z,\eta;\varrho,\theta)),\\ \mu'(u_2\,dw) &:= \int_{\widetilde{\gamma}(z,\eta;\varrho,\theta)} u_2(z,w,\eta)\,dw.\end{aligned}$$

Note that the real n-dimensional chain $\widetilde{\gamma}(z,\eta;\varrho,\theta)$ in X becomes a product of closed paths where each path is homotopic to the circle in $\mathbb{C}_{w_i}$ $(i = 1,\dots,n)$, in particular, we have $\partial\tilde{\gamma} = \emptyset$. By the same claim as Lemma 3.4 (3), we get $\mu'(u_2\,dw) \in \sum_{\beta\in\mathcal{P}_d^\vee} \Gamma(\widehat{V}_{\tilde{\boldsymbol{\kappa}}}^{(\beta)}; \mathscr{O}_{\widehat{X}})$. Hence we have obtained

$$\mu(u\,dw) = \mu'(u_2\,dw) \mod \sum_{\beta\in\mathcal{P}_d^\vee} \Gamma(\widehat{V}_{\tilde{\boldsymbol{\kappa}}}^{(\beta)}; \mathscr{O}_{\widehat{X}}).$$

Step 3. As ψ_3^c is an isomorphism, there exists P_3dw with $P_2dw = \psi_3^c(P_3dw)$. Then we can take a representative

$$u_3\,dw = \{u_3^{(\alpha,\beta,\beta')}dw\}_{(\alpha,\beta,\beta')\in\Lambda_{n+d}^3} \in Z^{n+d}(\widehat{\mathfrak{W}}_3; \mathscr{O}_{\widehat{X}^2}^{(0,n,0)}) \subset \sum_{(\alpha,\beta,\beta')\in\Lambda_{n+d}^3} \Gamma(\widehat{W}_3^{(\alpha,\beta,\beta')}; \mathscr{O}_{\widehat{X}^2}^{(0,n,0)})$$

of P_3dw, where we set

$$\Lambda_{n+d}^3 := \{(\alpha,\beta,\beta') \in \mathcal{P}_n \times \mathcal{P}_d \times \mathcal{P}_d;\ \#\alpha + \#\beta + \#\beta' = n+d\}.$$

Since the covering $\{\widehat{W}_2^{(\alpha,\,\beta)}\}$ is finer than $\{\widehat{W}_3^{(\alpha,\,\beta,\,\beta')}\}$, we get

$$u_2\,dw - u_3^{(*,*,\emptyset)}dw \in \sum_{(\alpha,\beta)\in\Lambda} \Gamma(\widehat{W}_2^{(\alpha,\,\beta)}; \mathscr{O}_{\widehat{X}^2}^{(0,n,0)}),$$

and thus, we obtain

$$\mu'(u_2\,dw) = \mu'(u_3^{(*,*,\emptyset)}dw) \mod \sum_{\beta\in\mathcal{P}_d^\vee} \Gamma(\widehat{V}_{\tilde{\boldsymbol{\kappa}}}^{(\beta)}; \mathscr{O}_{\widehat{X}})$$

for which we have:

Lemma A.2. *The following holds*:

$$\mu'(u_3^{(*,\emptyset,*)}dw) = \mu'(u_3^{(*,*,\emptyset)}dw) \mod \sum_{\beta\in\mathcal{P}_d^\vee} \Gamma(\widehat{V}_{\tilde{\boldsymbol{\kappa}}}^{(\beta)}; \mathscr{O}_{\widehat{X}}).$$

Proof. We set $*^{\vee k} := * \smallsetminus \{k\}$. By the cocycle condition for $u_3\,dw$, we have

$$(-1)^{n+d}(u_3^{(*,*,\emptyset)}dw - u_3^{(*,*^{\vee d},\{d\})}dw) + \sum_{i=1}^n (-1)^i u_3^{(*^{\vee i},*,\{d\})}dw + \sum_{i=1}^n (-1)^{n+i} u_3^{(*,*^{\vee i},\{d\})}dw = 0.$$

Hence, by the same claim as Lemma 3.4 (2), we obtain

$$\mu'(u_3^{(*,*,\emptyset)}dw) = \mu'(u_3^{(*,*^{\vee d},\{d\})}dw) \mod \sum_{\beta\in\mathcal{P}_d^\vee} \Gamma(\widehat{V}_{\tilde{\boldsymbol{\kappa}}}^{(\beta)}; \mathscr{O}_{\widehat{X}}).$$

By repeating the same argument, we obtain the result. □

Summing up, we have

$$\mu(u\,dw) = \mu'(u_3^{(*,\emptyset,*)}dw) \mod \sum_{\beta\in\mathcal{P}_d^\vee} \Gamma(\widehat{V}_{\tilde{\boldsymbol{\kappa}}}^{(\beta)}; \mathscr{O}_{\widehat{X}}).$$

Step 4. Set $P_4\,dw := \psi_4^c(P_3\,dw)$. Clearly $u_4\,dw := \psi_4(u_3\,dw)$ is given by $u_3^{(*,\emptyset,*)}dw$ which is a representative of P_4. By the previous step, we have

$$\mu(u\,dw) = \mu'(u_4\,dw) \mod \sum_{\beta\in\mathcal{P}_d^\vee} \Gamma(\widehat{V}_{\tilde{\boldsymbol{\kappa}}}^{(\beta)}; \mathscr{O}_{\widehat{X}}).$$

The Final Step. ψ_5^c is given by the residue morphism. Then the subsets $\widehat{U}'$, $\widehat{G}_4$ and the chain $\tilde{\gamma}(z,\eta;\varrho,\theta)$ satisfy the geometrical situation under which the following Lemma A.3 holds. Hence it follows from the lemma that the representative of $\psi_5^c(v_4\,dw)$ is given by $\mu'(u_4\,dw)$. Therefore we have the conclusion that a representative of $\mu^c(P\,dw)$ is given by $\mu(u\,dw)$. This completes the proof for the proposition. □

We first clarify a geometrical situation. Let $X := \mathbb{C}_z^\ell$ and $Y := \mathbb{C}_w^n$. Let Z (resp. U) be a closed (resp. a Stein open) subset in X, and let K_i (resp. W_i) be a closed (resp. a Stein open) subset in $X \times \mathbb{C}_{w_i}$ $(i = 1, \dots, n)$. The mappings $\pi\colon X \times Y \to X$, $\pi_i\colon X\times Y \to X\times\mathbb{C}_{w_i}$ and $\tau_i\colon X\times\mathbb{C}_{w_i} \to X$ denote the canonical projections respectively. In this situation, the following conditions are also assumed.

(i) The subset $U \smallsetminus Z \subset X$ has a covering $\{U^{(j)}\}_{j=1}^m$ of Stein open subsets for an $m \le \ell$.

(ii) The subset $W_i \smallsetminus K_i$ is Stein in $X \times \mathbb{C}_{w_i}$ for $1 \le i \le n$.

(iii) The mapping $\tau_i\colon K_i \cap W_i \to X$ is proper for $1 \le i \le n$.

(iv) $U \subset \pi(\bigcup_{i=1}^n \pi_i^{-1}(W_i))$.

Set $V^{(i)} := \pi_i^{-1}(W_i \smallsetminus K_i)$ and

$$K := \pi^{-1}(Z) \cap \bigcap_{i=1}^{n} \pi_i^{-1}(K_i), \quad W := \pi^{-1}(U) \cap \bigcap_{i=1}^{n} \pi_i^{-1}(W_i),$$
$$W^{(\alpha,\beta)} := \pi^{-1}(U^{(\alpha)}) \cap V^{(\beta)} \quad (\alpha \in \mathcal{P}_m,\ \beta \in \mathcal{P}_n).$$

Here, for $\alpha = \{j_1, \ldots, j_k\} \in \mathcal{P}_m$ with $k \le m$, we set $U^{(\alpha)} := U^{(j_1)} \cap \cdots \cap U^{(j_k)}$ and $U^{(*)} := U^{(\{1,2,\ldots,m\})}$ as usual. Similarly subsets $V^{(\beta)}$, $V^{(*)}$ and $W^{(*,*)}$ are defined.

We also denote by $\gamma_i(z) \subset \mathbb{C}_{w_i}$ a closed path in $\tau_i^{-1}(z) \cap W_i$ (regarded as a subset in $\mathbb{C}_{w_i}$) turning around each component of $\tau_i^{-1}(z) \cap K_i$ once with anti-clockwise direction.

Lemma A.3. *Under the situation described above, there exists the following commutative diagram*:

$$\begin{array}{ccc} H^{m+n}_{K\cap W}(W; \mathscr{O}^{(0,n)}_{X\times Y}) & \xrightarrow{\ \mu^c\ } & H^m_{Z\cap U}(U; \mathscr{O}_X) \\ \| & & \| \\ \dfrac{\Gamma(W^{(*,*)}; \mathscr{O}^{(0,n)}_{X\times Y})}{\sum_{(\alpha,\beta)\in(\mathcal{P}_m\times\mathcal{P}_n)^\vee} \Gamma(W^{(\alpha,\beta)}; \mathscr{O}^{(0,n)}_{X\times Y})} & \xrightarrow{\ \mu\ } & \dfrac{\Gamma(U^{(*)}; \mathscr{O}_X)}{\sum_{\alpha\in\mathcal{P}_m^\vee} \Gamma(U^{(\alpha)}; \mathscr{O}_X)}. \end{array}$$

Here μ is defined by

$$u(z,w)\,dw \mapsto \int_{\gamma_1(z)\times\cdots\times\gamma_n(z)} u(z,w)\,dw,$$

and $(\mathcal{P}_m \times \mathcal{P}_n)^\vee$ denotes $\{(\alpha,\beta) \in \mathcal{P}_m \times \mathcal{P}_n;\ \#\alpha + \#\beta = m+n-1\}$.

Proof. The lemma immediately follows from [**16**, Corollary 3.1.4]. However, for the reader's convenience, we will give a proof in what follows.

Let $Y' = \mathbb{C}^{n-1}_{w'}$ with coordinates $w' = (w_1, \ldots, w_{n-1})$, and let $\pi' : X \times Y \to X \times Y'$ (resp. $\pi'' : X \times Y' \to X$) be the canonical projection defined by $(x,w) \to (x,w')$ (resp. by $(x,w') \to (x)$). We also denote by $\pi''_i : X \times Y' \to X \times \mathbb{C}_{w_i}$ $(1 \le i \le n-1)$ the canonical projection by $(x,w') \to (x,w_i)$. Clearly we have $\pi = \pi'' \circ \pi'$ and we also see that the cohomological residue mapping μ^c coincides with the following composition of the cohomological residue mappings along the fibers of π' and π'':

$$H^{m+n}_{K\cap W}(W; \mathscr{O}^{(0,n)}_{X\times Y}) \xrightarrow{\mu'^c} H^{m+n-1}_{K'\cap W'}(W'; \mathscr{O}^{(0,n-1)}_{X\times Y'}) \xrightarrow{\mu''^c} H^m_{Z\cap U}(U; \mathscr{O}_X),$$

where μ'^c (resp. μ''^c) is the cohomological residue mapping along the fiber of π' (resp. π'') and the subsets K' and W' are given by

$$K' := \pi''^{-1}(Z) \cap \bigcap_{i=1}^{n-1} \pi_i''^{-1}(K_i), \quad W' := \pi''^{-1}(U) \cap \bigcap_{i=1}^{n-1} \pi_i''^{-1}(W_i).$$

Hence, if we can show the lemma for $n = 1$, then the result follows for any $n \in \mathbb{N}$ by the induction on n. Therefore we may assume from the beginning that $n = 1$; i.e. $Y = \mathbb{C}$.

Let us show the claim by the induction on $m \geq 0$.

First we prove the lemma for $m = 0$. Let $\mathcal{S}^{\bullet}_{X\times\mathbb{C}}$ (resp. $\mathcal{S}^{\bullet}_{X}$) be the $\bar{\partial}$ complex of $\mathscr{O}^{(0,1)}_{X\times\mathbb{C}}$ (resp. $\mathscr{O}_X$) with coefficients in the sheaf of distributions on $X \times \mathbb{C} = \mathbb{R}^{2\ell} \times \mathbb{R}^2$ (resp. $X = \mathbb{R}^{2\ell}$). Then we have the following diagram:

$$\begin{array}{ccc} H^1_{K\cap W}(W;\mathscr{O}^{(0,1)}_{X\times\mathbb{C}}) & \xleftarrow[\delta']{\sim} & \Gamma(W\smallsetminus K;\mathscr{O}^{(0,1)}_{X\times\mathbb{C}})/\Gamma(W;\mathscr{O}^{(0,1)}_{X\times\mathbb{C}}) \\ \iota\downarrow & & \downarrow \\ \varinjlim_{\widetilde{K}} H^1\Gamma_{\widetilde{K}\cap W}(W;\mathcal{S}^{\bullet}_{X\times\mathbb{C}}) & \xleftarrow{\delta} & \varinjlim_{\widetilde{K}} \Gamma(W\smallsetminus \widetilde{K};\mathscr{O}^{(0,1)}_{X\times\mathbb{C}})/\Gamma(W;\mathscr{O}^{(0,1)}_{X\times\mathbb{C}}) \\ \int_c\downarrow & & \mu\downarrow \\ H^0\Gamma(U;\mathcal{S}^{\bullet}_X) & \Longleftrightarrow & \Gamma(U;\mathscr{O}_X). \end{array} \tag{A.5}$$

Here $\widetilde{K}$ ranges through closed subsets in W such that $K \subset \operatorname{Int}\widetilde{K}$ and $\pi|_W\colon \widetilde{K} \to X$ is proper. Here $\operatorname{Int}\widetilde{K}$ denotes the interior of $\widetilde{K}$. The morphism δ is given by $u \mapsto \bar{\partial}\tilde{u}$, where $\tilde{u}$ is a distribution extension of u to W with $u = \tilde{u}$ on $W \smallsetminus \widetilde{K}$. The morphism $\int_c$ is nothing but the integration along the fiber of $\pi\colon X \times \mathbb{C} \to X$ for distributions. Note that the element in $H^1\Gamma_{\widetilde{K}\cap W}(W;\mathcal{S}^{\bullet}_{X\times\mathbb{C}})$ is a real differential 2-form as $\mathcal{S}^{\bullet}_{X\times\mathbb{C}}$ is the $\bar{\partial}$ complex of $\mathscr{O}^{(0,1)}_{X\times\mathbb{C}}$. Then the commutativity of the lower square in (A.5) comes from the Stokes formula. Let $\mathcal{B}^{\bullet}_{X\times\mathbb{C}}$ be a $\bar{\partial}$ complex of $\mathscr{O}^{(0,1)}_{X\times\mathbb{C}}$ with coefficients in the sheaf of hyperfunctions on $X \times \mathbb{C} = \mathbb{R}^{2\ell} \times \mathbb{R}^2$. Then we have $H^1_{K\cap W}(W;\mathscr{O}^{(0,1)}_{X\times\mathbb{C}}) = H^1\Gamma_{K\cap W}(W;\mathcal{B}^{\bullet}_{X\times\mathbb{C}})$. The morphism δ' is given by $u \mapsto \bar{\partial}\tilde{u}$, where $\tilde{u}$ is an extension of u to W as an element of the flabby sheaf. The morphism ι is the composition of morphisms

$$H^1\Gamma_{K\cap W}(W;\mathcal{B}^{\bullet}_{X\times\mathbb{C}}) \to \varinjlim_{\widetilde{K}} H^1\Gamma_{\widetilde{K}\cap W}(W;\mathcal{B}^{\bullet}_{X\times\mathbb{C}}) \xleftarrow{\sim} \varinjlim_{\widetilde{K}} H^1\Gamma_{\widetilde{K}\cap W}(W;\mathcal{S}^{\bullet}_{X\times\mathbb{C}}).$$

Let $\tilde{u}'$ (resp. $\tilde{u}$) be a distribution (resp. hyperfunction) extension of $u \in \Gamma(W \smallsetminus K;\mathscr{O}^{(0,1)}_{X\times\mathbb{C}})$ to W with $\tilde{u}' = u$ on $W \smallsetminus \widetilde{K}$ (resp. $\tilde{u} = u$ on $W \smallsetminus K$). Since $\operatorname{supp}(\tilde{u} - \tilde{u}') \subset \widetilde{K}$, we have

$$\partial\tilde{u} - \partial\tilde{u}' = 0 \in \varinjlim_{\widetilde{K}} H^1\Gamma_{\widetilde{K}\cap W}(W;\mathcal{B}^{\bullet}_{X\times\mathbb{C}}),$$

which implies the commutativity of the upper square in (A.5). Hence, as the residue morphism μ^c is the composition $\int_c \circ\iota$ by definition, we have obtained the claim of the lemma for the case $m = 0$. Now suppose that the claim of the lemma is true for $0, \dots, m-1$. We will show the lemma for m. Let us consider the commutative diagram between exact sequences:

$$\begin{array}{ccccccc} H^m_{K'\cap W}(W;\mathscr{O}^{(0,1)}_{X\times\mathbb{C}}) & \to & H^m_{K'\cap W^{(m)}}(W^{(m)};\mathscr{O}^{(0,1)}_{X\times\mathbb{C}}) & \to & H^{m+1}_{K\cap W}(W;\mathscr{O}^{(0,1)}_{X\times\mathbb{C}}) & \to & 0 \\ \mu^c\downarrow & & \mu^c\downarrow & & \mu^c\downarrow & & \\ H^{m-1}_{Z'\cap U}(U;\mathscr{O}_X) & \longrightarrow & H^{m-1}_{Z'\cap U^{(m)}}(U^{(m)};\mathscr{O}_X) & \longrightarrow & H^m_{Z\cap U}(U;\mathscr{O}_X) & \longrightarrow & 0, \end{array} \tag{A.6}$$

where $W^{(m)} := W \cap \pi^{-1}(U^{(m)})$, $Z' := U \smallsetminus \bigcup_{i=1}^{m-1} U^{(i)}$ and $K' := \pi^{-1}(Z') \cap \pi_1^{-1}(K_1)$. We also have the commutative diagram between exact sequences:

$$\begin{array}{ccccccc}
\dfrac{\Gamma(W'^{(*,1)};\mathscr{O}^{(0,1)}_{X\times\mathbb{C}})}{\displaystyle\sum_{\alpha\in\mathcal{P}^\vee_{m-1}}\Gamma(W'^{(\alpha,1)};\mathscr{O}^{(0,1)}_{X\times\mathbb{C}})} & \to & \dfrac{\Gamma(W''^{(*,1)};\mathscr{O}^{(0,1)}_{X\times\mathbb{C}})}{\displaystyle\sum_{\alpha\in\mathcal{P}^\vee_{m-1}}\Gamma(W''^{(\alpha,1)};\mathscr{O}^{(0,1)}_{X\times\mathbb{C}})} & \to & \dfrac{\Gamma(W^{(*,1)};\mathscr{O}^{(0,1)}_{X\times\mathbb{C}})}{\displaystyle\sum_{\alpha\in\mathcal{P}^\vee_{m}}\Gamma(W^{(\alpha,1)};\mathscr{O}^{(0,1)}_{X\times\mathbb{C}})} & \to 0 \\
{\scriptstyle\mu}\downarrow & & {\scriptstyle\mu}\downarrow & & {\scriptstyle\mu}\downarrow & \\
\dfrac{\Gamma(U'^{(*)};\mathscr{O}_X)}{\displaystyle\sum_{\alpha\in\mathcal{P}^\vee_{m-1}}\Gamma(U'^{(\alpha)};\mathscr{O}_X)} & \longrightarrow & \dfrac{\Gamma(U''^{(*)};\mathscr{O}_X)}{\displaystyle\sum_{\alpha\in\mathcal{P}^\vee_{m-1}}\Gamma(U''^{(\alpha)};\mathscr{O}_X)} & \longrightarrow & \dfrac{\Gamma(U^{(*)};\mathscr{O}_X)}{\displaystyle\sum_{\alpha\in\mathcal{P}^\vee_{m}}\Gamma(U^{(\alpha)};\mathscr{O}_X)} & \longrightarrow 0.
\end{array} \tag{A.7}$$

Here $\{W'^{(\alpha,1)}\}$, $\{W''^{(\alpha,1)}\}$, $\{U'^{(\alpha)}\}$, $\{U''^{(\alpha)}\}$ are the corresponding the coverings of $W \setminus K'$, $W^{(m)} \setminus K'$, $U \setminus Z'$, $U^{(m)} \setminus Z'$ respectively. By the induction hypothesis, the first and the second μ^c and μ in (A.6) and (A.7) coincide. Hence the third ones in the both diagrams also coincide. The proof is complete. □

B. General construction of $C^{\mathbb{R}}_{Y|X,z_0^*}$.

In this appendix, we will extend theories developed in Sections 2 and 3 to a general family of Čech coverings, which enables us to define the symbol mapping σ in a general complex manifold. We continue to use the same notation as those in Section 2 unless we specify them. Let X be an n-dimensional complex manifold with a system of local coordinates $z = (z_1, \dots, z_n)$, and Y a closed complex submanifold of X which is defined locally by $\{z' = 0\}$ where $z = (z', z'')$ with $z' := (z_1, \dots, z_d)$ for some $1 \le d \le n$. Set $\widehat{X} := X \times \mathbb{C}$, and let $\pi_\eta \colon \widehat{X} \ni (z, \eta) \mapsto z \in X$ be the canonical projection. Let $z_0 = (0, z_0'') \in Y$ and $z_0^* = (z_0''; \zeta_0') \in T^*_Y X$ with $\zeta_0' \ne 0$.

Let $\chi = \{f_1(z), \dots, f_d(z)\}$ be a sequence of holomorphic functions in an open neighborhood of z_0 satisfying the following conditions:

(1) $df_1(z_0) \wedge df_2(z_0) \wedge \cdots \wedge df_d(z_0) \ne 0$.

(2) $f_1, \dots, f_d$ belong to the defining ideal $\mathscr{I}_Y$ of Y.

(3) We have

$$ {}^{\mathrm{t}}\!\left[\frac{\partial f}{\partial z}(z_0)\right] \boldsymbol{e} = (\zeta_0', 0) \in (T^*X)_{z_0}$$

where $f(z) := (f_1(z), \dots, f_d(z))$ and $\boldsymbol{e} := (1, 0, \dots, 0) \in \mathbb{C}^d$.

We denote by $\Xi(z_0^*)$ the set of sequences satisfying the conditions above. Set

$$\begin{gathered} f(z) := (f_1(z), f_2(z), \dots, f_d(z)) = (f_1(z), f'(z)), \\ G^\chi_{\varrho,L} := \{z \in X;\ \varrho^2 |f'(z)| \le |f_1(z)|,\ f_1(z) \in L\}, \end{gathered}$$

where $\varrho > 0$ and $L \subset \mathbb{C}$ is a closed convex cone with $L \subset \{\tau \in \mathbb{C};\ \operatorname{Re}\tau > 0\} \cup \{0\}$. We also set, for an open neighborhood U of z_0 in X,

$$\widehat{G}^\chi_{\varrho,L} := \{(z, \eta) \in \widehat{X};\ \varrho |f'(z)| \le |\eta|,\ f_1(z) \in L\},$$

$$\widehat{U}^{\chi}_{\varrho,r,\theta} := \{(z,\eta) \in U \times S_{r,\theta};\ |f_1(z)| < \varrho|\eta|\}.$$

Now we define

$$\widehat{C}^{\mathbb{R},\chi}_{Y|X,z_0^*} := \varinjlim_{\varrho,r,\theta,L,U} H^d_{\widehat{G}^{\chi}_{\varrho,L}\cap \widehat{U}^{\chi}_{\varrho,r,\theta}}(\widehat{U}^{\chi}_{\varrho,r,\theta}; \mathscr{O}_{\widehat{X}}),$$

$$C^{\mathbb{R},\chi}_{Y|X,z_0^*} := \operatorname{Ker}(\partial_\eta \colon \widehat{C}^{\mathbb{R},\chi}_{Y|X,z_0^*} \to \widehat{C}^{\mathbb{R},\chi}_{Y|X,z_0^*}).$$

Then, by the same reasoning as that in Section 2, we have the isomorphisms

$$\mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*} \xleftarrow{\sim} \varinjlim_{\varrho,L,U} H^d_{G^{\chi}_{\varrho,L}\cap U}(U; \mathscr{O}_X) \xrightarrow{\sim} C^{\mathbb{R},\chi}_{Y|X,z_0^*},$$

where these isomorphisms are associated with the natural inclusions of sets and the canonical morphism $\pi_\eta^{-1}\mathscr{O}_X \to \mathscr{O}_{\widehat{X}}$ as we have seen in Section 2. Hence, for any χ_1 and χ_2 in $\Xi(z_0^*)$, two modules $C^{\mathbb{R},\chi_1}_{Y|X,z_0^*}$ and $C^{\mathbb{R},\chi_2}_{Y|X,z_0^*}$ are isomorphic through $\mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*}$. Using this fact, we replace the definition of $C^{\mathbb{R}}_{Y|X,z_0^*}$ introduced in Section 2 with a slightly generalized one. From now on, we write $M^{\mathbb{R},\chi}_{Y|X,z_0^*} := \varinjlim_{\varrho,L,U} H^d_{G^{\chi}_{\varrho,L}\cap U}(U; \mathscr{O}_X)$ for short.

Definition B.1. We denote by $C^{\mathbb{R}}_{Y|X,z_0^*}$ the isomorphism class $\{C^{\mathbb{R},\chi}_{Y|X,z_0^*}\}_{\chi\in\Xi(z_0^*)}$ consisting of $C^{\mathbb{R},\chi}_{Y|X,z_0^*}$ indexed by $\chi \in \Xi(z_0^*)$. In the same way, the isomorphism class $M^{\mathbb{R}}_{Y|X,z_0^*}$ is defined by $\{M^{\mathbb{R},\chi}_{Y|X,z_0^*}\}_{\chi\in\Xi(z_0^*)}$.

By a direct consequence of the construction above, we have the morphism of $C^{\mathbb{R}}_{Y|X,z_0^*}$ associated with a coordinate transformation. Let $w = (w', w'')$ be a system of local coordinates of a copy of X where Y is locally defined by $w' = 0$, and $z = \Phi(w)$ a local coordinate transformation in an open neighborhood of $w_0 \in Y$ satisfying $\Phi(Y) \subset Y$ and $z_0 = \Phi(w_0)$. Set $\widehat{\Phi} := \Phi \times \mathrm{id}_\eta$. Then it induces the sheaf morphism

$$\widehat{\Phi}^{-1}\mathscr{O}_{\widehat{X}} \ni \varphi \mapsto \varphi \circ \widehat{\Phi} \in \mathscr{O}_{\widehat{X}}.$$

Let $w_0^* := (w_0; {}^{\mathsf{t}}\big[(\partial\Phi/\partial w)(w_0)\big](\zeta_0', 0)) \in T^*_Y X$. It is easy to see

$$\chi \circ \Phi := \{f_1 \circ \Phi, \dots, f_d \circ \Phi\} \in \Xi(w_0^*)$$

for any $\chi = \{f_1, \dots, f_d\} \in \Xi(z_0^*)$. Hence we have the morphism $C^{\mathbb{R},\chi}_{Y|X,z_0^*} \to C^{\mathbb{R},\chi\circ\Phi}_{Y|X,w_0^*}$ defined by $[u(z,\eta)] \to [u(\Phi(w),\eta)]$, which gives $\widehat{\Phi}^* \colon C^{\mathbb{R}}_{Y|X,z_0^*} \to C^{\mathbb{R}}_{Y|X,w_0^*}$. This morphism is compatible with the morphism $\Phi^* \colon \mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*} \to \mathscr{C}^{\mathbb{R}}_{Y|X,w_0^*}$ associated with the coordinate transformation Φ because the both morphisms are induced from the same coordinate transformation of holomorphic functions.

Next we consider a Čech representation of $C^{\mathbb{R},\chi}_{Y|X,z_0^*}$ for $\chi = \{f_1, \dots, f_d\} \in \Xi(z_0^*)$. Set

$$U^{\chi}_{\boldsymbol{\kappa}} := \bigcap_{i=2}^{d} \{z = (z', z'') \in X;\ |f_1(z)| < \varrho r,\ |f_i(z)| < r',\ \|z'' - z_0''\| < r'\},$$

$$\widehat{U}^{\chi}_{\boldsymbol{\kappa}} := \bigcap_{i=2}^{d} \{(z,\eta) = (z',z'',\eta) \in X \times S_{\boldsymbol{\kappa}};\ |f_1(z)| < \varrho|\eta|,\ |f_i(z)| < r',\ \|z'' - z''_0\| < r'\},$$

where $\|z''\|$ denotes $\max\{|z_{d+1}|,\dots,|z_n|\}$. We also define

$$\begin{aligned}
V^{\chi,(1)}_{\boldsymbol{\kappa}} &:= \left\{ z \in U^{\chi}_{\boldsymbol{\kappa}};\ \frac{\pi}{2} - \theta < \arg f_1(z) < \frac{3\pi}{2} + \theta \right\},\\
V^{\chi,(i)}_{\boldsymbol{\kappa}} &:= \{ z \in U^{\chi}_{\boldsymbol{\kappa}};\ \varrho^2 |f_i(z)| > |f_1(z)| \} \qquad (2 \le i \le d),\\
\widehat{V}^{\chi,(1)}_{\boldsymbol{\kappa}} &:= \left\{ (z,\eta) \in \widehat{U}^{\chi}_{\boldsymbol{\kappa}};\ \frac{\pi}{2} - \theta < \arg f_1(z) < \frac{3\pi}{2} + \theta \right\},\\
\widehat{V}^{\chi,(i)}_{\boldsymbol{\kappa}} &:= \{ (z,\eta) \in \widehat{U}^{\chi}_{\boldsymbol{\kappa}};\ \varrho |f_i(z)| > |\eta| \} \qquad (2 \le i \le d).
\end{aligned}$$

Then it follows from the same arguments in Section 2 that we have

$$\begin{aligned}
\widehat{C}^{\mathbb{R},\chi}_{Y|X,z_0^*} &= \varinjlim_{\boldsymbol{\kappa}} \Gamma(\widehat{V}^{\chi,(*)}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}}) \Big/ \sum_{\alpha \in \mathcal{P}_d^{\vee}} \Gamma(\widehat{V}^{\chi,(\alpha)}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}}),\\
C^{\mathbb{R},\chi}_{Y|X,z_0^*} &= \varinjlim_{\boldsymbol{\kappa}} \left\{ u \in \Gamma(\widehat{V}^{\chi,(*)}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}}) \Big/ \sum_{\alpha \in \mathcal{P}_d^{\vee}} \Gamma(\widehat{V}^{\chi,(\alpha)}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}});\ \partial_{\eta} u = 0 \right\},\\
M^{\mathbb{R},\chi}_{Y|X,z_0^*} &= \varinjlim_{\boldsymbol{\kappa}} \Gamma(V^{\chi,(*)}_{\boldsymbol{\kappa}}; \mathscr{O}_X) \Big/ \sum_{\alpha \in \mathcal{P}_d^{\vee}} \Gamma(V^{\chi,(\alpha)}_{\boldsymbol{\kappa}}; \mathscr{O}_X).
\end{aligned}$$

Let us recall the definitions of the paths $\gamma_1(z,\eta;\varrho,\theta)$ and $\gamma_i(z,\eta;\varrho)$ in $\mathbb{C}$ which were given in Section 2. In this appendix, we take slightly modified paths. Set

$$\gamma_1(\eta;\varrho,\theta) := -\gamma_1(0,\eta;\varrho,\theta), \quad \gamma_i(\eta;\varrho) := \gamma_i(0,\eta;\varrho)\ (i > 1).$$

We define the real d-dimensional chain in $\mathbb{C}^d$

$$\gamma(\eta;\varrho,\theta) := \gamma_1(\eta;\varrho,\eta) \times \gamma_2(\eta;\varrho) \times \cdots \times \gamma_d(\eta;\varrho).$$

Then, for any $(0,z'') \in Y$ near z_0, we also define the real d-dimensional chain in $\mathbb{C}^d_{z'}$ by

$$\gamma^{\chi}(z'',\eta;\varrho,\theta) := \{ z' \in \mathbb{C}^d;\ f(z',z'') \in \gamma(\eta;\varrho,\theta) \},$$

where $f(z) := (f_1(z),\dots,f_d(z)) \colon \mathbb{C}^n \to \mathbb{C}^d$ for $\chi = \{f_1,\dots,f_d\} \in \Xi(z_0^*)$ and the orientation of γ^{χ} is determined by that of γ through f.

Let us introduce the symbol spaces

$$\begin{aligned}
\mathfrak{S}_{Y|X,z_0^*} &:= \varinjlim_{\Omega,S} \mathfrak{S}_{Y|X}(\Omega;S) \supset \mathfrak{N}_{Y|X,z_0^*} := \varinjlim_{\Omega,S} \mathfrak{N}_{Y|X}(\Omega;S),\\
\mathscr{S}_{Y|X,z_0^*} &:= \varinjlim_{\Omega \ni z_0^*} \mathscr{S}_{Y|X}(\Omega) \supset \mathscr{N}_{Y|X,z_0^*} := \varinjlim_{\Omega \ni z_0^*} \mathscr{N}_{Y|X}(\Omega).
\end{aligned}$$

Here $\Omega \underset{\text{conic}}{\Subset} T^*_Y X$ ranges through open conic neighborhoods of z_0^*, and the inductive limits with respect to S are taken by r_0, $\theta \to 0$. The sets $\mathfrak{S}_{Y|X}(\Omega; S)$, $\mathfrak{N}_{Y|X}(\Omega; S)$, $\mathscr{S}_{Y|X}(\Omega)$ and $\mathscr{N}_{Y|X}(\Omega)$ are defined in the same way as in Section 4. Then we can define the mapping $\hat{\sigma}^{\chi}\colon C^{\mathbb{R},\chi}_{Y|X,z_0^*} \to \mathfrak{S}_{Y|X,z_0^*}/\mathfrak{N}_{Y|X,z_0^*}$ by

$$\hat{\sigma}^{\chi}([u])(z'', \zeta', \eta) := \int_{\gamma^{\chi}(z'',\eta;\varrho,\theta)} u(z', z'', \eta)\, e^{-\langle z', \zeta' \rangle} dz'$$

for $u(z', z'', \eta) \in \Gamma(\widehat{V}^{\chi,(*)}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}})$ with a suitable $\boldsymbol{\kappa}$. Similarly we get the mapping $M^{\mathbb{R},\chi}_{Y|X,z_0^*} \to \mathscr{S}_{Y|X,z_0^*}/\mathscr{N}_{Y|X,z_0^*}$ by

$$\sigma^{\chi}([v])(z'', \zeta') := \int_{\gamma^{\chi}(z'',\eta_0;\varrho,\theta)} v(z', z'')\, e^{-\langle z', \zeta' \rangle} dz'$$

for $v(z', z'') \in \Gamma(V^{\chi,(*)}_{\boldsymbol{\kappa}}; \mathscr{O}_X)$ with a suitable $\boldsymbol{\kappa}$ and a sufficiently small fixed $\eta_0 > 0$.

Now we have the following theorem.

Theorem B.2. *The morphisms $\hat{\sigma}^{\chi}$ and σ^{χ} induce the well-defined mappings $\hat{\sigma}\colon C^{\mathbb{R}}_{Y|X,z_0^*} \to \mathfrak{S}_{Y|X,z_0^*}/\mathfrak{N}_{Y|X,z_0^*}$ and $\sigma\colon M^{\mathbb{R}}_{Y|X,z_0^*} \to \mathscr{S}_{z_0^*}/\mathscr{N}_{z_0^*}$ respectively. To be more precise, if χ_1, $\chi_2 \in \Xi(z_0^*)$ and $[u_1] \in C^{\mathbb{R},\chi_1}_{Y|X,z_0^*}$ and $[u_2] \in C^{\mathbb{R},\chi_2}_{Y|X,z_0^*}$ determining the same element in $\mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*}$, it follows that $\hat{\sigma}^{\chi_1}([u_1]) = \hat{\sigma}^{\chi_2}([u_2]) \in \mathfrak{S}_{Y|X,z_0^*}/\mathfrak{N}_{Y|X,z_0^*}$. Similarly, for $[v_1] \in M^{\mathbb{R},\chi_1}_{Y|X,z_0^*}$ and $[v_2] \in M^{\mathbb{R},\chi_2}_{Y|X,z_0^*}$ giving the same element in $\mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*}$, it follows that $\sigma^{\chi_1}([v_1]) = \sigma^{\chi_2}([v_2]) \in \mathscr{S}_{Y|X,z_0^*}/\mathscr{N}_{Y|X,z_0^*}$.*

Proof. By a linear coordinate transformation, we may assume $z_0^* = (0; 1, 0, \dots, 0) \in T^*_Y X$ and $z_0 = 0 \in X$. We need the following easy lemma.

Lemma B.3. *Let $g(z)$ be a holomorphic function in an open neighborhood of z_0. Assume that $g(z) \in \mathscr{I}_Y$ and ${}^{\mathrm{t}}\big[(\partial g/\partial z)(z_0)\big] = (1, 0, \dots, 0)$. Then, for ϱ and θ, there exists a sufficiently small $\varepsilon > 0$ such that*

$$\operatorname{Re} g(z) \geq \varepsilon|\eta| \qquad (z' \in \partial\gamma(\eta; \varrho, \theta),\ \eta \in S_{\boldsymbol{\kappa}},\ |z''| \leq \varepsilon,\ |\eta| < \varepsilon),$$

where $\partial\gamma$ denotes the boundary of $\gamma(\eta; \varrho, \theta)$.

Proof. The Taylor expansion of $g(z)$ along Y is given by

$$g(z) = \psi_1(z'')z_1 + \psi_2(z'')z_2 + \cdots + \psi_d(z'')z_d + O(|z'|^2)$$

with $\psi_1(0) = 1$ and $\psi_k(0) = 0$ $(k \geq 2)$. The claim immediately follows from this. □

Let $\chi = \{f_1, \dots, f_d\} \in \Xi(z_0^*)$. Set $f(z) := (f_1(z), \dots, f_d(z))$ and, for z'' near 0, we write by $f_{z''}(z')$ the mapping $f(z', z'')$ regarded as a mapping of the variable z' with a fixed z''. Then, by the coordinate transformation, we have

$$\hat{\sigma}^{\chi}([u])(z'', \zeta', \eta) = \int_{\gamma(\eta;\varrho,\theta)} u(f_{z''}^{-1}(w'), z'', \eta)\, e^{-\langle f_{z''}^{-1}(w'),\, \zeta'\rangle} \det[\partial_{w'} f_{z''}^{-1}]\, dw',$$
$$\sigma^{\chi}([v])(z'', \zeta') = \int_{\gamma(\eta_0;\varrho,\theta)} v(f_{z''}^{-1}(w'), z'')\, e^{-\langle f_{z''}^{-1}(w'),\, \zeta'\rangle} \det[\partial_{w'} f_{z''}^{-1}]\, dw'.$$

Therefore, by applying Lemma B.3 to the first coordinate function of $f_{z''}^{-1}(w')$, we have the commutative diagram below:

$$\begin{array}{ccc} M^{\mathbb{R},\chi}_{Y|X,z_0^*} & \xrightarrow{\sigma^{\chi}} & \mathscr{S}_{Y|X,z_0^*}/\mathscr{N}_{Y|X,z_0^*} \\ \big\downarrow\wr & & \big\downarrow \\ C^{\mathbb{R},\chi}_{Y|X,z_0^*} & \xrightarrow{\hat{\sigma}^{\chi}} & \mathfrak{S}_{Y|X,z_0^*}/\mathfrak{N}_{Y|X,z_0^*}\,. \end{array}$$

Since the first down-arrow $M^{\mathbb{R},\chi}_{Y|X,z_0^*} \xrightarrow{\sim} C^{\mathbb{R},\chi}_{Y|X,z_0^*}$ is isomorphic, to show the theorem, it suffices to prove the last claim in the theorem. We first consider a special case.

LEMMA B.4. *Let* $\chi_1 = \{f_1, \dots, f_d\}$, $\chi_2 = \{f_1, \dots, f_{d-1}, g\} \in \Xi(z_0^*)$. *Then the last claim in Theorem* B.2 *holds for these* χ_1 *and* χ_2.

PROOF. Let $v_1 \in \Gamma(V_{\boldsymbol{\kappa}}^{\chi_1,(*)}; \mathscr{O}_X)$ and $v_2 \in \Gamma(V_{\boldsymbol{\kappa}}^{\chi_2,(*)}; \mathscr{O}_X)$ with some $\boldsymbol{\kappa}$ which give the same element in $\mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*}$. Let us consider the coordinate transformation

$$w = (w', w'') = f(z) = (f_1(z), \dots, f_d(z), z_{d+1}, \dots, z_n),$$

and let $w_0 = f(z_0) = 0$, $w_0^* = (0; 1, 0, \dots, 0)$. Clearly the coordinate transformation changes χ_1 and χ_2 to $\tilde{\chi}_1 = (w_1, \dots, w_d)$ and $\tilde{\chi}_2 = (w_1, \dots, w_{d-1}, h)$ with $h(w) = g \circ f^{-1}$ respectively. Further, we have

$$\begin{aligned} \sigma^{\chi_1}([v_1])(w'', \zeta') &= \int_{\gamma(\eta_0;\varrho,\theta)} v_1(f^{-1}(w))\, e^{-\langle f_{w''}^{-1}(w'),\, \zeta'\rangle} \det[\partial_{w'} f_{w''}^{-1}]\, dw', \\ \sigma^{\chi_2}([v_2])(w'', \zeta') &= \int_{f_{w''}(\gamma^{\chi_2}(w'',\eta_0;\varrho,\theta))} v_2(f^{-1}(w))\, e^{-\langle f_{w''}^{-1}(w'),\, \zeta'\rangle} \det[\partial_{w'} f_{w''}^{-1}]\, dw'. \end{aligned} \tag{B.1}$$

It follows from definitions that $v_1(f^{-1}(w))$ and $v_2(f^{-1}(w))$ are holomorphic in $V_{\boldsymbol{\kappa}}^{(*)} = V_{\boldsymbol{\kappa}}^{\tilde{\chi}_1,(*)}$ and $V_{\boldsymbol{\kappa}}^{\tilde{\chi}_2,(*)}$ respectively. We can also see

$$f_{w''}(\gamma^{\chi_2}(w'', \eta_0; \varrho, \theta)) = \gamma^{\tilde{\chi}_2}(w'', \eta_0; \varrho, \theta).$$

Since $\tilde{\chi}_2$ belongs to $\Xi(w_0^*)$, we have $\partial_{w_d} h(w_0) \neq 0$. This implies that, by keeping η_0 of $\gamma_d(\eta_0; \rho)$ unchanged and by taking η_0 of other γ_k $(1 \le k \le d-1)$ so small if needed, the chain $\gamma(\eta_0; \rho, \theta) \times \{w''\}$ belongs the common domain of $V_{\boldsymbol{\kappa}}^{\tilde{\chi}_1,(*)}$ and $V_{\boldsymbol{\kappa}}^{\tilde{\chi}_2,(*)}$ for a sufficiently small w''. In particular, we can replace $f_{w''}(\gamma^{\chi_2}(w'', \eta_0; \varrho, \theta))$ in (B.1) with $\gamma(\eta_0; \varrho, \theta)$. Therefore the problem can be reduced to the case where $\chi_1 = \{z_1, \dots, z_d\}$, $\chi_2 = \{z_1, \dots, z_{d-1}, g\}$ and the morphisms σ^{χ_1} and σ^{χ_2} are replaced with the same morphism

$$\sigma(v)(z'',\zeta') := \int_{\gamma(\eta_0;\varrho,\theta)} v(z',z'')\, e^{-\langle\varphi(z),\,\zeta'\rangle} dz' \tag{B.2}$$

for some $\varphi(z) = (\varphi_1(z),\dots,\varphi_d(z))$ with $\{\varphi_1(z),\dots,\varphi_d(z)\} \in \varXi(z_0^*)$ ($z_0^* = (0;1,0,\dots,0)$). Let us show the lemma for this case. We have the diagram

$$M^{\mathbb{R},\chi_1}_{Y|X,z_0^*} \xleftarrow{\sim} \varinjlim_{\varrho,L,U} H^d_{G^{\chi_1}_{\varrho,L}\cap G^{\chi_2}_{\varrho,L}\cap U}(U;\mathscr{O}_X) \xrightarrow{\sim} M^{\mathbb{R},\chi_2}_{Y|X,z_0^*}. \tag{B.3}$$

The corresponding diagram of Čech representations is given by

$$\varinjlim_{\boldsymbol{\kappa}} \varGamma(V^{\chi_1,(*)}_{\boldsymbol{\kappa}};\mathscr{O}_X)\Big/\sum_{\alpha\in\mathcal{P}_d^\vee}\varGamma(V^{\chi_1,(\alpha)}_{\boldsymbol{\kappa}};\mathscr{O}_X) \underset{\iota_1}{\xleftarrow{\sim}} \varinjlim_{\boldsymbol{\kappa}} Z^d(\mathfrak{T}_{\boldsymbol{\kappa}};\mathscr{O}_X)\Big/B^d(\mathfrak{T}_{\boldsymbol{\kappa}};\mathscr{O}_X)$$
$$\underset{\iota_2}{\xrightarrow{\sim}} \varinjlim_{\boldsymbol{\kappa}} \varGamma(V^{\chi_2,(*)}_{\boldsymbol{\kappa}};\mathscr{O}_X)\Big/\sum_{\alpha\in\mathcal{P}_d^\vee}\varGamma(V^{\chi_2,(\alpha)}_{\boldsymbol{\kappa}};\mathscr{O}_X),$$

where the covering $\mathfrak{T}_{\boldsymbol{\kappa}} = \{T^{(i)}_{\boldsymbol{\kappa}}\}_{i=1}^{d+1}$ is given by

$$T^{(i)}_{\boldsymbol{\kappa}} = V^{\chi_1,(i)}_{\boldsymbol{\kappa}} \cap V^{\chi_2,(i)}_{\boldsymbol{\kappa}}\ (1\le i\le d-1),\ \ T^{(d)}_{\boldsymbol{\kappa}} = V^{\chi_1,(d)}_{\boldsymbol{\kappa}}\cap U^{\chi_2}_{\boldsymbol{\kappa}},\ \ T^{(d+1)}_{\boldsymbol{\kappa}} = V^{\chi_2,(d)}_{\boldsymbol{\kappa}}\cap U^{\chi_1}_{\boldsymbol{\kappa}}.$$

We also note that, for $v = \{v^{(\beta)}\} \in Z^d(\mathfrak{T}_{\boldsymbol{\kappa}};\mathscr{O}_X) \subset \sum_{\beta\in\varLambda_d}\varGamma(T^{(\beta)}_{\boldsymbol{\kappa}};\mathscr{O}_X)$, we have

$$v_1 := \iota_1(v) = v^{(\{1,\dots,d\})},\ v_2 := \iota_2(v) = v^{(\{1,\dots,d-1,d+1\})}.$$

Hence, to complete the proof, it suffices to show $\sigma([v_1]) = \sigma([v_2])$. Since v satisfies a cocycle condition, we have

$$v_2 - v_1 = (-1)^d \sum_{1\le k<d} (-1)^k v^{(*^{\vee k})}.$$

By modifying the path of the integration, we obtain $\sigma(v^{(\{2,\dots,d+1\})}) \in \mathscr{N}_{Y|X,z_0^*}$. Furthermore we get $\sigma(v^{(*^{\vee k})}) = 0$ if $2\le k<d$. Hence we have obtain that $\sigma(v_2) = \sigma(v_1) \in \mathscr{S}_{Y|X,z_0^*}/\mathscr{N}_{Y|X,z_0^*}$. □

By repeated application of Lemma B.4, we can show the last claim of the theorem for the case $\chi_1 = \{f_1,f_2,\dots,f_d\}$, $\chi_2 = \{f_1,g_2,\dots,g_d\} \in \varXi(z_0^*)$. Hence the theorem immediately follows from the lemma below. This completes the proof. □

Lemma B.5. *Let $\chi_1 = \{f_1,f_2,\dots,f_d\}$, $\chi_2 = \{g,f_2,\dots,f_d\} \in \varXi(z_0^*)$. Then the last claim in Theorem* B.2 *holds for these χ_1 and χ_2.*

Proof. By the same argument as that of the proof of Lemma B.4 and by noticing Lemma B.3, the problem can be reduced to the case $\chi_1 = (z_1,z_2,\dots,z_d)$ and $\chi_2 = (g,z_2,\dots,z_n)$ with the morphism σ defined by (B.2). Then we have the diagram (B.3) and the corresponding one by Čech representations is given by

$$\varinjlim_{\boldsymbol{\kappa}} \Gamma(V_{\boldsymbol{\kappa}}^{\chi_1,(*)};\mathscr{O}_X)\Big/\sum_{\alpha\in\mathcal{P}_d^\vee}\Gamma(V_{\boldsymbol{\kappa}}^{\chi_1,(\alpha)};\mathscr{O}_X) \underset{\iota_1}{\xleftarrow{\sim}} \varinjlim_{\boldsymbol{\kappa}} Z^d(\mathfrak{T}_{\boldsymbol{\kappa}}^2;\mathscr{O}_X)/B^d(\mathfrak{T}_{\boldsymbol{\kappa}}^2;\mathscr{O}_X)$$

$$\underset{\iota_2}{\xrightarrow{\sim}} \varinjlim_{\boldsymbol{\kappa}} \Gamma(V_{\boldsymbol{\kappa}}^{\chi_2,(*)};\mathscr{O}_X)\Big/\sum_{\alpha\in\mathcal{P}_d^\vee}\Gamma(V_{\boldsymbol{\kappa}}^{\chi_2,(\alpha)};\mathscr{O}_X),$$

where the covering $\mathfrak{T}_{\boldsymbol{\kappa}}^2 = \{T_{\boldsymbol{\kappa}}^{(i,j)}\}_{i,j=1}^d$ is defined by

$$T_{\boldsymbol{\kappa}}^{(i,j)} := V_{\boldsymbol{\kappa}}^{\chi_1,(i)} \cap V_{\boldsymbol{\kappa}}^{\chi_2,(j)} \cap U_{\boldsymbol{\kappa}}^{\chi_1} \cap U_{\boldsymbol{\kappa}}^{\chi_2}.$$

Furthermore the morphisms ι_1 and ι_2 are given by

$$v_1 := \iota_1(v) = v^{(*,\emptyset)},\ v_2 := \iota_2(v) = v^{(\emptyset,*)}$$

respectively for $v = \{v^{(\alpha,\beta)}\} \in Z^d(\mathfrak{T}_{\boldsymbol{\kappa}}^2;\mathscr{O}_X) \subset \sum_{(\alpha,\beta)\in\Lambda_d}\Gamma(T_{\boldsymbol{\kappa}}^{(\alpha,\beta)};\mathscr{O}_X)$. Then, by employing the same argument in the proof of Lemma A.2 in Appendix A, we have $\sigma(v_1) = \sigma(v_2) \in \mathscr{S}_{Y|X,z_0^*}/\mathscr{N}_{Y|X,z_0^*}$. The proof is complete. □

Now we compute behavior of a symbol by a coordinate transformation. Let $z = (z', z'') = (\Phi'(w), \Phi''(w)) = \Phi(w)$ be a local coordinate transformation near $w_0 \in Y$ with $\Phi(Y) \subset Y$ and $z_0 = \Phi(w_0)$ where Y is also defined by $w' = 0$ under the system of local coordinates $w = (w', w'')$. We denote by $\Phi'_{w''}(w')$ the mapping $z' = \Phi'(w', w'')$ regarded as a mapping of the variable w' with a fixed w''. Set $w_0^* := (w_0; d\Phi(w_0)(\zeta_0', 0)) \in T_Y^*X$. Let $\chi = \{z_1, \dots, z_d\}$ and $[u] \in C_{Y|X,z_0^*}^{\mathbb{R},\chi}$ with $u(z,\eta) \in \Gamma(\widehat{V}_{\boldsymbol{\kappa}}^{\chi,(*)};\mathscr{O}_{\widehat{X}})$ for some $\boldsymbol{\kappa}$. Then we get $\widehat{\Phi}^*([u]) = [u(\Phi(w),\eta)] \in C_{Y|X,w_0^*}^{\mathbb{R},\chi\circ\Phi}$. Hence we have obtained

$$\hat{\sigma}(\widehat{\Phi}^*([u]))(w'',\lambda',\eta) = \int_{\gamma^{\chi\circ\Phi}(w'',\eta;\varrho,\theta)} u(\Phi(w),\eta)\, e^{-\langle w',\,\lambda'\rangle} dw'$$

$$= \int_{\gamma(\eta;\varrho,\theta)} u(z', \Phi''(\Phi'^{-1}_{w''}(z'), w''), \eta)\, e^{-\langle \Phi'^{-1}_{w''}(z'),\,\lambda'\rangle} \det[\partial_{z'}\Phi'^{-1}_{w''}]\, dz'.$$

Let us consider the corresponding generalization of $E_{X,z_0^*}^{\mathbb{R}}$. Hereafter we follow the same notations as those in Section 3. Let X be an n-dimensional complex manifold. Set $X^2 := X \times X$ with a system of local coordinates (z, w) and $\widehat{X}^2 := X^2 \times \mathbb{C}$ with local coordinates (z, w, η). Let $\Delta \subset X^2$ be the diagonal set identified with X and $z_0^* = (z_0; \zeta_0) \in T^*X = T_\Delta^*X^2$ with $\zeta_0 \neq 0$. Let $\{f_1(z), \dots, f_n(z)\}$ be a sequence of holomorphic functions in an open neighborhood of z_0 of X satisfying the conditions:

(1) $df_1(z_0) \wedge \cdots \wedge df_n(z_0) \neq 0$.

(2) We have $f(z_0) = 0$ and ${}^{\mathrm{t}}\big[(\partial f/\partial z)(z_0)\big]\boldsymbol{e} = \zeta_0 \in (T^*X)_{z_0}$ where $f(z) := (f_1(z), \dots, f_n(z))$ and $\boldsymbol{e} := (1, 0, \dots, 0) \in \mathbb{C}^n$.

We denote by $\Xi_\Delta(z_0^*)$ the set of such a sequence. Let $\chi = \{f_1, \dots, f_n\} \in \Xi_\Delta(z_0^*)$ and set $f_{\Delta,i}(z,w) := f_i(z) - f_i(w)$,

$$f_\Delta(z,w) = (f_{\Delta,1}(z,w), \dots, f_{\Delta,n}(z,w)) := (f_{\Delta,1}(z,w), f'_\Delta(z,w)).$$

Define, for an open neighborhood $U \subset X^2$ of (z_0, z_0) and a closed convex cone $L \subset \mathbb{C}$ with $L \subset \{\tau \in \mathbb{C};\ \operatorname{Re}\tau > 0\} \cup \{0\}$,

$$G^\chi_{\Delta,\varrho,L} := \{(z,w) \in X^2;\ \varrho^2 |f'_\Delta(z,w)| \le |f_{\Delta,1}(z,w)|,\ f_{\Delta,1}(z,w) \in L\},$$

and

$$\begin{aligned} \widehat{U}^\chi_{\Delta,\varrho,r,\theta} &:= \{(z,w,\eta) \in U \times S_{r,\theta};\ |f_{\Delta,1}(z,w)| < \varrho|\eta|\}, \\ \widehat{G}^\chi_{\Delta,\varrho,L} &:= \{(z,w,\eta) \in \widehat{X}^2;\ \varrho|f'_\Delta(z,w)| \le |\eta|,\ f_{\Delta,1}(z,w) \in L\}. \end{aligned}$$

We also define

$$\begin{aligned} \widehat{E}^{\mathbb{R},\chi}_{X,z_0^*} &:= \varinjlim_{\varrho,r,\theta,L,U} H^n_{\widehat{G}^\chi_{\Delta,\varrho,L} \cap \widehat{U}^\chi_{\Delta,\varrho,r,\theta}}(\widehat{U}^\chi_{\Delta,\varrho,r,\theta}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2}), \\ E^{\mathbb{R},\chi}_{X,z_0^*} &:= \operatorname{Ker}(\partial_\eta \colon \widehat{E}^{\mathbb{R},\chi}_{X,z_0^*} \to \widehat{E}^{\mathbb{R},\chi}_{X,z_0^*}), \\ M^{\mathbb{R},\chi}_{X,z_0^*} &:= \varinjlim_{\varrho,L,U} H^n_{G_{\Delta,\varrho,L} \cap U}(U; \mathscr{O}^{(0,n)}_{X^2}). \end{aligned}$$

Then we obtain isomorphisms

$$\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \xleftarrow{\sim} M^{\mathbb{R},\chi}_{X,z_0^*} \xrightarrow{\sim} E^{\mathbb{R},\chi}_{X,z_0^*}.$$

Hence, by the same reasoning as that for $\mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*}$, we can introduce the following definition.

Definition B.6. We denote by $E^{\mathbb{R}}_{X,z_0^*}$ (resp. $M^{\mathbb{R}}_{X,z_0^*}$) the isomorphism class $\{E^{\mathbb{R},\chi}_{X,z_0^*}\}_{\chi \in \Xi_\Delta(z_0^*)}$ (resp. $\{M^{\mathbb{R},\chi}_{X,z_0^*}\}_{\chi \in \Xi_\Delta(z_0^*)}$).

We also give Čech representations of these cohomology groups. Set

$$\widehat{U}^\chi_{\Delta,\boldsymbol{\kappa}} := \bigcap_{i=2}^n \{(z,w,\eta) \in \widehat{X}^2;\ \|f(z)\| < r',\ \eta \in S_{r,\theta},\ |f_{\Delta,1}(z,w)| < \varrho|\eta|,\ |f_{\Delta,i}(z,w)| < r'\},$$

$$U^\chi_{\Delta,\boldsymbol{\kappa}} := \bigcap_{i=2}^n \{(z,w) \in X^2;\ \|f(z)\| < r',\ |f_{\Delta,1}(z,w)| < \varrho r,\ |f_{\Delta,i}(z,w)| < r'\}.$$

We also set

$$\begin{aligned} \widehat{V}^{\chi,(1)}_{\Delta,\boldsymbol{\kappa}} &:= \left\{(z,w,\eta) \in \widehat{U}^\chi_{\Delta,\boldsymbol{\kappa}};\ \frac{\pi}{2} - \theta < \arg f_{\Delta,1}(z,w) < \frac{3\pi}{2} + \theta\right\}, \\ \widehat{V}^{\chi,(i)}_{\Delta,\boldsymbol{\kappa}} &:= \{(z,w,\eta) \in \widehat{U}^\chi_{\Delta,\boldsymbol{\kappa}};\ \varrho|f_{\Delta,i}(z,w)| > |\eta|\} \quad (2 \le i \le n), \end{aligned}$$

$$V^{\chi,(1)}_{\Delta,\boldsymbol{\kappa}} := \left\{(z,w) \in U^{\chi}_{\Delta,\boldsymbol{\kappa}};\ \frac{\pi}{2} - \theta < \arg f_{\Delta,1}(z,w) < \frac{3\pi}{2} + \theta\right\},$$
$$V^{\chi,(i)}_{\Delta,\boldsymbol{\kappa}} := \{(z,\, w) \in U^{\chi}_{\Delta,\boldsymbol{\kappa}};\ \varrho^2|f_{\Delta,i}(z,w)| > |f_{\Delta,1}(z,w)|\} \quad (2 \le i \le n).$$

Then we get

$$\widehat{E}^{\mathbb{R},\chi}_X = \varinjlim_{\boldsymbol{\kappa}} \Gamma(\widehat{V}^{\chi,(*)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2}) \Big/ \sum_{\alpha \in \mathcal{P}^{\vee}_n} \Gamma(\widehat{V}^{\chi,(\alpha)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2}),$$
$$E^{\mathbb{R},\chi}_X = \varinjlim_{\boldsymbol{\kappa}} \left\{K \in \Gamma(\widehat{V}^{\chi,(*)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2}) \Big/ \sum_{\alpha \in \mathcal{P}^{\vee}_n} \Gamma(\widehat{V}^{\chi,(\alpha)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2});\ \partial_\eta K = 0\right\},$$
$$M^{\mathbb{R},\chi}_X = \varinjlim_{\boldsymbol{\kappa}} \Gamma(V^{\chi,(*)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n)}_{X^2}) \Big/ \sum_{\alpha \in \mathcal{P}^{\vee}_n} \Gamma(V^{\chi,(\alpha)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n)}_{X^2}).$$

Let $\gamma(z,\eta;\varrho,\theta)$ be an n-dimension real chain in $\mathbb{C}^n$ defined in Section 3. Then we define the n-dimensional real chain in $\mathbb{C}^n$ by

$$\gamma^{\chi}(z,\eta;\varrho,\theta) = f^{-1}_{\Delta,z}(-\gamma(0,\eta;\varrho,\theta)) = f^{\,-1}(\gamma(f(z),\eta;\varrho,\theta)),$$

where $f_{\Delta,z}(w)$ is the morphism $f_{\Delta}(z,w) = f(z) - f(w)$ regarded as a function of w for a fixed z and the orientation of γ^{χ} is induced from that of γ by f^{-1}. Then we can define the mapping $\hat{\sigma}^{\chi}\colon E^{\mathbb{R},\chi}_{X,z_0^*} \to \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*}$ by

$$\hat{\sigma}^{\chi}([K dw])(z,\zeta,\eta) := \int_{\gamma^{\chi}(z,\eta;\varrho,\theta)} K(z,w,\eta)\, e^{\langle w-z,\,\zeta\rangle} dw$$

for $K(z,w,\eta)\, dw \in \Gamma(\widehat{V}^{\chi,(*)}_{\Delta,\boldsymbol{\kappa}};\ \mathscr{O}^{(0,n,0)}_{\widehat{X}^2})$ with a suitable $\boldsymbol{\kappa}$. Similarly we have the mapping $M^{\mathbb{R},\chi}_{X,z_0^*} \to \mathscr{S}_{z_0^*}/\mathscr{N}_{z_0^*}$ by

$$\sigma^{\chi}([K dw])(z,\zeta) := \int_{\gamma^{\chi}(z,\eta_0;\varrho,\theta)} K(z,w)\, e^{\langle w-z,\,\zeta\rangle} dw$$

for $K(z,w)\, dw \in \Gamma(V^{\chi,(*)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n)}_{X^2})$ with a suitable $\boldsymbol{\kappa}$ and a sufficiently small fixed $\eta_0 > 0$. As an immediate consequence of Theorem B.2, we have obtained the following corollary.

Corollary B.7. *There exist the well-defined symbol morphisms*

$$\hat{\sigma}\colon E^{\mathbb{R}}_{X,z_0^*} \to \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*}, \qquad \sigma\colon M^{\mathbb{R}}_{X,z_0^*} \to \mathscr{S}_{z_0^*}/\mathscr{N}_{z_0^*}\,,$$

induced by $\hat{\sigma}^{\chi}$ *and* σ^{χ} *respectively.*

Let us consider a coordinate transformation. Let $z = \Phi(w)$ be a local coordinate transformation of X near $w_0 \in X$ with $z_0 = \Phi(w_0)$. We take (z,z',η) and (w,w',η) as the corresponding systems of local coordinates of $\widehat{X}^2$ respectively and the associated local coordinate transformation $\widehat{\Phi}$ of $\widehat{X}^2$ is defined by $(z, z', \eta) = (\Phi(w), \Phi(w'), \eta)$. Set $w_0^* := (w_0; d\Phi(w_0)(\zeta_0)) \in T^*X$. Let $\chi = \{z_1, \dots, z_n\}$ and $[K dz'] \in E^{\mathbb{R},\chi}_{X,z_0^*}$ with $K(z, z', \eta)\, dz' \in \Gamma(\widehat{V}^{\chi,(*)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2})$ for some $\boldsymbol{\kappa}$. Then, by the same argument as in $C^{\mathbb{R},\chi}_{Y|X,z_0^*}$, we get

$$\widehat{\Phi}^*([K dz']) = [\widehat{\Phi}^*(K dz')] = [K(\Phi(w), \Phi(w'), \eta)\, \det[\partial_{w'}\Phi(w')]\, dw'] \in E^{\mathbb{R},\chi\circ\Phi}_{X,w_0^*}.$$

Hence we have obtained

$$\begin{aligned}\hat{\sigma}(\widehat{\Phi}^*([K dz']))(w, \lambda, \eta) &= \int_{\gamma^{\chi\circ\Phi}(w,\eta;\varrho,\theta)} e^{\langle w'-w,\lambda\rangle} \widehat{\Phi}^*(K dz') \\ &= \int_{\gamma(z,\eta;\varrho,\theta)} K(z, z', \eta)\, e^{\langle \Phi^{-1}(z')-\Phi^{-1}(z),\lambda\rangle} dz'. \qquad \text{(B.4)}\end{aligned}$$

Finally we shall consider the action on $C^{\mathbb{R}}_{Y|X,z_0^*}$ associated with $E^{\mathbb{R}}_{X,z_0^*}$. Let $z_0^* = (z_0; \zeta_0) = (0, z_0''; \zeta_0', 0) \in \dot{T}^*_Y X \subset \dot{T}^*X$, $\chi_C \in \Xi(z_0^*)$ and $\chi_E \in \Xi_\Delta(z_0^*)$. Assume $\chi_C \subset \chi_E$; that is, χ_C and χ_E are given by $\{f_1, \dots, f_d\}$ and $\{f_1, \dots, f_d, \dots, f_n\}$ respectively. Note that, for any $\chi_C \in \Xi(z_0^*)$, we can always find a $\chi_E \in \Xi_\Delta(z_0^*)$ with $\chi_C \subset \chi_E$. Let $[u] \in C^{\mathbb{R},\chi_C}_{Y|X,z_0^*}$ with $u(w, \eta) \in \Gamma(V^{\chi_C,(*)}_{\boldsymbol{\kappa}}; \mathscr{O}_X)$ and $[K dw] \in E^{\mathbb{R},\chi_E}_{X,z_0^*}$ with $K(z, w, \eta)\, dw \in \Gamma(V^{\chi_E,(*)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2})$ for some $\boldsymbol{\kappa}$. Then we have the morphism

$$\mu^{\chi_E} : E^{\mathbb{R},\chi_E}_{X,z_0^*} \underset{\mathbb{C}}{\otimes} C^{\mathbb{R},\chi_C}_{Y|X,z_0^*} \ni [K dw] \otimes [u] \to \left[\int_{\gamma^{\chi_E}(z,\eta;\varrho,\theta)} K(z, w, \eta)\, u(w, \eta)\, dw\right] \in C^{\mathbb{R},\chi_C}_{Y|X,z_0^*},$$

which is well-defined. Indeed, by the coordinate transformation $(\tilde{z}, \tilde{w}) = (f(z), f(w))$, the situation can be reduced to one studied in Section 3.

THEOREM B.8. *The family $\{\mu^{\chi_E}\}_{\chi_E \in \Xi_\Delta(z_0^*)}$ of morphisms constructed above induces the well-defined morphism μ: $E^{\mathbb{R}}_{X,z_0^*} \otimes_{\mathbb{C}} C^{\mathbb{R}}_{Y|X,z_0^*} \to C^{\mathbb{R}}_{Y|X,z_0^*}$. Furthermore μ coincides with the action of $\mathscr{E}^{\mathbb{R}}_{X,z_0^*}$ on $\mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*}$.*

PROOF. It suffices to show that, for $\chi_C \subset \chi_E$, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{E}^{\mathbb{R}}_{X,z_0^*} \underset{\mathbb{C}}{\otimes} \mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*} & \xrightarrow{\mu^c} & \mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*} \\ \downarrow & & \downarrow \\ E^{\mathbb{R},\chi_E}_{X,z_0^*} \underset{\mathbb{C}}{\otimes} C^{\mathbb{R},\chi_C}_{Y|X,z_0^*} & \xrightarrow{\mu^{\chi_E}} & C^{\mathbb{R},\chi_C}_{Y|X,z_0^*}. \end{array}$$

We denote by ι the isomorphism $\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \xrightarrow{\sim} E^{\mathbb{R},\chi_E}_{X,z_0^*}$ and by the same symbol the one $\mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*} \xrightarrow{\sim} C^{\mathbb{R},\chi_C}_{Y|X,z_0^*}$. Let $u(w, \eta) \in \Gamma(V^{\chi_C,(*)}_{\boldsymbol{\kappa}}; \mathscr{O}_X)$ and $K(z, w, \eta)dw \in \Gamma(V^{\chi_E,(*)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{\widehat{X}^2})$ for some $\boldsymbol{\kappa}$. We define the coordinate transformations

$$\Phi(z) = \Phi_1(z) = f^{-1}(\widetilde{z}), \qquad \Phi_2(z,w) = (f^{-1}(\widetilde{z}), f^{-1}(\widetilde{w})),$$
$$\widehat{\Phi}_1(z,\eta) = (f^{-1}(\widetilde{z}),\eta), \qquad \widehat{\Phi}_2(z,w,\eta) = (f^{-1}(\widetilde{z}), f^{-1}(\widetilde{w}),\eta).$$

It follows from the fact $\chi_E \circ \Phi = \{\widetilde{z}_1, \dots, \widetilde{z}_n\}$ and Theorem 3.9 in Section 3 that we have

$$\mu^c(\iota^{-1}(\widehat{\Phi}_2^*[Kdw]) \otimes \iota^{-1}(\widehat{\Phi}_1^*[u])) = \iota^{-1} \circ \mu^{\chi_E \circ \Phi}(\widehat{\Phi}_2^*([Kdw]) \otimes \widehat{\Phi}_1^*([u])).$$

By the coordinate transformation law of the integration, we get

$$\mu^{\chi_E \circ \Phi}(\widehat{\Phi}_2^*([Kdw]) \otimes \widehat{\Phi}_1^*([u])) = \widehat{\Phi}_1^* \circ \mu^{\chi_E}([Kdw] \otimes [u]).$$

Furthermore, it follows from functorial properties that $\iota^{-1} \circ \widehat{\Phi}_k^* = \Phi_k^* \circ \iota^{-1}$ $(k = 1, 2)$ and μ^c and Φ^* commute. Hence we have obtained

$$\Phi_1^* \circ \mu^c(\iota^{-1}([Kdw]) \otimes \iota^{-1}([u])) = \Phi_1^* \circ \iota^{-1} \circ \mu^{\chi_E}([Kdw] \otimes [u]),$$

which implies $\mu^c(\iota^{-1}([Kdw]) \otimes \iota^{-1}([u])) = \iota^{-1} \circ \mu^{\chi_E}([Kdw] \otimes [u])$. This completes the proof. □

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