J. Math. Soc. Japan
Vol. 69, No. 2 (2017) pp. 801–818
doi: 10.2969/jmsj/06920801

Lifting puzzles and congruences of Ikeda and Ikeda–Miyawaki lifts

By Neil Dummigan

(Received Aug. 13, 2015)

Abstract. We show how many of the congruences between Ikeda lifts and non-Ikeda lifts, proved by Katsurada, can be reduced to congruences involving only forms of genus 1 and 2, using various liftings predicted by Arthur's multiplicity conjecture. Similarly, we show that conjectured congruences between Ikeda–Miyawaki lifts and non-lifts can often be reduced to congruences involving only forms of genus 1, 2 and 3.

1. Introduction.

For $k, g \geq 2$ even, let $f \in S_{2k-g}(\mathrm{SL}(2,\mathbb{Z}))$ be a normalised Hecke eigenform. Duke and Imamoglu conjectured the existence of a cuspidal Hecke eigenform $F \in S_k(\mathrm{Sp}_g(\mathbb{Z}))$ (a Siegel modular form of genus g) such that its standard L-function

$$L(s, F, \mathrm{St}) = \zeta(s) \prod_{i=1}^{g} L(f, s + (k - i)).$$

The existence of this F was proved by Ikeda [**14**], who gave its Fourier expansion, and we call it the Ikeda lift. In the case $g = 2$ it was already known, as the Saito–Kurokawa lift. Katsurada [**16**] proved that if $k \geq 2g + 4$ and $q > 2k$ is a prime number such that, for some divisor $\mathfrak{q} \mid q$ in a sufficiently large number field,

$$\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(f, k) \prod_{i=1}^{(g/2)-1} L_{\mathrm{alg}}(2i + 1, f, \mathrm{St})) > 0,$$

then, under certain weak conditions, there is a congruence mod $\mathfrak{q}$ of Hecke eigenvalues, between F and some Hecke eigenform, in the same space $S_k(\mathrm{Sp}_g(\mathbb{Z}))$, that is not an Ikeda lift. Here the L-values have been normalised by dividing them by particular choices of Deligne periods. This generalises his earlier work on congruences for Saito–Kurokawa lifts (for which only the factor $L(f, k)$ appears), and similarly it uses a pullback formula for an Eisenstein series of genus $2g$ to which a certain differential operator has been applied. The L-values arise as factors in a formula for the Petersson norm of F, which had been proved by Kohnen and Skoruppa for Saito–Kurokawa lifts, and for $g > 2$ was conjectured by Ikeda and proved by Katsurada and Kawamura. For $g = 2$, congruences were proved independently by Brown [**5**], who used them to construct elements in Selmer

---

2010 *Mathematics Subject Classification.* Primary 11F46.
Key Words and Phrases. congruences of automorphic forms, Ikeda lift, Ikeda–Miyawaki lift.

groups supporting the Bloch–Kato conjecture applied to the critical value $L(f,k)$, which for $g=2$ is immediately to the right of the central point.

As g increases, the value $s=k$ migrates further and further to the right in the critical range $1 \leq s \leq 2k-g$. (Of course, we must adjust k if we want to keep the weight $2k-g$ the same to look at a fixed f.) Prime divisors of the algebraic parts of these critical values appear as the moduli of congruences conjectured by Harder [**11**], [**26**], which support the Bloch–Kato conjecture for these critical values. These congruences of Hecke eigenvalues involve vector-valued Siegel modular forms of genus 2, and may be viewed as being congruences of Hecke eigenvalues between cuspidal automorphic representations of $\mathrm{GSp}_2(\mathbb{A})$ and representations induced from the Levi subgroup $\mathrm{GL}_1 \times \mathrm{GL}_2$ of the Siegel parabolic subgroup [**3**, Section 7]. The Hecke eigenvalues of these induced representations involve those of f. Faber and van der Geer [**10**] computed many Hecke eigenvalues of vector-valued Siegel modular forms of genus 2, providing numerical evidence for many instances of Harder's conjecture. The original example, with $41 \mid L_{\mathrm{alg}}(f,14)$, for f of weight 22, has been proved by Chenevier and Lannes [**6**].

Prime divisors of $L_{\mathrm{alg}}(2i+1,f,\mathrm{St})$ also appear as moduli of conjectural congruences of Hecke eigenvalues involving only genus 2 forms, in general vector-valued, in fact this applies to $L_{\mathrm{alg}}(r,f,\mathrm{St})$ for all odd r from 3 to $2k-g-1$. The congruences are between cusp forms and Klingen–Eisenstein series, and again may be viewed as being between cuspidal and induced automorphic representations of $\mathrm{GSp}_2(\mathbb{A})$, this time for the Klingen parabolic subgroup [**3**, Section 6]. The first example, for $q=71$ and f of weight 20, was proved by Kurokawa [**20**], and Mizumoto proved a more general result [**22**]. Their work involved scalar-valued forms of genus 2, and the rightmost critical value of $L(s,f,\mathrm{St})$. One deals with critical values further to the left by increasing the "vector part" j of the weight. Satoh proved a congruence mod 343 in a $j=2$ case [**24**], and further instances, for other j, were proved in [**9**].

Poor, Ryan and Yuen [**23**] computed the Euler factors at 2 of the standard L-functions of the seven cuspidal Hecke eigenforms in $S_{16}(\mathrm{Sp}_4(\mathbb{Z}))$ (genus 4). Two of these forms are Ikeda lifts, while another two are lifts of pairs of genus 1 forms, of a type conjectured by Miyawaki and proved by Ikeda. The remaining three were more mysterious, but the Euler 2-factors of their standard L-functions factored in such a way as to suggest that they were lifts of some previously unknown kind. A. Mellit suggested to T. Ibukiyama that one of them should be lifted from a vector-valued Siegel modular form of genus 2, whose spinor L-function would appear in the standard L-function of the lift. Ibukiyama [**12**] then made two conjectures on scalar-valued genus 4 lifts of genus 2 vector-valued forms, in whose standard L-functions the spinor and standard L-functions of the lifted form, respectively, would appear. For the "standard" lift, a genus 1 form is also involved. He checked that these conjectures produce precisely the Euler 2-factors computed by Poor, Ryan and Yuen, and generalised the conjectures to predict scalar-valued lifts, to higher genus, of genus 1 and (vector-valued) genus 2 forms.

Reconsidering Katsurada's congruences between Ikeda lifts and non-Ikeda lifts, the occurrence of the same L-values in conjectural congruences involving only genus 1 and genus 2 forms, and the apparent existence of scalar-valued, higher genus lifts of such forms, suggest the question of whether these things are related. Could the non-Ikeda lifts in Katsurada's congruences actually be lifts of the type proposed by Ibukiyama? For

$L(f,k)$, Ibukiyama's "standard lift" indeed explains Katsurada's congruence as a "lift" of Harder's. If $4 \mid g$ then for $L((g/2)+1, f, \mathrm{St})$ (the factor for $i = g/4$), Ibukiyama's "spinor lift" likewise explains Katsurada's congruence as a lift of a congruence of Kurokawa–Mizumoto type. In fact, generalising the spinor lift to lift the genus 1 form as well as a genus 2 form, we may similarly account for congruences involving $L(2i+1, f, \mathrm{St})$, for $g/4 \leq i \leq (g/2)-1$, i.e. for about half the values of i.

We consider also congruences between Ikeda–Miyawaki lifts and non-Ikeda–Miyawaki lifts, conjectured by Ibukiyama, Katsurada, Poor and Yuen [**13**]. They could be proved in the same manner as those between Ikeda lifts and non-Ikeda-lifts, if one knew a conjecture of Ikeda on the Petersson norm of an Ikeda–Miyawaki lift. The moduli are large prime divisors of $L_{\mathrm{alg}}(f \otimes \mathrm{Sym}^2 h, 2k+2n) \prod_{i=1}^{n-1} L_{\mathrm{alg}}(2i+1, f, \mathrm{St})$, where f and h are genus 1 forms of weights $2k$ and $k+n+1$ respectively, and the Ikeda–Miyawaki lift is of genus $2n+1$, weight $k+n+1$. Again, it appears that in many cases the non-Ikeda–Miyawaki lift should in fact be some other kind of lift. For $L_{\mathrm{alg}}(f \otimes \mathrm{Sym}^2 h, 2k+2n)$ we "lift" a genus 3 generalisation of Harder's conjecture, worked out by Harder himself in collaboration with the authors of [**4**], in which it is Conjecture 10.8. Their computations of genus 3 Hecke eigenvalues, together with L-value approximations by Mellit (subsequently confirmed by exact computations in [**13**]), provided numerical support for their conjecture in seventeen cases. For $L_{\mathrm{alg}}(2i+1, f, \mathrm{St})$, with $\lceil n/2 \rceil \leq i \leq n-1$, we again lift congruences of Kurokawa–Mizumoto type.

We may now appear to have a proliferation of unsupported conjectures on the existence of various lifts. But we show how they all fit into Arthur's endoscopic classification of the discrete spectrum of $\mathrm{Sp}_g(\mathbb{Q}) \backslash \mathrm{Sp}_g(\mathbb{A})$, and would be consequences of his conjectural multiplicity formula. Actually, for certain groups including Sp_g, Arthur has proved a version of his multiplicity formula [**1**, Theorem 1.5.2]. But its equivalence to the version applied here is dependent on an as-yet unproved equivalence between two ways of defining and parametrising an L-packet at ∞, as explained following [**7**, Conjecture 3.23][1].

After preliminaries on Arthur's endoscopic classification and multiplicity formula, in Sections 3 and 4, we apply them in Section 5 to obtain all the various lifts (including those of Ikeda and Ikeda–Miyawaki), conditional on the as yet unproved multiplicity formula. The compatibility of the Ikeda lift with Arthur's conjecture was already mentioned in [**14**, Section 14], and Ibukiyama looked at the Arthur parameters of his proposed lifts in [**12**, Section 3.4], without checking the multiplicity formula. In Section 6 we look at the congruences between Ikeda lifts and non-Ikeda lifts proved by Katsurada, and those between Ikeda–Miyawaki lifts and non-Ikeda–Miyawaki lifts conjectured in [**13**]. Finally, in Section 7 we describe in more detail how some of these congruences can be accounted for in the manner indicated above.

The Hecke algebra for Siegel modular forms of genus g is generated by Hecke operators for each prime p, traditionally denoted $T(p)$ and $T_i(p^2)$ for $1 \leq i \leq g$. Strictly speaking, our approach only accounts for congruences between Hecke eigenvalues for the $T_i(p^2)$, not the $T(p)$. This is because we produce Arthur parameters for $G = \mathrm{Sp}_g$ (with $\hat{G} = \mathrm{SO}(g+1, g)$) rather than for $G = \mathrm{GSp}_g$ (with $\hat{G} = \mathrm{Spin}(g+1, g)$). The Siegel

[1] A proof of this equivalence has now appeared in a preprint of Arancibia, Moeglin and Renard, so the constructions in this paper are now unconditional.

modular forms we consider are all eigenforms for the $T(p)$ as well as the $T_i(p^2)$, but we cannot deduce from this the congruence of the $T(p)$ Hecke eigenvalues.

2. Symplectic and special orthogonal groups.

Let $G = \mathrm{Sp}_g = \{h \in M_{2g} : {}^t hJh = J\}$, where

$$J_{i,2g+1-i} = \begin{cases} 1 & \text{if } 1 \leq i \leq g; \\ -1 & \text{if } g+1 \leq i \leq 2g, \end{cases}$$

and all other entries are 0. It has a maximal torus T comprising elements of the form $\mathrm{diag}(t_1, \dots, t_g, t_g^{-1}, \dots, t_1^{-1})$, which is mapped to t_i by characters e_i, for $1 \leq i \leq g$, which span the character group $X^*(T)$. The cocharacter group $X_*(T)$ is spanned by $\{f_1, \dots, f_g\}$, where $f_1 : t \mapsto \mathrm{diag}(t, 1, \dots, 1, t^{-1})$, etc. so $\langle e_i, f_j \rangle = \delta_{ij}$. We can order the roots so that the positive roots are $\Phi_G^+ = \{e_i - e_j : i < j\} \cup \{2e_i : 1 \leq i \leq g\} \cup \{e_i + e_j : i < j\}$, and the simple roots $\Delta_G = \{e_1 - e_2, e_2 - e_3, \dots, e_{g-1} - e_g, 2e_g\}$. The simple coroots (in order) are $\{f_1 - f_2, \dots, f_{g-1} - f_g, f_g\}$.

Let $\hat{G} = \mathrm{SO}(g+1, g) = \{h \in M_{2g+1} : {}^t h\tilde{J}h = \tilde{J}, \det(h) = 1\}$, with

$$\tilde{J}_{i,2g+2-i} = \begin{cases} 1 & \text{if } i \neq g+1; \\ 2 & \text{if } i = g+1, \end{cases}$$

and all other entries 0. It has a maximal torus $\hat{T}$ comprising elements of the form $\mathrm{diag}(t_1, \dots, t_g, 1, t_g^{-1}, \dots, t_1^{-1})$, which is mapped to t_i by characters $\tilde{e}_i$, for $1 \leq i \leq g$, which span $X^*(\hat{T})$. The cocharacter group $X_*(\hat{T})$ is spanned by $\{\tilde{f}_1, \dots, \tilde{f}_g\}$, where $\tilde{f}_1 : t \mapsto \mathrm{diag}(t, 1, \dots, 1, t^{-1})$, etc. so $\langle \tilde{e}_i, \tilde{f}_j \rangle = \delta_{ij}$. We can order the roots so that $\Phi_{\hat{G}}^+ = \{\tilde{e}_i - \tilde{e}_j : i < j\} \cup \{\tilde{e}_i : 1 \leq i \leq g\} \cup \{\tilde{e}_i + \tilde{e}_j : i < j\}$, and $\Delta_{\hat{G}} = \{\tilde{e}_1 - \tilde{e}_2, \tilde{e}_2 - \tilde{e}_3, \dots, \tilde{e}_{g-1} - \tilde{e}_g, \tilde{e}_g\}$. The simple coroots (in order) are $\{\tilde{f}_1 - \tilde{f}_2, \dots, \tilde{f}_{g-1} - \tilde{f}_g, 2\tilde{f}_g\}$. Note that for any root β with coroot $\check{\beta}$, we have $\langle \beta, \check{\beta} \rangle = 2$.

We see then that the root systems of G and $\hat{G}$ are dual to each other, so $\hat{G}$ is, as the notation indicates, the Langlands dual of G. The isomorphisms $X^*(\hat{T}) \simeq X_*(T)$ and $X^*(T) \simeq X_*(\hat{T})$ are such that $\tilde{e}_i \leftrightarrow f_i$ and $e_i \leftrightarrow \tilde{f}_i$, respectively.

Let $\mathfrak{H}_g$ be the Siegel upper half space of g by g complex symmetric matrices with positive-definite imaginary part. For $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_g(\mathbb{Z})$ and $Z \in \mathfrak{H}_g$, let $M\langle Z \rangle := (AZ+B)(CZ+D)^{-1}$ and $J(M, Z) := CZ+D$. Let V be the space of a representation ρ of $\mathrm{GL}(g, \mathbb{C})$. A holomorphic function $f : \mathfrak{H}_g \to V$ is said to belong to the space $M_\rho(\mathrm{Sp}_g(\mathbb{Z}))$ of Siegel modular forms of genus g and weight ρ if

$$f(M\langle Z \rangle) = \rho(J(M, Z))f(Z) \quad \forall M \in \mathrm{Sp}_g(\mathbb{Z}), Z \in \mathfrak{H}_g,$$

and, in the case $g = 1$, if it is holomorphic at the cusps. If $g > 1$, the Siegel operator Φ on $M_\rho(\mathrm{Sp}_g(\mathbb{Z}))$ is defined by

$$\Phi f(z) = \lim_{t\to\infty} f\left(\begin{bmatrix} z & 0 \\ 0 & it \end{bmatrix}\right) \quad \text{for } z \in \mathfrak{H}_{g-1}, t \in \mathbb{R}.$$

The kernel of Φ, denoted $S_\rho(\mathrm{Sp}_g(\mathbb{Z}))$, is the space of Siegel cusp forms of genus g and weight ρ. When $\rho = \det^k$, the forms are scalar valued, of weight k, and $S_\rho(\mathrm{Sp}_g(\mathbb{Z}))$ is denoted $S_k(\mathrm{Sp}_g(\mathbb{Z}))$.

3. Arthur's endoscopic classification.

Let $G = \mathrm{Sp}_g$ as above, so $\hat{G} = \mathrm{SO}(g+1, g)$. Let $\mathrm{St} : \hat{G} \to \mathrm{SL}(2g+1)$ be the standard inclusion homomorphism. Let $\mathcal{X}(\hat{G})$ be the set of (c_v), indexed by places v of $\mathbb{Q}$, such that for finite p, c_p is a semisimple conjugacy class in $\hat{G}(\mathbb{C})$, and c_∞ is a semisimple conjugacy class in $\mathrm{Lie}(\hat{G}(\mathbb{C}))$. Let $\Pi(G)$ be the set of irreducible representations π of $G(\mathbb{A})$ such that π_∞ is unitary and each π_p, for finite primes p, is smooth and unramified, i.e. has a non-zero $G(\mathbb{Z}_p)$-fixed vector. Let $\Pi_{\mathrm{disc}}(G)$ be the subset of those occurring discretely in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Given $\pi \in \Pi_{\mathrm{disc}}(G)$, let $c(\pi) = (c_v(\pi))$, where for finite p, $c_p(\pi)$ is the Satake parameter of π_p, and $c_\infty(\pi)$ is the infinitesimal character of π_∞. We may do something similar with G replaced by $\mathrm{PGL}(m)$ and $\hat{G}$ by $\widehat{\mathrm{PGL}(m)} = \mathrm{SL}(m)$, or with G replaced by $\mathrm{SO}(g+1, g)$ and $\hat{G}$ by Sp_g, $\mathrm{St} : \mathrm{Sp}_g \to \mathrm{SL}(2g)$, or with G and $\hat{G}$ both replaced by $\mathrm{SO}(g, g)$, $\mathrm{St} : \mathrm{SO}(g, g) \to \mathrm{SL}(2g)$.

As an example, if π_f is the cuspidal automorphic representation of $\mathrm{PGL}(2)(\mathbb{A})$ associated with a normalised, cuspidal Hecke eigenform $f = \sum_{n=1}^\infty a_n q^n$ of weight k for $\mathrm{SL}(2, \mathbb{Z})$, then $c_p(\pi_f) = \mathrm{diag}(\alpha_p, \alpha_p^{-1})$, where $a_p = p^{(k-1)/2}(\alpha_p + \alpha_p^{-1})$, and $c_\infty(\pi_f) = \mathrm{diag}((k-1)/2, -(k-1)/2)$. We have $L(f, s + (k-1)/2) = \prod_p \det(I - c_p(\pi_f)p^{-s})^{-1}$. In this example we may also think of $\mathrm{PGL}(2)$ as $\mathrm{SO}(2, 1)$, and $\mathrm{SL}(2)$ as $\widehat{\mathrm{SO}(2, 1)} = \mathrm{Sp}_1$. If instead we consider the cuspidal automorphic representation π_f^{st} of $\mathrm{Sp}_1(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A})$ associated with f then $c_p(\pi_f^{\mathrm{st}}) = \mathrm{diag}(\alpha_p^2, 1, \alpha_p^{-2}) \in \mathrm{SO}(2, 1)(\mathbb{C})$, and $\prod_p \det(I - \mathrm{St}(c_p(\pi_f^{\mathrm{st}}))p^{-s})^{-1}$ is the standard L-function $L(s, f, \mathrm{St}) = L(s + (k-1), \mathrm{Sym}^2 f)$, while $c_\infty(\pi_f^{\mathrm{st}}) = \mathrm{diag}(k-1, 0, 1-k)$, which can be thought of as $(k-1)e_1$.

By Arthur's symplectic-orthogonal alternative [**7**, Theorem* 3.9], given any $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}(m))$ (the subset of cuspidal representations in $\Pi_{\mathrm{disc}}(\mathrm{PGL}(m))$), there is a

$$G^\pi = \begin{cases} \mathrm{Sp}_{(m-1)/2} & \text{if } m \text{ is odd;} \\ \mathrm{SO}(m/2, m/2) \text{ or } \mathrm{SO}((m/2)+1, m/2) & \text{if } m \text{ is even,} \end{cases}$$

and $\pi' \in \pi_{\mathrm{disc}}(G^\pi)$ such that $c(\pi) = \mathrm{St}(c(\pi'))$.

Following [**7**, Section 3.11] (where more generally G is a classical semisimple group over $\mathbb{Z}$), let $\Psi_{\mathrm{glob}}(G)$ be the set of quadruples $(k, (n_i), (d_i), (\pi_i))$, where $1 \leq k \leq 2g+1$, k an integer, $n_i \geq 1$ are integers with $\sum_{i=1}^k n_i = 2g+1$, $d_i \mid n_i$ and each $\pi_i \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}(n_i/d_i))$ is a self-dual, cuspidal, automorphic representation of $\mathrm{PGL}(n_i/d_i)(\mathbb{A})$. There are two conditions:

1. if $(n_i, d_i) = (n_j, d_j)$ with $i \neq j$, then $\pi_i \neq \pi_j$;
2. d_i is odd if $\widehat{G^{\pi_i}}$ is orthogonal, while d_i is even if $\widehat{G^{\pi_i}}$ is symplectic.

An element $\psi \in \Psi_{\text{glob}}(G)$ is called a global Arthur parameter. We write

$$\underline{\psi} = \pi_1[d_1] \oplus \pi_2[d_2] \oplus \cdots \oplus \pi_k[d_k],$$

where there is an equivalence relation, such that for the equivalence class $\underline{\psi}$ of ψ the order of the summands is unimportant. If π_i is the trivial representation we just write $[d_i]$ for $\pi_i[d_i]$, and we just write π_i for $\pi_i[1]$.

To a global Arthur parameter $\psi \in \Psi_{\text{glob}}(G)$, we associate a homomorphism

$$\rho_\psi : \prod_{i=1}^{k} (\mathrm{SL}(n_i/d_i) \times \mathrm{SL}(2)) \to \mathrm{SL}_{2g+1},$$

well-defined up to conjugation in $\mathrm{SL}_{2g+1}(\mathbb{C})$, namely $\bigoplus_{i=1}^{k}(\mathbb{C}^{n_i/d_i} \otimes \mathrm{Sym}^{d_i-1}(\mathbb{C}^2))$. Hence we get a map

$$\rho_\psi : \prod_{i=1}^{k} (\mathcal{X}(\mathrm{SL}(n_i/d_i)) \times \mathcal{X}(\mathrm{SL}(2))) \to \mathcal{X}(\mathrm{SL}_{2g+1}).$$

Let $e = c(1) \in \mathcal{X}(\mathrm{SL}(2))$, where $1 \in \Pi_{\text{disc}}(\mathrm{PGL}(2))$ is the trivial representation. Then $e_p = \mathrm{diag}(p^{1/2}, p^{-1/2})$ and $e_\infty = (1/2, -1/2)$.

THEOREM 3.1. (*Arthur's Endoscopic Classification* [**7**, Theorem* 3.12], [**1**, Theorem 1.5.2]). *Given* $\pi \in \Pi_{\text{disc}}(G)$, *there is* $\psi(\pi) \in \Psi_{\text{glob}}(G)$ (*the global Arthur parameter of* π) *such that*

$$\mathrm{St}(c(\pi)) = \rho_{\psi(\pi)}\left(\prod_{i=1}^{k} c(\pi_i) \times e\right).$$

As an example, if π_f is the cuspidal automorphic representation of $\mathrm{PGL}(2)(\mathbb{A})$ associated with a normalised, cuspidal Hecke eigenform $f = \sum_{n=1}^{\infty} a_n q^n$ of weight $2k-2$ for $\mathrm{SL}(2,\mathbb{Z})$, with k even, if F, a cusp form of weight k for $\mathrm{Sp}_2(\mathbb{Z})$, is the Saito–Kurokawa lift of f, and if π_F is the associated cuspidal automorphic representation of $\mathrm{Sp}_2(\mathbb{A})$, then $\psi(\pi_F) = \pi_f[2] \oplus [1]$, with $c_\infty(\pi_F) = \mathrm{diag}(k-1, k-2, 0, 2-k, 1-k)$, $c_p(\pi_F) = \mathrm{diag}(\alpha_p p^{1/2}, \alpha_p p^{-1/2}, 1, \alpha_p^{-1} p^{1/2}, \alpha_p^{-1} p^{-1/2})$ and standard L-function $L(s, F, \mathrm{St}) = \prod_p (\det(I - \mathrm{St}(c_p(\pi_F))p^{-s}))^{-1} = \zeta(s) L(f, s + (k-1)) L(f, s + (k-2))$.

At this point we should say a little more about the relation between Siegel modular forms and automorphic representations. Asgari and Schmidt [**2**] describe how to get a cuspidal automorphic representation π'_F of $\mathrm{PGSp}_g(\mathbb{A})$, holomorphic discrete series at ∞, from a Hecke eigenform F in $S_k(\mathrm{Sp}_g(\mathbb{Z}))$, with $k \geq g+1$, and something similar works for vector-valued forms [**25**, Section 5.2]. From this π'_F one can get a cuspidal automorphic representation π_F of $\mathrm{Sp}_g(\mathbb{A})$, whose Satake parameters are obtained from those of π'_F by applying the 2-to-1 covering map from $\mathrm{Spin}(g+1, g)$ to $\mathrm{SO}(g+1, g)$. Conversely, given $\pi \in \Pi_{\text{disc}}(\mathrm{Sp}_g)$ with $c_\infty(\pi) = \mathrm{diag}(k-1, \ldots, k-g, 0, g-k, \ldots, 1-k)$ and π_∞ holomorphic discrete series, it comes from some $\pi' \in \Pi_{\text{disc}}(\mathrm{PGSp}_g(\mathbb{A}))$ (by [**7**, Proposition 4.7]), which is actually in $\Pi_{\text{cusp}}(\mathrm{PGSp}_g(\mathbb{A}))$ (by [**25**, Remark 5.2.3]). This is then of the form π'_F for some Hecke eigenform (for the $T(p)$ as well as the $T_i(p^2)$) $F \in S_k(\mathrm{Sp}_g(\mathbb{Z}))$, as explained in [**25**, Section 5.2].

4. Arthur's multiplicity formula.

Closely related to ρ_ψ above is

$$r_\psi : \prod_{i=1}^{k} (\widehat{G^{\pi_i}} \times \mathrm{SL}(2)) \to \widehat{G} = \mathrm{SO}(g+1, g).$$

Then $\mathrm{St} \circ r_\psi$ is a direct sum $\oplus_{i=1}^k V_i$, where V_i is an irreducible n_i-dimensional representation of $\widehat{G^{\pi_i}} \times \mathrm{SL}(2)$. Following [**7**, Section 3.20], let C_ψ be the centraliser of $\mathrm{im}(r_\psi)$ in $\hat{G}$. This is an elementary abelian 2-group generated by $Z(\hat{G})$ and elements s_i for those i such that n_i is even, where $\mathrm{St}(s_i)$ acts as -1 on V_i, and as $+1$ on V_j for all $j \neq i$.

Arthur [**1**] defined a character $\epsilon_\psi : C_\psi \to \{\pm 1\}$, where ϵ_ψ is trivial on $Z(\hat{G})$ and

$$\epsilon_\psi(s_i) = \prod_{j \neq i} \epsilon(\pi_i \times \pi_j)^{\min(d_i, d_j)},$$

$\epsilon(\pi_i \times \pi_j) = \pm 1$ being the global epsilon factor appearing in the functional equation of $L(s, \pi_i \times \pi_j)$, which in our case, where $\pi_i \times \pi_j$ will be unramified at all finite primes, is just the local factor $\epsilon_\infty(\pi_i \times \pi_j)$.

Given $\pi \in \Pi(G)$ such that $c(\pi) = \psi \in \Psi_{\mathrm{alg}}$ (a certain subset of $\Psi_{\mathrm{glob}}(G)$, see [**7**, Definition 3.15]), we can ask whether π actually occurs in $\Pi_{\mathrm{disc}}(G)$. Arthur's multiplicity conjecture answers this question. The answer depends on comparing ϵ_ψ with another character which depends on how all the π_p and π_∞ sit in their L-packets. Since all the π_p are unramified, their L-packets are trivial, i.e. they are uniquely determined up to isomorphism by their $c_p(\pi)$. Therefore we only need consider π_∞, which we want to be the holomorphic discrete series representation within its L-packet. There is an associated Shelstad parameter $\chi_{\mathrm{hol}} : C_{\psi_\infty} \to \mathbb{C}^\times$, where C_{ψ_∞} is a certain group which can be viewed as a 2-torsion subgroup of $\hat{T}$, such that $C_\psi \subseteq C_{\psi_\infty}$, and the requirement of Arthur's multiplicity formula is that $\chi_{\mathrm{hol}}|_{C_\psi} = \epsilon_\psi$. By [**7**, Lemma 9.3], χ_{hol} is the restriction of either $\sum_{\mathrm{odd}\ i=1}^{g} \tilde{e}_i$ or $\sum_{\mathrm{even}\ i=1}^{g} \tilde{e}_i \in X^*(\hat{T})$, and the restrictions to C_ψ are the same [**7**, Lemma 9.5], so we act as if $\chi_{\mathrm{hol}} = \sum_{\mathrm{odd}\ i=1}^{g} \tilde{e}_i$. Note that although C_ψ and C_{ψ_∞} are only well-defined up to conjugacy, there is a natural way of viewing one inside the other, compatible with the above view of C_{ψ_∞} inside $\hat{T}[2]$, and the explicit realisation in $\hat{T}[2]$ of the various $s_i \in C_\psi$ in the proofs in the next section.

5. Application to various lifts.

All the propositions in this section are conditional upon Arthur's multiplicity conjecture.

5.1. Ikeda lifts.

For $k, g \geq 2$ even, and $f \in S_{2k-g}(\mathrm{SL}(2, \mathbb{Z}))$ a Hecke eigenform, let π_f be the associated cuspidal, automorphic representation of $\mathrm{PGL}(2)(\mathbb{A})$, and consider $\pi_f[g] \oplus [1] \in \Psi_{\mathrm{alg}}$.

PROPOSITION 5.1. *There exists $\pi \in \Pi_{\rm disc}(\mathrm{Sp}_g)$ such that $\psi(\pi) = \pi_f[g] \oplus [1]$.*

PROOF. Since $n_1 = 2g$ is even, but $n_2 = 1$ is odd, C_ψ is generated by $Z(\hat{G})$ and $s_1 =: s_f$. We have $\epsilon_\psi(s_f) = \epsilon_\infty(\pi_f \times 1)^1 = \epsilon_\infty(\pi_f)$. Note that $c_\infty(\pi_f) = \mathrm{diag}((2k-g-1)/2, (1-g-2k)/2)$. The associated motive (twisted to have weight 0) would have Hodge type $\{(p,q),\ (q,p)\}$, with $p = (1-g-2k)/2$ and $q = (2k-g-1)/2$. Putting this in the formula i^{q-p+1} in the table in [**8**, Section 5.3], we recover the well-known $\epsilon_\infty(\pi_f) = i^{2k-g} = (-1)^{k-(g/2)} = (-1)^{g/2}$. Of course we would have to make a half-integral twist to really have a motive, with integral Hodge weights, but since we are only interested in the difference $q-p$, we can ignore this.

On the other hand $\chi_{\rm hol} = \tilde{e}_1 + \cdots + \tilde{e}_{g-1}$ (odd subscripts), which has $g/2$ terms, and $s_f = \mathrm{diag}(\underbrace{-1,\ldots,-1}_{g \text{ times}}, 1, \underbrace{-1,\ldots,-1}_{g \text{ times}})$, so $\chi_{\rm hol}(s_f) = (-1)^{g/2}$. Since this is the same as $\epsilon_\psi(s_f)$, π exists. □

Note that $c_\infty(\pi) = \mathrm{diag}(k-1, k-2, \ldots, k-g, 0, g-k, \ldots, 2-k, 1-k)$ matches $c_\infty(\pi_F)$, where π_F is the automorphic representation of $\mathrm{Sp}_g(\mathbb{A})$ associated with a cuspidal Hecke eigenform $F \in S_k(\mathrm{Sp}_g(\mathbb{Z}))$, and since π_∞ is holomorphic discrete series, π is of the form π_F. From $\psi(\pi_F)$ we can read off the standard L-function $L(s, F, \mathrm{St}) = \zeta(s)\prod_{i=1}^g L(f, s+(k-i))$, and we recognise F as the Ikeda lift of f [**14**].

5.2. Standard lifts.

Let k, g, f be as in the previous section, and let F be a cuspidal Hecke eigenform for $\mathrm{Sp}_2(\mathbb{Z})$, of weight $\det^\kappa \otimes \mathrm{Sym}^j(\mathbb{C}^2)$, with $(\kappa, j) = (k-g+2, g-2)$ (so we must impose $k > g-2$). To F we associate an automorphic representation $\pi_F^{\rm st}$ of $\mathrm{Sp}_2(\mathbb{A})$, with $c_\infty(\pi_F) = \mathrm{diag}(j+\kappa-1, \kappa-2, 0, 2-\kappa, 1-j-\kappa) = \mathrm{diag}(k-1, k-g, 0, g-k, 1-k)$. To get $\mathrm{diag}(k-1, k-2, \ldots, k-g, 0, g-k, \ldots, 2-k, 1-k)$ (seen in the previous section) from $\mathrm{diag}(k-1, k-g, 0, g-k, 1-k)$, we need to fill in the gaps using $(g-2)$ copies of $c_\infty(\pi_f) = \mathrm{diag}((2k-g-1)/2, (1-g-2k)/2)$, shifted to left and right. So we consider $\psi = \pi_F^{\rm st} \oplus \pi_f[g-2] \in \Psi_{\rm alg}$. Note that we have abused notation somewhat; $\pi_F^{\rm st}$ is a representation of $\mathrm{Sp}_2(\mathbb{A})$, but we are using the same notation for its lift to $\mathrm{PGL}(5)(\mathbb{A})$, via $\mathrm{St} : \mathrm{SO}(3,2) \to \mathrm{SL}(5)$. We must insist that we are in a situation where this lift is cuspidal, so we must exclude the case where $g = 2$ and F is a Saito–Kurokawa lift. (Similar remarks apply in subsequent sections.) In fact, we may as well exclude the case $g = 2$, in which F is already scalar-valued, and π below would be just the same as $\pi_F^{\rm st}$.

PROPOSITION 5.2. *There exists $\pi \in \Pi_{\rm disc}(\mathrm{Sp}_g)$ such that $\psi(\pi) = \pi_F^{\rm st} \oplus \pi_f[g-2]$.*

PROOF. Since $n_1 = 5$ is odd, but $n_2 = 2(g-2)$ is even, C_ψ is generated by $Z(\hat{G})$ and $s_2 =: s_f$. We have $\epsilon_\psi(s_f) = \epsilon_\infty(\pi_f \times \pi_F^{\rm st})^1 = \epsilon_\infty(\pi_f \times \pi_F^{\rm st})$. Since $c_\infty(\pi_f) = \mathrm{diag}((2k-g-1)/2, (1-g-2k)/2)$ and $c_\infty(\pi_F) = \mathrm{diag}(k-1, k-g, 0, g-k, 1-k)$, the associated motive (twisted to have weight 0) would have Hodge type a union of $\{(-q,q),\ (q,-q)\}$, where $2q$ runs through $\{2k-g-1+2(k-1) = 4k-g-3, 4k-3g-1, 2k-g-1, g-1, g-1\}$. Putting this in the formula $i^{q-p+1} = i^{2q+1}$, we find that

$$\epsilon_\infty(\pi_f \times \pi_F^{\rm st}) = i^{4k-g-2+4k-3g+2k-g+g+g} = i^{g+2} = (-1)^{(g/2)+1}.$$

On the other hand $s_f = \mathrm{diag}(1, \underbrace{-1,\dots,-1}_{g-2 \text{ times}}, 1, 1, 1 \underbrace{-1,\dots,-1}_{g-2 \text{ times}}, 1)$. In the left half, $(g/2)-1$ of the -1s are in odd position, so $\chi_{\mathrm{hol}}(s_f) = (-1)^{(g/2)+1}$. Since this is the same as $\epsilon_\psi(s_f)$, π exists. □

As already noted, $c_\infty(\pi) = \mathrm{diag}(k-1, k-2, \dots, k-g, 0, g-k, \dots, 2-k, 1-k)$, so as in the previous section $\pi = \pi_G$ for some cuspidal Hecke eigenform $G \in S_k(\mathrm{Sp}_g(\mathbb{Z}))$. This time $L(s, G, \mathrm{St}) = L(s, F, \mathrm{St}) \prod_{i=1}^{g-2} L(f, s + (k-g+i))$. The existence of such a G is precisely [**12**, Conjecture 3.2].

5.3. Spinor lifts.

Now $k, g \geq 2$ even, $f \in S_{2k-g}(\mathrm{SL}(2,\mathbb{Z}))$, and F is a cuspidal Hecke eigenform for $\mathrm{Sp}_2(\mathbb{Z})$, of weight $\det^\kappa \otimes \mathrm{Sym}^j(\mathbb{C}^2)$, with $(\kappa, j) = (r+1, 2k-g-1-r)$ (so we impose $k > (g/2)+r+1$), for some fixed odd r with $(g/2)+1 \leq r < g$. To F we associate an automorphic representation π_F^{spin} of $\mathrm{PGSp}_2(\mathbb{A}) \simeq \mathrm{SO}(3,2)(\mathbb{A})$, with

$$\begin{aligned} c_\infty(\pi_F^{\mathrm{spin}}) &= \mathrm{diag}\left(\frac{j+2\kappa-3}{2}, \frac{j+1}{2}, -\frac{j+1}{2}, -\frac{j+2\kappa-3}{2}\right) \\ &= \mathrm{diag}\left(\frac{2k-g+r-2}{2}, \frac{2k-g-r}{2}, -\frac{2k-g-r}{2}, -\frac{2k-g+r-2}{2}\right). \end{aligned}$$

Then

$$\begin{aligned} &c_\infty(\pi_F^{\mathrm{spin}}[g+1-r]) \\ &\quad = \mathrm{diag}(k-1, \dots, k+r-g-1, k-r, \dots, k-g, g-k, \dots, r-k, 1+g-r-k, \dots, 1-k), \end{aligned}$$

where the dots denote unbroken sequences of consecutive integers. To fill in the gaps we use $\pi_f[2r-g-2]$, then to put 0 in the middle we use [1]. Thus

$$\begin{aligned} &c_\infty(\pi_F^{\mathrm{spin}}[g+1-r] \oplus \pi_f[2r-g-2] \oplus [1]) \\ &\quad = \mathrm{diag}\left(k-1, k-2, \dots, k-g, 0, g-k, \dots, 2-k, 1-k\right). \end{aligned}$$

Note that since $r > 2$ and $j > 0$, there are no entries in $c_\infty(\pi_F^{\mathrm{spin}})$ differing by 1, so in the Arthur parameter of π_F^{spin}, all $d_i = 1$. The possibility that π_F^{spin} is endoscopic is ruled out, since there are no holomorphic Yoshida lifts at level 1. Hence the lift of π_F^{spin} to $\mathrm{PGL}(4)(\mathbb{A})$, which is what is really meant above by π_F^{spin}, must be cuspidal, as desired.

Proposition 5.3. *If $4 \mid g$, there exists $\pi \in \Pi_{\mathrm{disc}}(\mathrm{Sp}_g)$ such that $\psi(\pi) = \pi_F^{\mathrm{spin}}[g+1-r] \oplus \pi_f[2r-g-2] \oplus [1]$.*

Proof. This time $n_1 = 4(g+1-r)$ and $n_2 = 2(2r-g-2)$ are even, while $n_3 = 1$ is odd, so we must consider $s_1 =: s_F$ and $s_2 =: s_f$. Since $\widehat{G^{\pi_f}}$ and $\widehat{G^{\pi_F^{\mathrm{spin}}}}$ are both symplectic, it follows from a theorem of Arthur (see [**7**, Section 3.20]) that $\epsilon(\pi_f \times \pi_F^{\mathrm{spin}}) = 1$. Hence $\epsilon_\psi(s_f) = \epsilon_\infty(\pi_f \times 1)^1 = \epsilon_\infty(\pi_f) = (-1)^{g/2}$ as before, and likewise $\epsilon_\psi(s_F) = \epsilon_\infty(\pi_F^{\mathrm{spin}}) = i^{(2k-g-r+1)+(2k-g+r-1)} = (-1)^{g/2}$.

$$s_f = \mathrm{diag}(\underbrace{1,\dots,1}_{g+1-r},\underbrace{-1,\dots,-1}_{2r-g-2},\underbrace{1,\dots,1}_{2g+3-2r},\underbrace{-1,\dots,-1}_{2r-g-2},\underbrace{1,\dots,1}_{g+1-r}),$$

and on the left side the number of -1s in odd position is $r-(g/2)-1$, so $\chi_{\mathrm{hol}}(s_f) = (-1)^{r-(g/2)-1} = (-1)^{g/2}$, since r is odd.

$$s_F = \mathrm{diag}(\underbrace{-1,\dots,-1}_{g+1-r},\underbrace{1,\dots,1}_{2r-g-2},\underbrace{-1,\dots,-1}_{g+1-r},1,\underbrace{-1,\dots,-1}_{g+1-r},\underbrace{1,\dots,1}_{2r-g-2},\underbrace{-1,\dots,-1}_{g+1-r}),$$

and on the left side the number of -1s in odd position is $g+1-r$, which is even, so $\chi_{\mathrm{hol}}(s_F) = 1$. Thus, though $\chi_{\mathrm{hol}}(s_f) = \epsilon_\psi(s_f)$, for $\chi_{\mathrm{hol}}(s_F) = \epsilon_\psi(s_F)$ we need the condition $4 \mid g$. □

As already noted, $c_\infty(\pi) = \mathrm{diag}(k-1, k-2, \dots, k-g, 0, g-k, \dots, 2-k, 1-k)$, so as before, $\pi = \pi_G$ for some cuspidal Hecke eigenform $G \in S_k(\mathrm{Sp}_g(\mathbb{Z}))$. This time $L(s, G, \mathrm{St}) = \zeta(s)\prod_{i=1}^{g+1-r} L(s-i+(g-r+2)/2, F, \mathrm{spin})\prod_{i=1}^{2r-g-2} L(f, s+(k-r+i))$, where the spinor L-function is in its automorphic normalisation, centred at $s=1/2$. In the special case $r=(g/2)+1$ (in which case f does not actually appear), the existence of such a G is precisely [**12**, Conjecture 3.1].

5.4. Ikeda–Miyawaki lifts.

Consider Hecke eigenforms $f \in S_{2k}(\mathrm{SL}(2,\mathbb{Z}))$, $h \in S_{k+n+1}(\mathrm{SL}(2,\mathbb{Z}))$, where $k+n+1$ is even. Let π_f be the associated cuspidal, automorphic representation of $\mathrm{PGL}(2)(\mathbb{A})$, and π_h^{st} the cuspidal automorphic representation of $\mathrm{Sp}_1(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A})$ associated with h. Recall that $c_p(\pi_h^{\mathrm{st}}) = \mathrm{diag}(\alpha_p^2, 1, \alpha_p^{-2}) \in \mathrm{SO}(2,1)(\mathbb{C})$ (where $a_p(h) = p^{(k+n)/2}(\alpha_p + \alpha_p^{-1})$), and $c_\infty(\pi_h^{\mathrm{st}}) = \mathrm{diag}(k+n, 0, -k-n)$. Since $c_\infty(\pi_f) = \mathrm{diag}((2k-1)/2, (1-2k)/2)$, we see that $c_\infty(\pi_h^{\mathrm{st}} \oplus \pi_f[2n]) = \mathrm{diag}(k+n, \dots, k-n, 0, n-k, \dots, -n-k)$, where the dots denote unbroken sequences of consecutive integers. This is of the form $\mathrm{diag}\,(\kappa-1, \kappa-2, \dots, \kappa-g, 0, g-\kappa, \dots, 2-\kappa, 1-\kappa)$, where $\kappa = k+n+1$ and $g = 2n+1$.

PROPOSITION 5.4. *There exists $\pi \in \Pi_{\mathrm{disc}}(\mathrm{Sp}_{2n+1})$ such that $\psi(\pi) = \pi_h^{\mathrm{st}} \oplus \pi_f[2n]$.*

PROOF. Since $n_1 = 3$ is odd, while $n_2 = 4n$ is even, we consider $s_2 =: s_f$. First, $\epsilon_\psi(s_f) = \epsilon_\infty(\pi_h^{\mathrm{st}} \times \pi_f)$. The associated motive (twisted to have weight 0) would have Hodge type a union of $\{(-q,q),\ (q,-q)\}$, where $2q$ runs through $\{2k-1+2(k+n) = 4k+2n-1, 2k-1, 2n+1\}$. Putting this in the formula $i^{q-p+1} = i^{2q+1}$, we find that

$$\epsilon_\infty(\pi_f) = i^{4k+2n+2k+2n+2} = i^{2k+2} = (-1)^{k+1}.$$

Now $s_f = \mathrm{diag}(1, \underbrace{-1,\dots,-1}_{2n}, 1, \underbrace{-1,\dots,-1}_{2n}, 1)$, and in the left half, n of the -1s are in odd position, so $\chi_{\mathrm{hol}}(s_f) = (-1)^n$, which is the same as $(-1)^{k+1}$, since $n+k+1$ is even. □

As already noted, $c_\infty(\pi) = \mathrm{diag}(k+n, \dots, k-n, 0, n-k, \dots, -n-k)$, so $\pi = \pi_G$ for some cuspidal Hecke eigenform $G \in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbb{Z}))$. Also $L(s, G, \mathrm{St}) =$

$L(s,h,\mathrm{St})\prod_{i=1}^{2n}L(f,s+(k-n-1+i))$, and we recognise G as a lift whose existence was conjectured by Miyawaki and proved by Ikeda [**21**], [**15**].

5.5. Lifts from genus 3 and 1.

Let f be as in the previous section, with $k+n+1$ still even. Let F be a vector-valued cuspidal Hecke eigenform of genus 3 such that if π_F^{st} is the associated automorphic representation of $\mathrm{Sp}_3(\mathbb{A})$ then $c_\infty(\pi_F^{\mathrm{st}})=\mathrm{diag}(k+n,k+n-1,k-n,0,n-k,-n-k+1,-n-k)$. In the language of [**4**, Sections 4.1, 7], $(a,b,c)=(k+n-3,k+n-3,k-n-1)$. To fill in the gaps of length $2n-2$, we consider $\psi=\pi_F^{\mathrm{st}}\oplus\pi_f[2n-2]$. We may as well exclude the case $n=1$, in which F is already scalar-valued and π below would be just the same as π_F^{st}.

Proposition 5.5. *There exists $\pi\in\Pi_{\mathrm{disc}}(\mathrm{Sp}_{2n+1})$ such that $\psi(\pi)=\pi_F^{\mathrm{st}}\oplus\pi_f[2n-2]$.*

Proof. Since $n_1=7$ is odd, while $n_2=4n-4$ is even, we consider $s_2=:s_f$.

$$\epsilon_\psi(s_f)=\epsilon_\infty(\pi_F^{\mathrm{st}}\times\pi_f)=i^{(4k+2n)+(4k+2n-2)+(4k-2n)+2k+2n+(2n+2)+2n}=i^{2k}=(-1)^k.$$
$$s_f=\mathrm{diag}(1,1,\underbrace{-1,\dots,-1}_{2n-2},1,0,1,\underbrace{-1,\dots,-1}_{2n-2},1,1),$$

with $n-1$ of -1s in the left half in odd position, so $\chi_{\mathrm{hol}}(s_F)=(-1)^{n-1}$, which is the same as $(-1)^k$, since $k+n+1$ is even. □

As before, $\pi=\pi_G$ for some cuspidal Hecke eigenform $G\in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbb{Z}))$. We read off $L(s,G,\mathrm{St})=L(s,F,\mathrm{St})\prod_{i=1}^{2n-2}L(f,s+k-n+i)$.

5.6. Lifts from genus 1, 2 and 1.

As in Section 5.4, consider Hecke eigenforms $f\in S_{2k}(\mathrm{SL}(2,\mathbb{Z}))$, $h\in S_{k+n+1}(\mathrm{SL}(2,\mathbb{Z}))$, where $k+n+1$ is even. Let π_f be the associated cuspidal, automorphic representation of $\mathrm{PGL}(2)(\mathbb{A})$, and π_h^{st} the cuspidal automorphic representation of $\mathrm{Sp}_1(\mathbb{A})=\mathrm{SL}_2(\mathbb{A})$ associated with h. Let F be a cuspidal Hecke eigenform for $\mathrm{Sp}_2(\mathbb{Z})$, of weight $\det^\kappa\otimes\mathrm{Sym}^j(\mathbb{C}^2)$, with $(\kappa,j)=(r+1,2k-1-r)$, for some fixed odd r with $n+1\le r\le 2n-1$. To F we associate an automorphic representation π_F^{spin} of $\mathrm{PGSp}_2(\mathbb{A})\simeq\mathrm{SO}(3,2)(\mathbb{A})$, with

$$\begin{aligned}c_\infty(\pi_F^{\mathrm{spin}})&=\mathrm{diag}\left(\frac{j+2\kappa-3}{2},\frac{j+1}{2},-\frac{j+1}{2},-\frac{j+2\kappa-3}{2}\right)\\&=\mathrm{diag}\left(\frac{2k+r-2}{2},\frac{2k-r}{2},-\frac{2k-r}{2},-\frac{2k+r-2}{2}\right).\end{aligned}$$

Then

$$\begin{aligned}&c_\infty(\pi_F^{\mathrm{spin}}[2n+1-r])\\&\quad=\mathrm{diag}(k+n-1,\dots,k+r-n-1,k+n-r,\dots,k-n,\\&\qquad\qquad n-k,\dots,r-n-k,1+n-r-k,\dots,1-k-n),\end{aligned}$$

where the dots denote unbroken sequences of consecutive integers. To fill in the gaps we use $\pi_f[2r-2n-2]$, and we also add $c_\infty(\pi_h^{\mathrm{st}}) = \mathrm{diag}(k+n, 0, -n-k)$. Thus

$$\begin{aligned} &c_\infty(\pi_h^{\mathrm{st}} \oplus \pi_F^{\mathrm{spin}}[2n+1-r] \oplus \pi_f[2r-2n-2]) \\ &\quad = \mathrm{diag}\,(k+n, k+n-1, \dots, k-n, 0, n-k, \dots, 1-n-k, -n-k)\,. \end{aligned}$$

PROPOSITION 5.6. *There exists $\pi \in \Pi_{\mathrm{disc}}(\mathrm{Sp}_{2n+1})$ such that $\psi(\pi) = \pi_h^{\mathrm{st}} \oplus \pi_F^{\mathrm{spin}}[2n+1-r] \oplus \pi_f[2r-2n-2]$.*

PROOF. This time $n_2 = 4(2n+1-r)$ and $n_3 = 2(2r-2n-2)$ are even, while $n_1 = 3$ is odd, so we must consider $s_2 =: s_F$ and $s_3 =: s_f$. Since $\widehat{G^{\pi_f}}$ and $\widehat{G^{\pi_F^{\mathrm{spin}}}}$ are both symplectic, it follows from a theorem of Arthur (see [**7**, Section 3.20]) that $\epsilon(\pi_f \times \pi_F^{\mathrm{spin}}) = 1$. Hence

$$\epsilon_\psi(s_f) = \epsilon_\infty(\pi_f \times \pi_h^{\mathrm{st}})^1 = i^{2k+(2n+2)+(4k+2n)} = (-1)^{k+1},$$

and likewise

$$\begin{aligned} \epsilon_\psi(s_F) &= \epsilon_\infty(\pi_F^{\mathrm{spin}} \times \pi_h^{\mathrm{st}}) \\ &= i^{(2k+r-1)+(2k-r+1)+(2n+r+1)+(2n-r+3)+(4k+2n+r-1)+(4k+2n-r+1)} = 1. \\ s_f &= \mathrm{diag}(\underbrace{1,\dots,1}_{2n+1-r}, \underbrace{-1,\dots,-1}_{2r-2n-2}, \underbrace{1,\dots,1}_{4n+3-2r}, \underbrace{-1,\dots,-1}_{2r-2n-2}, \underbrace{1,\dots,1}_{2n+1-r}), \end{aligned}$$

and on the left side the number of -1s in odd position is $r-n-1$, so $\chi_{\mathrm{hol}}(s_f) = (-1)^{r-n-1} = (-1)^n$, since r is odd. This is the same as $(-1)^{k+1}$, since $n+k+1$ is even.

$$s_F = \mathrm{diag}(\underbrace{-1,\dots,-1}_{2n+1-r}, \underbrace{1,\dots,1}_{2r-2n-2}, \underbrace{-1,\dots,-1}_{2n+1-r}, 1, \underbrace{-1,\dots,-1}_{2n+1-r}, \underbrace{1,\dots,1}_{2r-2n-2}, \underbrace{-1,\dots,-1}_{2n+1-r}),$$

and on the left side the number of -1s in odd position is $2n+1-r$, which is even, so $\chi_{\mathrm{hol}}(s_F) = 1$. □

We have $\pi = \pi_G$ for some cuspidal Hecke eigenform $G \in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbb{Z}))$, and we get $L(s, G, \mathrm{St})$

$$= L(s, h, \mathrm{St}) \prod_{i=1}^{2n+1-r} L\left(s + \frac{2n-r}{2} + 1 - i, F, \mathrm{spin}\right) \prod_{j=1}^{2r-2n-2} L(f, s+k+n-r+j).$$

Note that in the case $r = n+1$, f does not appear.

6. Congruences between lifts and "non-lifts".

6.1. Congruences between Ikeda lifts and non-Ikeda lifts.

The following is Theorem 4.7 of [**16**]. The proof makes use of the proof by Katsurada and Kawamura [**18**] of a conjecture of Ikeda on the Petersson norm of his lift. The normalised L-values $L_{\mathrm{alg}}(f, k)$ and $L_{\mathrm{alg}}(2i+1, f, \mathrm{St})$ are obtained from $L(f, k)$ and $L(2i+$

$1, f, \mathrm{St})$ by dividing by suitably normalised Deligne periods, as explained in [**3**, Section 4]. For $L_{\mathrm{alg}}(f,k)$, the Deligne period is as constructed in [**16**, Section 4], using parabolic cohomology with integral coefficients. (Since $q > 2k$, we may ignore various factorials of small numbers.) For $L_{\mathrm{alg}}(2i+1, f, \mathrm{St})$ it is essentially a product $\Omega^+\Omega^-$ of normalised Deligne periods for $L(f,s)$ [**9**, Lemma 5.1], but given the condition (2) below, this is as good as the $\langle f, f\rangle$ used by Katsurada (see condition (3) in [**16**, Theorem 4.7]).

THEOREM 6.1. *For $k, g \geq 2$ even, and $f \in S_{2k-g}(\mathrm{SL}(2,\mathbb{Z}))$ a Hecke eigenform, let $F \in S_k(\mathrm{Sp}_g(\mathbb{Z}))$ be the Ikeda lift, as in Section 5.1 above. Suppose that $k \geq 2g+4$ and that $q > 2k$ is a prime number such that, for some divisor $\mathfrak{q} \mid q$ in a sufficiently large number field,*

$$\mathrm{ord}_{\mathfrak{q}}\Big(L_{\mathrm{alg}}(f,k)\prod_{i=1}^{(g/2)-1} L_{\mathrm{alg}}(2i+1,f,\mathrm{St})\Big) > 0.$$

Suppose further that

1. *for some even integer t with $k+2 \leq t \leq 2k-2g-2$, and some fundamental discriminant D with $(-1)^{g/2}D > 0$,*

$$\mathrm{ord}_{\mathfrak{q}}\left(\frac{\zeta(t+g-k)}{\pi^{t+g-k}}\left(\prod_{i=1}^{g} L_{\mathrm{alg}}(f,t+i-1)\right)L_{\mathrm{alg}}(f,(k-2g)/2,\chi_D)D\right) = 0,$$

   where χ_D is the associated quadratic character, and the Dirichlet L-value is normalised as in [**16**];

2. *there is not a congruence mod $\mathfrak{q}$ of Hecke eigenvalues between f and another Hecke eigenform in $S_{2k-g}(\mathrm{SL}(2,\mathbb{Z}))$;*

3. *if $g > 2$, $q \nmid \prod_{p \leq (2k-g)/12, p\ prime}(1+p+p^2+\cdots+p^{g-1})$.*

Then there exists a Hecke eigenform $G \in S_k(\mathrm{Sp}_g(\mathbb{Z}))$, not the Ikeda lift of any Hecke eigenform $h \in S_{2k-g}(\mathrm{SL}(2,\mathbb{Z}))$, such that for any prime p, corresponding Hecke eigenvalues for F and G, for all the Hecke operators $T(p)$ and $T_i(p^2)$ $(1 \leq i \leq g)$, are congruent mod $\mathfrak{q}$.

Ikeda proved only that F is a Hecke eigenform for the $T_i(p^2)$ (defined in [**16**, Section 2]), which generate a Hecke algebra associated with the pair $(\mathrm{Sp}_g(\mathbb{Q}_p), \mathrm{Sp}_g(\mathbb{Z}_p))$, but Katsurada [**16**, Proposition 4.1] extended this to $T(p)$, which with the $T_i(p^2)$ generates a Hecke algebra associated with $(\mathrm{GSp}_g(\mathbb{Q}_p), \mathrm{GSp}_g(\mathbb{Z}_p))$. (See also the final paragraph of Section 3 above.) If we ignore the $T(p)$ then the congruence in the theorem is equivalent to a congruence (for all p) of Satake parameters

$$c_p(\pi_F) \equiv c_p(\pi_G) \pmod{\mathfrak{q}},$$

(or strictly speaking $p^{kg-g(g+1)/2}c_p(\pi_F) \equiv p^{kg-g(g+1)/2}c_p(\pi_G) \pmod{\mathfrak{q}}$), with

$$c_p(\pi_F) = \mathrm{diag}(\alpha_{1,F},\ldots,\alpha_{g,F},1,\alpha_{g,F}^{-1},\ldots,\alpha_{1,F}^{-1}) \in \hat{T}(\mathbb{C}),$$

and likewise for G. We should interpret the congruence as being between $c_p(\pi_F)$ and some element in the orbit of $c_p(\pi_G)$ under the action of a Weyl group that can permute the indices $1,\dots,g$ and switch pairs $\alpha_{i,F}$ and $\alpha_{i,F}^{-1}$, in fact $c_p(\pi_F)$ really should be thought of as a conjugacy class in $\hat{G}(\mathbb{C})$, represented by the above element of $\hat{T}(\mathbb{C})$. To include $T(p)$ as well, we would need to consider also $\alpha_{0,F}$ with $\alpha_{0,F}^2\prod_{i=1}^g \alpha_{i,F}=1$, for each p.

6.2. Congruences between Ikeda–Miyawaki lifts and non-Ikeda–Miyawaki lifts.

The following is taken from Conjecture B and Problem B' of [**13**], which are inspired by a conjecture of Ikeda on the Petersson norm of the Ikeda–Miyawaki lift. The normalised L-values $L_{\mathrm{alg}}(2i+1,f,\mathrm{St})$ are as above. The meaning of $L_{\mathrm{alg}}(f\otimes \mathrm{Sym}^2 h, 2k+2n)$ in [**13**] is left a little vague. In theory we take it as in [**3**, Section 4]. Ibukiyama, Katsurada, Poor and Yuen use a practical substitute when they prove an instance of the congruence in [**13**, Section 5].

CONJECTURE 6.2. *For Hecke eigenforms $f\in S_{2k}(\mathrm{SL}(2,\mathbb{Z}))$, $h\in S_{k+n+1}(\mathrm{SL}(2,\mathbb{Z}))$, where $k+n+1$ is even, let $F\in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbb{Z}))$ be the Ikeda–Miyawaki lift, as in Section 5.4. Suppose that $q>2k+2n-2$ is a prime number such that, for some divisor $\mathfrak{q}\mid q$ in a sufficiently large number field,*

$$\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(f\otimes \mathrm{Sym}^2 h, 2k+2n)\prod_{i=1}^{n-1} L_{\mathrm{alg}}(2i+1,f,\mathrm{St}))>0.$$

Then there exists a Hecke eigenform $G\in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbb{Z}))$, not the Ikeda–Miyawaki lift of any Hecke eigenforms $f'\in S_{2k}(\mathrm{SL}(2,\mathbb{Z}))$, $h'\in S_{k+n+1}(\mathrm{SL}(2,\mathbb{Z}))$, such that for any prime p, corresponding Hecke eigenvalues for F and G, for all the Hecke operators $T(p)$ and $T_i(p^2)$ $(1\le i\le g)$, are congruent mod $\mathfrak{q}$.

Remarks about congruences of Satake parameters, similar to the previous subsection, apply.

7. Accounting for some of the congruences.

7.1. Ikeda lifts and standard lifts: $L_{\mathrm{alg}}(f,k)$.

We have $2k-g=j+2\kappa-2$, $k=j+\kappa$, if $(\kappa,j)=(k+2-g,g-2)$, in agreement with Section 5.2 above. Harder's conjecture [**11**], [**26**] may be formulated, given $\mathfrak{q}\mid q$ with $q>2k-g$ and $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(f,k))>0$, as the existence of a Hecke eigenform F for $\mathrm{Sp}_2(\mathbb{Z})$, of weight $\det^{\kappa}\otimes\mathrm{Sym}^j(\mathbb{C}^2)$, such that if π_F^{st} is the associated automorphic representation of $\mathrm{Sp}_2(\mathbb{A})$ then for all primes p,

$$c_p(\pi_F^{\mathrm{st}})\equiv \mathrm{diag}(\alpha_p p^{(g-1)/2},\alpha_p p^{(1-g)/2},1,\alpha_p^{-1}p^{(g-1)/2},\alpha_p^{-1}p^{(1-g)/2}) \pmod{\mathfrak{q}},$$

where $c_p(\pi_f)=\mathrm{diag}(\alpha_p,\alpha_p^{-1})$. The $(g-1)/2=(j+1)/2$ is what we called s in [**3**]. Note that if we let $\alpha_{1,F}=\alpha_p p^s$, $\alpha_{2,F}=\alpha_p p^{-s}$ and $\alpha_{0,F}=\alpha_p^{-1}$ (so $\alpha_0^2\alpha_1\alpha_2=1$) then

$$\alpha_{0,F}+\alpha_{0,F}\alpha_{1,F}+\alpha_{0,F}\alpha_{2,F}+\alpha_{0,F}\alpha_{1,F}\alpha_{2,F}=\alpha_p+\alpha_p^{-1}+p^{-s}+p^s,$$

which when scaled by $p^{(j+2\kappa-3)/2}$ gives the familiar $a_p(f) + p^{\kappa-2} + p^{j+\kappa-1}$ on the right hand side of Harder's conjecture (as a Hecke eigenvalue for $T(p)$ on an induced representation). For simplicity we actually ignore $T(p)$, and consider only the Hecke algebra generated by $T_1(p^2)$ and $T_2(p^2)$. This is because we are looking at an automorphic representation of $\mathrm{Sp}_2(\mathbb{A})$ rather than of $\mathrm{GSp}_2(\mathbb{A})$. In [**3**, Section 7], we looked at Harder's conjecture as a congruence of Hecke eigenvalues between a cuspidal automorphic representation of $\mathrm{GSp}_2(\mathbb{A})$ and a representation induced from the Levi subgroup $(\mathrm{GL}_1 \times \mathrm{GL}_2)(\mathbb{A})$ of the Siegel maximal parabolic (and worked it out explicitly only for $T(p)$). Here we can either restrict to $\mathrm{Sp}_2(\mathbb{A})$ or just consider directly Sp_2 with the Levi subgroup $\mathrm{GL}_1 \times \mathrm{SL}_2$ of its Siegel parabolic.

Now

$$c_p(\pi_f[g]) = \mathrm{diag}(\alpha_p p^{(g-1)/2}, \alpha_p p^{(g-3)/2}, \dots, \alpha_p p^{(1-g)/2}, \alpha_p^{-1} p^{(g-1)/2}, \dots, \alpha_p^{-1} p^{(1-g)/2}),$$

and

$$c_p(\pi_f[g-2]) = \mathrm{diag}(\alpha_p p^{(g-3)/2}, \dots, \alpha_p p^{(3-g)/2}, \alpha_p^{-1} p^{(g-3)/2}, \dots, \alpha_p^{-1} p^{(3-g)/2}),$$

so the congruence can be read as

$$c_p(\pi_F^{\mathrm{st}} \oplus \pi_f[g-2]) \equiv c_p(\pi_f[g] \oplus [1]) \pmod{\mathfrak{q}}.$$

Comparing with Section 5.1 and Section 5.2, we see that in the case of $\mathfrak{q} \mid L_{\mathrm{alg}}(f,k)$, we can explain the congruence between the Ikeda lift and a non-Ikeda lift in Theorem 6.1 as a congruence between the Ikeda lift and a "standard lift" as constructed in Section 5.2. So the congruence in Theorem 6.1 is derived from that in Harder's conjecture via lifting to scalar-valued large genus forms. In the excluded case $g = 2$, Harder's conjecture is replaced by its degeneration, a congruence between a Saito–Kurokawa lift and non-lift, which does not require further lifting.

7.2. Ikeda lifts and spinor lifts: $L_{\mathrm{alg}}(2i+1, f, \mathrm{St})$.

If $r = 2i+1$ then as i runs from 1 to $(g/2) - 1$, r runs through odd numbers from 3 to $g-1$. We shall only be able to account for the congruence in Conjecture 6.1 if $4 \mid g$ and $(g/2)+1 \le r \le g-1$. We also require $q > 4k - 2g$. Let $(\kappa, j) = (r+1, 2k-g-1-r)$, so $\kappa + j = 2k - g$ and $r = s+1$, where $s = \kappa - 2$ as in [**3**, Section 6]. Then a conjectural congruence of Kurokawa–Mizumoto type (instances of which were proved in [**9**], [**20**], [**22**], [**24**]) may be formulated, given $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(r, f, \mathrm{St})) > 0$, as the existence of a Hecke eigenform F for $\mathrm{Sp}_2(\mathbb{Z})$, of weight $\det^\kappa \otimes \mathrm{Sym}^j(\mathbb{C}^2)$, such that if π_F^{spin} is the associated automorphic representation of $\mathrm{SO}(3,2)(\mathbb{A})$ then for all primes p,

$$c_p(\pi_F^{\mathrm{spin}}) \equiv \mathrm{diag}(\alpha_p p^{s/2}, \alpha_p p^{-s/2}, \alpha_p^{-1} p^{s/2}, \alpha_p^{-1} p^{-s/2}) \pmod{\mathfrak{q}},$$

where $c_p(\pi_f) = \mathrm{diag}(\alpha_p, \alpha_p^{-1})$. Note that the trace of the right hand side, when scaled by $p^{(j+2\kappa-3)/2}$, becomes the familiar $a_p(f)(1 + p^{\kappa-2})$. Recalling that $s = r-1$, this would imply that $c_p(\pi_F^{\mathrm{spin}}[g+1-r])$

$$\equiv \mathrm{diag}(\alpha_p p^{(g-1)/2}, \dots, \alpha_p p^{(2r-g-1)/2}, \alpha_p p^{(1+g-2r)/2}, \dots, \alpha_p p^{(1-g)/2},$$

$$\alpha_p^{-1}p^{(g-1)/2},\dots,\alpha_p^{-1}p^{(2r-g-1)/2},\alpha_p^{-1}p^{(1+g-2r)/2},\dots,\alpha_p^{-1}p^{(1-g)/2}).$$

The right hand side is the "difference" between $c_p(\pi_f[g])$ and $c_p(\pi_f[2r-g-2])$. Thus we can read the congruence as

$$c_p(\pi_F^{\mathrm{spin}}[g+1-r]\oplus\pi_f[2r-g-2]\oplus[1])\equiv c_p(\pi_f[g]\oplus[1]),$$

i.e. as a congruence between the Ikeda lift and one of the "spinor lifts" constructed in Section 5.3. In the case of $\mathfrak{q}\mid L_{\mathrm{alg}}(2i+1,f,\mathrm{St})$, with $4\mid g$, $g/4\le i\le(g/2)-1$ and $q>4k-2g$, we can explain the congruence between the Ikeda lift and a non-Ikeda lift in Theorem 6.1 (at least if we ignore $T(p)$) as a congruence between the Ikeda lift and a spinor lift. Thus the congruence in Theorem 6.1 is derived from that of Kurokawa–Mizumoto type via lifting to scalar-valued, large genus forms. Note that we have had to impose a stronger lower bound for q.

7.3. Ikeda–Miyawaki lifts: $L_{\mathrm{alg}}(f\otimes\mathrm{Sym}^2h,2k+2n)$.

Recall that we consider Hecke eigenforms $f\in S_{2k}(\mathrm{SL}(2,\mathbb{Z}))$, $h\in S_{k+n+1}(\mathrm{SL}(2,\mathbb{Z}))$, where $k+n+1$ is even. Let $a_p(f)=p^{(2k-1)/2}(\alpha_p+\alpha_p^{-1})$ and $b_p(h)=p^{(k+n)/2}(\beta_p+\beta_p^{-1})$. Let $(a,b,c)=(k+n-3,k+n-3,k-n-1)$, as in Section 5.5 above. Then $b+c+4=2k$, $a+4=k+n+1$ (the weights of f and h), $a+b+6=2k+2n$ and $s:=(b-c+1)/2=(2n-1)/2$. Comparing with [**3**, Section 8, Case 2], the conjecture there (see also [**4**, Conjecture 10.8]), given $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(f\otimes\mathrm{Sym}^2h,2k+2n)>0$ with $q>a+b+2c+8=4k$, can be formulated (ignoring $T(p)$) as the existence of a cuspidal Hecke eigenform F for $\mathrm{Sp}_3(\mathbb{Z})$, vector-valued of type (a,b,c), such that

$$c_p(\pi_F^{\mathrm{st}})\equiv\mathrm{diag}(\alpha_pp^s,\alpha_p^{-1}p^s,\beta_p^2,1,\beta_p^{-2},\alpha_pp^{-s},\alpha_p^{-1}p^{-s})\pmod{\mathfrak{q}}.$$

To get the diagonal entries, apply the cocharacters $f_1,f_2,f_3,0,-f_3,-f_2,-f_1$ to $\chi_p+s\tilde{\alpha}=-\log_p(\alpha_p)(e_1-e_2)-\log_p(\beta_p)+s(e_1+e_2)$ in [**3**, Section 8], omitting e_0 since we are really dealing with $G=\mathrm{Sp}_3$, $M\simeq\mathrm{GL}_2\times\mathrm{SL}_2$.

Since $c_p(\pi_h^{\mathrm{st}})=\mathrm{diag}(\beta_p^2,1,\beta_p^{-2})$, and since $s=(2n-1)/2$, we can read this as

$$c_p(\pi_F^{\mathrm{st}}\oplus\pi_f[2n-2])\equiv c_p(\pi_h^{\mathrm{st}}\oplus\pi_f[2n])\pmod{\mathfrak{q}},$$

i.e. as a congruence between the Ikeda–Miyawaki lift and one of the lifts constructed in Section 5.5. Thus the congruence in Conjecture 6.2, between the Ikeda–Miyawaki lift and a non-Ikeda–Miyawaki lift, can be derived from the conjectured genus 3 Eisenstein congruence, via lifting to scalar-valued, large genus forms. Again, we have had to impose a stronger lower bound for q. In the excluded case $n=1$, the Eisenstein congruence degenerates to a congruence between an Ikeda–Miyawaki lift and a non-Ikeda–Miyawaki lift, without any further lifting.

7.4. Ikeda–Miyawaki lifts: $L_{\mathrm{alg}}(2i+1,f,\mathrm{St})$.

If $r=2i+1$ then as i runs from 1 to $n-1$, r runs through odd numbers from 3 to $2n-1$. We shall only be able to account for the congruence in Theorem 6.2 if $n+1\le r\le 2n-1$. We also require $q>4k$. Let $(\kappa,j)=(r+1,2k-1-r)$, so $\kappa+j=2k$ and $r=s+1$, where $s=\kappa-2$ as in [**3**, Section 6]. Then a conjecture of Kurokawa–

Mizumoto type, given $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(r,f,\mathrm{St}))>0$, predicts the existence of a Hecke eigenform F for $\mathrm{Sp}_2(\mathbb{Z})$, of weight $\det^{\kappa}\otimes\mathrm{Sym}^j(\mathbb{C}^2)$, such that if π_F^{spin} is the associated automorphic representation of $\mathrm{SO}(3,2)(\mathbb{A})$ then for all primes p,

$$c_p(\pi_F^{\mathrm{spin}})\equiv \mathrm{diag}(\alpha_p p^{s/2},\alpha_p p^{-s/2},\alpha_p^{-1}p^{s/2},\alpha_p^{-1}p^{-s/2}) \pmod{\mathfrak{q}},$$

where $c_p(\pi_f)=\mathrm{diag}(\alpha_p,\alpha_p^{-1})$. Recalling that $s=r-1$, this would imply that

$$\begin{aligned}&c_p(\pi_F^{\mathrm{spin}}[2n+1-r])\\ &\quad\equiv \mathrm{diag}(\alpha_p p^{(2n-1)/2},\dots,\alpha_p p^{(2r-2n-1)/2},\alpha_p p^{(1+2n-2r)/2},\dots,\alpha_p p^{(1-2n)/2},\\ &\qquad\quad \alpha_p^{-1}p^{(2n-1)/2},\dots,\alpha_p^{-1}p^{(2r-2n-1)/2},\alpha_p^{-1}p^{(1+2n-2r)/2},\dots,\alpha_p^{-1}p^{(1-2n)/2}).\end{aligned}$$

The right hand side is the “difference” between $c_p(\pi_f[2n])$ and $c_p(\pi_f[2r-2n-2])$. Thus we can read the congruence as

$$c_p(\pi_h^{\mathrm{st}}\oplus\pi_F^{\mathrm{spin}}[2n+1-r]\oplus\pi_f[2r-2n-2])\equiv c_p(\pi_h^{\mathrm{st}}\oplus\pi_f[2n]),$$

i.e. as a congruence between the Ikeda–Miyawaki lift and one of the lifts constructed in Section 5.6. In the case of $\mathfrak{q}\mid L_{\mathrm{alg}}(2i+1,f,\mathrm{St})$, with $\lceil n/2\rceil\le i\le n-1$ and $q>4k$, we can explain the congruence between the Ikeda–Miyawaki lift and a non-Ikeda–Miyawaki lift in Conjecture 6.2 (at least if we ignore $T(p)$) as a congruence between the Ikeda–Miyawaki lift and a lift from Section 5.6. Thus the congruence in Conjecture 6.2 is derived from that of Kurokawa–Mizumoto type via lifting to scalar-valued, large genus forms. Again, we have had to impose a stronger lower bound for q.

References

[1] J. Arthur, The endoscopic classification of representations. Orthogonal and symplectic groups, Amer. Math. Soc. Colloquium Publications, **61**, Amer. Math. Soc., Providence, RI, 2013, xviii+590 pp.

[2] M. Asgari and R. Schmidt, Siegel modular forms and representations, Manuscripta Math., **104** (2001), 173–200.

[3] J. Bergström and N. Dummigan, Eisenstein congruences for split reductive groups, Selecta Math., **22** (2016), 1073–1115.

[4] J. Bergström, C. Faber and G. van der Geer, Siegel modular forms of degree three and the cohomology of local systems, Selecta Math. (N.S.), **20** (2014), 83–124.

[5] J. Brown, Saito–Kurokawa lifts and applications to the Bloch–Kato conjecture, Compos. Math., **143** (2007), 290–322.

[6] G. Chenevier and C. Lannes, Formes automorphes et voisins de Kneser des réseaux de Niemeier, 388 pages, arXiv:1409.7616, 26/09/2014.

[7] G. Chenevier and D. Renard, Level one algebraic cuspforms of classical groups of small rank, Mem. Amer. Math. Soc., **237** (2015), no. 1121.

[8] P. Deligne, Valeurs de Fonctions L et Périodes d'Intégrales, AMS Proc. Symp. Pure Math., **33** (1979), part 2, 313–346.

[9] N. Dummigan, Symmetric square L-functions and Shafarevich–Tate groups, II, Int. J. Number Theory, **5** (2009), 1321–1345.

[10] C. Faber and G. van der Geer, Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes, I, II, C. R. Math. Acad. Sci. Paris, **338** (2004), 381–384 and 467–470.

[11] G. Harder, A congruence between a Siegel and an elliptic modular form, manuscript, 2003, reproduced in The 1-2-3 of Modular Forms (ed. K. Ranestad), 247–262, Springer-Verlag, Berlin Heidelberg, 2008.

[12] T. Ibukiyama, Lifting conjectures from vector valued Siegel modular forms of degree two, Comment. Math. Univ. St. Pauli, **61** (2012), 87–102.

[13] T. Ibukiyama, H. Katsurada, C. Poor and D. Yuen, Congruences to Ikeda–Miyawaki lifts and triple L-values of elliptic modular forms, J. Number Theory, **134** (2014), 142–180.

[14] T. Ikeda, On the lifting of elliptic cusp forms to Siegel cusp forms of degree 2n, Ann. of Math. (2), **154** (2001), 641–681.

[15] T. Ikeda, Pullback of lifting of elliptic cusp forms and Miyawaki's conjecture, Duke Math. J., **131** (2006), 469–497.

[16] H. Katsurada, Congruence between Duke–Imamoglu–Ikeda lifts and non–Duke–Imamoglu–Ikeda lifts, Comment. Math. Univ. St. Pauli, **64** (2015), 109–129.

[17] H. Katsurada, Congruence of Siegel modular forms and special values of their standard zeta functions, Math. Z., **259** (2008), 97–111.

[18] H. Katsurada and K. Kawamura, Ikeda's conjecture on the period of the Duke–Imamoglu–Ikeda lift, Proc. London Math. Soc. (3), **111** (2015), 445–483.

[19] H. Katsurada and S. Mizumoto, Congruences for Hecke eigenvalues of Siegel modular forms, Abh. Math. Semin. Univ. Hambg., **82** (2012), 129–152.

[20] N. Kurokawa, Congruences between Siegel modular forms of degree 2, Proc. Japan Acad., **55A** (1979), 417–422.

[21] I. Miyawaki, Numerical examples of Siegel cusp forms of degree 3 and their zeta functions, Mem. Fac. Sci. Kyushu Univ., **46** (1992), 307–339.

[22] S. Mizumoto, Congruences for eigenvalues of Hecke operators on Siegel modular forms of degree two, Math. Ann., **275** (1986), 149–161.

[23] C. Poor, N. Ryan and D. S. Yuen, Lifting puzzles in degree four, Bull. Aust. Math. Soc., **80** (2009), 65–82.

[24] T. Satoh, On certain vector valued Siegel modular forms of degree two, Math. Ann., **274** (1986), 335–352.

[25] O. Taibi, Dimensions of spaces of level one automorphic forms for split classical groups using the trace formula, preprint, 2014, http://arxiv.org/abs/1406.4247.

[26] G. van der Geer, Siegel modular forms and their applications, in The 1-2-3 of Modular Forms (ed. K. Ranestad), 181–245, Springer-Verlag, Berlin Heidelberg, 2008.

Neil DUMMIGAN
University of Sheffield
School of Mathematics and Statistics
Hicks Building, Hounsfield Road
Sheffield, S3 7RH, U.K.
E-mail: n.p.dummigan@shef.ac.uk