J. Math. Soc. Japan
Vol. 69, No. 2 (2017) pp. 477–501
doi: 10.2969/jmsj/06920477

Value distribution of leafwise holomorphic maps on complex laminations by hyperbolic Riemann surfaces

By Atsushi Atsuji

Dedicated to Professor Shigeo Kusuoka on his 60th birthday

(Received Dec. 2, 2014)
(Revised June 5, 2015)

Abstract. We discuss the value distribution of Borel measurable maps which are holomorphic along leaves of complex laminations. In the case of complex lamination by hyperbolic Riemann surfaces with an ergodic harmonic measure, we have a defect relation appearing in Nevanlinna theory. It gives a bound of the number of omitted hyperplanes in general position by those maps.

We first consider general cases. Let $(M, \mathcal{L}, S)$ be a (possibly singular) complex lamination in a compact Polish space $\overline{M}$ with $\overline{M} \subset N$ where N is a complex manifold. S denotes a set of singular points of the lamination such that $(M, \mathcal{L})$ is a smooth complex lamination with $M = \overline{M} \setminus S$. We also write $(M, \mathcal{L})$ for $(M, \mathcal{L}, S)$ when $S = \emptyset$.

We assume that leaves of the lamination are complex manifolds with Hermitian metrics and their complex structure is compatible with N, but we do not assume complex structure of M and $\overline{M}$. We assume some dependence of the metric and its derivatives of all orders along leaves on M (They should be measurable or continuous on M as mentioned later). We also assume that the leafwise Ricci curvature defined from the metric is bounded on M.

We note the cases when $S = \emptyset$ include the cases of minimal sets of a singular holomorphic foliation and Levi-flat surfaces. When $S \neq \emptyset$, a typical example is a holomorphic foliation generated by holomorphic vector fields on $\overline{M} = N$.

We say a Borel measurable map f from M to $\mathbf{P}^m(\mathbf{C})$ is leafwise holomorphic if f is holomorphic along leaves.

In this paper we discuss intersection of the image of leaves by such maps with hypersurfaces in $\mathbf{P}^m(\mathbf{C})$. We see this property of the lamination from measure theoretical and probabilistic points of view. Harmonic measure introduced by L.Garnett ([**17**]) is an essential tool. Harmonic measures defined by Garnett are associated with Laplace–Beltrami operator defined from a Riemann metric on leaves. Instead of these measures we use harmonic measures associated with a complex Laplacian defined from a Hermitian

---

2010 *Mathematics Subject Classification.* Primary 32H30; Secondary 58J65.
Key Words and Phrases. Nevanlinna theory, complex lamination, value distribution theory, holomorphic diffusion.
The author was partly supported by the Grant-in-Aid for Scientific Research (C) 24540192, Japan Society for the Promotion of Science.

metric. We call such a measure A-harmonic measure where A is a complex Laplacian (see Section 1 for precise definition).

Let m be an A-harmonic probability measure on the lamination. We say some property holds for *almost all* leaves (abr. *a.a.* leaves) if there exists a saturated set $G \subset M$ such that $m(G) = 1$ and the property holds for all leaves included in G. We remark that the notion of A-harmonic measure coincides with harmonic measure in the sense of L.Garnett in the case of $dim_{\mathbf{C}} L = 1$ ($L \in \mathcal{L}$). We also note that an A-harmonic measure always exists if M is compact (cf. [**6**], [**17**]).

When $S = \emptyset$, we first have the following results:

Theorem 1. *Let $(M, \mathcal{L})$ be a compact complex lamination with a leafwise Hermitian metric whose derivatives of all orders along leaves depend continuously on M. Let m be an A-harmonic measure and $f : M \to \mathbf{P}^m(\mathbf{C})$ be a nonconstant leafwise holomorphic map. For a.a. $L \in \mathcal{L}$, $f(L)$ intersects almost all hyperplanes in $\mathbf{P}^m(\mathbf{C})$ with respect to the measure defined from Fubini–Study metric on $\mathbf{P}^m(\mathbf{C})^*$.*

This can be strengthened as follows:

Theorem 1′. *Under the assumption of the above theorem, for a.a. $L \in \mathcal{L}$, $f(L)$ intersects q.e. algebraic hypersurfaces in $\mathbf{P}^m(\mathbf{C})$.*

Here "q.e." means quasi-everywhere with respect to a capacity defined by Molzon ([**22**]).

The above theorems generally hold even when $S \neq \emptyset$ if we have an A-harmonic probability measure. In singular case Fornæss and Sibony ([**14**]) showed that there exists a harmonic measure when $\overline{M} = \mathbf{P}^2(\mathbf{C})$ and S is hyperbolic. It is well-known that smooth Levi flats and minimal sets do not exist in $\mathbf{P}^n(\mathbf{C})$ ($n \geq 3$) ([**20**], [**28**]). So the singular case is much important while the problem is still open in case of $n = 2$.

Next let us consider a rather special case that the leaves are of one dimensional, namely, Riemann surfaces. We say that an open Riemann surface is hyperbolic if its universal covering is equivalent to a unit disc. Otherwise the Riemann surfaces are called parabolic. Remark that this usage is different from the classical theory of Riemann surfaces. For a lamination by hyperbolic Riemann surfaces each leaf has a Hermitian metric form defined from the Poincaré metric on the unit disc. Let this form be denoted by ω_P and let ω be the Hermitian metric form on each leaf induced from N. These two metrics play important roles in our results.

In singular case we also use harmonic currents on the laminations. It is known that a positive harmonic current corresponds to a harmonic measure of some complex Laplacian ([**11**], [**14**]). Let T be a positive harmonic current on $(M, \mathcal{L}, S)$. Then $m_P := T \wedge \omega_P$ is a harmonic measure. Under this setting we can see the following result.

We say a point $a \in S$ is linearizable if there exists a holomorphic coordinate around a such that leaves are integral curves of the vector field

$$V = \sum_{j=1}^{n} \lambda_j z_j \frac{\partial}{\partial z_j},$$

where λ_j are non-zero complex numbers. We say S is linearizable if all points of S are linearizable. It is known that there exists a positive harmonic current T on $(M, \mathcal{L}, S)$ and $T \wedge \omega_P$ gives a finite harmonic measure if S is linearizable ([**5**], [**11**]). We denote this measure by $m_P := T \wedge \omega_P$.

THEOREM 2. *Let $(M, \mathcal{L}, S)$ be a singular complex foliation by hyperbolic Riemann surfaces and $f : M \to \mathbf{P}^m(\mathbf{C})$ be a nonconstant leafwise holomorphic map. Assume that S is linearizable and let T be a positive harmonic current on $(M, \mathcal{L}, S)$. For a.a.$L \in \mathcal{L}$ with respect to m_P, $f(L)$ intersects q.e. algebraic hypersurfaces in $\mathbf{P}^m(\mathbf{C})$.*

In some singular cases we can see more precise feature for intersection with hyperplanes if the harmonic measure is ergodic.

We say a leafwise holomorphic map f is degenerate along a.a. leaves if $f(L)$ is contained in a hyperplane for a.a. $L \in \mathcal{L}$ with respect to m_P. Our main theorem is the following.

THEOREM 3. *Let $(M, \mathcal{L}, S)$ be a singular complex foliation by hyperbolic Riemann surfaces and $f : M \to \mathbf{P}^m(\mathbf{C})$ a nonconstant leafwise holomorphic map. Assume that S is linearizable and there exists a positive harmonic current T on $(M, \mathcal{L}, S)$ such that m_P is ergodic. Let $\alpha := \int_M T \wedge f^*\omega_1$ $(\le \infty)$ where ω_1 is the Fubini–Study metric on $\mathbf{P}^m(\mathbf{C})$.*

If $H_1, \dots, H_q$ are hyperplanes in $\mathbf{P}^m(\mathbf{C})$ in general position and

$$q > m + 1 + \frac{m(m+1)}{4\pi\alpha},$$

then for a.a. $L \in \mathcal{L}$ with respect to m_P,

$$f(L) \cap (H_1 \cup \cdots \cup H_q) \neq \emptyset,$$

or f is degenerate along a.a. leaves.

In the above we set $m(m+1)/4\pi\alpha = 0$ when $\alpha = \infty$. More precisely, we obtain a defect relation for f as classical Nevanlinna theory: If f is non-degenerate along *a.a.* leaves, then we have

$$\sum_{i=1}^{q} \delta(H_i) \le m + 1 + \frac{m(m+1)}{4\pi\alpha}, \tag{0.1}$$

where $\delta(H_i)$ is a defect satisfying $0 \le \delta(H_i) \le 1$ and $\delta(H) = 0$ if f omits H.

We note Fornæss and Sibony ([**14**]) showed that m_P is an ergodic harmonic measure when $\overline{M} = \mathbf{P}^2(\mathbf{C})$, S is hyperbolic and there are no algebraic leaves in $\mathcal{L}$.

Remark that if L is parabolic and f is non-degenerate, the bound of the number of omitted hyperplanes by $f(L)$ is $m+1$. This is just a consequence of value distribution theory of holomorphic curves due to H. Cartan (cf. [**15**], [**24**], [**26**]). We can get a more refined result than the above theorem using Fujimoto's calculus with our method as mentioned in Section 3.

We also note that Feres and Zeghib ([**13**]) discussed existence and non-existence of nonconstant continuous leafwise holomorphic functions on compact laminations. They obtained some results including Liouville type theorems for leafwise holomorphic functions. If leaves are Kähler, then a holomorphic function is a harmonic function along leaves. Liouville type theorem along *a.a.* leaves follows from Garnett's fundamental lemma which says that any bounded leafwise harmonic function must be constant along leaves ([**17**]). It seems natural to think that we could obtain some results on more precise value distribution of leafwise holomorphic maps.

We use Nevanlinna theory with some leafwise diffusion processes on the lamination to show these results. We first introduce a suitable leafwise diffusion process for our problem and see its basic properties in the next section. In Section 2 we remark a relationship between Nevanlinna theory and our diffusion process, and give a proof of Theorem 1, 1′ and 2. Our probabilistic setting also works in the proof of Theorem 3. Namely it enables us to link the traditional Nevanlinna theory with leafwise holomorphic maps and to use ergodic theorems. Then we can obtain the defect relations as mentioned above and give the proof of Theorem 3 in the last section.

We would note that we owe our basic idea using a Dirichlet form to [**11**]. The author would thank Professor Nessim Sibony for indicating [**11**] and their related works to him and for giving valuable comments on the first draft.

1. Leafwise holomorphic diffusions.

We first assume each leaf L is equipped with a Hermitian metric. We first assume that the metric is smooth along L and continuous on M, and its derivatives of all orders depend continuously on M. If we take the induced metric from N as the Hermitian metric on L, these properties are satisfied. We note that this continuity assumption can be relaxed to Borel measurability for our purpose as mentioned later. We also assume that the Ricci curvature of leaves is uniformly bounded on M. If M is compact, this assumption is automatically satisfied.

We introduce some class of diffusion processes called holomorphic diffusions. As for basic properties of these diffusion process, see [**18**], [**19**] (Kähler case), [**25**] (complex Brownian motion and stochastic calculus) and [**27**]. Let $g = (g_{\alpha,\overline{\beta}})$ be the Hermitian metric on L. We have a diffusion process (X_t, P_x) on each L whose generator

$$A = 2g^{\alpha,\overline{\beta}} \frac{\partial^2}{\partial z_\alpha \overline{\partial} z_\beta} \ : \ \text{complex Laplacian with respect to } g, \tag{1.1}$$

where $(g^{\alpha,\overline{\beta}}) = g^{-1}$. P_x denotes the law of X_t with $X_0 = x$. We note that if g is a Kähler metric, then $A = \Delta_L/2$ where Δ_L denotes the Laplace–Beltrami operator defined from the Riemannian metric. In this case the associated diffusion is called a Brownian motion on a Kähler manifold L. In general Hermitian case $A = \Delta_L/2 + b$ for some vector field b.

We emphasize that conservativeness of (X_t, P_x), which is called *stochastic completeness* in Kähler case, is much important rather than geodesic completeness in our method.

Proposition 4. (X_t, P_x) *is a conservative diffusion process on each leaf. Namely* $P_x(X_t \in L_x \ \forall t > 0) = 1$.

This follows from the boundedness of Ricci curvature on leaves.
It is known that

Proposition 5 (cf. [**19**], [**25**]). *If $f : L \to \mathbf{P}^1(\mathbf{C})$ is a holomorphic map, then $f(X_t)$ is a time-changed Brownian motion on $\mathbf{P}^1(\mathbf{C})$ associated with Fubini–Study metric.*

We call a diffusion process satisfying the above property a holomorphic diffusion. Precisely, a diffusion process X_t is a holomorphic diffusion if X_t is a holomorphic martingale, that is, X_t satisfies the following property: for any open set $U \subset L$ and a holomorphic function f on U, $Ref(X_t)$ is a local martingale while X_t stays on U. In our case holomorphic martingale property comes from $ARef = 0$ if f is holomorphic.
Let L_x denote the leaf through x. We immediately have the following.

Corollary 6. *Let D be the zeros of a non-constant holomorphic function f on L_x. Then D is a polar set for X_t starting at x if $x \notin D$. Namely if $x \notin D$,*

$$P_x(X_t \notin D \ (\forall t > 0)) = 1.$$

Proof. $f(X_t)$ is a time-changed complex Brownian motion on $\mathbf{C}$ (cf. [**18**]). $0 \in \mathbf{C}$ is a polar set for complex Brownian motion starting at a different point from 0 (cf. [**19**], [**25**]). □

Throughout this paper the expectation by P_x is denoted by E_x. Namely, for f a bounded measurable function on $\Omega(L) := C([0,\infty) \to L)$

$$E_x[f] = \int_{\Omega(L)} f(\omega) dP_x(\omega).$$

(X_t, P_x) defines a Markov semigroup acting on bounded continuous function on M by

$$D_t u(x) := E_x[u(X_t)] \text{ for } u \in C_b(M).$$

Candel showed that D_t is a diffusion semigroup on $C(M)$ if M is compact ([**6**], [**8**]). It is easy to see that D_t can be extended to a diffusion semigroup on the space of bounded Borel measurable functions. From this fact we can see that $x \mapsto P_x(B)$ is Borel measurable on M if $B \in \mathcal{B}(\Omega(M))$: a collection of natural Borel sets of $\Omega(M)$ where $\Omega(M) = C([0,\infty) \to M)$. Thus (X_t, P_x) can be regarded as a diffusion process on M.

Candel [**6**] (see also [**8**]) associated these diffusions with harmonic measures. We say a function is leafwise C^2 (resp. C^2_o) if it is continuous on M (resp. in $C_o(M)$), twice differentiable along leaves and all of its derivatives are continuous on M. $C^2_{\mathcal{L}}(M)$ (resp. $C^2_{\mathcal{L},o}(M)$) denotes the set of all leafwise C^2 (resp. C^2_o) functions. By assumption Au is continuous on M for $u \in C^2_{\mathcal{L}}(M)$.
We define A-harmonic measure (simply called harmonic measure if there is no fear of confusion) m if m is a probability measure on M and

$$\int_M Au(x) dm(x) = 0$$

for all $u \in C^2_{\mathcal{L},o}(M)$.
If M is compact, A-harmonic measure always exists ([**6**], [**17**]).

It is known ([**6**], [**8**], [**17**]) that

Proposition 7. *If m is an A-harmonic measure, then m is an invariant measure of D_t (equivalently, of (X_t, P_x)).*

When M is noncompact, these arguments may not be available. But we can obtain a leafwise diffusion satisfying the same properties as above in the case of singular complex laminations by hyperbolic Riemann surfaces. Dinh–Nyugen–Sibony ([**11**]) constructed a diffusion semigroup in this case and we can also construct a desired diffusion process directly under the situation of [**11**]. Let $\mathbb{D}$ denote the unit disc and $\phi_a : \mathbb{D} \to L_a$ be the covering map such that $\phi_a(0) = a$.

We take a Poincaré metric $\tilde{\omega}_P$ of curvature -1 on $\mathbb{D}$:

$$\tilde{\omega}_P = \frac{4}{(1-|z|^2)^2}\frac{i}{2}dz \wedge d\overline{z}.$$

ϕ_a pushes $\tilde{\omega}_P$ to a Hermitian metric ω_P on L_a. It is known that ω_P is Borel measurable on M(cf. Proposition 3.1 of [**11**]) and in some case it is continuous on M ([**7**], [**9**]).

Let Z_t be the Brownian motion starting from o associated with the Poincaré metric $\tilde{\omega}_P$ on $\mathbb{D}$. Z_t is defined by the diffusion process whose generator is half of hyperbolic Laplacian Δ_{hyp}:

$$\Delta_{hyp} = (1-|z|^2)^2\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{(1-|z|^2)^2}{4}\Delta_{\mathbf{R}^2},$$

where $\Delta_{\mathbf{R}^2}$ is the Laplacian associated with standard Euclidean metric on $\mathbf{R}^2$. Define (X_t, P_a) by $X_t = \phi_a(Z_t)$ and P_a is the law of X_t, namely $P_a(X_t \in A) = \mathbb{P}_o(\phi_a(Z_t) \in A)$ for $A \in \mathcal{B}(M)$ where $\mathbb{P}_o$ is the law of Z_t with $Z_0 = o$. Remark that Z_t is conservative. Note that the universal covering map is unique up to an automorphism on $\mathbb{D}$ and the Poincaré metric is invariant under the action of automorphisms on $\mathbb{D}$. From this property we have for $C \in \mathcal{B}(\mathbb{D})$

$$\mathbb{P}_u(Z_t \in C) = \mathbb{P}_o(\alpha_{o,u}(Z_t) \in C), \tag{1.2}$$

where $\alpha_{o,u} \in Aut(\mathbb{D})$ with $\alpha_{o,u}(o) = u$. It is clear that the above definition of (X_t, P_a) is independent of the choice of ϕ_a.

Proposition 8. *For any leaf L, (X_t, P_a) $(a \in L)$ is a conservative holomorphic diffusion staying on L.*

Proof. Conservativeness follows from that of Z_t. From the definition it is clear that (X_t, P_a) is a holomorphic martingale on leaves, namely $Ref(X_t)$ is a local martingale for suitable stochastic intervals (cf. [**12**], [**25**]) for any holomorphic function f defined locally on each leaf. We have only to note Markov property of X_t. We have to show

$$P_x(\{X_{t+s} \in A\} \cap B) = E_x[P_{X_s}(X_t \in A) : B]$$

for any $A \in \mathcal{B}(M)$ and $B \in \mathcal{F}_s := \sigma(X_t : 0 \le t \le s)$, where $E_x[f : B] = \int_B f dP_x$. By Markov property of Z_t

$$P_x(\{X_{t+s} \in A\} \cap B) = \mathbb{P}_o(\{Z_{t+s} \in \phi_x^{-1}(A)\} \cap B) = E_x[\mathbb{P}_{Z_s}(Z_t \in \phi_x^{-1}(A)) : B].$$

On the other hand, by (1.2) we have

$$\begin{aligned}\mathbb{P}_u(\phi_x(Z_t) \in A) &= \mathbb{P}_o(\phi_x \circ \alpha_{o,u}(Z_t) \in A)\\ &= \mathbb{P}_o(\phi_{\phi_x(u)}(Z_t) \in A)\\ &= P_{\phi_x(u)}(X_t \in A).\end{aligned}$$

Then, taking $u = Z_s$, we have

$$\mathbb{P}_{Z_s}(Z_t \in \phi_x^{-1}(A)) = P_{X_s}(X_t \in A) \quad P_x\text{- a.s.}$$

Let $a, b \in M$ and U be a small neighborhood of a involving b. Take $\tilde{U} \subset \phi_a^{-1}(U)$ and $\tilde{b} \in \tilde{U}$ such that $0 \in \tilde{U}$ and $\phi_a(\tilde{b}) = b$. Then $\tilde{b} \to 0$ as $b \to a$. Since $\phi_b = \phi_a \circ \alpha_{0,\tilde{b}}$, with (1.2)

$$P_b(X_t \in A) = \mathbb{P}_o(\phi_b(Z_t) \in A) = \mathbb{P}_o(\phi_a \circ \alpha_{o,\tilde{b}}(Z_t) \in A) = \mathbb{P}_{\tilde{b}}(\phi_a(Z_t) \in A).$$

Since Z_t has strong Feller property, $P_b(X_t \in A) \to P_a(X_t \in A)$ as $b \to a$. This shows the strong Feller property of X_t and consequently strong Markov property of X_t. □

We have to check that we can regard (X_t, P_a) $(a \in M)$ as a diffusion on M. To do this we employ another construction of a holomorphic diffusion and we will identify them.

We here introduce a positive harmonic current T on $(M, \mathcal{L}, S)$. The differentials and differential forms appearing below are differential and forms on leaves parameterized on transversals. We say T is of bidegree (p, p) or dimension $l-p$ if T acts leafwise $(l-p, l-p)$ differential forms where $l = dim_{\mathbf{C}} L$ $(L \in \mathcal{L})$. For details about currents on laminations, see Sullivan [**29**]. T is a harmonic current if $i\partial\overline{\partial}T = 0$. If T is of bidegree (p, p), this means

$$\int_M i\partial\overline{\partial}\phi \wedge T = 0$$

for any leafwise smooth $(l-p-1, l-p-1)$ form ϕ.

We note that for a complex lamination $(M, \mathcal{L}, S)$, there exists an open covering $\{U\}$ of M such that U is homeomorphic to $\mathbb{B} \times \mathbb{T}$, which is called a *flow box*, where $\mathbb{B}$ is a domain of $\mathbf{C}^n$ and $\mathbb{T}$ is a topological space. Take an arbitrary flow box $U \cong \mathbb{B} \times \mathbb{T}$. It is known that T is a positive harmonic current on $(M, \mathcal{L}, S)$ if and only if T has a local expression as

$$T = h(a, b)[\mathbb{B} \times \{b\}]d\mu(b), \tag{1.3}$$

where $[\mathbb{B} \times \{b\}]$ is a current of integration on $\mathbb{B} \times \{b\}$, $d\mu$ is a measure on $\mathbb{T}$ and $h(a, b)$ is a positive harmonic function in a for μ-a.e. b. If S is locally pluripolar on N, then a positive harmonic current exists ([**5**]).

In [**11**], they use a bilinear form:

$$\mathcal{E}(u,v) = -i \int_M v\partial\overline{\partial}u \wedge T \quad (u,v \in C^2_{\mathcal{L},o}(M)), \tag{1.4}$$

where T is a positive harmonic current on $(M,\mathcal{L},S)$. Let $m_P := T \wedge \omega_P$. Define $\Delta_P u$ by Radon–Nikodym derivative $2i\partial\overline{\partial}u \wedge T/dm_P$. We also write

$$\mathcal{E}(u,v) = -2\pi \int_M v dd^c u \wedge T \quad (u,v \in C^2_{\mathcal{L},o}(M)),$$

where d^c denotes $(i/4\pi)(\overline{\partial} - \partial)$.

Theorem 9. *$\mathcal{E}(u,v)$ defines a regular Dirichlet form $(\mathcal{E},\mathcal{H}^1(T))$ on $L^2(m_P)$, where $\mathcal{H}^1(T)$ is the completion of $C^2_{\mathcal{L},o}(M)$ by the norm $||\cdot||_{\mathcal{H}^1}$ defined by*

$$||u||^2_{\mathcal{H}^1} = 2\pi \int_M du \wedge d^c u \wedge T + \int_M u^2 dm_P.$$

Remark 10. i) *While this is essentially due to* [**11**]*, we remark that this holds for the more general cases. We need the existence of positive harmonic current and the uniform boundedness of leafwise Ricci curvature as the following arguments. Thus the assumption on S may be able to be relaxed.*

ii) *ω_P is Borel measurable, so we do not use here continuity assumption about metrics and its derivatives mentioned in the beginning of this section.*

We first note an elementary fact:

Lemma 11. *$C^2_{\mathcal{L},o}(M)$ is dense in $L^2(m_P)$ and $C_o(M)$.*

Proof. We can localize the problem. Let $\{U\}$ be a covering of M and take a flow box $U \cong \mathbb{B} \times \mathbb{T}$. Set $\tilde{C}_o(\mathbb{B} \times \mathbb{T}) = \{f(x)g(y) | f \in C^2_o(\mathbb{B}),\ g \in C_o(\mathbb{T})\}$. Then $\tilde{C}_o(\mathbb{B} \times \mathbb{T})$ is dense in $C_o(\mathbb{B} \times \mathbb{T})$. In fact, $\tilde{C}_o(\mathbb{B} \times \mathbb{T})$ is subalgebra of $C_o(\mathbb{B} \times \mathbb{T})$ satisfying that $\tilde{C}_o$ separates points and there exists $f \in \tilde{C}_o$ such that $f(x) \neq 0$ for any $x \in \mathbb{B} \times \mathbb{T}$. Then Stone–Weierstrass theorem implies the denseness on $C_o(\mathbb{B} \times \mathbb{T})$. As for L^2 it is easy by the previous fact. □

Proof of Theorem 9. Note that this form is not symmetric in general since T is not necessarily closed. We will remark that their bilinear form (1.4) defines a regular Dirichlet form as follows. Let

$$\tilde{\mathcal{E}}(u,v) := 2\pi \int_M du \wedge d^c v \wedge T \quad (u,v \in C^2_{\mathcal{L},o}(M)).$$

Then $\tilde{\mathcal{E}}$ is a symmetric bilinear form on $C^2_{\mathcal{L},o}(M)$ and $\mathcal{E}(u,u) = \tilde{\mathcal{E}}(u,u)$ since $dd^c u^2 = 2udd^c u + 2du \wedge d^c u$ and T is harmonic. Then $\mathcal{E}$ is positive definite. We first note that $\tilde{\mathcal{E}}$ is closable on $L^2(m_P)$. We can obtain a symmetric operator S such that $(Su,v)_{L^2(m_P)} = -\tilde{\mathcal{E}}(u,v)$. We can see S takes the form of $S = \Delta_P + V$ for a vector field V on $C^2_{\mathcal{L},o}(M)$.

Note that V comes from dT since

$$d(ud^c v \wedge T) = du \wedge d^c v \wedge T + udd^c v \wedge T + ud^c v \wedge dT \quad (u, v \in C^2_{\mathcal{L},o}(M)).$$

Let $(1,0)$-form τ such that $\partial T = \tau \wedge T$. τ can be written as $\tau = \partial h/h$ locally where h is a positive harmonic function appearing in the local expression (1.3) of T. Since each leaf is of constant curvature, by Yau's gradient estimate ([**30**]) we can see that τ is bounded uniformly. Namely we have

$$i\tau \wedge \overline{\tau} \wedge T \leq const.\omega_P \wedge T, \tag{1.5}$$

equivalently V is bounded. The coefficients of Δ_P is locally bounded since ω_P is so (Proposition 3.1 of [**11**]). Thus the coefficients of S belong to $L^2_{loc}(m_P)$. Hence $\tilde{\mathcal{E}}$ is closable as seen in (1.1.3) of [**16**].

From (1.5) and Cauchy–Schwarz inequality, we have that for $u, v \in C^2_{\mathcal{L},o}(M)$

$$|\mathcal{E}(u,v) - \tilde{\mathcal{E}}(u,v)| \leq const.||u||_{L^2(m_P)}\tilde{\mathcal{E}}(v,v)^{1/2}.$$

From this it is easy to see that $\mathcal{E}(u,v)$ satisfies the weak sector condition: there exists a constant $C > 0$ such that

$$|\mathcal{E}_1(u,v)| \leq C\mathcal{E}_1(u,u)^{1/2}\mathcal{E}_1(v,v)^{1/2} \quad (u, v \in \mathcal{H}^1(T)).$$

As we mentioned above,

$$||u||^2_{\mathcal{H}^1} = \mathcal{E}_1(u,u),$$

where $\mathcal{E}_1(u,u) = \mathcal{E}(u,u) + (u,u)$. We also note that $C^2_{\mathcal{L},o}(M)$ is dense in $C_o(M)$. These observations imply that $(\mathcal{E}, \mathcal{H}^1(T))$ is a regular Dirichlet form. Obviously it is also local. Then by the theory of Dirichlet forms (cf. [**21**]) we have a diffusion process corresponding to $(\mathcal{E}, \mathcal{H}^1(T))$. □

REMARK 12. *The above argument is available for general leafwise holomorphic diffusion on noncompact complex lamination with complete Kähler leaves under the assumption that the leafwise Ricci curvature is uniformly bounded from below if we have an appropriate harmonic measure. Namely we can construct a leafwise holomorphic diffusion process corresponding to a regular Dirichlet form*

$$\mathcal{E}(u,v) = -\int_M Au \cdot v dm,$$

where A is a complex Laplacian and dm is an A-harmonic measure.

PROPOSITION 13. *The diffusion process defined by $(\mathcal{E}, \mathcal{H}^1(M))$ coincides with (X_t, P_x) on L_y for m_P-a.e. y. Namely the diffusion process starting at x defined as the above stays in L_x a.s. and its distribution coincides with the one of (X_t, P_x).*

PROOF. Let $\phi_y : \mathbb{D} \to L_y$ be a universal covering map. We will see

$$\Delta_{hyp}(u \circ \phi_y) = (\Delta_P u) \circ \phi_y \text{ on } \mathbb{D} \tag{1.6}$$

for $u \in C^2_{\mathcal{L},o}(M)$. This implies that the generator of X_t is $\Delta_P/2$. Take a flow box $U \cong \mathbb{B} \times \mathbb{T}$. Then T has a local expression as in (1.3)

$$T = h(a,b)[\mathbb{B} \times \{b\}]d\mu(b),$$

where $[\mathbb{B} \times \{b\}]$ is a current of integration on $\mathbb{B} \times \{b\}$, $d\mu$ is a measure on $\mathbb{T}$ and $h(a,b)$ is a positive harmonic function in a for μ-a.e. b. Then we can check (1.6) on $\phi^{-1}(\mathbb{B} \times \{b\})$ with y in this plaque. Let $v \in C^\infty_o(\mathbb{D})$. Recall $\phi^*_y \omega_P = \tilde{\omega}_P$. In fact,

$$\begin{aligned}\int_{\phi^{-1}(\mathbb{B}\times\{b\})} v(x)\phi^*_y h(x,b)\Delta_{hyp}(u\circ\phi_y)(x)\tilde{\omega}_P &= 2i\int_{\phi^{-1}(\mathbb{B}\times\{b\})} v(x)\phi^*_y h(x,b)\partial\overline{\partial}(u\circ\phi_y)\\ &= 2i\int_{\phi^{-1}(\mathbb{B}\times\{b\})} v(x)\phi^*_y h(x,b)\phi^*_y(\partial\overline{\partial}u)\\ &= \int_{\phi^{-1}(\mathbb{B}\times\{b\})} v(x)\phi^*_y h(x,b)(\Delta_P u)\circ\phi_y(x)\phi^*_y\omega_P.\end{aligned}$$

Let us consider a heat equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta_P u, \quad u(0,x) = \phi(x),$$

where $\phi \in C_o(M)$. When one consider this equation on L_x, he has a unique bounded solution $u(t,x) = E_x[\phi(X_t)]$ since X_t is conservative. On the other hand, $u(t,x)$ can be written as $u(t,x) = \hat{E}_x[\phi(\hat{X}_t)]$ on M where $(\hat{X}_t, \hat{P}_x)$ is the diffusion process uniquely corresponding to $(\mathcal{E}, \mathcal{H}^1(M))$. □

From this lemma (X_t, P_x) can be regarded as a diffusion process on M. In particular, the map $x \mapsto P_x(A)$ is Borel measurable on M for $A \in \mathcal{B}(\Omega(M))$. Since there is no fear of confusion for our application, we use (X_t, P_x) for $(\hat{X}_t, \hat{P}_x)$.

It is known that if S is linearizable, m_P is a finite measure (Proposition 4.2 of [**11**]). $\Delta_P m_P = 0$ follows by definition. Then we have

Proposition 14. *If S is linearizable, m_P is a harmonic measure in our sense, namely m_P is the finite invariant measure of (X_t, P_x).*

The following ergodic theorems are powerful tools for our proofs. We say an A-harmonic probability measure m is ergodic if $m(B) = 0$ or 1 for any saturated measurable set B.

Lemma 15 ([**6**], [**8**], [**17**]). *Let m be a A-harmonic measure.*
1) *If $f \in L^1(m)$,*

$$\frac{1}{t}\int_0^t E_x[f(X_s)]ds \xrightarrow[t\to\infty]{} \exists f^*(x) \ m\text{-a.e. } x,$$

where f^* is constant along leaves and

$$\int_M f^*(x)dm(x) = \int_M f(x)dm(x).$$

2)

$$P_x\left(\frac{1}{t}\int_0^t f(X_s)ds \xrightarrow[t\to\infty]{} f^*(x)\right) = 1$$

m-a.e. x.
3) *Moreover if m is ergodic, $f^*(x) = \int_M f(x)dm(x)$.*

2. Nevanlinna theory with leafwise holomorphic diffusions.

In this section we introduce Nevanlinna theory for leafwise holomorphic maps using holomorphic diffusions introduced in the previous section. We first introduce a formula similar to the first main theorem of Nevanlinna theory. As the previous section we assume (X_t, P_x) is conservative and there exists an A-harmonic measure m which is an invariant measure of (X_t, P_x).

Let σ be a holomorphic section of degree d on a complex line bundle on $\mathbf{P}^m(\mathbf{C})$ and $f : M \to \mathbf{P}^m(\mathbf{C})$ a nonconstant leafwise holomorphic map. Set

$$u_\sigma(x) := \log \frac{||f(x)||^d}{|\sigma \circ f|(x)}, \tag{2.1}$$

where $||x|| = (|x_0|^2 + \cdots + |x_n|^2)^{1/2}$ if $x = (x_0, \ldots, x_n)$ in a homogeneous coordinate. Let D denote the divisor defined from zeros of σ. We use the same notation D as the support of divisor D if no fear of confusion. If the line bundle is hyperplane bundle, then D is regarded as a hyperplane H in $\mathbf{P}^m(\mathbf{C})$. In this case we write u_H for u_σ.

Definition 16 ([**2**]). *Let $x \in M$ such that $|\sigma \circ f(x)| \neq 0$. Assume $f(L) \not\subset D$. For a stopping time T with $T < \infty$ a.s., we define*

$$\begin{aligned}
\tilde{m}_x(T, u_\sigma) &:= E_x[u_\sigma(X_T)] \\
\tilde{N}_x(T, u_\sigma) &:= \lim_{\lambda\to\infty} \lambda P_x\left(\sup_{0\le s\le T} u_\sigma(X_s) > \lambda\right) \\
\tilde{T}_x(T) &:= \frac{1}{d} E_x\left[\int_0^T Au_\sigma(X_s)ds\right],
\end{aligned}$$

where A denotes the generator of (X_t, P_x), provided that these quantities make sense.

Note that u_σ is plurisubharmonic on $L \setminus f^{-1}(D)$ and then $Au_\sigma \ge 0$ on $L \setminus f^{-1}(D)$. Hence $\int_0^t Au_\sigma(X_s)ds$ makes sense for $t > 0$ since $f^{-1}(D) \cap L$ is a polar set for X_t. Since $A \log|\sigma \circ f| = 0$ locally outside $f^{-1}(D)$ and $f^{-1}(D) \cap L$ is a polar set, $\tilde{T}_x(T)$ is independent of D and d.

Since the Ricci curvature of leaves is bounded, by Proposition 8 in [**2**] we can see that if T is deterministic or the first exit time from a relatively compact domain,

$$\tilde{N}_x(T, u_\sigma) = 0 \quad \text{if} \quad f(L_x) \cap D = \emptyset,$$

provided that $\tilde{T}_x(T) < \infty$.

We note an expression of $\tilde{T}_x(T)$ in a special case: laminations by hyperbolic Riemann surfaces. We introduce uniformization functions.

Definition 17. *Define a function η by*

$$\eta \omega_P = \omega, \tag{2.2}$$

where ω is the induced metric from N.

If $N = \mathbf{P}^n(\mathbf{C})$, we can take the Fubini–Study metric form on $\mathbf{P}^n(\mathbf{C})$ as ω given by

$$\omega = dd^c \log ||z||^2 = \frac{i}{2\pi} \partial \overline{\partial} \log ||z||^2. \tag{2.3}$$

η is called an uniformization function of $(M, \mathcal{L}, S)$ (cf. [**7**], [**9**], [**11**]). It is known that η and ω_P are Borel measurable on M ([**11**]). Candel and Gomez-Mont showed that η is positive on M and continuous on $\mathbf{P}^n(\mathbf{C})$ with $\eta = 0$ on S if $N = \overline{M} = \mathbf{P}^n(\mathbf{C})$, S is hyperbolic and linearizable ([**9**]).

To see the following expression of our characteristic function we introduce an energy density ζ of f with respect to ω defined by

$$\zeta \omega = f^* \omega_1, \tag{2.4}$$

where ω_1 is the Fubini–Study metric on $\mathbf{P}^m(\mathbf{C})$.

Proposition 18. *Let $(M, \mathcal{L}, S)$ be a singular complex foliation by hyperbolic Riemann surfaces. Let (X_t, P_x) be the holomorphic diffusion associated with ω_P and η the uniformization function defined in* (2.2). *Then*

$$\tilde{T}_x(T) = \pi E_x \left[\int_0^T \zeta \cdot \eta(X_s) ds \right].$$

Proof. In this case $Au = \Delta_P u/2 = i\partial\overline{\partial} u \wedge T/dm_P$. Then

$$\begin{aligned} \frac{1}{2} \Delta_P \log ||f|| &= i \partial \overline{\partial} \log ||f|| \wedge T/dm_P \\ &= \pi f^* \omega_1 \wedge T/dm_P \\ &= \pi \zeta \omega \wedge T/dm_P \\ &= \pi \zeta \eta \end{aligned}$$

outside $\phi^{-1}(D)$. □

We have an analogy to the first main theorem of Nevanlinna theory:

Proposition 19. *Assume $|\sigma \circ f|(x) \neq 0$ and $f^{-1}(D) \cap L$ is a polar set for X_t.*

For any stopping time T with $\tilde{T}_x(T) < \infty$,

$$\tilde{m}_x(T, u_\sigma) - u_\sigma(x) + \tilde{N}_x(T, u_\sigma) = \tilde{T}_x(T).$$

We remark that from this first main theorem we immediately have a Casorati–Weierstrass type theorem using the following leafwise zero-one law under general situations.

Define $\theta_t : \Omega(M) \to \Omega(M)$ by $(\theta_t\omega)(s) = \omega(t+s)$ and $\mathcal{I} := \{A \in \mathcal{B}(\Omega(M)) | 1_A \circ \theta_t = 1_A \ a.s.\ \forall t > 0\}$, called θ-invariant σ-field.

LEMMA 20. *For $A \in \mathcal{I}$, $P_x(A) = 0$ or 1 ($\forall x \in L,\ a.a.L$).*

PROOF. Note that $x \mapsto P_x(A)$ ($A \in \mathcal{I}$) is A-harmonic on each L and Borel measurable on M. It is easy to see that a Garnett's theorem (Theorem 1(b) of [**17**]) holds for A-harmonic case. From this $P_x(A)$ is constant along a.a.leaves. Set $c = P_x(A)$ on L_x. $\exists A_n \in \mathcal{F}_n := \sigma(X_s; s \le n) \subset \mathcal{B}(\Omega(M))$ s.t. $A_n \to A$ with $P_x(A_n \triangle A) \to 0$ $(n \to \infty)$. Then $P_x(A_n \cap A) \to c$ $(n \to \infty)$.

$$\begin{aligned} P_x(A_n \cap A) &= E_x[1_{A_n} 1_A] \\ &= E_x[1_{A_n} E_x[1_A | \mathcal{F}_n]] \\ &= E_x[1_{A_n} E_x[1_A \circ \theta_n | \mathcal{F}_n]] \\ &= E_x[1_{A_n} E_{X_n}[1_A]] \text{ (by Markov property)} \\ &= c P_x(A_n) \to c^2. \end{aligned}$$

Hence $c^2 = c$. □

PROOF OF THEOREMS 1, $1'$ AND 2.

In Theorems 1 and $1'$ we assume that M is compact and the leafwise Hermitian metric is leafwise smooth and continuous on M. Then Candel's argument in [**6**] based on Hahn–Banach theorem for existence of A-harmonic probability measure is available. Since S is linearizable in Theorem 2, then we have a harmonic probability measure as mentioned before. In both cases we have A-harmonic probability measure. Since the argument in the previous section is also available under the assumptions of these theorems, then we can construct a holomorphic diffusion whose generator is A and its invariant measure is A-harmonic measure. Thus Theorems 1, $1'$ and 2 can be reduced to the following theorem. □

THEOREM 21. *Let $(M, \mathcal{L})$ be a (possibly singular) complex lamination which supports a leafwise holomorphic diffusion with its invariant probability measure m and $f : M \to \mathbf{P}^m(\mathbf{C})$ be a nonconstant leafwise holomorphic map. For $a.a.L \in \mathcal{L}$ with respect to m, $f(L)$ intersects q.e. algebraic hypersurfaces in $\mathbf{P}^m(\mathbf{C})$.*

Here "q.e." means quasi-everywhere with respect to a capacity defined by Molzon ([**22**]).

PROOF. By Lemma 20 there exists a measurable saturated set $G \subset M$ with $\mu(G) =$

1 such that $P_x(A) = 0$ or 1 for $A \in \mathcal{I}$ and $\forall x \in G$. Suppose $f(L_{x_o})$ omits K: a set of σ of positive capacity in the sense of Molzon for $x_o \in G$. Then it is known ([**22**]) that there exists a probability measure ν on K such that

$$U_K(x) := \int_K u_\sigma(x) d\nu(\sigma)$$

is bounded on L_{x_o}. Let $D_n \subset L_{x_o}$ $(n = 1, 2, \dots)$ be an exhaustion of L_{x_o} and $\tau_n = \inf\{t > 0 | X_t \notin D_n\}$. Note that conservativeness of X_t implies $\tau_n \uparrow \infty$ $(n \to \infty)$ a.s. Since $\tilde{N}_{x_o}(t \wedge \tau_n, u_\sigma) = 0$, by Proposition 19 and Fubini's theorem

$$\begin{aligned} T_{x_o}(t \wedge \tau_n) &= \int_K \tilde{m}_{x_o}(t, u_\sigma) d\nu(\sigma) - \int_K \tilde{m}_{x_o}(0, u_\sigma) d\nu(\sigma) \\ &= E_{x_o}[U_K(X_{t \wedge \tau_n})] - U_K(x_o). \end{aligned}$$

Letting $n \to \infty$, we see that $T_{x_o}(t)$ is bounded in t. Since $\mathbf{P}^m(\mathbf{C})$ is a compact Kähler manifold, $\lim_{t \to \infty} T_{x_o}(t) < \infty$ implies that for any Kähler form k on $\mathbf{P}^m(\mathbf{C})$

$$\int_0^\infty k(df(X_s), df(X_s)) < \infty \quad P_{x_o}\text{-a.s.}$$

By martingale convergence theorem on manifolds (cf. [**1**], [**10**], [**12**]), this implies that $\lim_{t \to \infty} f(X_t)$ exists in $\mathbf{P}^m(\mathbf{C})$, P_{x_o}-a.s. Since $\lim_{t \to \infty} f(X_t)$ is $\mathcal{I}$-measurable and $x_o \in G$, Lemma 20 implies that this limit point is a constant point of $\mathbf{P}^m(\mathbf{C})$ P_{x_o}-a.s. Let this point be denoted by z_o. Take σ_o such that $\sigma_o(z_o) = 0$. Proposition 19 implies that $m(t, u_{\sigma_o})$ is bounded. Letting $t \to \infty$, $u_\sigma(X_t) \to \infty$ a.s. By Fatou's lemma this leads a contradiction. □

3. Proof of Theorem 3.

In this section we consider a singular complex lamination $(M, \mathcal{L}, S)$ by hyperbolic Riemann surfaces as mentioned in the introduction.

As mentioned in Section 1, we can take the Hermitian metric ω_P defined from the Poincare metric on the unit disc. It is known to be measurable on M under the assumptions of the main theorem (cf. [**11**]) and the uniformization function η is also measurable. We consider the Laplacian Δ_P associated with ω_P as introduced in Section 1. Let ϕ_x be the covering map from the unit disc $\mathbb{D}$ to L_x such that $\phi(o) = x$ and (X_t, P_x) the holomorphic diffusion whose generator is $\Delta_P/2$. Recall that

$$X_t = \phi_x(Z_t) \text{ and } P_x(X_t \in B) = \mathbb{P}_o(\phi_x(Z_t) \in B),$$

for $B \in \mathcal{B}(M)$ where $(Z_t, \mathbb{P}_z)$ is the hyperbolic Brownian motion on $\mathbb{D}$ defined in Section 1.

Let T be a positive harmonic current on $(M, \mathcal{L}, S)$ and $m_P = T \wedge \omega_P$. Under the assumption of Theorem 3 m_P is a finite measure. We assume that m_P is an ergodic probability measure. Recall that m_P is the invariant measure of (X_t, P_x).

Let f be a leafwise holomorphic map, namely f is a Borel measurable map from M

to $\mathbf{P}^m(\mathbf{C})$ and is holomorphic along leaves. We first look at the degeneracy of f.

LEMMA 22. *Let f be a leafwise holomorphic map from M to $\mathbf{P}^m(\mathbf{C})$. If m_P is ergodic, then f is degenerate along a.a. leaves or nondegenerate along a.a. leaves.*

PROOF. We first note that f is a degenerate holomorphic map on a leaf L if and only if the Wronskian of f defined on L vanishes on L. We remark that the Wronskian is measurable on M since all derivatives of f are Borel measurable on M thanks to Cauchy's theorem and measurability of f. The set of leaves where f is degenerate coincides with the set of leaves where the Wronskian vanishes m_P-a.e. Then the former set is measurable and by ergodicity this set has full measure or null measure. □

Thus from now on we assume that f is nondegenerate along a.a.leaves in this section.

Regarding $f \circ \phi_x$ as a holomorphic map from $\mathbb{D}$ to $\mathbf{P}^m(\mathbf{C})$, we consider the value distribution of $f \circ \phi_x$. We borrow a technique of Nevanlinna theory on holomorphic curves due to Fujimoto ([**15**]). We give a small modification adjusting his second main theorem to the case of holomorphic map from the unit disc.

We introduce some notations. For the sake of simplicity of notation we often write ϕ for $f \circ \phi_x$ from now on in this section. Let k-th derived curve of ϕ from $\mathbb{D}$ be denoted by ϕ^k and Φ_k be the map in $\mathbf{C}^{N_k+1}$ associated to ϕ^k such that $\phi^k = \pi_k^* \Phi_k$ where $\pi_k : \mathbf{C}^{N_k+1} \setminus \{0\} \to \mathbf{P}^{N_k}(\mathbf{C})$ is the canonical projection and $N_k = \binom{m+1}{k+1} - 1$. Set $\Omega_k := dd^c \log ||\Phi_k||^2$ on $\{\Phi_k \neq 0\}$ and define h_k by $\Omega_k = h_k dd^c|z|^2$.

Let $H_1, \dots, H_q$ be hyperplanes in $\mathbf{P}^m(\mathbf{C})$ in N-subgeneral position.

DEFINITION 23. *We assume that $\phi(o) \notin \cup H_j$ for simplicity. Define*

$$T_\phi^k(r) = \int_0^r \frac{dt}{t} \int_{|z| \le t} h_k dd^c|z|^2 \ (k = 0, 1, \dots, n). \tag{3.1}$$

Let $\{\zeta_j\}$ be the zeros of $\langle \phi, H \rangle$ for hyperplane H. We define

$$N_\phi(r, H)^{[m]} = \sum_j \log \frac{r}{|\zeta_j|}$$

counting with multiplicity up to m times.

In the above definition we write simply $T_\phi(r) = T_\phi^0(r)$ when $k = 0$ and $N_\phi(r, H) = N_\phi(r, H)^{[\infty]}$ when $m = \infty$. We also note that

$$T_\phi(r) = \tilde{T}_x(\tau_r), \qquad \text{and} \qquad N_\phi(r, H) = \tilde{N}_x(\tau_r, u_H)$$

defined in Definition 16, where $\tau_r = \{t > 0 | X_t \notin \phi_x(\{|z| < r\})\}$.

REMARK. In the above definition the assumption that $\phi(o) \notin \cup H_j$ is not important, differently from the case of holomorphic curve from $\mathbf{C}$. When $\phi(o) \in \cup H_j$, this contributes only $O(1)$ term in the following second main theorem while it gives $O(\log r)$

term in the case of holomorphic curve from **C** (cf. [**15**]). So it is harmless in our case. We have the following.

Proposition 24. *Let $H_1, \dots, H_q$ be hyperplanes in $\mathbf{P}^m(\mathbf{C})$ in N-subgeneral position and let $w(j)$ and θ be Nochka weights and a Nochka constant respectively for these hyperplanes where $q > 2N - m + 1$. Then for every $\epsilon > 0$, $\delta > 0$ there exists a set $E \subset (0,1)$ with $\int_E 1/(1-r) < \infty$ such that for $r \in (0,1) \setminus E$,*

$$\theta(q-2N+m-1-\epsilon)T_\phi(r) \le \sum_{j=1}^{q} w(j)N_\phi(r,H_j)^{[m]} + \frac{m(m+1)}{2}(1+\delta)\log\frac{1}{1-r} + O(\log T_\phi(r)). \tag{3.2}$$

If $H_1, \dots, H_q$ is in general position, we can take $N = m, w(j) = \theta = 1$. Then (3.2) becomes

$$(q-m-1-\epsilon)T_\phi(r) \le \sum_{j=1}^{q} N_\phi(r,H_j)^{[m]} + \frac{m(m+1)}{2}(1+\delta)\log\frac{1}{1-r} + O(\log T_\phi(r)).$$

To show Proposition 24 we need some lemmas. Let $\tilde{\omega}_P$ denote the Poincaré metric as in Section 1 defined by

$$\tilde{\omega}_P = \frac{4}{(1-|z|^2)^2}\frac{i}{2}dz \wedge d\overline{z} = \frac{4\pi}{(1-|z|^2)^2}dd^c|z|^2.$$

We say that a function u has mild singularities on $\mathbb{D}$ (cf. [**15**]) if there exists a countable set $\{\zeta_j\} \subset \mathbb{D}$ without accumulation points inside $\mathbb{D}$ such that u is smooth on $\mathbb{D} \setminus \{\zeta_j\}$, and on a neighborhood of each ζ_j u takes such a form as

$$|u(z)| = |z-\zeta_j|^\sigma v(z) \prod_{k=1}^{l} |\log|g_k(z)||^{\tau_k}$$

with some real numbers σ, τ_j, nonzero holomorphic function g_j and positive C^∞-function v. For this function we define

$$N(r,u) = \int_{\mathbb{D}} g_r(o,z)d\nu_u,$$

where

$$\nu_u = \sum_j \pi\sigma\delta(\zeta_j) \tag{3.3}$$

with $\delta(\zeta)$: Dirac mass on ζ and $g_r(w,z)$ is the Green's function of $\Delta_{\mathbf{R}^2}$ on $\mathbb{D}_r$ with Dirichlet boundary condition. It is known

$$g_r(o,z) = \frac{1}{\pi}\log\frac{r}{|z|} \tag{3.4}$$

Let $\tau_r = \inf\{t > 0 \ : \ |Z_t| > r\}$ and $\mathbb{E}_x$ denote the expectation by $\mathbb{P}_x$ (the law of Z_t).

LEMMA 25. *Let u be a function on $\mathbb{D}$ with mild singularities defined as above. Assume that $x \notin \{\zeta_j\}$ and $g_j(x) \neq 0$. Then*

$$\mathbb{E}_x[\log|u|(Z_T)] - \log|u|(x) + N_x(T, \log|u|) = \frac{1}{2}\mathbb{E}_x\left[\int_0^T \Delta_{hyp} \log|u|(Z_s)ds\right], \qquad (3.5)$$

for any stopping time T such that the right hand side is finite, where

$$N_x(T, v) = \lim_{\lambda\to\infty} \lambda \mathbb{P}_x\left(\sup_{0\leq s\leq T} v^+(Z_s) > \lambda\right) - \lim_{\lambda\to\infty} \lambda \mathbb{P}_x\left(\sup_{0\leq s\leq T} v^-(Z_s) > \lambda\right)$$

with $v^+ = \max\{0, v\}$ and $v^- = \min\{0, v\}$.
In particular, if $T = \tau_r$ and $x = o$, we have

$$\begin{aligned}\mathbb{E}_o[\log|u|(Z_{\tau_r})] - \log|u|(o) + N(r, u) &= \frac{1}{2}\int_{\mathbb{D}} g_r(o, z)\Delta_{hyp} \log|u|(z)\tilde{\omega}_P \qquad (3.6)\\ &= 2\pi \int_{\mathbb{D}_r} g_r(o, z) dd^c \log|u|(z).\end{aligned}$$

PROOF. We first remark that $\{\zeta_j\}$ and zeros of g_j are polar for Z_t. Thus $\Delta_{hyp} \log|u|(Z_s)$ makes sense for all $s > 0$ with probability 1.

As the case of Proposition 19 we apply Ito formula and Tanaka formula (cf. [**25**]) to $\log^+|u|$ and $\log^-|u|$. Then we have

$$\begin{aligned}&\mathbb{E}_x[\log^+|u|(Z_{T_n})] - \log^+|u|(x) + N_x(T_n, \log^+|u|)\\ &= \frac{1}{2}\mathbb{E}_x\left[\int_0^{T_n} (\Delta_{hyp} \log|u|)^+(Z_s)ds\right] + \mathbb{E}_x[L_{T_n}]\\ &- \mathbb{E}_x[\log^-|u|(Z_{T_n})] + \log^-|u|(x) - N_x(T_n, \log^-|u|)\\ &= -\frac{1}{2}\mathbb{E}_x\left[\int_0^{T_n} (\Delta_{hyp} \log|u|)^-(Z_s)ds\right] - \mathbb{E}_x[L_{T_n}],\end{aligned}$$

where L_t is a local time at 0 of the semimartingale $\log|u|(Z_t)$ (cf. [**25**]) and T_n is a stopping time such that $T_n \uparrow T$ a.s. and $\mathbb{E}_x[L_{T_n}] < \infty$. Adding the above equations side by side and letting $n \to \infty$, we have (3.5). As for (3.6), we only note that $\partial^2/(\partial z\partial\overline{z}) = \Delta_{\mathbf{R}^2}/4$ and the Green's function of Δ_{hyp} with respect to $\tilde{\omega}_P$ coincides with g_r. Also note (cf. [**3**]) that

$$N(r, u) = N_o(\tau_r, \log|u|). \qquad \square$$

The following lemma is a hyperbolic version of the basic lemma often used in Nevanlinna theory originally due to Borel.

LEMMA 26. *Let u be a nonnegative, locally integrable function on $\mathbb{D}$. Assume u is bounded on a neighborhood of o. For any $\delta > 0$ there exists a set $E \subset [0, \infty)$ of finite*

Lebesgue measure such that

$$\mathbb{E}_o[u(Z_{\tau_r})] \le 2^{\delta-(1+\delta)^2}\left(\frac{r}{1-r^2}\right)^{\delta}\left(\mathbb{E}_o\left[\int_0^{\tau_r} u(Z_s)ds\right]\right)^{(1+\delta)^2}$$

for $\rho := \log(1+r)/(1-r) \notin E$.

Note that ρ is the hyperbolic length from o to z ($|z| = r$).

Proof. This can be shown by simple calculations as follows. By the assumption on u, the right hand side makes sense for $r \in [0,1)$. Noting that $\pi dd^c|z|^2$ gives Lebesgue measure on $\mathbb{D}$ and by coarea formula

$$\begin{aligned}\mathbb{E}_o\left[\int_0^{\tau_r} u(Z_s)ds\right] &= \int_{\mathbb{D}_r} g_r(o,z)u(z)\tilde{\omega}_P \\ &= \pi\int_{\mathbb{D}_r} g_r(o,z)u(z)\frac{4}{(1-|z|^2)^2}dd^c|z|^2 \\ &= \frac{1}{\pi}\int_0^r \log\frac{r}{t}\int_0^{2\pi} u(te^{i\theta})d\theta\frac{4t}{(1-t^2)^2}dt.\end{aligned}$$

Set

$$\begin{aligned}G(r) &:= \frac{1}{2\pi}\int_0^r \log\frac{r}{t}\int_0^{2\pi} u(te^{i\theta})d\theta\frac{4t}{(1-t^2)^2}dt, \\ V(r) &:= \int_0^r M(t)\frac{4t}{(1-t^2)^2}dt, \\ M(r) &:= \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta.\end{aligned}$$

And also $\tilde{G}(\rho) := G(r)$, $\tilde{V}(\rho) := V(r)$ and $\tilde{M}(\rho) := M(r)$ with $\rho = \log(1+r)/(1-r)$. It is easy to see

$$G(r) = \int_0^r \frac{V(t)}{t}dt.$$

Then

$$\tilde{G}'(\rho) = G'(r)\frac{dr}{d\rho} = \frac{1-r^2}{2r}V(r).$$

For $\delta > 0$ there exists $E \subset (0,\infty)$ of finite Lebesgue measure such that for $\rho \notin E$

$$\tilde{V}'(\rho) \le \tilde{V}(\rho)^{1+\delta} \text{ and } \tilde{G}'(\rho) \le \tilde{G}(\rho)^{1+\delta}.$$

Also

$$\tilde{V}'(\rho) = V'(r)\frac{1-r^2}{2} = \tilde{M}(\rho)\frac{4r}{(1-r^2)^2}\frac{1-r^2}{2} = \tilde{M}(\rho)\frac{2r}{1-r^2}.$$

Combining these equations, for $\rho \notin E$ with $\rho = \log(1+r)/(1-r)$ we have

$$\tilde{M}(\rho)\frac{2r}{1-r^2} \leq \tilde{V}(\rho)^{1+\delta} = \left(\frac{2r}{1-r^2}\tilde{G}'(\rho)\right)^{1+\delta} \leq \left(\frac{2r}{1-r^2}\right)^{1+\delta}\tilde{G}(\rho)^{(1+\delta)^2}. \qquad \square$$

From this we have an estimate of $T_\phi^k(r)$.

Lemma 27. *For any $\delta > 0$ there exists a set $E \subset (0,1)$ with $\int_E 1/(1-r)dr < \infty$ such that*

$$T_\phi^k(r) \leq T_\phi(r) + C\delta \log \frac{1}{1-r} + O(\log T_\phi(r))$$

Proof. From Theorem 3.2.2 in [**15**] with Lemma 26 we have that for any $\delta > 0$ there exists a set $E \subset (0,1)$ with $\int_E 1/(1-r)dr < \infty$ such that

$$T_\phi^{k-1}(r) - 2T_\phi^k(r) + T_\phi^{k+1}(r) \leq \delta \log \frac{1}{1-r} + O(\log T_\phi^k(r))$$

for $r \notin E$. By the argument of Proposition 3.2.8 in [**15**] we have the desired result. $\square$

Proof of Proposition 24. We follow the proof by Fujimoto in [**15**] with some modification adjusting to our case. We here pick up only the modified points. For a hyperplane H, a vector v corresponds to H such that H is given by the equation: $\langle w, v\rangle = 0$. We identify v with H and we use the same notation H for v. Namely the hyperplane H is given by the equation $\langle w, H\rangle = 0$. Set $\Phi_k(H) := \langle \Phi_k, H\rangle$ and

$$\varphi_k(H)(z) := \frac{||\Phi_k(H)||^2(z)}{||\Phi_k||^2(z)}.$$

Let

$$\varphi = \frac{|W(\phi_0, \ldots, \phi_m)|}{||\Phi(H_1)||^{w(1)} \cdots ||\Phi(H_q)||^{w(p)}},$$

where $W(\phi_0, \ldots, \phi_m)$ is the Wronskian of ϕ. We define functions $\hat{h}$ and h^* as follows. Choose a holomorphic function g_k such that $\nu_{g_k} = \nu_{|\Phi_k|}$ (cf. (3.3) and set $\tilde{\Phi}_k := \Phi_k/g_k$ $(k = 0, 1, \ldots, m)$. Set

$$\hat{h} = \frac{\prod_{k=0}^{n-1} ||\tilde{\Phi}_k||^{2\epsilon}}{\prod_{k=0}^{m-1}\prod_{j=1}^{q} \log^{2w(j)}(a/\varphi_k(H_j))}$$

for $\epsilon > 0, a > 0$. Define h^* by

$$dd^c \log \hat{h} = h^* dd^c |z|^2.$$

Corollary 2.5.5 in [**15**] says that for every $\epsilon > 0$, taking suitable $a > 0$,

$$(h^*)^{m(m+1)/2} \geq C \frac{||\Phi_0||^{2\theta(q-2N+m-1)}||\Phi_n||^2 \hat{h}}{||\tilde{\Phi}_0||^{2\epsilon} \cdots ||\tilde{\Phi}_{n-1}||^{2\epsilon} \prod_{j=1}^{q} ||\Phi(H_j)||^{2w(j)}}$$

for some positive constant C. Taking expectation $\mathbb{E}_o$ of both sides after taking log of both sides, we have (3.2.15) in [**15**]:

$$\begin{aligned}
&\theta(q-2N+m-1)T_\phi(r) + N(r,\varphi) - \epsilon(T_\phi^0(r) + \cdots + T_\phi^{m-1}(r)) + \frac{1}{2}\mathbb{E}_o[\log \hat{h}(Z_{\tau_r})] \\
&\quad \leq \frac{m(m+1)}{2}\frac{1}{2}\mathbb{E}_o[\log h^*(Z_{\tau_r})] + O(1).
\end{aligned}$$

Let us estimate $\mathbb{E}_o[\log h^*(Z_{\tau_r})]$.

$$\begin{aligned}
&\mathbb{E}_o[\log h^*(Z_{\tau_r})] \\
&\quad = \mathbb{E}_o[\log(1-|Z_{\tau_r}|^2)^2 h^*(Z_{\tau_r})] + 2\log\frac{1}{1-r^2} \\
&\quad \leq \log \mathbb{E}_o[(1-|Z_{\tau_r}|^2)^2 h^*(Z_{\tau_r})] + 2\log\frac{1}{1-r^2} \text{ (by Jensen's inequality)} \\
&\quad \leq (1+\delta)^2 \log \mathbb{E}_o\left[\int_0^{\tau_r}(1-|Z_s|^2)^2 h^*(Z_s)ds\right] + (2+\delta)\log\frac{1}{1-r^2} \text{ (by Lemma 26)} \\
&\quad = (1+\delta)^2 \log \mathbb{E}_o\left[\int_0^{\tau_r}\Delta_{hyp}\log\hat{h}(Z_s)ds\right] + (2+\delta)\log\frac{1}{1-r^2} + O(1) \\
&\quad = (1+\delta)^2 \log^+ \mathbb{E}_o[\log\hat{h}(Z_{\tau_r})ds] + (2+\delta)\log\frac{1}{1-r^2} + O(1) \text{ (by Lemma 25)}
\end{aligned}$$

for $r \notin E_1 \subset (0,1)$ with $\int_{E_1} 1/(1-r)dr < \infty$. The last equality comes from $N(r,\hat{h}) = 0$ by definition of $\hat{h}$. By Lemma 3.2.13 in [**15**] note that

$$N_\phi(r,\varphi) \geq -\sum_{j=1}^{q} w(j)N_\phi(r,H_j)^{[m]}.$$

$\mathbb{E}_o[\log\hat{h}(Z_{\tau_r})]$ is bounded from below so that $\log^+\mathbb{E}_o[\log\hat{h}(Z_{\tau_r})] - \mathit{const}.\mathbb{E}_o[\log\hat{h}(Z_{\tau_r})]$ is bounded from above. We also note that by Lemma 27 there exist a set E_2 and $C > 0$ such that

$$\epsilon T_\phi^k(r) \leq \epsilon T_\phi + C\epsilon\delta\log\frac{1}{1-r}$$

except for $r \in E_2$ with $\int_{E_2} 1/(1-r)dr < \infty$. Taking $E = E_1 \cup E_2$ and writing δ again for $\delta/2 + mC\epsilon\delta$, we have the desired conclusion. □

Finally we apply the ergodic theorem to the Second main theorem. Let η be the uniformization function, $\rho_t = \log(1+|Z_t|)/(1-|Z_t|)$ and $\tilde{\tau}_\rho = \inf\{t > 0 \mid \rho_t \geq \rho\}$. This is the first exit time from $\mathbb{D}_r$ when $\rho = \log(1+r)/(1-r)$. Namely

$$\tilde{\tau}_\rho = \tau_r.$$

Recall that η is independent of choice of the universal covering map ϕ_x. As mentioned at Proposition 18,

$$T_\phi(r) = \tilde{T}_x(\tilde{\tau}_\rho) = \pi E_x \left[\int_0^{\tilde{\tau}_\rho} \zeta \cdot \eta(X_s) ds \right] = \pi \mathbb{E}_o \left[\int_0^{\tau_r} \zeta \circ \phi_x(Z_s) \eta \circ \phi_x(Z_s) ds \right],$$

where ζ is the energy density of f with respect to ω defined in (2.4). We use the following lemma:

LEMMA 28. i) $\rho_t/t \to 1/2$ $(t \to \infty)$ *a.s.*
ii) $\tilde{\tau}_\rho/\rho \to 2$ $(\rho \to \infty)$ *a.s. and in* $L^1(\mathbb{P}_z)$ $(z \in \mathbb{D})$.

PROOF. i) Set $k(z) = 2 \log 1/(1 - |z|^2)$. Then $\frac{1}{2}\Delta_{hyp} k(z) = 1$. By Ito formula

$$k(Z_t) - k(Z_0) = M_t + t, \tag{3.7}$$

where $M_t = b(a_t)$ with $a_t = \int_0^t ||\nabla k||^2_{hyp}(Z_s) ds$ and b_t is one-dimensional standard Brownian motion. By direct calculation $||\nabla k||^2_{hyp}(z) = 4|z|^2 \leq 4$. From law of iterated logarithm of Brownian motion (cf. [**25**]) $|M_t| \leq 2\sqrt{t \log\log 4t}$ for large t a.s. Thus $k(Z_t)/t \to 1$ a.s.
ii) i) implies that $\tilde{\tau}_\rho/\rho \to 2$ $(\rho \to \infty)$ a.s.
By (3.7)

$$2\rho - 4\log(1 + |Z_{\tilde{\tau}_\rho}|) - k(Z_0) = M_{\tilde{\tau}_\rho} + \tilde{\tau}_\rho.$$

Taking expectation of both sides,

$$\mathbb{E}_z[\tilde{\tau}_\rho] = 2\rho + O(1),$$

since $\mathbb{E}_z[M_{\tilde{\tau}_\rho}] = 0$.
On the other hand

$$\left| \frac{\tilde{\tau}_\rho}{\rho} - 2 \right| \leq \frac{M^*_{\tilde{\tau}_\rho}}{\rho} + \frac{4\log 2 + k(Z_0)}{\rho},$$

where $M^*_{\tilde{\tau}_\rho} = \sup_{0 \leq t \leq \tilde{\tau}_\rho} |M_t|$. By Burkholder inequality (cf. [**25**])

$$\mathbb{E}_z[M^*_{\tilde{\tau}_\rho}] \leq const.\mathbb{E}_z[\tilde{\tau}_\rho^{1/2}] \leq O(\sqrt{\rho}).$$

Hence

$$\mathbb{E}_z \left[\left| \frac{\tilde{\tau}_\rho}{\rho} - 2 \right| \right] \leq O\left(\frac{1}{\sqrt{\rho}} \right). \qquad \square$$

Let

$$\xi := \zeta \cdot \eta. \tag{3.8}$$

If ξ is integrable with respect to m_P, by the ergodic theorem (Lemma 15) there exists a measurable set $G \subset M$ such that $m_P(G) = 1$ and for any $x \in G$

$$\frac{1}{t}\int_0^t \xi(X_s)ds \xrightarrow[t\to\infty]{} \xi^*(x)\ P_x\text{-a.s.} \tag{3.9}$$

with a leafwise constant function ξ^* satisfying

$$\int_M \xi^*(x)dm_P = \int_M \xi dm_P. \tag{3.10}$$

Then

$$\frac{\tilde{\tau}_\rho}{\rho}\cdot\frac{1}{\tilde{\tau}_\rho}\int_0^{\tilde{\tau}_\rho} \xi\circ\phi_x(Z_s)ds \xrightarrow[\rho\to\infty]{} 2\xi^*(x)\ \mathbb{P}_o\text{-a.s.}.$$

Take $x \in G$ and ϕ_x. We first consider the case when ξ is bounded. We note

$$\begin{aligned}
&\left|\frac{\tilde{\tau}_\rho}{\rho}\cdot\frac{1}{\tilde{\tau}_\rho}\int_0^{\tilde{\tau}_\rho} \xi\circ\phi_x(Z_s)ds - 2\xi^*(x)\right| \\
&\leq \left|\frac{\tilde{\tau}_\rho}{\rho}\cdot\frac{1}{\tilde{\tau}_\rho}\int_0^{\tilde{\tau}_\rho} \xi\circ\phi_x(Z_s)ds - 2\frac{1}{\tilde{\tau}_\rho}\int_0^{\tilde{\tau}_\rho} \xi\circ\phi_x(Z_s)ds\right| + \left|2\frac{1}{\tilde{\tau}_\rho}\int_0^{\tilde{\tau}_\rho} \xi\circ\phi_x(Z_s)ds - 2\xi^*(x)\right| \\
&\leq \sup_{x\in M}\xi\cdot\left|\frac{\tilde{\tau}_\rho}{\rho} - 2\right| + 2\left|\frac{1}{\tilde{\tau}_\rho}\int_0^{\tilde{\tau}_\rho} \xi\circ\phi_x(Z_s)ds - \xi^*(x)\right|.
\end{aligned}$$

Since ξ is bounded and $\tilde{\tau}_\rho/\rho \to 2$ in $L^1(\mathbb{P}_o)$, by dominated convergence theorem we have

$$\frac{1}{\log 1/(1-r)}T_\phi(r) \xrightarrow[r\to 1]{} 2\pi\xi^*(x). \tag{3.11}$$

From this with Proposition 24 we have the following defect relations.
Define

$$\delta_\phi(H)^{[m]} := \liminf_{r\to 1}\left(1 - \frac{N(r,H)^{[m]}}{T_\phi(r)}\right)$$

and

$$\delta_\phi(H) := \liminf_{r\to 1}\left(1 - \frac{N(r,H)}{T_\phi(r)}\right).$$

Obviously $\delta_\phi(H) \leq \delta_\phi(H)^{[m]}$. Then a defect relation for the former implies the defect relation (0.1) for the latter mentioned in the introduction.

Theorem 29. *Assume* $(M, \mathcal{L}, S)$*,* T *and* f *satisfy the assumption of Theorem* 3 *and assume* $\phi_x : \mathbb{D} \to L_x$ *is the universal covering map with* $\phi(o) = x$ *and* $\phi = f\circ\phi_x$*. Let* $H_1, \ldots, H_q$ *with* $q > 2N - m + 1$ *be hyperplanes in* $\mathbf{P}^m(\mathbf{C})$ *located in* N*-subgeneral position and* ξ *be defined in* (3.8)*. If* ξ *is integrable with respect to* m_P*,*

$$\sum_{j=1}^{q} w(j)\delta_\phi(H_j)^{[m]} \le m+1+\frac{m(m+1)}{4\pi\xi^*(x)},$$

where $w(j)$ $(j=1,\dots q)$ *are Nochka weights and* $\xi^*(x)$ *is a leafwise constant function appearing in* (3.9). *In particular, if* $H_1,\dots,H_q$ *are in general position,*

$$\sum_{j=1}^{q} \delta_\phi(H_j)^{[m]} \le m+1+\frac{m(m+1)}{4\pi\xi^*(x)}. \tag{3.12}$$

Proof. If ξ is bounded, this follows immediately from Proposition 24 and (3.11). When ξ may be unbounded, we set $\xi_l=\min\{\xi,l\}$ $(l=1,2,\dots)$ and define

$$T_\phi^{(l)}(r)=E_x\left[\int_0^{\tilde\tau_\rho}\xi_l(X_s)ds\right].$$

By the above argument, there exists a leafwise constant function $\xi_l^*(x)$ such that

$$\frac{1}{\log 1/(1-r)}T_\phi^{(l)}(r)\xrightarrow[r\to 1]{} 2\pi\xi_l^*(x)$$

for m_P-a.e. x. Obviously $\xi_l^*(x)\le\xi_{l+1}^*(x)\le\xi^*(x)$ $(l=1,2,\dots)$ so that $\lim_{l\to\infty}\xi_l^*(x)\le\xi^*(x)$. By (3.10) and monotone convergence theorem

$$\int_M \xi^*(x)dm_P=\int_M\xi(x)dm_P=\lim_{l\to\infty}\int_M\xi_l(x)dm_P=\int_M\lim_{l\to\infty}\xi_l^*(x)dm_P.$$

These imply $\lim_{l\to\infty}\xi_l^*(x)=\xi^*(x)$ m_P-a.e. x. Nochka weight satisfies (cf. [**15**, p. 71])

$$\sum_{j=1}^{q} w(j)=\theta(q-2N+m-1)m+1,$$

if $q>2N-m+1$. From Proposition 24 with this, we have

$$\sum_{j=1}^{q} w(j)\left(1-\frac{N_\phi(r,H_j)^{[m]}}{T_\phi(r)}\right)$$
$$\le m+1+\frac{m(m+1)}{2}(1+\delta)\log\frac{1}{1-r}/T_\phi^{(l)}(r)+O(\log T_\phi(r))/T_\phi(r).$$

Hence we have

$$\sum_{j=1}^{q} w(j)\delta_\phi(H_j)^{[m]} \le m+1+\frac{m(m+1)}{4\pi\xi_l^*(x)}.$$

Letting $l\to\infty$, we have the desired inequality. □

The above theorem looks much better than Theorem 1 if $\xi^*(x)>0$. Under the assumption of Theorem 1 it may happen that $\eta^*(x)=0$ for some points of positive measure. Ergodicity of harmonic measure ensures this property.

Corollary 30. *Assume the assumption of Theorem* 3. *The defect relation* (3.12) *holds with*

$$\xi^*(x) = \int_M \xi dm_P > 0.$$

If $\int_M \xi dm_P = \infty$,

$$\sum_{j=1}^{q} \delta_\phi(H_j)^{[m]} \leq m+1.$$

Proof. We note that if $\int_M \xi dm_P = \infty$,

$$\lim_{r\to 1} \frac{1}{\log 1/(1-r)} T_\phi(r) = \infty.$$

In fact,

$$\liminf_{r\to 1} \frac{1}{\log 1/(1-r)} T_\phi(r) \geq \lim_{r\to 1} \frac{1}{\log 1/(1-r)} T_\phi^{(l)}(r) = 2\pi \int_M \xi_l dm_P.$$

Let $l \to \infty$. Together with Proposition 24, we have the last assertion. □

Theorem 3 is shown immediately from these defect relations with noticing that if ϕ omits H, $\delta_\phi(H)^{[m]} = 1$ and $\int_M \xi dm_P = \alpha$.

Remark. 1. Noguchi gave a simplified proof to Cartan–Nochka second main theorem for holomorphic curves ([**23**]). We may be able to use his method instead of Fujimoto's. He used a lemma of logarithmic derivative due to Vitter. We remark that we can obtain a lemma of logarithmic derivative for holomorphic maps from unit disc using our method.

2. We may also be able to improve the above theorems in the case when the image of f is degenerate using a method for degenerate holomorphic curves developed by Min Ru [**26**]. We can modify his method similarly to this note. The key point is to use Lemma 26 and the ergodic theorem.

References

[1] A. Atsuji, A Casorati–Weierstrass theorem for holomorphic maps and invariant $\sigma-$fields of holomorphic diffusions, Bull. Sci. Math., **123** (1999), 371–383.

[2] A. Atsuji, On the number of omitted values by a meromorphic function of finite energy and heat diffusions, Jour. Geom. Anal., **20** (2010), 1008–1025.

[3] A. Atsuji, A second main theorem of Nevanlinna theory for meromorphic functions on complete Kähler manifolds, J. Math. Soc. Japan, **60** (2008), 471–493.

[4] R. Bass, Probabilistic techniques in analysis, Springer, New-York, 1995.

[5] B. Berndtsson and N. Sibony, The $\overline{\partial}$-equation on a positive current, Invent. Math., **147** (2002), 371–428.

[6] A. Candel, The harmonic measures of Lucy Garnett, Adv. Math., **176** (2003), 187–247.

[7] A. Candel, Uniformization of surface laminations, Ann. Sci. Ecole Norm. Sup. (4), **26** (1993), 489–516.

[8] A. Candel and L. Conlon, Foliations. II, Graduate Studies in Mathematics, **60**, Amer. Math. Soc., Providence, RI, 2003.

[9] A. Candel and X. Gomez-Mont, Uniformization of the leaves of a rational vector field, Ann. Inst. Fourier, **45** (1995), 1123–1133.

[10] R. W. R. Darling, Convergence of martingales on a Riemannian manifold, Publ. Res. Inst. Math. Kyoto Univ., **19** (1983), 753–763.

[11] T.-C. Dinh, V.-A. Nguyen and N. Sibony, Heat equation and ergodic theorems for Riemann surface laminations, Math. Ann., **354** (2012), 331–376.

[12] M. Emery, Stochastic Calculus in Manifolds, Springer, 1989.

[13] R. Feres and A. Zeghib, Leafwise holomorphic functions, Proc. Amer. Math. Soc., **131** (2003), 1717–1725 (electronic).

[14] J. E. Fornæss and N. Sibony, Unique ergodicity of harmonic currents on singular foliations of $\mathbf{P}^2$, Geom. Funct. Anal., **19** (2010), 1334–1377.

[15] H. Fujimoto, Value Distribution Theory of the Gauss Map of Minimal Surfaces in $\mathbf{R}^m$, Aspects of Mathematics **21**, Vieweg, 1993.

[16] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet Forms and Symmetric Markov Processes, Walter de Gruyter, 1994.

[17] L. Garnett, Foliations, the ergodic theorem and Brownian motion, J. Funct. Anal., **51** (1983), 285–311.

[18] R. K. Getoor and M. J. Sharpe, Conformal martingales, Invent. Math., **16** (1972), 271–308.

[19] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes, Second edition, North-Holland Math. Library, **24**, North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1989.

[20] A. Lins Neto, A note on projective Levi-flats and minimal sets of algebraic foliations, Ann. Inst. Fourier, **49** (1999), 1369–1385.

[21] Z.-M. Ma and M. Röckner, Introduction to the Theory of (Non-Symmetric) Dirichlet Forms, Universitext, Springer, 1992.

[22] R. E. Molzon, Potential theory on complex projective space: Application to characterization of pluripolar sets and growth of analytic varieties, Illinois J. Math., **28** (1984), 103–119.

[23] J. Noguchi, A note on entire pseudo-holomorphic curves and the proof of Cartan–Nochka's theorem, Kodai Math. J., **28** (2005), 336–346.

[24] J. Noguchi and T. Ochiai, Geometric Function Theory in Several Complex Variables, Amer. Math. Soc., Providence, RI, 1997.

[25] D. Revuz and M. Yor, Continuous martingales and Brownian motion, Springer, Berlin, 1991.

[26] M. Ru, Nevanlinna Theory and It's Relation to Diophantine Approximation, World Scientific Pub., 2001.

[27] L. Schwartz, Semi-martingales sur des variétés analytiques complexes, Lecture notes in Math. **780**, Springer, 1980.

[28] Y.-T. Siu, Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimension ≥ 3, Ann. Math., **151** (2000), 1217–1243.

[29] D. Sullivan, Cycles for the dynamical study of foliated manifolds and complex manifolds, Invent. Math., **36** (1976), 225–255.

[30] S.-T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure and App. Math., **28** (1975), 201–228.

Atsushi Atsuji
Keio University
Department Mathematics
3-14-1 Hiyoshi
Yokohama 223-8522, Japan
E-mail: atsuji@math.keio.ac.jp