An index formula for a bundle homomorphism of the tangent bundle into a vector bundle of the same rank, and its applications

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Abstract. In a previous work, the authors introduced the notion of 'coherent tangent bundle', which is useful for giving a treatment of singularities of smooth maps without ambient spaces. Two different types of Gauss–Bonnet formulas on coherent tangent bundles on 2-dimensional manifolds were proven, and several applications to surface theory were given.

Let M^n ($n \geq 2$) be an oriented compact n-manifold without boundary and TM^n its tangent bundle. Let $\mathcal{E}$ be a vector bundle of rank n over M^n, and $\varphi : TM^n \to \mathcal{E}$ an oriented vector bundle homomorphism. In this paper, we show that one of these two Gauss–Bonnet formulas can be generalized to an index formula for the bundle homomorphism φ under the assumption that φ admits only certain kinds of generic singularities.

We shall give several applications to hypersurface theory. Moreover, as an application for intrinsic geometry, we also give a characterization of the class of positive semi-definite metrics (called Kossowski metrics) which can be realized as the induced metrics of the coherent tangent bundles.

1. Introduction.

Let M^n be an oriented closed n-manifold and $(\mathcal{E}, \langle \ , \ \rangle, D)$ an oriented vector bundle of rank n having inner product $\langle \ , \ \rangle$ and a metric connection D, that is

$$X \langle \xi_1, \xi_2 \rangle = \langle D_X \xi_1, \xi_2 \rangle + \langle \xi_1, D_X \xi_2 \rangle$$

holds, where ξ_i ($i = 1, 2$) are sections of M^n into $\mathcal{E}$ and X is a vector field of M^n. A bundle homomorphism

$$\varphi : TM^n \to (\mathcal{E}, \langle \ , \ \rangle, D)$$

is called a *coherent tangent bundle* if it satisfies

$$D_X \varphi(Y) - D_Y \varphi(X) = \varphi([X, Y]) \tag{1.1}$$

for any two vector fields X, Y on M^n. When $n = 2$, the authors proved in [19] and [20] that the two different types of Gauss–Bonnet formulas ($\chi_{\mathcal{E}}$ is the Euler characteristic of the oriented vector bundle $\mathcal{E}$)

$$\chi_{\mathcal{E}} = \left(\frac{1}{2\pi} \int_{M^2} K \, d\hat{A}_\varphi = \right) \chi(M_+^2) - \chi(M_-^2) + S_\varphi^+ - S_\varphi^-, \qquad (1.2)$$

$$2\pi\chi(M^2) = \int_{M^2} K \, dA_\varphi + 2 \int_{\Sigma_\varphi} \kappa_\varphi \, d\tau_\varphi, \qquad (1.3)$$

under the assumption that the singular set of φ consists of A_2-points and A_3-points, where K is the Gaussian curvature of the induced metric $ds_\varphi^2 = \varphi^* \langle \ , \ \rangle$, the two subsets $M_\pm^2$ are defined in (2.23), $d\tau_\varphi$ is the length element on the φ-singular set with respect to ds_φ^2, and $S_\varphi^\pm$ are the numbers of positive and negative A_3-points of φ, respectively. If $f : M^2 \to \mathbf{R}^3$ is a wave front, and $(\varphi :=)df : TM^2 \to \mathcal{E}$ is the bundle homomorphism induced by f, then A_2-points (resp. A_3-points) correspond to cuspidal edges (resp. swallowtails). The precise definition of A_2 or A_3-points are given in Section 2. The authors gave several applications of this formula in [23] and [24] for surfaces in $\mathbf{R}^3$.

We remark that the second formula (1.3) depends on the metric connection D, but the first formula (1.2) does not need information about the inner product. So it is natural to expect that one can extend the formula (1.2) to higher dimensional cases. The purpose of this paper is to accomplish this for even dimensional manifolds without assuming condition (1.1) as follows: let $\varphi : TM^n \to \mathcal{E}$ be a homomorphism between the tangent bundle TM^n and an oriented vector bundle $\mathcal{E}$ of rank n on M^n. Suppose that φ admits only A_k-singular points (the definition of A_k-singular points ($k = 2, \ldots, n$) is given in Section 2). We denote by $\mathfrak{A}_k$ ($k = 2, \ldots, n$) the set of A_k-singular points. When k is odd, we can define the positivity and negativity of A_k-points (see Section 3). We denote by $\mathfrak{A}_k^+$ (resp. by $\mathfrak{A}_k^-$) the set of positive (resp. negative) A_k-singular points. When $n = 2m$ is an even number, the Euler characteristic $\chi_{\mathcal{E}}$ of the vector bundle $\mathcal{E}$ satisfies the following formula

$$\chi_{\mathcal{E}} = \chi(M_+^n) - \chi(M_-^n) + \sum_{j=1}^m \left(\chi(\mathfrak{A}_{2j+1}^+) - \chi(\mathfrak{A}_{2j+1}^-) \right), \qquad (1.4)$$

where $\chi(M_+^n)$ (resp. $\chi(M_-^n)$) is the Euler characteristic of the subset M_+^n (resp. M_-^n) of M^n at which the co-orientation induced by φ is (resp. is not) compatible with the orientation of TM^n (cf. (2.23)), the number $\chi(\mathfrak{A}_{2j+1}^+)$ (resp. $\chi(\mathfrak{A}_{2j+1}^-)$) is the Euler characteristic of $\mathfrak{A}_{2j+1}^+$ (resp. $\mathfrak{A}_{2j+1}^-$). In particular, $\chi(\mathfrak{A}_{2m+1}^+)$ (resp. $\chi(\mathfrak{A}_{2m+1}^-)$) is equal to the number $\#\mathfrak{A}_{2m+1}^+$ (resp. $\#\mathfrak{A}_{2m+1}^-$) of positive (resp. negative) A_{2m+1}-points (cf. Definition 2.2). For example, the formulas for $n = 2, 4$ are given by

$$\chi_{\mathcal{E}} = \chi(M_+^2) - \chi(M_-^2) + \#\mathfrak{A}_3^+ - \#\mathfrak{A}_3^-, \qquad (1.5)$$

$$\chi_{\mathcal{E}} = \chi(M_+^4) - \chi(M_-^4) + \chi(\mathfrak{A}_3^+) - \chi(\mathfrak{A}_3^-) + \#\mathfrak{A}_5^+ - \#\mathfrak{A}_5^-. \qquad (1.6)$$

Formula (1.5) is a generalization of (1.2). As pointed out by Saeki and Sakuma in [18], any closed orientable 4-manifold with vanishing signature admits C^∞-maps into

$\boldsymbol{R}^4$ having only fold or cusp singularities. The $\boldsymbol{Z}_2$-version of our formula (1.4) was given by Levine [**11**] (see [**17**, Remark 3.12]). If we set φ to be the derivative of a Morin map $f : M^n \to N^n$, then we get (6.1), which is proved by Nakai [**15**] and Dutertre–Fukui [**4**]. Index formulas in $\boldsymbol{Z}_2$-coefficients for globally defined Morin maps $f : M^n \to N^p$ ($n \geq p$) are given by Fukuda [**6**] and Saeki [**17**], and formula (1.4) is a generalization of them. Our proof is independent of those in [**15**] and [**4**]. More precisely, we apply the Poincaré–Hopf index formula for sections of oriented vector bundles. (In [**15**] and [**4**], Viro's integral calculus [**25**] is applied.) Our index formula does not rely on ambient spaces, and we can give applications even for a case without ambient space (cf. Section 7).

In fact, one of the important applications of (1.4) is for a certain class of positive semi-definite metrics. We define a class of positive semi-definite metrics on manifolds called 'Kossowski metrics' which was originally defined by Kossowski [**10**] for 2-manifolds. The induced metrics of wave fronts in $\boldsymbol{R}^{n+1}$ which admit only A_{k+1}-singularities ($k = 1, \ldots, n$) are all Kossowski metrics. Conversely, Kossowski [**10**] showed that germs of real analytic generic Kossowski metrics on 2-manifolds can be realized as the first fundamental forms of wave fronts in $\boldsymbol{R}^3$.

Let $(\mathcal{E}, \varphi, \langle \ , \ \rangle, D)$ be a coherent tangent bundle over an n-manifold M^n then the pull-back of $\langle \ , \ \rangle$ by φ gives a Kossowski metric on M^n whenever φ admits only non-degenerate singular points (Proposition 7.7). The converse assertion for $n = 2$ was proved in [**7**]. In this paper, we generalize this for $n \geq 3$, namely, we show that each Kossowski metric ds^2 induces a coherent tangent bundle $(\mathcal{E}, \varphi, \langle \ , \ \rangle, D)$ such that $ds^2 = \varphi^* \langle \ , \ \rangle$ and the pull-back of the connection D by φ coincides with the Levi–Civita connection on the regular set of φ (cf. Theorem 7.9). We then get an index formula (cf. Corollary 7.12) for Kossowski metrics on compact manifolds admitting only A_{k+1}-singularities ($k = 1, \ldots, n$).

To give other applications of formula (1.4), the case of $\mathcal{E} = TM^n$ is important. An arbitrarily given bundle automorphism $\varphi : TM^n \to TM^n$ can be identified with the set of $(1,1)$-tensors on M^n ($n = 2m$), and (1.4) reduces to the following identity:

$$2\chi(M^n_-) = \sum_{j=1}^{m} \left(\chi(\mathfrak{A}^+_{2j+1}) - \chi(\mathfrak{A}^-_{2j+1}) \right). \qquad (1.7)$$

In Section 6, we give several applications of (1.4) and (1.7) for geometry of hypersurfaces.

The paper is organized as follows: in Section 2, we give a precise definition of A_k-singularities. In Section 3, the well-definedness of the positivity and negativity of odd order A_{2k+1}-singular points is shown. Moreover, we define characteristic vector fields with respect to the homomorphism $\varphi : TM^n \to \mathcal{E}$ and show the existence of such a vector field X defined on M^n. It is well-known that the sum of all indices of zeros of a generic section Y of $\mathcal{E}$ is equal to the Euler characteristic $\chi_{\mathcal{E}}$ of the oriented vector bundle $\mathcal{E}$. Since the section $Y := \varphi(X)$ of $\mathcal{E}$ has finitely many zeros, it holds that

$$\chi_{\mathcal{E}} = \sum_{p \in M^n \setminus \Sigma^{n-1}} \text{ind}_p(Y) + \sum_{p \in \mathfrak{A}_2} \text{ind}_p(Y) + \cdots + \sum_{p \in \mathfrak{A}_{n+1}} \text{ind}_p(Y), \qquad (1.8)$$

where

$$\Sigma^{n-1} := \mathfrak{A}_2 \cup \cdots \cup \mathfrak{A}_{n+1}$$

is the singular set of φ. Using this, we prove (1.4) in Sections 4 and 5. In Section 6, we prove Theorems 6.3 and 6.6. Several other applications are given in Section 6 and Section 7.

2. Preliminaries.

Let M^n be an oriented n-manifold and $\varphi : TM^n \to \mathcal{E}$ a bundle homomorphism between the tangent bundle TM^n and a vector bundle $\mathcal{E}$ of rank n on M^n. Then a point $p \in M^n$ is called a *singular point* if the linear map $\varphi_p : T_p M^n \to \mathcal{E}_p$ has a non-trivial kernel, where $\mathcal{E}_p$ is the fiber of $\mathcal{E}$ at p. Since M^n is oriented, we can take a non-vanishing n-form Ω defined on M^n which is compatible with the orientation of M^n. We call Ω an *orientation* of M^n.

On the other hand, $\mathcal{E}$ is locally oriented, that is, there is a non-vanishing section μ of the determinant line bundle of the dual bundle $\mathcal{E}^*$ of $\mathcal{E}$ defined on a neighborhood $U(\subset M^n)$ of a given point $p \in M^n$. We call μ a *local orientation* of $\mathcal{E}$.

Then there is a (unique) C^∞-function $\lambda : U \to \mathbf{R}$ such that

$$\varphi^* \mu = \lambda \, \Omega, \qquad (2.1)$$

on U, where $\varphi^* \mu$ is the pull-back of μ by φ. A point $q \in U$ is a singular point if and only if $\lambda(q) = 0$. A singular point $q \in M^n$ is called *non-degenerate* if the exterior derivative $d\lambda$ does not vanish at q. The bundle homomorphism φ is called *non-degenerate* if all the singular points are non-degenerate. If φ is non-degenerate, the singular set

$$\Sigma^{n-1} := \{ q \in M^n \, ; \, \mathrm{Ker}(\varphi_q) \neq \{0\} \}$$

is an embedded hypersurface of M^n, where $\mathrm{Ker}(\varphi_q)$ is the kernel of the linear mapping $\varphi_q : T_q M^n \to \mathcal{E}_q$.

DEFINITION 2.1. Let U be an open subset of M^n. A function $h : U \to \mathbf{R}$ is called a φ-*function* if there exists a C^∞-function $\sigma : U \to \mathbf{R} \setminus \{0\}$ such that

$$h = \sigma \lambda \qquad (2.2)$$

on U, where λ is the function as in (2.1).

Needless to say, λ itself is a φ-function. However, λ depends on the choice of Ω and μ, and this ambiguity is just corresponding to the choice of φ-functions. In the following discussion, we may replace λ by an arbitrarily fixed φ-function.

Suppose that φ is non-degenerate. Then the kernel of φ at each singular point $p \in \Sigma^{n-1}$ is of dimension 1. In particular, there exists a smooth vector field $\tilde{\eta}$ defined on a sufficiently small neighborhood $U(\subset M^n)$ of p such that the restriction

$$\eta := \tilde{\eta}|_{U \cap \Sigma^{n-1}}$$

has the property that η_q is the generator of the kernel of φ_q for each $q \in U \cap \Sigma^{n-1}$. We call η a *null vector field* and $\tilde{\eta}$ an *extended null vector field* (cf. [**21**, p. 733]). For a given extended null vector field $\tilde{\eta}$, we often denote by η the restriction of $\tilde{\eta}$ to Σ^{n-1}. We set

$$\tilde{\eta}\lambda := d\lambda(\tilde{\eta}), \qquad \tilde{\eta}\tilde{\eta}\lambda(= \tilde{\eta}^2\lambda) := d\big(\tilde{\eta}\lambda\big)(\tilde{\eta}), \tag{2.3}$$

and

$$\tilde{\eta}^{k+1}\lambda := d\big(\tilde{\eta}^k\lambda\big)(\tilde{\eta}) \qquad (k = 0, 1, 2, \dots), \tag{2.4}$$

inductively. As a convention, we set $\tilde{\eta}^0\lambda := \lambda$.

DEFINITION 2.2.   Let $\varphi : TM^n \to \mathcal{E}$ be a non-degenerate bundle homomorphism and Σ^{n-1} its singular set. A point $p \in \Sigma^{n-1}$ is an A_{k+1}-point $(1 \le k \le n)$ if

(1) $\lambda(p) = \tilde{\eta}\lambda(p) = \cdots = \tilde{\eta}^{k-1}\lambda(p) = 0$, $\tilde{\eta}^k\lambda(p) \ne 0$, and

(2) the Jacobi matrix of the $\mathbf{R}^k$-valued C^∞-function

$$\Lambda := (\lambda, \tilde{\eta}\lambda, \dots, \tilde{\eta}^{k-1}\lambda)$$

   is of rank k at p.

We denote by $\mathfrak{A}_{k+1}$ the set of A_{k+1}-points on M^n.

Suppose that $\varphi : TM^n \to \mathcal{E}$ is a non-degenerate bundle homomorphism. If $k = 1$ (namely, for A_2-points), then $d\Lambda = d\lambda$ and the condition (2) of Definition 2.2 is automatically satisfied. Moreover, if $k = 2$, the condition (2) also follows from (1). In fact, the two differential forms $d\lambda$ and $d(\tilde{\eta}\lambda)$ are linearly independent at p, since $d\lambda(p) \ne 0$, $\tilde{\eta}\lambda(p) = 0$ and $\tilde{\eta}^2\lambda(p) \ne 0$. In other words, the second condition of Definition 2.2 comes into effect only for $k \ge 3$ if φ is non-degenerate.

Let $\varphi : TM^n \to \mathcal{E}$ be a bundle homomorphism. Suppose that $\varphi : TM^n \to \mathcal{E}$ is non-degenerate and the singular set Σ^{n-1} is non-empty. Then the map

$$\hat{\varphi} : T\Sigma^{n-1} \to \hat{\mathcal{E}} := \varphi(TM^n|_{\Sigma^{n-1}}) \tag{2.5}$$

is induced.

PROPOSITION 2.3.   *In this situation, $\hat{\mathcal{E}}$ is a vector bundle of rank $n-1$ on Σ^{n-1}, and $\hat{\varphi} : T\Sigma^{n-1} \to \hat{\mathcal{E}}$ is a bundle homomorphism.*

We call $\hat{\varphi}$ the *reduction* of φ. By Proposition 2.3, $\hat{\mathcal{E}}$ is a subbundle of codimension one of $\mathcal{E}$.

PROOF.   We fix a point $p \in \Sigma^{n-1}$ arbitrarily. It is sufficient to show the existence of linearly independent local sections $s_1, \dots, s_{n-1}$ of $\hat{\mathcal{E}}$ defined on a neighborhood of p in Σ^{n-1}. Since φ is non-degenerate, there exists an extended null vector field $\tilde{\eta}$ defined on a local coordinate neighborhood $(U; x_1, \dots, x_n)$ centered at p. Without loss of generality,

we may assume that $\eta = \partial/\partial x_n$ holds at p. Since the kernel of φ is one dimensional, $s_j := \varphi\,(\partial/\partial x_j)\ (j = 1, \ldots, n-1)$ has the desired property. $\qquad\qquad\square$

The following two assertions (cf. Theorems 2.4 and 2.5) gives fundamental properties of the reduction homomorphism.

THEOREM 2.4.   *Let Σ^{n-1} be the singular set of a non-degenerate bundle homomorphism $\varphi : TM^n \to \mathcal{E}$. Let $\lambda : U \to \mathbf{R}$ and $\tilde\eta$ be a φ-function and an extended null vector field defined on an open subset $U(\subset M^n)$, respectively. Then the following assertions hold*:

(1) *The singular set $\Sigma^{n-2} \cap U$ of $\hat\varphi|_U$ satisfies*

$$\Sigma^{n-2} \cap U = \{q \in \Sigma^{n-1} \cap U \,;\, \tilde\eta_q \in T_q\Sigma^{n-1}\} = \{q \in \Sigma^{n-1} \cap U \,;\, \lambda = \tilde\eta\lambda = 0\}.$$

(2) *$\tilde\eta\lambda$ is a $\hat\varphi$-function defined on $\Sigma^{n-1} \cap U$.*

PROOF.   It can be easily checked that $\tilde\eta_q \in T_q\Sigma^{n-1}$ if and only if $q \in \Sigma^{n-2}$ for each $q \in \Sigma^{n-1} \cap U$. Thus, we get the equality

$$\Sigma^{n-2} \cap U = \{q \in \Sigma^{n-1} \cap U \,;\, \tilde\eta_q \in T_q\Sigma^{n-1}\} = \{q \in \Sigma^{n-1} \cap U \,;\, \tilde\eta\lambda(q) = 0\},$$

proving the assertion (1).

If $\hat\varphi$ has no singular points on $\Sigma^{n-1} \cap U$, then $\tilde\eta_q \notin T_q\Sigma^{n-1}$ for all $q \in \Sigma^{n-1} \cap U$, and thus $\tilde\eta\lambda$ has no zeros, so the assertion (2) is trivially true. So we may assume that the singular set $\Sigma^{n-2} \cap U$ of $\hat\varphi$ is not empty.

We now fix a point $p \in \Sigma^{n-2} \cap U$, and take a local coordinate system $(V; y_1, \ldots, y_n)$ centered at p such that $V \subset U$. Since φ is non-degenerate, we may assume that $\partial\lambda/\partial y_1 \neq 0$ at p. By the implicit function theorem, there exists a function $y_1(y_2, \ldots, y_n)$ such that $y_1(0, \ldots, 0) = 0$ and

$$\lambda\big(y_1(y_2, \ldots, y_n), y_2, \ldots, y_n\big) = 0.$$

If we set

$$x_1 := \lambda, \quad x_j := y_j \qquad (j = 2, \ldots, n), \tag{2.6}$$

then $(W; x_1, \ldots, x_n)$ gives a new local coordinate system at p if we choose $W(\subset V)$ sufficiently small. We can write

$$\tilde\eta = b\partial_1 + \sum_{j=2}^n c_j\partial_j \tag{2.7}$$

on $\Sigma^{n-1} \cap W$, where we set

$$\partial_j := \partial/\partial x_j \qquad (j = 1, 2, \ldots, n).$$

Then we have that

$$b(q) = \tilde{\eta}\lambda(q) \qquad (q \in \Sigma^{n-1} \cap W). \tag{2.8}$$

Since $\tilde{\eta}\lambda(p) = 0$ and $\tilde{\eta}(p) \neq 0$, we may assume that $c_2(p) \neq 0$ without loss of generality. If we set $\boldsymbol{e}_i := \varphi(\partial_i)$ $(i = 1, 2, \ldots, n)$, then

$$\boldsymbol{e}_2 = -\frac{\tilde{\eta}\lambda}{c_2}\boldsymbol{e}_1 - \sum_{j=3}^{n} \frac{c_j}{c_2}\boldsymbol{e}_j \tag{2.9}$$

holds on $\Sigma^{n-1} \cap W$ for a sufficiently small W. We fix an inner product $\langle\ ,\ \rangle$ on $\mathcal{E}$. We can take a local unit section $\boldsymbol{u}$ of $\mathcal{E}$ defined on $\Sigma^{n-1} \cap W$ such that $\boldsymbol{u}$ is orthogonal to $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. Then $\hat{\mathcal{E}}$ defined by (2.5) is equal to the subbundle of $\mathcal{E}$ which is orthogonal to $\boldsymbol{u}$. Let μ be a local orientation of $\mathcal{E}$ on W. It is obvious that $\boldsymbol{u}, \boldsymbol{e}_1, \boldsymbol{e}_3, \ldots, \boldsymbol{e}_n$ are linearly independent on $\Sigma^{n-1} \cap W$, and so we may assume that

$$\delta := \mu(\boldsymbol{u}, \boldsymbol{e}_1, \boldsymbol{e}_3, \ldots, \boldsymbol{e}_n)$$

is a positive valued function on $\Sigma^{n-1} \cap W$. Since $\hat{\mathcal{E}}$ is the subbundle of $\mathcal{E}$ which is orthogonal to $\boldsymbol{u}$,

$$\hat{\mu}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}) := \mu(\boldsymbol{u}_q, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}) \qquad (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1} \in \hat{\mathcal{E}}_q,\ q \in W)$$

gives a local orientation of $\hat{\mathcal{E}}$, and a $\hat{\varphi}$-function $\hat{\lambda} : W \to \boldsymbol{R}$ of $\hat{\mathcal{E}}$ is given by

$$\hat{\lambda} := \hat{\mu}(\boldsymbol{e}_2, \ldots, \boldsymbol{e}_n) = \mu(\boldsymbol{u}, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n)$$

$$= -\frac{\tilde{\eta}\lambda}{c_2}\mu(\boldsymbol{u}, \boldsymbol{e}_1, \boldsymbol{e}_3, \ldots, \boldsymbol{e}_n) = -\frac{\delta}{c_2}\tilde{\eta}\lambda,$$

which proves the assertion (2), since p is an arbitrarily fixed point of $\Sigma^{n-1} \cap U$.　　□

Moreover, the following assertion holds.

THEOREM 2.5.　*Let k be an integer satisfying $1 \leq k \leq n$. Under the same assumptions as in Theorem 2.4, $p \in U$ is an A_{k+1}-point of φ if and only if p (is a non-degenerate singular point of $\hat{\varphi}$ and) is an A_k-point of $\hat{\varphi}$, where A_1-points mean regular points.*

The restriction of the null vector field $\tilde{\eta}$ to Σ^{n-1} is not tangent to Σ^{n-1} in general. To prove Theorem 2.5, we now construct an extended null vector field $\tilde{\zeta}$ as a modification of $\tilde{\eta}$ as follows: as in the proof of Theorem 2.4, we fix a point p. Let $(W; x_1, \ldots, x_n)$ be the local coordinate system centered at p given in the proof of Theorem 2.4. By (2.7) and (2.8),

$$\tilde{\zeta} = \tilde{\eta} - (\tilde{\eta}\lambda)\partial_1 \left(= \sum_{j=2}^{n} c_j \partial_j \right) \tag{2.10}$$

gives an extended null vector field of $\hat{\varphi}$ on $\Sigma^{n-1} \cap W$. Let $\mu_1, \ldots, \mu_r$ be fixed smooth functions on W. For two C^∞-functions f, g on W, we write

$$f \equiv g \mod (\mu_1, \ldots, \mu_r)$$

if there exist C^∞-functions $h_1, \ldots, h_r$ defined on W such that

$$f - g = h_1 \mu_1 + \cdots + h_r \mu_r.$$

The following lemma is obvious:

LEMMA 2.6. *If* $f \equiv g \mod (\mu_1, \ldots, \mu_r)$, *then it holds that*

$$\tilde\eta f \equiv \tilde\eta g \mod (\mu_1, \ldots, \mu_r, \tilde\eta \mu_1, \ldots, \tilde\eta \mu_r).$$

We prove the following assertion.

PROPOSITION 2.7. *The equalities*

$$\tilde\eta^{j+1} \lambda \equiv \tilde\zeta^j(\tilde\eta \lambda) \mod (\tilde\eta \lambda, \ldots, \tilde\eta^j \lambda) \qquad (j = 1, \ldots, k-1) \qquad (2.11)$$

hold on W.

PROOF.   We prove the assertion by induction on j. If $j = 1$, we have that (cf. (2.10))

$$\tilde\zeta(\tilde\eta \lambda) = \big(\tilde\eta - (\tilde\eta \lambda)\partial_1\big)(\tilde\eta \lambda) = \tilde\eta(\tilde\eta \lambda) - \tilde\eta \lambda (\tilde\eta \lambda)_{x_1} \equiv \tilde\eta^2 \lambda \mod (\tilde\eta \lambda).$$

So we now assume that (2.11) holds and consider the case of $j + 1$. It holds that

$$\tilde\zeta^{j+1}(\tilde\eta \lambda) = \tilde\zeta(\tilde\zeta^j(\tilde\eta \lambda)) = \tilde\eta(\tilde\zeta^j(\tilde\eta \lambda)) - \tilde\eta \lambda (\tilde\zeta^j(\tilde\eta \lambda))_{x_1}.$$

In particular

$$\tilde\zeta^{j+1}(\tilde\eta \lambda) \equiv \tilde\eta(\tilde\zeta^j(\tilde\eta \lambda)) \mod (\tilde\eta \lambda). \qquad (2.12)$$

On the other hand, applying Lemma 2.6 to (2.11), we have

$$\tilde\eta(\tilde\zeta^j(\tilde\eta \lambda)) \equiv \tilde\eta(\tilde\eta^{j+1} \lambda) \mod (\tilde\eta \lambda, \ldots, \tilde\eta^{j+1} \lambda). \qquad (2.13)$$

By (2.12) and (2.13), we get the assertion for $j + 1$. $\square$

PROOF OF THEOREM 2.5.   By (1) of Theorem 2.4, $p \in \Sigma^{n-1}$ is an A_2-point if and only if $\tilde\eta \lambda(p) \neq 0$. By (2) of Theorem 2.4, $\tilde\eta \lambda$ is a $\hat\varphi$-function, and thus $\tilde\eta \lambda(p) \neq 0$ if and only if p is a regular point of $\hat\varphi$. This proves the assertion for $k = 1$. So we now consider the case that $k \geq 2$. We set $\lambda_1 := \tilde\eta \lambda$. Since $k \geq 2$, we have

$$\lambda(p) = \lambda_1(p) = 0. \qquad (2.14)$$

Under this assumption (2.14), p satisfies (1) of Definition 2.2 if and only if

$$\tilde\eta \lambda_1(p) = \cdots = \tilde\eta^{k-2} \lambda_1(p) = 0, \quad \tilde\eta^{k-1} \lambda_1(p) \neq 0. \qquad (2.15)$$

By Proposition 2.7, this is equivalent to the condition

$$\tilde{\zeta}\lambda_1(p) = \cdots = \tilde{\zeta}^{k-2}\lambda_1(p) = 0, \quad \tilde{\zeta}^{k-1}\lambda_1(p) \neq 0. \tag{2.16}$$

On the other hand, we can take a local coordinate system $(x_1, \ldots, x_n)$ centered at p such that (cf. (2.6))

(1) $\lambda_{x_1}(p) \neq 0$ and $\lambda_{x_2}(p) = \cdots = \lambda_{x_n}(p) = 0$,

(2) $(x_2, \ldots, x_n)$ gives a local coordinate system of Σ^{n-1} at p.

The existence of this coordinate system yields that p satisfies (2) of Definition 2.2 if and only if the Jacobi matrix of the $\boldsymbol{R}^{k-1}$-valued C^∞-function

$$\Lambda_1 := (\tilde{\eta}\lambda, \ldots, \tilde{\eta}^{k-1}\lambda) = (\lambda_1, \tilde{\eta}\lambda_1, \ldots, \tilde{\eta}^{k-2}\lambda_1)$$

is of rank $k-1$ at p. By Proposition 2.7, Λ_1 has the same rank as the function

$$\hat{\Lambda}_1 := (\lambda_1, \tilde{\zeta}\lambda_1, \ldots, \tilde{\zeta}^{k-2}\lambda_1)$$

at p. Together with (2.16), we get the assertion. $\qquad\square$

For the sake of simplicity, we denote $\tilde{\eta}\lambda$ as in (2.3) by $\dot{\lambda}$, and

$$\dot{\lambda} := \tilde{\eta}\lambda, \qquad \ddot{\lambda} := \tilde{\eta}^2\lambda, \quad \ldots, \quad \lambda^{(k)} := \tilde{\eta}^k\lambda \tag{2.17}$$

from now on.

Let p be an A_{k+1}-point of a non-degenerate homomorphism $\varphi : TM^n \to \mathcal{E}$. We fix an extended null vector field $\tilde{\eta}$ defined on a neighborhood U of p. Then for each $j = 1, \ldots, k-1$, it holds that (cf. Definition 2.14)

(1-j) $\lambda(p) = \dot{\lambda}(p) = \cdots = \lambda^{(j-1)}(p) = 0$, and

(2-j) the Jacobi matrix of the $\boldsymbol{R}^j$-valued C^∞-map $\Lambda := (\lambda, \dot{\lambda}, \ldots, \lambda^{(j-1)})$ is of rank j at p.

By the implicit function theorem, there exists a neighborhood $V_j(\subset U)$ of p and an $(n-j)$-dimensional submanifold S^{n-j} such that

$$S^{n-j} = \{q \in V_j \,;\, \lambda(q) = \dot{\lambda}(q) = \cdots = \lambda^{(j-1)}(q) = 0\}. \tag{2.18}$$

So we set $V := \bigcap_{j=1}^{k} V_j$.

LEMMA 2.8.   *The restriction $\varphi|_V : TV \to \mathcal{E}|_V$ of φ induces the j-th non-degenerate reduction homomorphism*

$$(\varphi|_V)^{(j)} : T\Sigma_V^{n-j} \longrightarrow \mathcal{E}^{(j)} \qquad (j = 1, \ldots, k)$$

such that the singular set Σ_V^{n-j-1} of $(\varphi|_V)^{(j)}$ satisfies

$$\Sigma_V^{n-j-1} = S^{n-j-1} \tag{2.19}$$

and $\lambda^{(j)} : V \to \boldsymbol{R}$ gives a $(\varphi|_V)^{(j)}$-function, where Σ_V^{n-j} is the singular set of $(\varphi|_V)^{(j-1)}$.

PROOF.   When $j = 1$, Theorem 2.4 implies the assertion. We show the assertion inductively. We assume that the $(j-1)$-th reduction $(\varphi|_V)^{(j-1)} : T\Sigma_V^{n-j+1} \to \mathcal{E}^{(j-1)}$ exists and the equality

$$\Sigma_V^{n-j} = S^{n-j} \tag{2.20}$$

holds and $\lambda^{(j-1)}$ is a $(\varphi|_V)^{(j-1)}$-function. Since p is an A_{k+1}-point, Theorem 2.5 yields that p is an A_{k-j+2}-point of $(\varphi|_V)^{(j-1)}$. Since $k \geq j$, the reduction

$$(\varphi|_V)^{(j)} : T\Sigma_V^{n-j} \longrightarrow \mathcal{E}^{(j)},$$

is non-degenerate if we choose a sufficiently small V, where Σ_V^{n-j-1} is the singular set of $(\varphi|_V)^{(j)}$. Then (1) of Theorem 2.4 implies that

$$\Sigma_V^{n-j-1} := \{q \in \Sigma_V^{n-j} \,;\, \eta_q \in T_q \Sigma_V^{n-j}\}.$$

Since $\eta_q \in T_q \Sigma_V^{n-j}$ holds if and only if

$$\lambda^{(j)}(q) = d\lambda^{(j-1)}(\eta_q) = 0,$$

we have that

$$\Sigma_V^{n-j-1} = \{q \in \Sigma_V^{n-j} \,;\, \eta_q \in T_q \Sigma_V^{n-j}\} = \{q \in \Sigma_V^{n-j} \,;\, \lambda^{(j)}(q) = 0\}.$$

Moreover, by (2.20),

$$\Sigma_V^{n-j-1} = \{q \in \Sigma_V^{n-j} \,;\, \lambda^{(j)}(q) = 0\} = \{q \in S^{n-j} \,;\, \lambda^{(j)}(q) = 0\} = S^{n-j-1}. \tag{2.21}$$

We fix a $(\varphi|_V)^{(j)}$-function $\lambda_j : \Sigma_V^{n-j} \to \boldsymbol{R}$. Since we have shown that $(\varphi|_V)^{(j)}$ is non-degenerate, $d\lambda_j \neq 0$ on Σ_V^{n-j-1}. By (2.21), the zeros of $\lambda^{(j)}$ coincide with those of λ_j. Then the division property of C^∞-functions yields that there exists a C^∞-function germ σ on Σ_V^{n-j} such that

$$\lambda^{(j)} = \sigma\lambda_j.$$

Since $d\lambda^{(j)}(p) \neq 0$ by (2) of Definition 2.2 we have $\sigma(p) \neq 0$, namely $\lambda^{(j)}$ is also a $(\varphi|_V)^{(j)}$-function. Thus we proved the j-th step of the induction procedure. □

Since the singular set of the j-th reduction $\varphi^{(j)}$ does not depend on the choice of λ and $\tilde{\eta}$, we get the following assertion.

PROPOSITION 2.9.   *Let p be an A_{k+1}-point of a non-degenerate homomorphism $\varphi : TM^n \to \mathcal{E}$, and $\tilde{\eta}$ an extended null vector field defined on a neighborhood U of p. Then there exists a neighborhood $V(\subset U)$ of p such that*

$$\Sigma_V^{n-j} := \{q \in V \,;\, \lambda(q) = \cdots = \lambda^{(j-1)}(q) = 0\}$$

$$= \{q \in \Sigma_V^{n-j+1} \, ; \, \tilde{\eta}_q \in T_q \Sigma_V^{n-j+1}\} \qquad (j = 1, \ldots, k)$$

is an $(n-j)$-dimensional submanifold of V. Moreover, each Σ_V^{n-j} does not depend on the choice of λ and $\tilde{\eta}$. Furthermore, the following equalities hold

$$\mathfrak{A}_2 \cap V = \Sigma_V^{n-1} \setminus \Sigma_V^{n-2}, \ldots, \mathfrak{A}_k \cap V = \Sigma_V^{n-k+1} \setminus \Sigma_V^{n-k}, \quad \mathfrak{A}_{k+1} \cap V = \Sigma_V^{n-k},$$

where $\mathfrak{A}_{j+1}$ $(j = 1, \ldots, k)$ is the set of A_k-points of φ.

In this paper, we mainly discuss on bundle homomorphisms having only A_{k+1}-singularities $(1 \leq k \leq n)$, so we give the following definition.

DEFINITION 2.10. A non-degenerate homomorphism $\varphi : TM^n \to \mathcal{E}$ is called a *Morin homomorphism* if the set of singular points of φ consists of A_k-points for $k = 2, 3, \ldots, n + 1$. A Morin homomorphism φ is called of *depth k* if A_{k+1}-points exist but there are no A_{k+2}-points on M^n.

The following assertion follows immediately from the definition of Morin homomorphisms.

PROPOSITION 2.11. *Let $\varphi : TM^n \to \mathcal{E}$ be a non-degenerate homomorphism and $p \in M^n$ an A_{k+1}-point. Then there exists a neighborhood U of p such that the restriction of φ into U gives a Morin homomorphism.*

PROOF. Take an extended null vector field $\tilde{\eta}$ defined on U. Since p is an A_{k+1}-point, there exists a neighborhood U of p such that

- $\lambda^{(k)} \neq 0$ on U, and

- the Jacobi matrix of Λ as in Definition 2.2 is of rank k on U,

where λ is a local φ-function defined on U. Let $q \in U$ be a singular point of φ. Then there exists a positive integer $j(\leq k)$ such that

$$\lambda^{(0)}(q) = \cdots = \lambda^{(j-1)}(q) = 0, \qquad \lambda^{(j)}(q) \neq 0.$$

Then q is an A_{j+1}-point, proving the assertion. $\qquad \square$

Moreover, as a corollary of Theorem 2.5, we get the following assertion.

PROPOSITION 2.12. *Let $\varphi : TM^n \to \mathcal{E}$ be a Morin homomorphism of depth $k(\geq 2)$. Then its reduction $\hat{\varphi} : T\Sigma^{n-1} \to \hat{\mathcal{E}}$ is a Morin homomorphism of depth $k - 1$.*

Suppose that $\varphi : TM^n \to \mathcal{E}$ is a Morin homomorphism of depth k. By Proposition 2.9,

$$\Sigma^{n-j} := \{p \in M^n \, ; \, \lambda(p) = \cdots = \lambda^{(j-1)}(p) = 0\} \qquad (j = 1, \ldots, k)$$

does not depend on the choice of a φ-function λ and the extended null vector field $\tilde{\eta}$, that is, it is well-defined as an $(n-j)$-dimensional submanifold of M^n, and

$$\mathfrak{A}_2 = \Sigma^{n-1} \setminus \Sigma^{n-2} \ , \ldots, \ \mathfrak{A}_k = \Sigma^{n-k+1} \setminus \Sigma^{n-k}, \quad \mathfrak{A}_{k+1} = \Sigma^{n-k}.$$

In this case, we give the following conventions

$$\mathfrak{A}_{k+j+1} = \Sigma^{n-k-j} = \emptyset \qquad (1 \le j \le n - k).$$

We now consider the case that $\mathcal{E}$ is orientable. Then, there is a non-vanishing section μ of the determinant line bundle of the dual bundle $\mathcal{E}^*$ of $\mathcal{E}$ defined on M^n. We call μ an *orientation* of $\mathcal{E}$. In this case, there is a unique C^∞-function $\lambda : M^n \to \boldsymbol{R}$ such that

$$\varphi^* \mu = \lambda \, \Omega, \tag{2.22}$$

where Ω is an *orientation* of M^n. We call λ the φ-*function* associated to μ and Ω defined on M^n. We set

$$M_+^n := \{ p \in M^n \, ; \, \lambda(p) > 0 \}, \qquad M_-^n := \{ p \in M^n \, ; \, \lambda(p) < 0 \}. \tag{2.23}$$

Then Σ^{n-1} coincides with the boundary $\partial M_+^n = \partial M_-^n$.

DEFINITION 2.13.   Let $\varphi : TM^n \to \mathcal{E}$ be a non-degenerate bundle homomorphism and λ a φ-function associated to μ and Ω. A φ-function $\tau : U \to \boldsymbol{R}$ defined on an open subset $U(\subset M^n)$ is called an *oriented* φ-*function* if there exists a positive valued function $\sigma \in C^\infty(U)$ such that $\tau = \sigma \lambda$ on U.

Our definition of Morin homomorphisms is motivated by the existence of the following two typical examples: Let m, n be two positive integers. Two differentiable map germs $f_i \colon (\boldsymbol{R}^m, p_i) \to (\boldsymbol{R}^n, q_i)$ $(i = 1, 2)$ are *right-left equivalent* if there exist diffeomorphism germs $\psi \colon (\boldsymbol{R}^m, p_1) \to (\boldsymbol{R}^m, p_2)$ and $\Psi : (\boldsymbol{R}^n, q_1) \to (\boldsymbol{R}^n, q_2)$ such that $\Psi \circ f_1 = f_2 \circ \psi$.

DEFINITION 2.14.   The *Morin-k-singularities* $(1 \le k \le n)$ are map germs which are right-left equivalent to

$$f(x_1, \ldots, x_n) = \big(x_1 x_n + x_2 (x_n)^2 + \cdots + x_{k-1} (x_n)^{k-1} + (x_n)^{k+1}, x_1, \ldots, x_{n-1} \big)$$

at the origin. The Morin-0-singularities mean regular points.

EXAMPLE 2.15.   Let M^n and N^n be oriented n-manifolds, and let $f : M^n \to N^n$ be a C^∞-map having only Morin singularities. Then the differential df of f canonically induces a Morin homomorphism (cf. Appendix of [21])

$$\varphi = df : TM^n \longrightarrow \mathcal{E}_f := f^* TN^n.$$

Let ω_{M^n} and μ_{N^n} be the fundamental n-forms of M^n and N^n, respectively. Then there exists a C^∞-function λ on M^n such that $f^* \mu_{N^n} = \lambda \omega_{M^n}$, which gives an oriented φ-function. The set M_+^n (resp. M_-^n) coincides with the set where $\lambda > 0$ (resp. $\lambda < 0$). The sign of λ coincides with the sign of the Jacobian of f with respect to oriented

local coordinate systems of M^n and N^n. In this case, Morin-k-points of the map f are A_{k+1}-points of the homomorphism $\varphi = df$ (see [**21**, Theorem A1]). When (N^n, ds^2) is a Riemannian manifold, then the pull-back bundle f^*TN^n on M^n has a canonical coherent tangent bundle structure (cf. [**24**]).

DEFINITION 2.16. The A_{k+1}-type singularity (or A_{k+1}-front singularity) is a map germ defined by

$$X \longmapsto \left((k+1)t^{k+2} + \sum_{j=2}^{k}(j-1)t^j x_j, -(k+2)t^{k+1} - \sum_{j=2}^{k} jt^{j-1}x_j, X_1 \right) \qquad (2.24)$$

at the origin, where $X = (t, x_2, \ldots, x_n)$ and $X_1 = (x_2, \ldots, x_n)$. Its image coincides with the discriminant set $\{F = F_t = 0\} \subset (\mathbf{R}^{n+1}; u_0, \ldots, u_n)$ of the versal unfolding

$$F(t, u_0, \ldots, u_n) := t^{k+2} + u_k t^k + \cdots + u_1 t + u_0. \qquad (2.25)$$

By definition, A_1-front singularities are regular points. A 3/2-cusp in a plane is an A_2-front singularity and a swallowtail in $\mathbf{R}^3$ is an A_3-front singularity.

EXAMPLE 2.17. Let $f : M^n \to \mathbf{R}^{n+1}$ be a wave front which admits only A_{k+1}-type singularities ($k = 1, \ldots, n$). Suppose that f is *co-orientable*, that is, there exists a globally defined unit normal vector field ν along f. Let $f^*T\mathbf{R}^{n+1}$ be the pull-back of $T\mathbf{R}^{n+1}$ by f, and consider the subbundle $\mathcal{E}_f$ of $f^*T\mathbf{R}^{n+1}$ whose fiber $\mathcal{E}_p$ at $p \in M^n$ is the orthogonal complement of ν_p. Then the differential df of f induces a bundle homomorphism

$$\varphi_f = df : TM^n \ni v \longmapsto df(v) \in \mathcal{E}_f$$

called the *first homomorphism* of f as in [**22**, Section 2], which gives a Morin homomorphism (cf. Appendix of [**21**]). Consider the function

$$\lambda := \det(f_{x_1}, \ldots, f_{x_n}, \nu),$$

where $f_{x_i} := \partial f/\partial x_i$ ($i = 1, \ldots, n$) and $(x_1, \ldots, x_n)$ is an oriented local coordinate system of M^n. Then λ is an oriented φ-function of $\mathcal{E}_f$, and the set M_+^n (resp. M_-^n) coincides with the set where $\lambda > 0$ (resp. $\lambda < 0$). Moreover, A_{k+1}-front singular points of the map f are A_{k+1}-points of the homomorphism $\varphi = df$ (see [**21**, Corollary 2.5]). As in the case of the previous example, $\mathcal{E}_f$ has a canonical coherent tangent bundle structure (cf. [**24**]).

REMARK 2.18. As seen in Examples 2.15 and 2.17, our definition of A_k-points gives a unified intrinsic treatment of singularities of both Morin maps of the same dimension and the A_k-singularities appearing in hypersurfaces in $\mathbf{R}^{n+1}$. In this intrinsic treatment, the usual k-th singular points for Morin maps and the A_{k+1}-points for wave fronts are both regarded as A_{k+1}-points of bundle homomorphisms. In other words, the order of singularities of Morin maps is not synchronized with the order of singularities of the corresponding bundle homomorphisms. For example, a fold (i.e. a Morin-1-singularity) and a cusp (i.e. a Morin-2-singularity) induce an A_2-point and an A_3-point of bundle

homomorphism, respectively.

3.   Characteristic vector fields.

We fix a Morin homomorphism $\varphi : TM^n \to \mathcal{E}$, where M^n is an oriented compact n-manifold. We now suppose that $\mathcal{E}$ is oriented, and fix an oriented φ-function $\lambda : M^n \to \mathbf{R}$. Then the singular set Σ^{n-j} $(j = 0, \ldots, n)$ of the $(j-1)$-th reduction $\varphi^{(j-1)}$ defined in the previous section is an orientable submanifold of M^n, unless it is empty.

PROPOSITION 3.1.   *If k $(2 \le k \le n)$ is even, then the sign of the function $\lambda^{(k)}$ does not depend on the choice of the extended null vector field $\tilde{\eta}$.*

PROOF.   Even if we change the extended null vector field $\tilde{\eta}$ to $-\tilde{\eta}$, the sign of the function $\lambda^{(k)}$ on the set Σ^{n-k} does not change, since k is even.   □

Hence, for each even integer k $(2 \le k \le n)$, we can set

$$\Sigma^{n-k}_+ := \{p \in \Sigma^{n-k} \,;\, \lambda^{(k)}(p) > 0\}, \quad \Sigma^{n-k}_- := \{p \in \Sigma^{n-k} \,;\, \lambda^{(k)}(p) < 0\}.$$

As a convention, we define $\Sigma^n_+ = M^n_+$ and $\Sigma^n_- = M^n_-$, where $M^n_\pm$ are as in (2.23). Also, the following assertion holds:

PROPOSITION 3.2.   *Let k be an odd positive integer, and p an A_{k+1}-point. Then the scalar multiple $\lambda^{(k)}\eta$ of the null vector field η along Σ^{n-k} points toward the domain Σ^{n-k+1}_+ at p, where $\Sigma^n := M^n$.*

PROOF.   We now take a Riemannian metric ds^2 on M^n. We denote by ds^2_{n-k+1} the Riemannian metric of Σ^{n-k+1} induced by ds^2. Then the hypersurface Σ^{n-k} embedded in Σ^{n-k+1} can be characterized as the level set $\lambda^{(k-1)} = 0$. Then we have that

$$ds^2_{n-k+1}\big(\tilde{\eta}_p, \operatorname{grad}(\lambda^{(k-1)})_p\big) = d\lambda^{(k-1)}_p(\tilde{\eta}_p) = \lambda^{(k)}(p),$$

where "grad" denotes the gradient of the function with respect to the metric ds^2_{n-k+1}. Thus $ds^2_{n-k+1}\big(\lambda^{(k)}\tilde{\eta}, \operatorname{grad}(\lambda^{(k-1)})\big)$ is positive at p. Since $\operatorname{grad}(\lambda^{(k-1)})$ gives a normal vector field along Σ^{n-k} pointing toward Σ^{n-k+1}_+, the assertion is proven.   □

DEFINITION 3.3.   Let $\varphi : TM^n \to \mathcal{E}$ be a Morin homomorphism and p an A_{2k+1}-point. Since the sign of $\lambda^{(2k)}(p)$ does not depend on the $\pm$-ambiguity of the choice of extended null vector field $\tilde{\eta}$ (cf. Proposition 3.1), we call p a *positive A_{2k+1}-point* (resp. a *negative A_{2k+1}-point*) if $\lambda^{(2k)}(p)$ is positive (resp. negative).

The set of positive (resp. negative) A_{2k+1}-points is denoted by $\mathfrak{A}^+_{2k+1}$ (resp. $\mathfrak{A}^-_{2k+1}$). Then the equalities

$$\begin{aligned}
\mathfrak{A}^+_{2k+1} &:= \{p \in \mathfrak{A}_{2k+1} \,;\, \lambda^{(2k)}(p) > 0\} = \Sigma^{n-2k}_+ \setminus \Sigma^{n-2k-1}, \\
\mathfrak{A}^-_{2k+1} &:= \{p \in \mathfrak{A}_{2k+1} \,;\, \lambda^{(2k)}(p) < 0\} = \Sigma^{n-2k}_- \setminus \Sigma^{n-2k-1}
\end{aligned} \tag{3.1}$$

hold. If $n = 2$ and $f : M^2 \to \mathbf{R}^3$ is a wave front, then positive (resp. negative) A_3-points

as in Example 2.17 correspond to positive (resp. negative) swallowtails.

Let X be a vector field of M^n which vanishes at $p \in M^n$. Take a local coordinate system $(U; x_1, \ldots, x_n)$ at p and write

$$X = \xi_1 \frac{\partial}{\partial x_1} + \cdots + \xi_n \frac{\partial}{\partial x_n}.$$

Then a zero p of X is called *generic* if the Jacobian of the map

$$U \ni q \longmapsto \big(\xi_1(q), \ldots, \xi_n(q)\big) \in \boldsymbol{R}^n$$

does not vanish at $q = p$. A vector field X defined on M^n is called *generic* if all its zeros are generic.

DEFINITION 3.4.   Let $\varphi \colon TM^n \to \mathcal{E}$ be a Morin homomorphism of depth k ($k = 1, \ldots, n$). A C^∞-vector field X defined on M^n is called a *characteristic vector field* of φ if it satisfies the following three conditions.

(i) X is a generic vector field on M^n which does not vanish at any point of Σ^{n-1}.

(ii) For each $j = n - k, \ldots, n - 1$, there exists a generic tangent vector field X_j of Σ^j such that the equality $\varphi(X) = \varphi(X_j)$ holds on Σ^j and X_j has no zeros on Σ^{j-1}.

(iii) For each A_{l+1}-point p ($l = 1, \ldots, k$) (namely, $p \in \Sigma^{n-l} \backslash \Sigma^{n-l-1}$) satisfying $\varphi(X_p) = 0$, there exists a neighborhood U of p of M^n such that the restriction of X to $U \cap \Sigma^{n-l+1}$ coincides with X_{n-l+1} on $U \cap \Sigma^{n-l+1}$ (cf. Figure 1). Moreover, if l is odd, X points into Σ_+^{n-l+1} at $p \in \Sigma^{n-l}$.

REMARK 3.5.   Let X be a characteristic vector field on M^n. If $k = n$, then $\varphi(X)$ must vanish at each A_{n+1}-point. (In fact, since any null vector fields are tangent to Σ^1 at each A_{n+1}-point p, the property (ii) yields that X_1 points in the null direction at p, and $X = X_1$ near p on Σ^1 by (iii).)

In this section, we shall construct a characteristic vector field, which will play a crucial role in proving formula (1.4) in the introduction:

PROPOSITION 3.6.   *Let M^n be a compact oriented manifold, and $\varphi : TM^n \to \mathcal{E}$ a Morin homomorphism. Suppose that $\mathcal{E}$ is oriented. Then, there exists a characteristic vector field defined on M^n associated to φ.*

To prove the assertion, we prepare the following:

LEMMA 3.7.   *Let M^n be a compact orientable manifold, and $\varphi \colon TM^n \to \mathcal{E}$ ($n \geq 1$) a Morin homomorphism of depth k ($k \geq 1$) and X a generic vector field on Σ^{n-1} such that it does not have any zero on a compact subset $C (\subset \Sigma^{n-1})$. (Here we are not assuming that $\mathcal{E}$ is orientable.) Then there exists a vector field $\tilde{X}$ satisfying the following properties:*

(1) *$\tilde{X}$ is a generic vector field on M^n which has no zeros on Σ^{n-1}.*

(2) *$\varphi(\tilde{X}) = \varphi(X)$ holds on Σ^{n-1}.*

(3) $\tilde{X} = X$ on C.

Proof.   We fix a Riemannian metric on M^n. Since M^n is orientable, we can take $\boldsymbol{n}$ as a normal vector field defined on Σ^{n-1}. Taking δ to be sufficiently small, there exists a canonical diffeomorphism

$$\exp : \Sigma^{n-1} \times [-\delta, \delta] \longrightarrow \overline{\mathcal{N}_\delta(\Sigma^{n-1})}$$

such that $t \mapsto \exp(q, t)$ is the normal geodesic of M^n with arclength parameter starting from each $q \in \Sigma^{n-1}$ in the direction $\boldsymbol{n}$. Here $\mathcal{N}_\delta(\Sigma^{n-1})$ is the δ-tubular neighborhood of Σ^{n-1} in M^n and $\overline{\mathcal{N}_\delta(Z^{n-1})}$ is its closure. Then

$$\tilde{\boldsymbol{n}}(q, s) := \frac{\partial \exp(q, s)}{\partial s}$$

gives a unit vector field defined on $\mathcal{N}_\delta(\Sigma^{n-1})$ as an extension of $\boldsymbol{n}$. Take an open neighborhood V of C as an open subset of Σ^{n-1} such that the closure $\overline{V}$ of V is compact and X has no zeros on $\overline{V}$. Without loss of generality, we may assume that the normal vector $\boldsymbol{n}$ is proportional to the null vector field on $\Sigma^{n-1} \setminus V$. Let $\rho : \Sigma^{n-1} \to [0, 1]$ be a smooth function such that

$$\rho(q) = \begin{cases} 1 & (\text{if } q \in C), \\ 0 & (\text{if } q \notin V). \end{cases}$$

Let W be the vector field on $\mathcal{N}_\delta(\Sigma^{n-1})$ obtained via parallel transport of X along each normal geodesic $s \mapsto \exp(q, s)$. We set

$$\tilde{W}(q, s) := W(q, s) + \left(s^2 \rho(q) + (1 - \rho(q))\right) \tilde{\boldsymbol{n}}(q, s), \tag{3.2}$$

which is a vector field on $\mathcal{N}_\delta(\Sigma^{n-1})$. Then $\tilde{W}$ has no zeros on $\mathcal{N}_\delta(\Sigma^{n-1})$ since X has no zeros on $\mathcal{N}_\delta(\Sigma^{n-1})$. We then apply Lemma A.1 in the appendix by setting $K = \overline{\mathcal{N}_\delta(\Sigma^{n-1})}$ and get a generic vector field $\tilde{X}$ defined on M^n such that $\varphi(\tilde{X})$ coincides with $\varphi(X)$ on Σ^{n-1}. It can be easily checked that $\tilde{X}$ is the desired vector field. $\square$
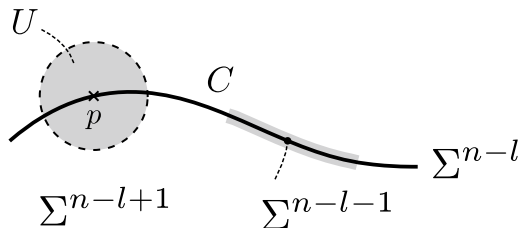

Figure 1.   Proof of Proposition 3.6.

Proof of Proposition 3.6.   We prove the assertion by induction of the depth k of the Morin homomorphism. So we firstly consider the case that $k = 1$. Suppose that

$n \geq 2$. Then Σ^{n-1} is positive dimensional. We take a generic vector field X on Σ^{n-1} and apply the previous lemma by setting C to be the empty set.

Next we consider the case that $n = 1$. Let $p_1, \ldots, p_m$ be A_2-points on M^1. Then we can take an extended null vector field $\tilde{\eta}_j$ defined on a neighborhood U_j of p_j which has no zeros on $\overline{V_j}(\subset U_j)$. We may assume that the V_j's are pairwise disjoint. Applying Lemma A.1 by setting $K = \overline{V_1} \cup \cdots \cup \overline{V_m}$, we can get a generic vector field X on M such that $X = \tilde{\eta}_j$ on V_j for $j = 1, 2, \ldots, m$, which gives the properties (i)–(iii).

We now assume that the assertion holds for $k - 1$. We fix an inner product $\langle \ , \ \rangle$ on $\mathcal{E}$. As in the proof of Theorem 2.4, we can take a unit section $\boldsymbol{u}$ such that the induced bundle $\hat{\mathcal{E}}$ defined by (2.5) is the subbundle of $\mathcal{E}$ which is orthogonal to $\boldsymbol{u}$. Using the assumption of induction, there exists a vector field X satisfying the properties (i)–(iii) on Σ^{n-1} for $\hat{\mathcal{E}}$ by taking $\dot{\lambda}$ to be a $\hat{\varphi}$-function. Let δ be a small positive number such that X has no zeros on $C := \overline{\mathcal{N}_\delta(\Sigma^{n-2})}$, where $\mathcal{N}_\delta(\Sigma^{n-2})$ is a δ-tubular neighborhood of Σ^{n-2} in the Riemannian manifold Σ^{n-1}. We apply Lemma 3.7 for X (see Figure 1), and we get the vector field $\tilde{X}$ satisfying the properties (1)–(3). Then $\tilde{X}$ satisfies (i), (ii) and (iii) by construction. The property (ii) follows from (2). $\qquad\square$

4. Adapted coordinate systems and the two dimensional case.

PROPOSITION 4.1. *Let $\varphi : TM^n \to \mathcal{E}$ be a Morin homomorphism on an n-manifold M^n. Then there exists a local coordinate system $(U; x_1, \ldots, x_n)$ centered at an A_{k+1}-point $p \in M^n$ $(k \geq 1)$ satisfying the following properties*:

(1) *For each $j = 1, \ldots, k-1$, the restriction of $\{\partial/\partial x_{j+1}, \ldots, \partial/\partial x_n\}$ spans the tangent space of Σ^{n-j} at p,*

(2) *$\partial/\partial x_k$ gives an extended null vector field on U.*

The local coordinate system $(x_1, \ldots, x_n)$ given in Proposition 4.1 is called a φ-*adapted coordinate system* at p.

PROOF. Let $\lambda : U \to \boldsymbol{R}$ be a φ-function defined on a local coordinate neighborhood $(U; y_1, \ldots, y_n)$ of p. Let $\tilde{\eta}$ be an extended null vector field on U and η its restriction to $\Sigma^{n-1} \cap U$. Then by (2) of Definition 2.2, we have that

$$\frac{\partial(\lambda, \dot{\lambda}, \ldots, \lambda^{(k-1)})}{\partial(y_1, \ldots, y_k)} \neq 0.$$

By the implicit function theorem, there exist functions $y_j(y_{k+1}, \ldots, y_n)$ $(j = 1, \ldots, k)$ such that $y_j(0, \ldots, 0) = 0$ and

$$\lambda^{(j-1)}(y_1(\hat{y}), \ldots, y_k(\hat{y}), \hat{y}) = 0 \qquad (j = 1, \ldots, k),$$

where $\hat{y} = (y_{k+1}, \ldots, y_n)$ and $\lambda^{(0)} := \lambda$. So if we set

$$x_1 := \lambda, \ x_2 := \dot{\lambda}, \ \ldots, \ x_k := \lambda^{(k-1)}, \quad x_l := y_l \qquad (l = k+1, \ldots, n),$$

then $\psi := (x_1, \ldots, x_n)$ gives a new local coordinate system at p satisfying the property (1). Then the restriction $\eta|_{\Sigma^{n-k}}$ of the null vector field is a tangent vector field of Σ^{n-k+1} along Σ^{n-k}, and can be written as

$$\eta|_{\Sigma^{n-k+1}} = \sum_{j=k}^{n} c_j \partial_j,$$

where $\partial_j := \partial/\partial x_j \; (j = k, \ldots, n)$. Since η is transversal to Σ^{n-k} at p, the coefficient c_k does not vanish. Let $\{g_t\}_{|t|<\epsilon} : V \to M^n$ be the local 1-parameter group of transformations generated by $\tilde{\eta}$, where $V (\subset U)$ is a neighborhood of p in M^n and $\epsilon > 0$ is a small positive number. Then

$$\Phi : (t_1, t_2, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_n) \mapsto g_{t_k}(\psi(t_1, t_2, \ldots, t_{k-1}, 0, t_{k+1}, \ldots, t_n))$$

gives a local diffeomorphism such that the equalities

$$d\Phi(\partial/\partial t_k) = \eta, \quad d\Phi(\partial/\partial t_l) = \partial/\partial x_l \qquad (l = k+1, \ldots, n)$$

hold, and they span the tangent space of Σ^{n-k+1} when $t_1 = t_2 = \cdots = t_k = 0$. Thus the inverse map Φ^{-1} gives the desired local coordinate system. $\qquad\square$

Here we prove formula (1.4) for $n = 2$. Although this formula was proved as a corollary of the Gauss–Bonnet type formula in [**23**] and [**24**], our proof in this section is new.

Let X be a characteristic vector field associated to a Morin homomorphism $\varphi : TM^2 \to \mathcal{E}$ of depth at most 2 on a compact oriented 2-manifold, and we assume that $\mathcal{E}$ is oriented. Take a section Y of $\mathcal{E}$ as $Y := \varphi(X)$. Then the following assertion holds:

PROPOSITION 4.2. *Let $Z(Y)$, $Z(X)$ be the set of zeros on M^2 of Y and X, respectively, and let $Z(X_1)$ be the zeros on Σ^1 of X_1 (as in Definition 3.4). Then it holds that*

$$Z(Y) \cap (M^2 \setminus \Sigma^1) = Z(X), \tag{4.1}$$

$$Z(Y) \cap (\Sigma^1 \setminus \Sigma^0) = Z(X_1) \subset \mathfrak{A}_2, \tag{4.2}$$

$$Z(Y) \cap \Sigma^0 = \mathfrak{A}_3. \tag{4.3}$$

PROOF.   Since $Y = \varphi(X)$, property (i) in Definition 3.4 implies that $Z(X) \subset Z(Y)$. Since $\varphi : T_p M^2 \to \mathcal{E}_p$ is a linear isomorphism when $p \in M^2 \setminus \Sigma^1$, we have (4.1). Since $Z(X_1) \cap \Sigma^0$ is the empty set, property (ii) of characteristic vector field yields

$$Z(Y) \cap (\Sigma^1 \setminus \Sigma^0) = Z(X_1).$$

Since $Y = \varphi(X_1)$ on Σ^1 and X_1 is proportional to a null vector at each A_3-point p, we obtain (4.3). $\qquad\square$

When $n = 2$, (1.8) reduces to

$$\chi_{\mathcal{E}} = \sum_{p \in M^2 \setminus \Sigma^1} \mathrm{ind}_p(Y) + \sum_{p \in \mathfrak{A}_2} \mathrm{ind}_p(Y) + \sum_{p \in \mathfrak{A}_3} \mathrm{ind}_p(Y). \tag{4.4}$$

PROPOSITION 4.3. *The first term of the right-hand side of* (4.4) *satisfies*

$$\sum_{p \in M^2 \setminus \Sigma^1} \mathrm{ind}_p(Y) = \chi(M_+^2) - \chi(M_-^2). \tag{4.5}$$

PROOF.   Let p be in $Z(Y) \setminus \Sigma^1$, and λ be an oriented φ-function on a neighborhood of p. We denote by $\mathrm{sgn}(\lambda(p))$ the sign of the function λ at the point p. Since $\mathrm{sgn}(\lambda(p)) = 1$ (resp. $\mathrm{sgn}(\lambda(p)) = -1$) if $\varphi_p : T_p M^2 \to \mathcal{E}_p$ is orientation preserving (resp. orientation reversing), we have that

$$\mathrm{ind}_p(Y) = \mathrm{sgn}(\lambda(p)) \, \mathrm{ind}_p(X) \qquad (p \in M^2 \setminus \Sigma^1).$$

We set

$$\bar{M}_+^2(\delta) := \overline{M_+^2 \setminus \mathcal{N}_\delta(\Sigma^1)}, \quad \bar{M}_-^2(\delta) := \overline{M_-^2 \setminus \mathcal{N}_\delta(\Sigma^1)} \qquad (\delta > 0),$$

where $\mathcal{N}_\delta(\Sigma^1)$ is the δ-tubular neighborhood of Σ^1 as in the proof of Lemma 3.7, and the overline means the closure operation. If we choose δ sufficiently small, then $Z(Y) \cap (M^2 \setminus \Sigma^1)$ is contained in $\bar{M}_+^2(\delta) \cup \bar{M}_-^2(\delta)$ and $\bar{M}_+^2(\delta)$ (resp. $\bar{M}_-^2(\delta)$) has the same homotopy type as M_+^2 (resp. M_-^2). In particular, the following identity holds

$$\sum_{p \in M^2 \setminus \Sigma^1} \mathrm{ind}_p(Y) = \sum_{p \in \bar{M}_+^2(\delta)} \mathrm{ind}_p(X) - \sum_{p \in \bar{M}_-^2(\delta)} \mathrm{ind}_p(X). \tag{4.6}$$

Here, $-X$ (resp. X) is an outward vector of $\bar{M}_+^2(\delta)$ (resp. $\bar{M}_-^2(\delta)$) by property (iii) of Definition 3.4 of the characteristic vector field X. Since the operation $X \mapsto -X$ is orientation preserving, applying the Poincaré–Hopf index formula (cf. [**14**]), we have that

$$\chi(M_+^2) = \chi(\bar{M}_+^2(\delta)) = \sum_{p \in \bar{M}_+^2(\delta)} \mathrm{ind}_p(-X) = \sum_{p \in \bar{M}_+^2(\delta)} \mathrm{ind}_p(X) = \sum_{p \in M_+^2} \mathrm{ind}_p(X).$$

Similarly, we can also show that

$$\chi(M_-^2) = \sum_{p \in M_-^2} \mathrm{ind}_p(X),$$

which proves the assertion.   □

PROPOSITION 4.4.   *The second term of the right-hand side of* (4.4) *satisfies*

$$\sum_{p \in \mathfrak{A}_2} \mathrm{ind}_p(Y) = 0. \tag{4.7}$$

PROOF.   We fix p in $Z(Y) \cap (\Sigma^1 \setminus \Sigma^0)$. Then p is an A_2-point. Let $(U; x_1, x_2)$ be a φ-adapted coordinate system as in Proposition 4.1 (for $n = 2$ and $k = 2$) around p which

is compatible with the orientation of M^2. Then by (2) of Proposition 4.1,

$$\tilde{\eta} := \partial/\partial x_1$$

gives an extended null vector field on U. Let μ be an orientation (i.e. a non-vanishing section of the determinant bundle of $\mathcal{E}^*$ which is compatible with the orientation of $\mathcal{E}$ defined on M^2) of $\mathcal{E}$, and set

$$\lambda := \mu\left(\varphi\left(\frac{\partial}{\partial x_1}\right), \varphi\left(\frac{\partial}{\partial x_2}\right)\right).$$

Then λ is an oriented φ-function with respect to the orientations of $\mathcal{E}$ and M^2. Since $\partial/\partial x_1$ is an extended null vector field, $\varphi(\partial/\partial x_1)$ vanishes on $\Sigma^1 \cap U = \{\lambda = 0\}$. Then by the well-known preparation theorem for C^∞-functions, there exists a section $\boldsymbol{e}_1$ of $\mathcal{E}$ such that $\varphi(\partial/\partial x_1) = \lambda \boldsymbol{e}_1$. On the other hand, we set $\boldsymbol{e}_2 := \varphi(\partial/\partial x_2)$. Then $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ gives a frame field of $\mathcal{E}$ on U which is compatible with the orientation of $\mathcal{E}$. In fact,

$$\lambda = \mu\left(\varphi\left(\frac{\partial}{\partial x_1}\right), \varphi\left(\frac{\partial}{\partial x_2}\right)\right) = \mu(\lambda \boldsymbol{e}_1, \boldsymbol{e}_2) = \lambda\mu(\boldsymbol{e}_1, \boldsymbol{e}_2),$$

and hence $\mu(\boldsymbol{e}_1, \boldsymbol{e}_2) = 1$. We set

$$X = \xi_1\frac{\partial}{\partial x_1} + \xi_2\frac{\partial}{\partial x_2} \qquad \text{and} \qquad Y = \alpha_1\boldsymbol{e}_1 + \alpha_2\boldsymbol{e}_2.$$

Then it holds that

$$\alpha_1 = \lambda\xi_1, \qquad \alpha_2 = \xi_2.$$

Since λ vanishes on Σ^1 and since $\partial/\partial x_2$ spans $T_p\Sigma^1$ (cf. (1) of Proposition 4.1), we have $\lambda(p) = \lambda_{x_2}(p) = 0$, where $\lambda_{x_2} := \partial\lambda/\partial x_2$. In particular, the equality $(\alpha_1)_{x_2} = \partial\alpha_1/\partial x_2 = 0$ holds at p. Since the equalities $(\alpha_1)_{x_1} = \lambda_{x_1}\xi_1 = \dot{\lambda}\xi_1$ also hold at p, we have that

$$\text{sgn}\left(\det\left(\begin{matrix}(\alpha_1)_{x_1} & (\alpha_1)_{x_2} \\ (\alpha_2)_{x_1} & (\alpha_2)_{x_2}\end{matrix}\right)\right) = \text{sgn}\left(\dot{\lambda}\xi_1(\alpha_2)_{x_2}\right)$$

$$= \text{sgn}\left(\dot{\lambda}\xi_1(\xi_2)_{x_2}\right) = \text{ind}_p(X_1)\,\text{sgn}\left(\dot{\lambda}\xi_1\right).$$

Here, we used the relation $\text{ind}_p(X_1) = \text{sgn}(\xi_2)_{x_2}$. In fact, by (1) of Proposition 4.1, one can parametrize Σ^1 around p as

$$\Sigma^1 \cap U = \{(x_1, x_2) = (f(t), t)\,;\, t \in I\},$$

where I is a sufficiently small interval including 0 and f is a smooth function on I such that $df(0)/dt = 0$. That is, t can be taken as a local coordinate system of Σ^1. Then there exists a smooth function $\hat{\xi} : I \to \boldsymbol{R}$ such that

$$X_1 = \hat{\xi}\frac{d}{dt} = \hat{\xi}\left(\frac{df}{dt}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right).$$

The condition (ii) of Definition 3.4 yields that

$$\hat{\xi} = \xi_2, \qquad \frac{d\hat{\xi}}{dt} = \frac{df}{dt}\frac{\partial \xi_2}{\partial x_1} + \frac{\partial \xi_2}{\partial x_2}.$$

Since $df(0)/dt = 0$, we have

$$\mathrm{ind}_p(X_1) = \mathrm{sgn}_{t=0}\left(\frac{d\hat{\xi}}{dt}\right) = \mathrm{sgn}_p\left(\frac{\partial \xi_2}{\partial x_2}\right).$$

Since $p \notin \mathcal{N}_\delta(\Sigma^0)$ for sufficiently small δ, the characteristic vector field X points in the direction of $M_+^2 = \{\lambda > 0\}$ at p. So the equality

$$\mathrm{sgn}(\xi_1) = \mathrm{sgn}(\dot{\lambda})$$

holds at p. Thus $\dot{\lambda}(p)\xi_1(p) > 0$ and

$$\mathrm{ind}_p(Y) = \mathrm{ind}_p(X_1).$$

Since $Z(X_1) \subset \mathfrak{A}_2$ and $\chi(\Sigma^1) = 0$, applying the Poincaré–Hopf index formula for the vector field X_1 on Σ^1, we get the assertion. $\square$

By (4.4), Proposition 4.3 and Proposition 4.4, formula (1.4) follows immediately from the following assertion:

PROPOSITION 4.5. *Let p be an arbitrarily given A_3-point. Then*

$$\mathrm{ind}_p(Y) = \begin{cases} 1 & (\textit{if } p \in \mathfrak{A}_3^+), \\ -1 & (\textit{if } p \in \mathfrak{A}_3^-). \end{cases}$$

PROOF. We take a φ-adapted coordinate system $(U; x_1, x_2)$ centered at p which is compatible with the orientation of M^2. In particular, $\tilde{\eta} := \partial/\partial x_2$ is an extended null vector field on U, and $(\partial/\partial x_2)_p \in T_p\Sigma^1$. Let μ be a local orientation of $\mathcal{E}$, and let $\lambda := \mu\big(\varphi(\partial/\partial x_1), \varphi(\partial/\partial x_2)\big)$. We set $\boldsymbol{e}_1 := \varphi(\partial/\partial x_1)$. Since $\varphi(\tilde{\eta})$ vanishes on Σ^1, there exists a section $\boldsymbol{e}_2$ of $\mathcal{E}$ on U such that $\varphi(\partial/\partial x_2) = \varphi(\tilde{\eta}) = \lambda\boldsymbol{e}_2$. Since

$$\lambda = \mu\left(\varphi\left(\frac{\partial}{\partial x_1}\right), \varphi\left(\frac{\partial}{\partial x_2}\right)\right) = \mu(\boldsymbol{e}_1, \lambda\boldsymbol{e}_2) = \lambda\mu(\boldsymbol{e}_1, \boldsymbol{e}_2),$$

we have $\mu(\boldsymbol{e}_1, \boldsymbol{e}_2) = 1$, which implies that $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ forms a frame field of $\mathcal{E}$ compatible with the orientation of $\mathcal{E}$. We set

$$X = \xi_1\frac{\partial}{\partial x_1} + \xi_2\frac{\partial}{\partial x_2}.$$

By (iii) and (i) of Definition 3.4, $X_p \in T_p\Sigma^1$ and $X_p \neq 0$, and hence we have $\xi_1(p) = 0$

and $\xi_2(p) \neq 0$. We now set

$$Y = \alpha_1 \boldsymbol{e}_1 + \alpha_2 \boldsymbol{e}_2. \tag{4.8}$$

Then it holds that $\alpha_1 = \xi_1$ and $\alpha_2 = \lambda \xi_2$. By (iii) of Definition 3.4, X is tangent to Σ^1 near p. Since λ vanishes along Σ^1, it holds that

$$0 = d\lambda(X) = \lambda_{x_1}\xi_1 + \dot{\lambda}\xi_2$$

on a sufficiently small neighborhood p in Σ^1, where we used the fact that $\lambda_{x_2} = \dot{\lambda}$ (cf. (2) of Proposition 4.1). Since $d\lambda(X)$ vanishes along Σ^1 and $\partial/\partial x_2 \in T\Sigma^1$ at p, the fact $\xi_1(p) = \dot{\lambda}(p) = 0$ yields that the equalities

$$0 = \frac{\partial d\lambda(X)}{\partial x_2} = \lambda_{x_1 x_2}\xi_1 + \lambda_{x_1}(\xi_1)_{x_2} + \ddot{\lambda}\xi_2 + \dot{\lambda}(\xi_2)_{x_2}$$

$$= \lambda_{x_1}(\xi_1)_{x_2} + \ddot{\lambda}\xi_2$$

hold at p. Since $d\lambda(p) \neq 0$ and $\lambda_{x_2}(p) = \dot{\lambda}(p) = 0$, we can conclude that $\lambda_{x_1}(p) \neq 0$. In particular, we have that

$$(\xi_1)_{x_2}(p) = -\frac{\ddot{\lambda}(p)\xi_2(p)}{\lambda_{x_1}(p)}.$$

Using the facts $\lambda_{x_2}(p) = \dot{\lambda}(p) = 0$, we have that

$$\text{ind}_p(Y) = \text{sgn}\left(\det\begin{pmatrix} (\xi_1)_{x_1}(p) & (\xi_1)_{x_2}(p) \\ \lambda_{x_1}(p)\xi_2(p) & \xi_2(p)\lambda_{x_2}(p) \end{pmatrix}\right)$$

$$= \text{sgn}\left(\det\begin{pmatrix} (\xi_1)_{x_1}(p) & (\xi_1)_{x_2}(p) \\ \lambda_{x_1}(p)\xi_2(p) & 0 \end{pmatrix}\right)$$

$$= -\text{sgn}\left(\xi_2(p)\lambda_{x_1}(p)\left(-\frac{\ddot{\lambda}(p)\xi_2(p)}{\lambda_{x_1}(p)}\right)\right) = \text{sgn}\left(\xi_2(p)^2\ddot{\lambda}(p)\right).$$

Since the sign of an A_3-point coincides with the sign of $\ddot{\lambda}$, Proposition 4.5 is proved. $\square$

5. The proof of the index formula.

In this section, we prove our formula (1.4) for n-manifolds ($n = 2m \geq 4$).

Let M^n be an oriented manifold, and X a characteristic vector field associated to a Morin homomorphism $\varphi : TM^n \to \mathcal{E}$. Suppose that $\mathcal{E}$ is oriented. Let $(U; x_1, \ldots, x_n)$ be a φ-adapted coordinate system centered at an A_2-point $p \in M^n$ (cf. Proposition 4.1), which is compatible with the orientation of M^n. Suppose that $Y := \varphi(X)$ vanishes at p. Then X has an expression

$$X = \xi_1\frac{\partial}{\partial x_1} + \cdots + \xi_n\frac{\partial}{\partial x_n}. \tag{5.1}$$

By a property of φ-adapted coordinate systems,

$$\eta := \frac{\partial}{\partial x_1}$$

gives a null vector field. By (ii) in Definition 3.4, $\xi_1 \neq 0$ holds. Moreover, the fact $\varphi(X_p) = 0$ yields that

$$\xi_1(p) \neq 0, \quad \xi_j(p) = 0 \qquad (j = 2, \ldots, n). \tag{5.2}$$

LEMMA 5.1. *It holds that*

$$\mathrm{ind}_p(Y) = \mathrm{sgn}\left(\xi_1(p)\dot{\lambda}(p)\right)\mathrm{ind}_p(X_{n-1}).$$

PROOF. Let μ be an orientation of $\mathcal{E}$, and set

$$\lambda := \mu\left(\varphi\left(\frac{\partial}{\partial x_1}\right), \ldots, \varphi\left(\frac{\partial}{\partial x_n}\right)\right),$$

which is an oriented φ-function on a neighborhood of p. We set

$$\boldsymbol{e}_j := \varphi(\partial/\partial x_j) \qquad (j = 2, \ldots, n).$$

Since $\tilde{\eta}_1 = \partial/\partial x_1$ is an extended null vector field, by the preparation theorem for C^∞-functions, we can write $\varphi(\partial/\partial x_1) = \lambda\boldsymbol{e}_1$, where $\boldsymbol{e}_1$ is a local section defined on a neighborhood of p. Since

$$\lambda = \mu\left(\varphi\left(\frac{\partial}{\partial x_1}\right), \ldots, \varphi\left(\frac{\partial}{\partial x_n}\right)\right) = \lambda\mu(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n),$$

we have $\mu(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) = 1$. In particular, $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ gives an oriented frame on the vector bundle $\mathcal{E}$ around p. So we can write

$$Y = \alpha_1\boldsymbol{e}_1 + \cdots + \alpha_n\boldsymbol{e}_n, \qquad \text{where} \qquad \alpha_j = \begin{cases} \xi_j & (j \neq 1), \\ \lambda\xi_1 & (j = 1). \end{cases} \tag{5.3}$$

We set

$$J := \det(\alpha_{ij})_{i,j=1,\ldots,n}, \qquad \alpha_{ij} := \frac{\partial\alpha_i}{\partial x_j}.$$

If $J(p) \neq 0$, it holds that

$$\mathrm{ind}_p(Y) = \mathrm{sgn}\big(J(p)\big). \tag{5.4}$$

By (5.2) and (5.3), we have that

$$(\alpha_1)_{x_1}(p) = \xi_1(p)\lambda_{x_1}(p) \neq 0, \qquad (\alpha_1)_{x_2}(p) = \cdots = (\alpha_1)_{x_n}(p) = 0. \tag{5.5}$$

Then (5.5) implies that

$$J(p) = \det \begin{pmatrix} (\alpha_1)_{x_1} & 0 & \dots & 0 \\ (\alpha_2)_{x_1} & (\alpha_2)_{x_2} & \dots & (\alpha_2)_{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ (\alpha_n)_{x_1} & (\alpha_n)_{x_2} & \dots & (\alpha_n)_{x_n} \end{pmatrix}$$

$$= (\alpha_1)_{x_1} \det \begin{pmatrix} (\alpha_2)_{x_2} & \dots & (\alpha_2)_{x_n} \\ \vdots & \ddots & \vdots \\ (\alpha_n)_{x_2} & \dots & (\alpha_n)_{x_n} \end{pmatrix}$$

$$= (\alpha_1)_{x_1} \det \begin{pmatrix} (\xi_2)_{x_2} & \dots & (\xi_2)_{x_n} \\ \vdots & \ddots & \vdots \\ (\xi_n)_{x_2} & \dots & (\xi_n)_{x_n} \end{pmatrix},$$

because of (5.3). Thus, by (5.2), (5.4) and (5.5), we have that

$$\mathrm{ind}_p(Y) = \mathrm{sgn}\big(J(p)\big) = \mathrm{sgn}\big((\alpha_1)_{x_1}\big) \, \mathrm{sgn} \, \frac{\partial(\xi_2, \dots, \xi_n)}{\partial(x_2, \dots, x_n)}$$

$$= \mathrm{sgn}\big(\xi_1(p)\lambda_{x_1}(p)\big) \, \mathrm{ind}_p(X_{n-1}). \qquad \square$$

We now prove the formula (1.4). Let M^n $(n = 2m)$ be a compact oriented n-manifold without boundary, and $\varphi : TM^n \to \mathcal{E}$ be a Morin homomorphism, where $\mathcal{E}$ is an oriented vector bundle. We fix a characteristic vector field X as in the previous section (cf. Proposition 3.6). Take a section Y of $\mathcal{E}$ as

$$Y := \varphi(X).$$

We denote by $Z(X)$ and $Z(Y)$ the set of zeros of X and Y, respectively. The following assertion can be proved as in Proposition 4.2.

PROPOSITION 5.2.   *Let $Z(X_{n-j})$ $(j = 0, 1)$ be the set of zeros for X_{n-j}, where $X_n = X$. Then it holds that*

$$Z(Y) \cap (M^n \setminus \Sigma^{n-1}) = Z(X), \tag{5.6}$$

$$Z(Y) \cap (\Sigma^{n-1} \setminus \Sigma^{n-2}) = Z(X_{n-1}). \tag{5.7}$$

By (1.8), it is sufficient to show the following assertion.

THEOREM 5.3.   *The following identity holds*

$$\sum_{p \in M^n} \mathrm{ind}_p(Y) = \chi(M_+^{2m}) - \chi(M_-^{2m}) + \sum_{j=1}^{m} \big(\chi(\mathfrak{A}_{2j+1}^+) - \chi(\mathfrak{A}_{2j+1}^-)\big).$$

We prove the theorem by induction on the dimension $n = 2m$. We have already shown that the formula holds for $m = 1$ in Section 4. So we now assume that the formula 1.4 holds for $m - 1$, and will prove the case for m. Let

$$\hat{\varphi} : T\Sigma^{n-1} \to \hat{\mathcal{E}}$$

be the reduction. Then it induces again the second reduction $\hat{\hat{\varphi}} : T\Sigma^{n-2} \to \hat{\hat{\mathcal{E}}}$. Since $\mathcal{E}$ is oriented, we can take an oriented φ-function $\lambda : M^n \to \boldsymbol{R}$ satisfying (2.22). By Proposition 3.1, $\ddot{\lambda}$ is an oriented $\hat{\hat{\varphi}}$-function of $\hat{\hat{\mathcal{E}}}$ defined on Σ^{n-2}. Since the restriction of X to Σ^{n-2} is a characteristic vector field of Σ^{n-2}, the induction assumption yields that

$$\sum_{p \in \Sigma^{n-2}} \operatorname{ind}_p(Y) = \sum_{j=1}^{m} \left(\chi(\mathfrak{A}_{2j+1}^+) - \chi(\mathfrak{A}_{2j+1}^-) \right).$$

On the other hand, as in Proposition 4.3, one can prove the following assertion:

PROPOSITION 5.4. *The first term of the right-hand side of* (1.8) *in the introduction satisfies*

$$\sum_{p \in M^n \setminus \Sigma^{n-1}} \operatorname{ind}_p(Y) = \chi(M_+^n) - \chi(M_-^n). \tag{5.8}$$

Now formula (1.4) for the $2m$-dimensional case reduces to the following assertion:

PROPOSITION 5.5. *The second term of the right-hand side of* (1.8) *in the introduction satisfies*

$$\sum_{p \in \Sigma^{n-1} \setminus \Sigma^{n-2}} \operatorname{ind}_p(Y) = 0. \tag{5.9}$$

PROOF. We fix a point $p \in \Sigma^{n-1} \setminus \Sigma^{n-2}$ satisfying $Y_p = 0$ arbitrarily. By property (iii) in Definition 3.4, there exists a vector field X_{n-1} on Σ^{n-1} such that $Z(X_{n-1}) = Z(Y) \cap (\Sigma^{n-1} \setminus \Sigma^{n-2})$. By Lemma 5.1, it holds that

$$\operatorname{ind}_p(Y) = \operatorname{ind}_p(X_{n-1}) \operatorname{sgn}\left(\dot{\lambda}(p)\xi_1(p)\right).$$

By (iii) of Definition 3.4, $\xi_1 \partial/\partial x_1$ points into M_+^n at p. Since $\partial/\partial x_1$ is an extended null vector field, $\dot{\lambda}\partial/\partial x_1$ points also into M_+^n at p (cf. Proposition 3.2). Hence

$$\operatorname{sgn}\left(\dot{\lambda}(p)\xi_1(p)\right) \geq 0,$$

and $\operatorname{ind}_p(Y) = \operatorname{ind}_p(X_{n-1})$ holds. Since $Z(X_{n-1}) = Z(Y) \cap (\Sigma^{n-1} \setminus \Sigma^{n-2})$ and Σ^{n-1} is odd dimensional, it holds that

$$\sum_{p \in \Sigma^{n-1} \setminus \Sigma^{n-2}} \operatorname{ind}_p(Y) = \sum_{p \in \Sigma^{n-1}} \operatorname{ind}_p(X_{n-1}) = \chi(\Sigma^{n-1}) = 0. \qquad \square$$

6. Applications.

In this section, we shall give several applications of the formula (1.7): recall that a C^∞-map $f : M^{2m} \to N^{2m}$ between $2m$-manifolds is called a *Morin map* if the corresponding bundle homomorphism $\varphi = df$ as in Example 2.15 admits only A_k-singularities

for $k = 2, \ldots, 2m + 1$ (cf. Remark 2.18).

THEOREM 6.1 ([**15**] and [**4**]).   *Let M^{2m} and N^{2m} be compact oriented 2m-manifolds, and let $f : M^{2m} \to N^{2m}$ be a Morin map. Then it holds that*

$$\deg(f)\chi(N^{2m}) = \chi(M_+^{2m}) - \chi(M_-^{2m}) + \sum_{j=1}^{m} \chi(\mathfrak{A}_{2j+1}^+) - \chi(\mathfrak{A}_{2j+1}^-), \qquad (6.1)$$

where $\deg(f)$ is the topological degree of the map f, and M_+^{2m} (resp. M_-^{2m}) is the set of points at which the Jacobian of f is positive (resp. negative).

This formula is a generalization of Quine's formula [**16**] for Morin maps between 2-manifolds (see also [**5**]). It should be remarked that the numbering of Morin singularities is different from the usual one (cf. Remark 2.18). For example, a fold (resp. a cusp) singularity is an A_2-singular point (resp. an A_3-singular point) in (6.1).

PROOF OF THEOREM 6.1.   Let $\mathcal{E}$ be the pull-back of the tangent bundle TN^{2m} of N^{2m} by f. Then the map f induces a bundle homomorphism $\varphi_f := df : TM^{2m} \to \mathcal{E}$ as in Example 2.15. Since f is a Morin map, φ_f has only A_k-points, and then the formula follows from (1.4) using the fact that $\chi_{\mathcal{E}} = \deg(f)\chi(N^{2m})$.   □

Next we give applications for immersed hypersurfaces in $\boldsymbol{R}^{2m+1}$. Let M^{2m} be a compact oriented 2m-manifold and $f : M^{2m} \to \boldsymbol{R}^{2m+1}$ a wave front. Suppose that there exists a unit normal vector field ν along f defined on M^{2m}. Then it induces the Gauss map into the unit 2m-sphere $\nu : M^{2m} \longrightarrow S^{2m}$, and a family of wave fronts

$$f_t := f + t\nu \qquad (t \in \boldsymbol{R}),$$

each of which is called a *parallel hypersurface* of f. The Gauss map of f_t is commonly equal to ν for all $t \in \boldsymbol{R}$. The Gauss map ν can be considered as the limit $\lim_{t \to \infty} f_t/t$.

COROLLARY 6.2.   *Let M^{2m} be a compact oriented 2m-manifold and $f : M^{2m} \longrightarrow \boldsymbol{R}^{2m+1}$ an immersion. Suppose that the Gauss map ν is a Morin map. Then the singular set of ν satisfies identity (1.7) in the introduction, where M_-^{2m} is the set of points at which the Gauss–Kronecker curvature of f (i.e. the determinant of the shape operator) is negative.*

This formula is a generalization of the Bleecker–Wilson formula for Gauss maps of immersed surfaces in $\boldsymbol{R}^3$.

PROOF OF COROLLARY 6.2.   We apply formula (6.1) for the Gauss map ν of the immersion f. Then we have that

$$2(\deg \nu) = \chi(M_+^{2m}) - \chi(M_-^{2m}) + \sum_{j=1}^{m} \left(\chi(\mathfrak{A}_{2j+1}^+) - \chi(\mathfrak{A}_{2j+1}^-) \right).$$

Since f is an immersion, it is well-known that $2(\deg \nu)$ is equal to $\chi(M^{2m})$.

Next, we show that M_+^{2m} (resp. M_-^{2m}) coincides with the set where the Gauss–Kronecker curvature is positive (resp. negative): Let ds^2 be the induced Riemannian metric on M^{2m} by the immersion f, and let $e_1, \ldots, e_{2m}$ be an oriented local orthonormal frame field on M^{2m} such that

$$d\nu(e_j) = -\mu_j df(e_j) \qquad (j = 1, \ldots, 2m),$$

that is, $e_1, \ldots, e_{2m}$ are eigenvector fields of the shape operator of f, and $\mu_1, \ldots, \mu_{2m}$ are principal curvatures. Then we have that

$$\lambda := \det\big(d\nu(e_1), \ldots, d\nu(e_{2m}), \nu\big) = \prod_{j=1}^{2m} \mu_j = K, \tag{6.2}$$

where $K := \mu_1 \cdots \mu_{2m}$ is the Gauss–Kronecker curvature of f. This λ is positive (resp. negative) if and only if $K > 0$ (resp. $K < 0$), which proves the assertion. $\square$

Next, we show the following.

THEOREM 6.3. *Let M^{2m} be a compact oriented $2m$-manifold and $f : M^{2m} \to \mathbf{R}^{2m+1}$ a wave front. Suppose that f admits only A_k-front singularities ($2 \le k \le 2m+1$), as defined in Definition 2.16. Then the singular set of f satisfies the identity*

$$2\deg(\nu) = \chi(M_+^{2m}) - \chi(M_-^{2m}) + \sum_{j=1}^{m} \left(\chi(\mathfrak{A}_{2j+1}^+) - \chi(\mathfrak{A}_{2j+1}^-)\right), \tag{6.3}$$

where $\deg(\nu)$ is the degree of the Gauss map $\nu \colon M^{2m} \to S^{2m}$ induced by f, and $\chi(M_+^{2m})$ (resp. $\chi(M_-^{2m})$) is the Euler characteristic of the subset M_+^{2m} (resp. M_-^{2m}) of M^{2m} at which

$$\lambda := \det(f_{x_1}, \ldots, f_{x_{2m}}, \nu)$$

is positive (resp. negative) for an oriented local coordinate system $(x_1, \ldots, x_{2m})$, where $f_{x_j} = \partial f / \partial x_j$.

This formula is independent of the index formula for the Gauss map ν (cf. Theorem 6.1). In fact, the singular set of f does not coincide with that of its Gauss map in general.

PROOF OF THEOREM 6.3. We apply (1.4) for the bundle homomorphism

$$\varphi_f := df : TM^{2m} \longrightarrow \mathcal{E}_f$$

as in Example 2.17. Then it is sufficient to show that $\chi_{\mathcal{E}_f}$ is equal to $2\deg(\nu)$. Let ξ be a vector field on the unit $2m$-sphere S^{2m}. By parallel transport, ξ_q ($q \in S^{2m}$) can be considered as a vector in $\mathcal{E}_p$ for $p \in \nu^{-1}(q)$. Thus, ξ induces a section $\tilde{\xi}$ of $\mathcal{E}$ defined on M^{2m}. Then the equalities

$$\chi_{\mathcal{E}_f} = \sum_{p \in M^{2m}} \mathrm{ind}_p(\tilde{\xi}) = \deg(\nu) \sum_{q \in S^{2m}} \mathrm{ind}_q(\xi) = \deg(\nu)\chi(S^{2m}) = 2\deg(\nu)$$

hold, which proves the identity. □

Next, we give an application to parallel hypersurfaces of strictly convex hypersurfaces.

THEOREM 6.4.   *Let S^{2m} be the unit 2m-sphere, and let $f : S^{2m} \longrightarrow \mathbf{R}^{2m+1}$ be a strictly convex immersion, that is, the Gauss map $\nu \colon S^{2m} \to S^{2m}$ is a diffeomorphism. Let $t \in \mathbf{R}$ be a value such that the parallel hypersurface*

$$f_t : S^{2m} \longrightarrow \mathbf{R}^{2m+1}$$

has only A_k-singularities ($k = 2, \ldots, 2m + 1$). Then the singular set of f_t satisfies (1.7) and $1/K_t$ can be extended as a C^∞-function on S^{2m} and gives an oriented φ_t-function for $\varphi_t = df_t$ (cf. Definition 2.1), where K_t is the Gauss–Kronecker curvature of f_t.

The corresponding assertion for a convex surface $f : S^2 \to \mathbf{R}^3$ is given by Martinez-Maure [13] under the generic assumption that the Gaussian curvature is unbounded at the singular set of f_t, and proved in [24] for the general case. The above formula is a generalization of it.

PROOF OF THEOREM 6.4.   We apply Theorem 6.3 for the bundle homomorphism $\varphi_t = df_t : TS^{2m} \to \mathcal{E}_{f_t}$. Since f is convex, the Gauss map $\nu \colon S^{2m} \to S^{2m}$ is of degree one. Since $f = f_0$ is an immersion, and the Gauss map ν is common in the parallel family $\{f_t\}_{t \in \mathbf{R}}$, we have that

$$\chi(M_+^{2m}) + \chi(M_-^{2m}) = \chi(S^{2m}) = 2\deg(\nu)$$

$$= \chi(M_+^{2m}) - \chi(M_-^{2m}) + \sum_{j=1}^{m} \left(\chi(\mathfrak{A}_{2j+1}^+) - \chi(\mathfrak{A}_{2j+1}^-) \right),$$

where $M_+^{2m} = S_+^{2m}$ (resp. $M_-^{2m} = S_-^{2m}$) is the set where $\lambda_t > 0$ (resp. $\lambda_t < 0$). Here, $\lambda = \lambda_t$ is the function as in the statement of Theorem 6.3. Moreover, since ν is an immersion, one can take the Riemannian metric $d\sigma^2$ on S^{2m} as the pull-back of the canonical metric of S^{2m} by ν, and let $\{e_1, \ldots, e_{2m}\}$ be an oriented local orthonormal frame field on S^{2m} with respect to $d\sigma^2$ such that

$$df(e_j) = -(1/\mu_j)d\nu(e_j) \qquad (j = 1, \ldots, 2m),$$

that is, $e_1, \ldots, e_{2m}$ are eigenvector fields of the shape operator of f. Since

$$df_t(e_j) = df(e_j) + td\nu(e_j) = -\left(\frac{1}{\mu_j} - t \right) d\nu(e_j),$$

the Gauss–Kronecker curvature K_t of f_t is expressed as

$$K_t = \left(\prod_{j=1}^{2m}\left(\frac{1}{\mu_j} - t\right)\right)^{-1}.$$

On the other hand,

$$\lambda_t := \det\big(df_t(\boldsymbol{e}_1),\ldots,df_t(\boldsymbol{e}_{2m}),\nu\big)$$

$$= \left(\prod_{j=1}^{2m}\left(\frac{1}{\mu_j} - t\right)\right)\det\big(d\nu(\boldsymbol{e}_1),\ldots,d\nu(\boldsymbol{e}_{2m}),\nu\big)$$

$$= \frac{1}{K_t}\det\big(d\nu(\boldsymbol{e}_1),\ldots,d\nu(\boldsymbol{e}_{2m}),\nu\big) = K\left(\frac{1}{K_t}\right),$$

which implies that $1/K_t$ is an oriented φ_t-function, since K is positive because of the convexity of f, where $\varphi_t = df_t$. $\square$

Now we consider the singularities of vector fields on M^{2m}. Let D be an arbitrary linear connection on M^{2m} and X a vector field defined on M^{2m}. One can apply (1.4) for the bundle homomorphism

$$\varphi_X : TM^{2m} \ni v \longmapsto D_v X \in TM^{2m}$$

if φ_X admits only A_k-singularities and get (1.7), where M_+^{2m} is the set of points where

$$(D_{v_1}X,\ldots,D_{v_{2m}}X)$$

forms a positive frame for a given locally defined positive frame $v_1,\ldots,v_{2m}$ on T_pM^{2m}. In [**23**], this map was introduced on a Riemannian 2-manifold, and we called the singular points of φ_X the *irrotational points* there. However, it would be better to call them the A_k-*singular points of the vector field* with respect to the connection D. In fact, the singular set of φ_X has no relation with the rotations of the vector fields.

At the end of this section, we give an application for the Blaschke normal maps for strictly convex hypersurfaces: we fix a strictly convex immersion

$$f : S^{2m} \longrightarrow \boldsymbol{R}^{2m+1}.$$

Then there exists a unique vector field ξ along f satisfying the following two properties, which is called the *affine normal vector field*:

(1) the linear map

$$S : TS^{2m} \ni v \longmapsto D_v\xi$$

gives an endomorphism on TS^{2m}, that is, $S(v) := D_v\xi$ is tangent to $f(S^{2m})$ for each v, where D is the canonical affine connection on $\boldsymbol{R}^{2m+1}$,

(2) there exists a unique covariant symmetric tensor h such that

$$D_X df(Y) - h(X,Y)\xi$$

gives a tangential vector field on $f(S^{2m})$ for any vector fields X and Y on S^{2m}. Since f is strictly convex, h is positive definite. Then the $2m$-form Ω defined by

$$\Omega(X_1, \ldots, X_{2m}) := \det\big(df(X_1), \ldots, df(X_{2m}), \xi\big)$$

coincides with the volume element associated to h, where $X_1, \ldots, X_{2m}$ are vector fields on S^{2m} and "det" denotes the canonical volume form of $\boldsymbol{R}^{2m+1}$.

The vector field ξ induces a map

$$\xi : S^{2m} \ni p \longmapsto \xi_p \in \boldsymbol{R}^{2m+1}, \tag{6.4}$$

which is called the *Blaschke normal map* of f. The following assertion holds as in the case of $m = 1$ (cf. [**23**, Lemma 3.1]).

LEMMA 6.5.   *The Blaschke normal map ξ gives a wave front.*

PROOF.   Consider a non-zero section

$$L : S^{2m} \ni p \longmapsto (\xi_p, \nu_p) \in T^*\boldsymbol{R}^{2m+1} = \boldsymbol{R}^{2m+1} \times (\boldsymbol{R}^{2m+1})^*,$$

where $(\boldsymbol{R}^{2m+1})^*$ is the dual vector space of $\boldsymbol{R}^{2m+1}$, and $\nu : S^{2m} \to (\boldsymbol{R}^{2m+1})^*$ is the map defined by

$$\nu_p(\xi_p) = 1, \qquad \nu_p(df(T_p S^{2m})) = \{0\} \qquad (p \in S^{2m}),$$

which is called the *conormal map* of f. By definition, L induces an isotropic map of S^{2m} into the projective cotangent bundle $P(T^*\boldsymbol{R}^{2m+1}) = \boldsymbol{R}^{2m+1} \times P^*(\boldsymbol{R}^{2m+1})$ with the canonical contact structure. Take a local coordinate system $(x_1, \ldots, x_{2m})$ of S^{2m}. Then we have that

$$\begin{aligned}
\nu_{x_i}(f_{x_j}) = (D_{\partial/\partial x_i} \nu)(f_{x_j}) &= \frac{\partial}{\partial x_i}\nu(f_{x_j}) - \nu(D_{\partial/\partial x_i} f_{x_j}) \\
&= -\nu(D_{\partial/\partial x_i} f_{x_j}) \\
&= -\nu\left(D_{\partial/\partial x_i} f_{x_j} - h\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\right) + h\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \\
&= h\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \qquad (i, j = 1, \ldots, 2m).
\end{aligned}$$

Since h is positive definite, one can show that $\nu_{x_1}, \ldots, \nu_{x_{2m}}$ are linearly independent. Moreover, since $\nu(T_p S^{2m}) = \{0\}$ for each $p \in S^{2m}$, $\nu, \nu_{x_1}, \ldots, \nu_{x_{2m}}$ are linearly independent. In particular, the map L induces a Legendrian immersion, which proves the assertion. $\square$

The following assertion is a hypersurface version of [**23**, Theorem 3.2].

THEOREM 6.6.   *Let S^{2m} be the $2m$-sphere and $f : S^{2m} \to \boldsymbol{R}^{2m+1}$ a strictly convex immersion. Suppose that the Blaschke normal map $\xi : S^{2m} \to \boldsymbol{R}^{2m+1}$ (cf. (6.4)) admits only A_k-front singularities for $2 \leq k \leq 2m+1$. Then the singular set of ξ satisfies (1.7),*

where $M_+^{2m}(= S_+^{2m})$ (*resp.* $M_-^{2m}(= S_-^{2m})$) *is the subset of* S^{2m} *at which the determinant of the affine shape operator* (*called the affine Gauss–Kronecker curvature*) *is positive* (*resp. negative*), *and* $\mathfrak{A}_{2j+1}^+$ (*resp.* $\mathfrak{A}_{2j+1}^-$) *is the set of positive* (*resp. negative*) A_{2j+1}*-front singular points of* ξ *for each* $j = 1, \ldots, m$.

PROOF. If the singular points of ξ consist only of A_k-points ($2 \leq k \leq 2m+1$), the affine shape operator

$$S : TS^{2m} \ni v \longmapsto D_v\xi \in f^*T\boldsymbol{R}^{2m+1}$$

gives a Morin homomorphism. Applying (1.4) for S, we get Theorem 6.6. $\qquad\square$

Finally, we give an example which illustrates Theorem 6.6: Consider a plane curve

$$\gamma(t) = (1 - 2\epsilon \sin t)\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \qquad \left(-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\right),$$

which lies on the upper-half plane and gives a convex curve if $0 \leq \epsilon < 1/4$. Rotating it around the horizontal axis, we get a rotationally symmetric strictly convex surface in $\boldsymbol{R}^3$. The left hand side of Figure 2 indicates the curve γ for $\epsilon = 17/80$, and the right hand side of Figure 2 gives the profile curve of the Blaschke normal map ξ of the surface for $\epsilon = 17/80$. As shown in Figure 2 (right), ξ has no swallowtails (i.e. it has no A_3-points), and our formula implies that the Euler number $\chi(M_-^2)$ vanishes. In fact, the set $\xi(M_-^2)$ gives a cylindrical strip if one rotates the profile curve of ξ around the horizontal axis.
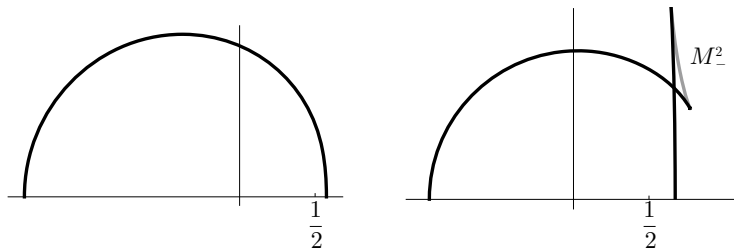

Figure 2. The curve γ (left) and the profile curve of ξ (right).

7. Coherent tangent bundles induced by Kossowski metrics.

In this section, we introduce a class of positive semi-definite metrics called Kossowski metrics describing the properties of wave fronts intrinsically. This class of metrics was defined by Kossowski [**10**] for 2-dimensional manifolds. In [**7**], it was shown that each Kossowski metric induces a coherent tangent bundle, and the formulas (1.2) and (1.3) for the metric were proved. Our purpose is to generalize the results of [**7**] to higher dimensional cases, that is, we will give an application of the formula (1.4) for Kossowski metrics.

We now fix an n-manifold M^n, and a positive semi-definite metric ds^2 on M^n. A point $p \in M^n$ is called a *singular point* of the metric ds^2 if the metric is not positive

definite at p. We denote by $\mathfrak{X}$ the set of smooth vector fields on M^n, and by $C^\infty(M^n)$ the set of $\boldsymbol{R}$-valued smooth functions on M^n.

We set $\langle X, Y \rangle := ds^2(X, Y)$. Kossowski [8] defined a map $\Gamma : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to C^\infty(M^n)$ as

$$
\begin{aligned}
\Gamma(X, Y, Z) := \frac{1}{2} \big(&X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle \\
&+ \langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle \big).
\end{aligned}
\tag{7.1}
$$

We call Γ the *Kossowski pseudo-connection*. (Kossowski [8] called Γ the dual connection of the Levi–Civita connection on $M^n \setminus \Sigma^{n-1}$, where Σ^{n-1} is the singular set of ds^2.) It was introduced by Kossowski (cf. [8], [9] and [10]), and plays an important role to show a realization theorem of generic singularities of Kossowski metrics as first fundamental forms of wave fronts in $\boldsymbol{R}^3$. If the metric ds^2 is positive definite, then the equality

$$
\Gamma(X, Y, Z) = \langle \nabla_X Y, Z \rangle
\tag{7.2}
$$

holds, where ∇ is the Levi–Civita connection of ds^2. One can easily check the following two identities (cf. [10])

$$
X \langle Y, Z \rangle = \Gamma(X, Y, Z) + \Gamma(X, Z, Y),
\tag{7.3}
$$

$$
\Gamma(X, Y, Z) - \Gamma(Y, X, Z) = \langle [X, Y], Z \rangle.
\tag{7.4}
$$

The equation (7.3) corresponds to the condition that ∇ is a metric connection, and the equation (7.4) corresponds to the condition that ∇ is torsion free. The following assertion can be also easily verified:

PROPOSITION 7.1 (Kossowski [8], [10]).   *For each $Y \in \mathfrak{X}$ and for each $p \in M^n$, the map*

$$
T_p M^n \times T_p M^n \ni (v_1, v_2) \longmapsto \Gamma(V_1, Y, V_2)(p) \in \boldsymbol{R}
$$

is a well-defined bi-linear map, where V_j $(j = 1, 2)$ are vector fields of M^n satisfying $v_j = V_j(p)$.

For each $p \in M^n$, the subspace

$$
N_p := \big\{ v \in T_p M^n \, ; \, ds^2(v, w) = 0 \text{ for all } w \in T_p M^n \big\}
\tag{7.5}
$$

is called the *null space* at p. A non-zero vector which belongs to N_p is called a *null vector* at p.

LEMMA 7.2 (Kossowski [8], see also [7]).   *Let p be a singular point of ds^2. Then the Kossowski pseudo-connection Γ induces a tri-linear map*

$$
\hat{\Gamma}_p : T_p M^n \times T_p M^n \times N_p \ni (v_1, v_2, v_3) \longmapsto \Gamma(V_1, V_2, V_3)(p) \in \boldsymbol{R},
$$

where V_j $(j = 1, 2, 3)$ are vector fields of M^n such that $v_j = V_j(p)$.

DEFINITION 7.3.   A singular point p of the metric ds^2 is called *admissible*[1] if $\hat{\Gamma}_p$ in Lemma 7.2 vanishes. If each singular point of ds^2 is admissible, then ds^2 is called an *admissible metric*.

DEFINITION 7.4.   An admissible metric ds^2 defined on M^n is called a *frontal metric*[2] if for each $p \in M^n$ there exists a local coordinate system $(U; x_1, \ldots, x_n)$ and a C^∞-function λ on U such that

$$\det(g_{ij}) = \lambda^2, \tag{7.6}$$

where $ds^2 = \sum_{i,j=1}^n g_{ij} dx_i dx_j$ is a local expression of the metric ds^2 on U and $\det(g_{ij})$ is the determinant of the $n \times n$ matrix $(g_{ij})_{i,j=1,\ldots,n}$.

We remark that the condition (7.6) is independent of the choice of local coordinate systems. If $f : M^n \to \boldsymbol{R}^{n+1}$ is a front, then the induced metric $ds^2(:= df \cdot df)$ on M^n is a frontal metric (cf. [**7**, Proposition 2.11]).

DEFINITION 7.5.   A singular point p of a given frontal metric is called *non-degenerate* or *generic* (cf. [**10**]) if its exterior derivative $d\lambda$ does not vanish at p, where λ is the function as in (7.6). A frontal metric ds^2 is called a *Kossowski metric* if all of the singular points of ds^2 are non-degenerate.

Since each singular point of a Kossowski metric is non-degenerate, and the singular set (denoted by Σ^{n-1}) consists of a hypersurface of M^n. Moreover, the function λ changes sign across Σ^{n-1}. In particular, a C^∞-function λ satisfying (7.6) is uniquely determined up to the sign.

DEFINITION 7.6 (cf. [**7**]).   Let ds^2 be a Kossowski metric on M^n. A local coordinate system $(U; x_1, \ldots, x_n)$ of M^n is called *adjusted* at a singular point $p \in U$ if

$$\partial_n := \partial/\partial x_n$$

belongs to N_p. Moreover, if $(U; x_1, \ldots, x_n)$ is adjusted at each singular point of U, it is called an *adapted local coordinate system* of M^n.

Since the singular set Σ^{n-1} of a Kossowski metric is a hypersurface in M^n, one can easily prove the existence of an adapted local coordinate system at each singular point. We are interested in the class of Kossowski metrics because of the following fact:

PROPOSITION 7.7.   *Let $(\mathcal{E}, \varphi, \langle,\rangle, D)$ be a coherent tangent bundle (see the introduction) on a manifold M^n. Then the induced metric $ds^2 := \varphi^* \langle,\rangle$ is a frontal metric. Moreover, if φ admits only non-degenerate singular points, then ds^2 is a Kossowski metric on M^n.*

PROOF.   The admissibility of the metric follows from the identity

---

[1]The notion of admissibility was introduced by Kossowski [**10**]. He called it $d(\langle,\rangle)$-*flatness*.
[2]It is called a *discriminant transverse metric* in [**10**].

$$\Gamma(X, Y, Z) = \langle D_X \varphi(Y), \varphi(Z) \rangle \qquad (X, Y, Z \in \mathfrak{X}).$$

On the other hand, for each $p \in M^n$, one can take an orthonormal frame field $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ of $\mathcal{E}$ on a coordinate neighborhood $(U; x_1, \dots, x_n)$ of p. Let $\theta_1, \dots, \theta_n$ be the dual frame field of $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$. Then $\mu := \theta_1 \wedge \cdots \wedge \theta_n$ gives an orientation of $\mathcal{E}$ on U, and there exists a smooth function $\lambda \in C^\infty(U)$ such that

$$\varphi^*\mu = \lambda dx_1 \wedge \cdots \wedge dx_n.$$

If we write $ds^2 = \sum_{i=1}^n g_{ij} dx_i dx_j$ on U, then we have that

$$|\lambda| = \sqrt{\det(g_{ij})}, \tag{7.7}$$

since $\varphi^*\mu$ gives a Riemannian volume element on $U \setminus \Sigma^{n-1}$. Thus λ^2 coincides with $\det(g_{ij})$, which implies that ds^2 is a frontal metric. Then the final assertion follows immediately by comparing the definitions of non-degeneracy of singular points for φ and for ds^2. $\qquad\qquad\square$

EXAMPLE 7.8.   A Riemannian n-manifold (M^n, g) $(n \geq 3)$ is called *conformally flat* if for each point $p \in M^n$, there exists a neighborhood $U(\subset M^n)$ of p and a C^∞-function σ on U such that $e^{2\sigma}g$ is a metric with vanishing sectional curvature. When $n \geq 4$, (M^n, g) is conformally flat if and only if the conformal curvature tensor

$$W_{ijkl} := R_{ijkl} + (B_{ik}g_{jl} - B_{il}g_{jk} + B_{jl}g_{ik} - B_{jk}g_{il}) + \frac{S_g}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{7.8}$$

vanishes identically on M^n, where $(x_1, \dots, x_n)$ is a local coordinate system of M^n,

$$B := \sum_{i,j=1}^n B_{ij} dx_i \otimes dx_j, \quad B_{ij} := \frac{1}{n-2}\left(R_{ij} - \frac{S_g g_{ij}}{2(n-1)}\right) \tag{7.9}$$

is called the *Schouten tensor*, g_{ij}, R_{ijkl}, R_{ij} are the components of the metric g, the curvature tensor of g, and the Ricci tensor of g respectively, and S_g denotes the scalar curvature. When $n = 3$, (M^3, g) is conformally flat if and only if B in (7.9) is a Codazzi tensor, that is, ∇B is a symmetric 3-tensor, where ∇ is the Levi–Civita connection of (M^3, g). (When $n \geq 4$, conformal flatness implies that B is a Codazzi tensor because of the second Bianchi identity.) We denote by $(g^{ij})_{i,j=1}^n$ the inverse matrix of $(g_{ij})_{i,j=1}^n$, and set

$$\check{B} := \sum_{i,j,a} g^{ia} B_{aj} \frac{\partial}{\partial x_i} \otimes dx_j \tag{7.10}$$

which gives a $(1,1)$-tensor of M^n, and it induces a bundle homomorphism

$$\check{B} : T_p M^n \ni v \mapsto \check{B}_p(v) \in T_p M^n \qquad (p \in M^n). \tag{7.11}$$

Since B in (7.9) is a Codazzi tensor, $\check{B}$ satisfies the torsion free condition (1.1) with

respect to ∇ (cf. [**12**]), In particular, $\check{B} : TM^n \to (TM^n, g, \nabla)$ gives a structure of a coherent tangent bundle. The pull-back of the Riemannian metric g by $\check{B}$ is given by

$$\check{g} := \sum_{i,j,a,b} B_{ia} B_{jb} g^{ab} \, dx_i dx_j. \tag{7.12}$$

It is a remarkable fact that $\check{g}$ gives a new conformally flat metric on $M^n \setminus \Sigma^{n-1}$ (cf. [**12**]). This new metric $\check{g}$ is called the *dual metric* of g. By Proposition 7.7, $\check{g}$ gives an example of a frontal metric. The points where $\check{g}$ is not positive definite correspond exactly to the singular points of the bundle homomorphism $\check{B}$. We call A_k-points of the bundle homomorphism $\check{B}$ the A_k-*points of the dual metric*.

As a converse of Proposition 7.7, the following assertion holds.

THEOREM 7.9. *Let ds^2 be a Kossowski metric on an n-manifold M^n. Then there exists a coherent tangent bundle $\varphi : TM^n \to (\mathcal{E}, \langle , \rangle, D)$ such that $\varphi^* \langle , \rangle$ coincides with ds^2.*

The case of $n = 2$ has already been proved in [**7**], and this theorem is a generalization of it. We fix an adapted local coordinate system $(U; x_1, \ldots, x_n)$ arbitrarily. We now carry out the Schmidt orthogonalization for the frame

$$\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n},$$

that is, we set

$$\hat{e}_1 := \partial_1, \quad e_1 := \hat{e}_1 / |\hat{e}_1|,$$

$$\hat{e}_j := \partial_j - \sum_{i=1}^{j-1} \langle \partial_j, e_i \rangle \, e_i, \quad e_j := \hat{e}_j / |\hat{e}_j| \qquad (j = 2, \ldots, n-1).$$

Then $e_1, \ldots, e_{n-1}$ are smooth vector fields on U. Finally, we set

$$\hat{e}_n := \partial_n - \sum_{i=1}^{n} \langle \partial_n, e_i \rangle \, e_i, \quad e_n := \frac{\hat{e}_n}{\lambda \prod_{j=1}^{n-1} |\hat{e}_j|}, \tag{7.13}$$

where λ is a C^∞-function on U satisfying (7.7). Then the resulting vector field e_n is defined only on $U \setminus \Sigma^{n-1}$, and $e_1, \ldots, e_n$ consists of an orthonormal frame on $U \setminus \Sigma^{n-1}$, which is called the *orthonormal frame field associated to* the adapted coordinate system $(x_1, \ldots, x_n)$.

We now set

$$\omega_{ij} := \sum_{k=1}^{n} \langle e_{\nabla_{\partial_k}} e_j, e_i \rangle \, dx_k = \sum_{k=1}^{n} \Gamma(\partial_k, e_j, e_i) \, dx_k \qquad (i, j = 1, \ldots, n), \tag{7.14}$$

on $U \setminus \Sigma^{n-1}$, where ∇ is the Levi–Civita connection of the metric ds^2 on $M^n \setminus \Sigma^{n-1}$ and Γ is the Kossowski pseudo-connection.

Lemma 7.10. *Each ω_{ij} $(i, j = 1, \ldots, n)$ can be extended to a smooth 1-form on U.*

Proof. If $1 \leq i, j \leq n - 1$, then ω_{ij} is trivially a smooth 1-form on U. So we consider the case $i = n$. By (7.13) and (7.14), it holds on $M^n \setminus \Sigma^{n-1}$ that

$$\Gamma(\partial_k, \boldsymbol{e}_j, \hat{\boldsymbol{e}}_n) = \lambda \omega_{nj}(\partial_k) \prod_{l=1}^{n-1} |\hat{\boldsymbol{e}}_l| \qquad (k = 1, \ldots, n, \ j = 1, \ldots, n-1).$$

Since ds^2 is admissible, the left hand side vanishes on $U \cap \Sigma^{n-1}$, there exists a smooth function $a_{kj} \in C^\infty(U)$ such that

$$\Gamma(\partial_k, \boldsymbol{e}_j, \hat{\boldsymbol{e}}_n) = \lambda a_{kj} \prod_{l=1}^{n-1} |\hat{\boldsymbol{e}}_l| \qquad (k = 1, \ldots, n, \ j = 1, \ldots, n-1).$$

In particular, we have that $\omega_{nj}(\partial_k) = a_{kj}$. We next consider the case $j = n$. Since

$$\Gamma(\partial_k, \hat{\boldsymbol{e}}_n, \boldsymbol{e}_i) = -\Gamma(\partial_k, \boldsymbol{e}_i, \hat{\boldsymbol{e}}_n) = 0$$

on $U \cap \Sigma^{n-1}$, one can easily see that $\omega_{in}(\partial_k)$ can also be extended as a C^∞-function on U. Finally, ω_{nn} vanishes on $U \setminus \Sigma^{n-1}$, and is trivially extended on U. $\qquad \square$

Proof of Theorem 7.9. Let $\{(U_a; x_1^a, \ldots, x_n^a)\}_{a \in \Lambda}$ be an atlas of M^n consisting of local adapted coordinate systems. Since ds^2 is a Kossowski metric, there exists a C^∞-function λ_a on U_a $(a \in \Lambda)$ such that

$$\det(g_{ij}^a) = (\lambda_a)^2,$$

where $ds^2 = \sum_{i,j=1}^n g_{ij}^a dx_i^a dx_j^a$.

We fix two indices $a, b \in \Lambda$ such that $U_a \cap U_b \neq \emptyset$, and set

$$(U; x_1, \ldots, x_n) := (U_a; x_1^a, \ldots, x_n^a), \qquad (V; y_1, \ldots, y_n) := (U_b; x_1^b, \ldots, x_n^b)$$

for the sake of simplicity. We denote by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ and $\tilde{\boldsymbol{e}}_1, \ldots, \tilde{\boldsymbol{e}}_n$ the orthonormal frame fields associated to the adapted coordinate systems $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, respectively. By the previous procedure of orthogonalization, there are upper triangular matrices $\mathcal{T}$ and $\tilde{\mathcal{T}}$ such that

$$(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)\mathcal{T}, \tag{7.15}$$

$$(\tilde{\boldsymbol{e}}_1, \ldots, \tilde{\boldsymbol{e}}_n) = \left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right)\tilde{\mathcal{T}}. \tag{7.16}$$

These two matrices $\mathcal{T}$ and $\tilde{\mathcal{T}}$ can be written as

$$\mathcal{T} = \begin{pmatrix} * & * \\ \boldsymbol{0} & d \end{pmatrix}, \tag{7.17}$$

$$\tilde{\mathcal{T}} = \begin{pmatrix} * & * \\ \mathbf{0} & \tilde{d} \end{pmatrix}, \tag{7.18}$$

where $*$ means a real valued (or a matrix valued) function which is smooth along Σ^{n-1} and $\mathbf{0}$ is the row zero vector in $\mathbf{R}^{n-1}$. On the other hand, d (resp. $\tilde{d}$) means a 'divergent function' which is not smooth along $U_a \cap \Sigma^{n-1}$ but $\lambda_a d$ (resp. $\lambda_b \tilde{d}$) is a C^∞-function on U_a (resp. U_b).

Since $\mathcal{T}$ and $\tilde{\mathcal{T}}$ are upper triangular matrices, one can easily check that

$$\lambda_a \det(\mathcal{T}) = *, \qquad \lambda_b \det(\tilde{\mathcal{T}}) = *. \tag{7.19}$$

On the other hand, there is a matrix valued function $\mathcal{J}$ such that the equality

$$\left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right) = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \mathcal{J} \tag{7.20}$$

holds on $U_a \cap U_b$. Since $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are adapted coordinate systems, we can write

$$\mathcal{J} = \begin{pmatrix} * & \mathbf{0} \\ * & * \end{pmatrix}, \tag{7.21}$$

where $\mathbf{0}$ is the column zero vector in $\mathbf{R}^{n-1}$. By (7.15), (7.16) and (7.20), we have that

$$(\tilde{\boldsymbol{e}}_1, \dots, \tilde{\boldsymbol{e}}_n) = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_n) \mathcal{T}^{-1} \mathcal{J} \tilde{\mathcal{T}}.$$

We now compute $\mathcal{T}^{-1} \mathcal{J} \tilde{\mathcal{T}}$ using the relations

$$* + * = *, \qquad d + * = d, \qquad * \times * = *, \qquad d \times * = d$$

on U_a and

$$* + * = *, \qquad \tilde{d} \times * = \tilde{d}, \qquad * \times * = *, \qquad \tilde{d} \times * = \tilde{d}$$

on U_b, where $\times$ means the usual multiplications of scalars and matrices. These relations follow from the definitions of divergent terms d and $\tilde{d}$. Here, $d \times *$ might not be divergent in general. The above convention $d \times * = d$ means that $d \times *$ can be a divergent term as a possibility. On the other hand, if the term $d \times d$ appears, then it is more dangerous than the divergent terms, since $\lambda d \times d$ is still a divergent term. Fortunately, such a term never appears in the calculation of $\mathcal{T}^{-1} \mathcal{J} \tilde{\mathcal{T}}$ as follows: the equalities (7.17) and (7.19) yield that

$$\mathcal{T}^{-1} = \lambda_a \begin{pmatrix} d & * \\ \mathbf{0} & * \end{pmatrix}.$$

We now set

$$\tau_{ab} := \mathcal{T}^{-1} \mathcal{J} \tilde{\mathcal{T}},$$

which gives a C^∞-function on $U_a \cap U_b \setminus \Sigma^{n-1}$. Since

$$\mathcal{T}^{-1}\mathcal{J} = \lambda_a \begin{pmatrix} d & * \\ \mathbf{0} & * \end{pmatrix} \begin{pmatrix} * & \mathbf{0} \\ * & * \end{pmatrix} = \lambda_a \begin{pmatrix} d & * \\ * & * \end{pmatrix},$$

we have that

$$\tau_{ab} = \lambda_a \begin{pmatrix} d & * \\ * & * \end{pmatrix} \begin{pmatrix} * & * \\ \mathbf{0} & \tilde{d} \end{pmatrix} = \lambda_a \begin{pmatrix} d\,d + \tilde{d} \\ * & \tilde{d} \end{pmatrix} = \begin{pmatrix} * & * \\ * & * \end{pmatrix},$$

since $\lambda_a = \lambda_b \times *$. So we can conclude that τ_{ab} can be smoothly extended on $U_a \cap U_b$. In particular, the co-cycle condition

$$\tau_{ab}\tau_{bc}\tau_{ca} = \mathrm{id} \tag{7.22}$$

holds on $U_a \cap U_b \cap U_c$, where id is the identity matrix. Thus there exists a vector bundle $\mathcal{E}$ with inner product $\langle\,,\,\rangle$ whose transition functions are $\{\tau_{ab}\}$. Let

$$\Omega^a := \begin{pmatrix} \omega_{11}^a & \cdots & \omega_{1n}^a \\ \vdots & \ddots & \vdots \\ \omega_{n1}^a & \cdots & \omega_{nn}^a \end{pmatrix}$$

be a matrix valued 1-from on U_a according to Lemma 7.10, which gives a connection form of the Levi–Civita connection of ds^2 on $U_a \setminus \Sigma^{n-1}$. In particular, Ω^a takes value in the set of skew-symmetric matrices. The family of matrix valued 1-form $\{\Omega^a\}_{a \in \Lambda}$ satisfies the identity

$$\Omega^b = \tau_{ab}^{-1}(d\tau_{ab}) + \tau_{ab}^{-1}\Omega^a \tau_{ab} \tag{7.23}$$

on $U_a \cap U_b \setminus \Sigma^{n-1}$. Then by continuity, (7.23) holds on $U_a \cap U_b$. Thus, it induces a metric connection D on $\mathcal{E}$. By the definition of $\mathcal{E}$, the bundle homomorphism

$$\varphi : TM^n \longrightarrow \mathcal{E}$$

is canonically induced so that $\varphi(e_1^a), \ldots, \varphi(e_n^a)$ consists of an orthonormal frame of $\mathcal{E}$ on $U_a \setminus \Sigma^{n-1}$, where $e_1^a, \ldots, e_n^a$ is the orthonormal frame field associated to $(x_1^a, \ldots, x_n^a)$. Then the restriction of the map φ into $M^n \setminus \Sigma^{n-1}$ gives a vector bundle isomorphism between the tangent bundle of $M^n \setminus \Sigma^{n-1}$ and $\mathcal{E}|_{M^n \setminus \Sigma^{n-1}}$, and $\varphi^* \langle\,,\,\rangle = ds^2$ holds on $M^n \setminus \Sigma^{n-1}$. Then by continuity, $\varphi^* \langle\,,\,\rangle = ds^2$ holds on all M^n. On the other hand, the pull-back connection of D coincides with the Levi–Civita connection of ds^2 on $M^n \setminus \Sigma^{n-1}$. In particular, (1.1) holds on $M^n \setminus \Sigma^{n-1}$. Then, by continuity, (1.1) also holds on all of M^n. Thus we get a coherent tangent bundle associated to the Kossowski metric ds^2. $\square$

A Kossowski metric is said to be *co-orientable* if one can choose the chart

$$\{(U_a; x_1^a, \ldots, x_n^a)\}_{a \in \Lambda}$$

of M^n such that

$$\mu := \lambda_a dx_1^a \wedge \cdots \wedge dx_n^a$$

gives a globally defined smooth n-form on M^n. It can be easily checked that the co-orientability of ds^2 corresponds to the fact that the induced bundle $\mathcal{E}$ is orientable (cf. [**7**, Proposition 2.11]). We remark that each λ_a $(a \in \Lambda)$ is a φ-function of the induced coherent tangent bundle.

DEFINITION 7.11.   A Kossowski metric ds^2 on M^n is called a *Morin metric* if its induced coherent tangent bundle admits only A_{k+1}-points $(k = 1, \ldots, n)$.

Then as an application of the formula (1.4), we get the following assertion.

COROLLARY 7.12.   *Let ds^2 be a co-orientable Morin metric defined on an oriented compact manifold M^{2m}. Then the identity (1.4) holds, where $\chi_\mathcal{E}$ is the Euler characteristic of the coherent tangent bundle $\mathcal{E}$ associated to ds^2.*

This corollary is a generalization of [**7**, Proposition 3.3]. The following assertion is the spacial case of this corollary if we set ds^2 to be the dual metric of the conformally flat metric as in Example 7.8.

THEOREM 7.13.   *Let (M^{2m}, g) be a compact orientable conformally flat manifold whose dual conformally flat metric $\check{g}$ admits only A_k-singularities for $2 \le k \le 2m + 1$. Then the singular set of the dual conformally flat metric $\check{g}$ satisfies (1.7), where M_+^{2m} (resp. M_-^{2m}) is the subset of M^{2m} at which the determinant of the Schouten tensor is positive (resp. negative), and $\mathfrak{A}_{2j+1}^+$ (resp. $\mathfrak{A}_{2j+1}^-$) is the set of positive (resp. negative) A_{2j+1}-points $(j = 1, \ldots, m)$ of the bundle homomorphism $\check{B}$.*

A.   Extension of generic vector fields.

We prove the following assertion, which is needed to prove the existence of a characteristic vector field associated to a given Morin homomorphism:

LEMMA A.1.   *Let M^n be a compact manifold and X a C^∞-vector field defined on an open subset of M^n containing a compact subset K such that X has no zeros on the boundary ∂K of K. Then there exists a C^∞-vector field $\tilde{X}$ defined on M^n such that $\tilde{X}$ coincides with X on K and has only generic zeros on $M^n \setminus K$.*

PROOF.   We may assume that X is defined on a neighborhood U of K. Take an open subset V such that

$$K \subset V \subset \overline{V} \subset U,$$

where $\overline{V}$ is the closure of V. Taking U sufficiently close to K, we may assume that X has no zeros on $U \setminus K^\circ$, where K° (possibly empty) is the set of the interior points of K. Then we can take C^∞-functions $\rho_j : M^n \to [0, 1]$ $(j = 1, 2)$ such that $\rho_1 = 1$ on K (resp. $\rho_2 = 1$ on $\overline{V}$) and $\rho_1 = 0$ on $M^n \setminus V$ (resp. $\rho_2 = 0$ on $M \setminus U$). We set $\hat{X} := \rho_2 X$, which is a vector field on M^n. It is well-known that there exists a sequence of generic vector fields $\{Z_j\}_{j=1,2,3,\ldots}$ on M^n converging to $\hat{X}$ with respect to the Whitney C^∞-topology.

We set

$$\tilde{X}_j := \rho_1 \hat{X} + (1 - \rho_1)Z_j.$$

Then $\tilde{X}_j$ coincides with X on K, because $\rho_1 = \rho_2 = 1$ on K. Since $\hat{X}$ has no zeros on the compact set $\overline{V} \setminus K^\circ$, $\tilde{X}_j$ has a zero at $p \in \overline{V} \setminus K^\circ$ if $\hat{X} = (1 - \rho_1^{-1})Z_j$ holds at p. This is impossible for sufficient large j, since $Z_j \to \hat{X}$ as $j \to \infty$ and $\rho_1 \in [0,1]$. Moreover, $\tilde{X}_j$ coincides with Z_j on $M^n \setminus V$, since $\rho_1 = 0$ on the complement of V. Thus it has only generic zeros on $M^n \setminus V$. In particular, $\tilde{X}_j$ has the desired property for sufficiently large j. $\hspace{2cm}\square$

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