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On stability of Leray's stationary solutions of the Navier–Stokes system in exterior domains

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Abstract. This paper studies the stability of a stationary solution of the Navier–Stokes system in 3-D exterior domains. The stationary solution is called a Leray's stationary solution if the Dirichlet integral is finite. We apply an energy inequality and maximal L^p-in-time regularity for Hilbert space-valued functions to derive the decay rate with respect to time of energy solutions to a perturbed Navier–Stokes system governing a Leray's stationary solution.

1. Introduction.

1.1. Purposes.

This paper has two purposes. The first one is to provide a new method for deriving the decay rate with respect to time of energy solutions to incompressible viscous fluid systems by using both an energy inequality and maximal L^p-in-time regularity for Hilbert space-valued functions. The second one is to investigate the decay rate with respect to time of global-in-time strong L^2-solutions of a perturbed Navier–Stokes system governing a small stationary solution by the method.

There are many literature on the stability for the Navier–Stokes flow. Especially, this paper considers the decay rate with respect to time of energy solutions to an incompressible viscous fluid system. Masuda [**32**] used an energy inequality and the Stokes operator to derive L^∞-decay for energy solutions of a perturbed Navier–Stokes system governing a stationary solution in exterior domains. Kato [**19**] applied L^p-L^q estimates for the heat kernel to investigate the decay rate with respect to time of global-in-time mild L^n-solutions of the Navier–Stokes system in the whole space $\mathbb{R}^n$. Schonbek [**38**], [**39**] made use of the Fourier transform and an energy inequality to derive L^2-decay for weak solutions of the Navier–Stokes system in the whole space $\mathbb{R}^3$ when the initial datum belongs to $L^1 \cap L^2$. In this paper we investigate both L^2-asymptotic stability and $L^r (2 < r \leq \infty)$-decay rate with respect to time of energy solutions to a perturbed Navier–Stokes system governing a stationary solution in an exterior domain. We introduce a new approach to derive L^r-decay for energy solutions of the incompressible viscous fluid system by applying an energy inequality and maximal L^p-in-time regularity for Hilbert space-valued functions. This method improves one from Masuda [**32**]. In our

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method we use neither the Fourier transform nor L^p-L^q estimates for the semigroup generated by a linear operator, so our approach of this paper enables us to derive L^r-decay for energy solutions of various incompressible viscous fluid systems.

Let Ω be an exterior domain with smooth boundary in $\mathbb{R}^3$. Let us consider the following initial-boundary value problem of the Navier–Stokes system:

$$\begin{cases} \partial_t u - \nu\Delta u + (u,\nabla)u + \nabla\Pi = \nabla\cdot F & \text{in } \Omega\times(0,\infty),\\ \nabla\cdot u = 0 & \text{in } \Omega\times(0,\infty),\\ u|_{\partial\Omega} = 0,\ \lim\limits_{|x|\to\infty} u = u_\infty,\ u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where the unknown function $u = u(x,t) = (u^1,u^2,u^3)$ is the velocity of the fluid, the unknown function $\Pi = \Pi(x,t)$ is the pressure of the fluid, while the given function $F = F(x) = (F_{jk}(x))_{j,k=1,2,3}$ is the external force, the given positive constant ν is the viscosity coefficient, the given constant $u_\infty = (u^1_\infty, u^2_\infty, u^3_\infty) \in \mathbb{R}^3$ is the velocity of the fluid at infinity, and $u_0 = u_0(x)$ is the given initial datum. Here we use the convention: $\Delta := \partial_1^2+\partial_2^2+\partial_3^2$ and $\nabla := (\partial_1,\partial_2,\partial_3)$. Note that $\nabla\cdot F = (\sum_{k=1}^3 \partial_k F_{jk})_{j=1,2,3}$. The model (1.1) illustrates the motion of an incompressible viscous fluid past an obstacle.

This paper studies the stability of the system (1.1) around solutions of the following stationary Navier–Stokes equations:

$$\begin{cases} -\nu\Delta w + (u_\infty,\nabla)w + (w,\nabla)w + \nabla\pi = \nabla\cdot F & \text{in } \Omega,\\ \nabla\cdot w = 0 & \text{in } \Omega,\\ w|_{\partial\Omega} = -u_\infty,\ \lim\limits_{|x|\to\infty} w = 0. \end{cases} \tag{1.2}$$

Here $w = w(x) = (w^1,w^2,w^3)$ and $\pi = \pi(x)$. A solution (w,π) of the system (1.2) is called a *Leray's stationary solution* if the Dirichlet integral $\int_\Omega |\nabla w(x)|^2 dx$ is finite and $w \in L^6(\Omega)$ (see Leray [**29**] and Heck–Kim–Kozono [**14**]). This paper discusses the stability of Leray's stationary solutions under the more general conditions:

Assumption 1.1. *The function w satisfies $\nabla\cdot w = 0$ in Ω and*

$$w \in [L^{3,\infty}(\Omega)\cap L^{p_1}(\Omega)\cap \dot{W}^{1,p_2}(\Omega)]^3 \text{ for some } p_1\in(3,\infty] \text{ and } p_2\in[2,\infty).$$

Here $L^{3,\infty}(\Omega)$ and $\dot{W}^{1,p_2}(\Omega)$ are the weak L^3-space and the homogeneous Sobolev space, respectively. Note that there exists a solution (w,π) of the system (1.2) satisfying Assumption 1.1. Kim–Kozono [**20**] and Heck–Kim–Kozono [**14**] showed the existence of a unique Leray's weak solution of (1.2) belonging to $L^{3,\infty}(\Omega)\cap L^6(\Omega)\cap\dot{W}^{1,2}(\Omega)$ when u_∞ is sufficiently small and F is sufficiently small in a suitable norm. Borchers–Miyakawa [**5**] and Novotný–Padula [**35**] constructed a solution (w,π) of (1.2) satisfying $\sup\{|x||w(x)|\} + \sup\{|x|^2|\nabla w(x)|\} < +\infty$ when $u_\infty = 0$ and f is sufficiently small in their weighted norm. Remark that $w\in L^{3,\infty}\cap L^{r_1}$ for $r_1>3$ if $\sup\{|x||w(x)|\}<+\infty$. Remark also that $\nabla w\in L^{3/2,\infty}\cap L^{r_2}$ for $r_2>3/2$ if $\sup\{|x|^2|\nabla w(x)|\}<+\infty$. Many researchers have been studying both the existence and the uniqueness of solutions to the system (1.2) since Leray [**29**] and Finn [**10**]. See [**14**], [**26**], [**41**], and the references given there.

In this paper we investigate both L^2-asymptotic stability and L^∞-decay of energy solutions to the following perturbed Navier–Stokes system governing a Leray's stationary solution:

$$\begin{cases} \partial_t v - \nu\Delta v + (u_\infty, \nabla)v + (w, \nabla)v + (v, \nabla)w + (v, \nabla)v + \nabla\mathfrak{p} = 0, \\ \nabla \cdot v = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega \times (0, \infty), \\ v|_{\partial\Omega} = 0, \quad \lim_{|x|\to\infty} v = 0, \;\; v|_{t=0} = v_0. \end{cases} \tag{1.3}$$

Note that we check that $(v, \mathfrak{p})$ formally satisfies the system (1.3) if we set $v = v(x, t) = (v^1, v^2, v^3) := u(x, t) - u_\infty - w(x)$, $\mathfrak{p} = \mathfrak{p}(x, t) := \Pi(x, t) - \pi(x)$, and $v_0 := u_0 - u_\infty - w$. We call a solution $(v, \mathfrak{p})$ of the system (1.3) an *energy solution* if there is $C > 0$ such that for all $0 \leq s < t$

$$\|v(t)\|_{L^2}^2 + C\int_s^t \|\nabla v(\tau)\|_{L^2}^2 d\tau \leq \|v(s)\|_{L^2}^2.$$

We easily see that $\|v(t)\|_{L^2}$ tends to a non-negative constant C_0 as $t \to \infty$ if $(v, \mathfrak{p})$ is an energy solution of the system (1.3), however, we do not know whether $C_0 = 0$ or not in general.

1.2. Known results.

In this paper we study the stability of L^2-solutions to the system (1.3). Heywood [**15**], [**16**] first investigated the stability of L^2-solutions to (1.3) under the conditions that $w \in W^{2,2}(\Omega)$ and $\sup\{|x||w(x)|\}$ is sufficiently small. He applied the Galerkin method and an energy inequality to show the existence of a unique global-in-time strong L^2-solution of (1.3) with the property that for each $\Omega' \subset \Omega$, $\|v(t)\|_{L^2(\Omega')} \to 0$ as $t \to \infty$. Masuda [**32**] and Miyakawa–Sohr [**33**] considered the stability of a weak solution of (1.3) when $\nabla w \in L^3$ and $\sup\{|x||w(x)|\}$ is sufficiently small. Masuda [**32**] made use of the Stokes operator and an energy inequality to show the existence of a weak solution of (1.3) satisfying L^∞-asymptotic stability. Miyakawa and Sohr [**33**] constructed a weak solution of (1.3) satisfying the strong energy inequality to derive L^2-asymptotic stability of their solution. Maremonti [**31**] studied the decay rate for L^2-solutions of (1.3) when $u_\infty = 0$, $w \in L^6 \cap W^{1,3} \cap \dot{W}^{1,p}$ for $p \in (3, \infty]$, and ν is sufficiently large. He applied an energy inequality and the Galerkin approach to construct a global-in-time L^2-solution of (1.3) satisfying $\|v(t)\|_{L^\infty} \leq Ct^{-1/2}$ as $t \to \infty$. Under the condition that the initial datum is in $L^\ell \cap L^2$ for some $1 < \ell < 2$, Borchers and Miyakawa [**4**] applied the bounded analytic semigroup generated by the Stokes operator to study L^2-decay for the Navier–Stokes flow in exterior domains, i.e. the system (1.3) when $u_\infty = 0$ and $w = 0$. Neustupa [**34**] derived a sufficient condition of the stability of solutions to (1.3) when $\nabla w \in L^{3/2} \cap L^3$. They made use of their assumptions on the semigroup generated by the main linear operator of the system (1.3) to show the existence of a unique global-in-time strong L^2-solution of (1.3), satisfying $\lim_{t\to\infty} \|\nabla v(t)\|_{L^2} = 0$. Koba [**22**] applied maximal L^p-regularity for Hilbert space-valued functions to investigate L^2-asymptotic stability of energy solutions to the generalized Navier–Stokes–Boussinesq system including (1.3).

This paper investigates the decay rate with respect to time of a global-in-time solution of the system (1.3). Iwashita [**18**] used L^p-L^q estimates for the Stokes semigroup to derive $L^r(r > 3)$-decay for a global-in-time mild solution of the Navier–Stokes system in an exterior domain. Kozono and Ogawa [**25**] constructed a unique global-in-time strong L^3-solution of (1.3) when $u_\infty = 0$, $w \in C(\overline{\Omega}) \cap L^3(\Omega) \cap \dot{W}^{1,3/2}(\Omega)$, and $\|w\|_{L^3} + \|\nabla w\|_{L^{3/2}}$ is sufficiently small. Under the condition that the initial datum belongs to $L^\ell \cap L^3$ for some $1 < \ell < 3$, they investigated the decay rate with respect to time of their solution by applying L^p-L^q estimates for the semigroup generated by the main linear operator of their system. Borchers and Miyakawa [**5**] studied the asymptotic stability of solutions of (1.3) under the restrictions that $w \in L^\infty(\Omega)$ and $\sup_{x\in\Omega}\{|x||w(x)|\} + \sup_{x\in\Omega}\{|x|^2|\nabla w(x)|\}$ is sufficiently small. They applied fundamental properties of the analytic semigroup generated by their main linear operator to derive the stability. Galdi–Heywood–Shibata [**12**] and Shibata [**37**] applied L^p-L^q estimates for the Oseen semigroup, which was obtained by Kobayashi–Shibata [**24**], to show the existence of a unique global-in-time mild L^3-solution, satisfying $L^r(r > 3)$-asymptotic stability, of (1.3) when $w \in W^{1,\infty}(\Omega)$ and $\sup\{(1 + |x|)(1 + |x| - x \cdot u_\infty/|u_\infty|)^\delta |w(x)|\} + \sup\{(1 + |x|)^{3/2}(1 + |x| - x \cdot u_\infty/|u_\infty|)^{1/2+\delta}|\nabla w(x)|\}$ is sufficiently small for some $\delta > 0$. Enomoto and Shibata [**8**], [**9**] improved the method from [**24**] to study the stability of (1.3). In [**9**], they derived L^∞-decay for a global-in-time mild L^3-solution of the system (1.3) when $\|w\|_{L^{3/(1-\epsilon)}}$, $\|w\|_{L^{3/(1+\epsilon)}}$, $\|\nabla w\|_{L^{3/(2+\epsilon)}}$, and $\|\nabla w\|_{L^{3/(2-\epsilon)}}$ are sufficiently small for some $\epsilon > 0$. Under the same assumptions in [**9**], Bae and Roh [**2**] investigated the decay rate with respect to time of mild L^3-solutions of (1.3) when the initial datum is in a weighted Lebesgue space.

In this paper, we consider the stability of solutions to the system (1.3) under the restriction that $\|w\|_{L^{3,\infty}}$ is sufficiently small. Kozono and Yamazaki [**27**] studied the system (1.3) when $u_\infty = 0$, $w \in L^{3,\infty}(\Omega) \cap L^\infty(\Omega) \cap \dot{W}^{1,r}(\Omega)$ for some $r > 3$, and both $\|v_0\|_{L^{3,\infty}}$ and $\|w\|_{L^{3,\infty}}$ are sufficiently small. They derived $L^{p,\infty}$-L^q estimates for the semigroup generated by the main linear operator of their system to prove the existence of a unique global-in-time strong $L^{3,\infty}$-solution of (1.3) satisfying L^r-asymptotic stability. Shibata [**40**] showed the existence of a unique global-in-time mild $L^{3,\infty}$-solution, satisfying $L^r(r > 3)$-asymptotic stability, of (1.3) when $\|v_0\|_{L^{3,\infty}}$ and $\|w\|_{L^{3,\infty}}$ are sufficiently small by applying L^p-L^q estimates for the Oseen semigroup and the real interpolation theory. Recently, Koba [**23**] made use of L^p-L^q type estimates for the Oseen semigroup to investigate the stability of $L^{3,\infty}$-solutions of the system (1.3) when $\|v_0\|_{L^{3,\infty}}$ and $\|w\|_{L^{3,\infty}}$ are sufficiently small. See Yamazaki [**43**] for the stability of $L^{n,\infty}$-solutions to the Navier–Stokes system with time-dependent external force and Hishida–Shibata [**17**] for the stability for the motion of an incompressible viscous fluid past a rotating obstacle in $L^{3,\infty}$-framework.

1.3. Main results and key ideas.

We now state main results.

Theorem 1.2. *Let $\nu > 0$, $u_\infty \in \mathbb{R}^3$, and let w be as in Assumption* 1.1. *Then there are $\delta_0 = \delta_0(\nu) > 0$ and $c_0 = c_0(\Omega, \nu, u_\infty, w) > 0$ such that if*

$$\|w\|_{L^{3,\infty}(\Omega)} < \delta_0 \tag{1.4}$$

and if

$$v_0 \in H^1_{0,\sigma}(\Omega) \text{ and } \|v_0\|_{W^{1,2}(\Omega)} < c_0,$$

then there exists a unique global-in-time strong solution $(v, \mathfrak{p})$:

$$v \in BC([0,\infty); H^1_{0,\sigma}(\Omega)) \cap C((0,\infty); [W^{2,2}(\Omega)]^3) \cap C^1((0,\infty); L^2_\sigma(\Omega)),$$
$$\nabla \mathfrak{p} \in C((0,\infty)); [L^2(\Omega)]^3),$$

of the system (1.3), *satisfying*

$$\lim_{t\to\infty} \|v(t)\|_{L^2(\Omega)} = 0. \tag{1.5}$$

Furthermore, if $p_1 \in [6,\infty]$ *and* $p_2 \in [2,6]$, *then*

$$\begin{aligned} \|v_t(t)\|_{L^2(\Omega)} &= O(t^{-1/4}) && \text{as } t \to \infty,\\ \|\nabla v(t)\|_{L^2(\Omega)} &= O(t^{-1/8}) && \text{as } t \to \infty,\\ \|v(t)\|_{L^\infty(\Omega)} &= O(t^{-1/16}) && \text{as } t \to \infty, \end{aligned}$$

and for each $2 < q < 6$

$$\|\nabla v(t)\|_{L^q(\Omega)} = O(t^{-(3/q-1/2)/8}) \quad \text{as } t \to \infty.$$

Here $L^2_\sigma(\Omega)$ *and* $H^1_{0,\sigma}(\Omega)$ *are the two solenoidal spaces defined by Section* 2, $\mathfrak{p}$ *is a pressure associated with* v, *and* $f(t) = O(t^{-\alpha})$ *as* $t \to \infty$ *means that there are* $C > 0$ *and* $T > 0$ *such that* $f(t) \le Ct^{-\alpha}$ *for* $t > T$.

Remark that δ_0 is the constant introduced in Lemma 3.5, and c_0 the constant introduced in Proposition 4.2. Remark also that we can choose $u_\infty = 0$ in the assumptions of Theorem 1.2. Note that our solution $(v, \mathfrak{p})$ is an energy solution of the system (1.3) (see Lemma 4.1). Combining Miyakawa–Sohr [**33**] and Theorem 1.2, we obtain

Corollary 1.3. *Let* $\nu > 0$, $u_\infty \in \mathbb{R}^3$, *and let* w *be as in Assumption* 1.1 *such that* (1.4) *holds. Assume that* $p_1 = \infty$ *and* $p_2 = 3$. *Then for every* $v_0 \in L^2_\sigma(\Omega)$ *there exists at least one weak solution of the system* (1.3) *satisfying* L^2*-asymptotic stability* (1.5). *Moreover, the weak solution is smooth with respect to time when time is sufficiently large.*

Miyakawa and Sohr [**33**] proved that for every $v_0 \in L^2_\sigma(\Omega)$ there exists at least one weak solution satisfying the strong energy inequality of the system (1.3) when $p_1 = \infty$ and $p_2 = 3$. From a weak-strong argument, we deduce Corollary 1.3. See [**33**] for the definition of weak solutions of the system (1.3). See also [**21**, Chapters 5 and 7].

Since the assumptions on w of Theorem 1.2 are weaker than those of ([**32**], [**31**], [**33**], [**5**], [**9**]), our results on both L^2-asymptotic stability and L^∞-decay of solutions to the system (1.3) are the generalization of a part of their results (see the previous subsection). Note that their results on the decay rate with respect to time of their solution cannot directly be compared with our decay results because our assumptions are different from their ones.

Now we compare our results with [**32**], [**31**] and [**9**]. Under the conditions that $\nabla w \in L^3$ and $\sup\{|x||w(x)|\}$ is sufficiently small, Masuda [**32**] showed that for every initial datum $v_0 \in L^2_\sigma$ there exists at least one weak solution of the system (1.3) satisfying $\|v(t)\|_{L^\infty} = O(t^{-1/8})$ as $t \to \infty$. Maremonti [**31**] proved the existence of a global-in-time L^2-solution of (1.3), satisfying $\|v(t)\|_{L^\infty} = O(t^{-1/2})$ as $t \to \infty$, when $u_\infty = 0$, $w \in L^6 \cap W^{1,3} \cap \dot{W}^{1,p}(p > 3)$, and ν is sufficiently large. Under the assumptions that $\|v_0\|_{L^3}$, $\|w\|_{L^{3/(1-\epsilon)}}$, $\|w\|_{L^{3/(1+\epsilon)}}$, $\|\nabla w\|_{L^{3/(2+\epsilon)}}$, and $\|\nabla w\|_{L^{3/(2-\epsilon)}}$ are sufficiently small for some $\epsilon > 0$, Enomoto–Shibata [**9**] established the existence a unique global-in-time mild L^3-solution of (1.3) satisfying $\|v(t)\|_{L^\infty} \leq Ct^{-1/2}$ and $\|\nabla v(t)\|_{L^3} \leq Ct^{-1/2}$ for $t > 0$. On the other hand, under our assumptions, this paper shows the existence of a unique global-in-time strong L^2-solution satisfying $\|v(t)\|_{L^\infty} = O(t^{-1/16})$ and $\|\nabla v(t)\|_{L^3} = O(t^{-1/16})$ as $t \to \infty$. Note that our solution $(v, \mathfrak{p})$ is a weak solution and a mild L^3-solution of the system (1.3).

Let us explain two difficulties in deriving L^2-asymptotic stability (1.5) and the decay rate with respect to time of solutions to the system (1.3). The first difficulty is that it is not easy to show that the semigroup generated by the main linear operator of (1.3) is a bounded analytic semigroup even if w is sufficiently small in $L^{3,\infty}$. The second one is that it is difficult to derive L^p-L^q estimates for the semigroup generated by our main linear operator under the restriction that $\|w\|_{L^{3,\infty}}$ is sufficiently small. To overcome these two difficulties, we apply an abstract theory on linear stability in [**22**](see Lemma 3.2) and make use of an energy inequality for v and maximal L^p-in-time regularity for Hilbert space-valued functions. The approach is new and improves the method from [**32**]. Remark that we cannot derive the decay rate with respect to time of solutions of (1.3) by just using the method from [**32**] since our assumptions on w are weaker than those of [**32**].

Let us sketch out our method for deriving the decay rate with respect to time of energy solutions to the system (1.3). We first change the system (1.3) into the following abstract system:

$$\begin{cases} v_t + Av = F(v, \nabla v), \quad t > T_0, \\ v|_{t=T_0} = v(T_0). \end{cases} \tag{1.6}$$

Here A is the Stokes operator and F is a map (see Sections 2 and 4 for details). Secondly, we make use of an energy inequality for v, the structure of the system (1.6), and the Hölder inequality to show that for $t > T_0$

$$\|v_t(t)\|_{L^2} \leq Ct^{-1/p} \left(\int_{T_0}^t \|v_t(s)\|_{L^2}^p ds \right)^{1/p}. \tag{1.7}$$

Thirdly, we apply maximal L^p-regularity for the Stokes operator and an energy inequality for v into the system (1.6) to observe that

$$\int_{T_0}^{\infty} \|v_t(s)\|_{L^2}^p ds < \text{Const.} < +\infty. \tag{1.8}$$

Combining (1.7) and (1.8) yields that for $t > T_0$

$$\|v_t(t)\|_{L^2} \leq Ct^{-1/p}. \tag{1.9}$$

From an energy inequality, the structure of the system (1.6), and (1.9), we derive the decay rate with respect to time of both $\|Av(t)\|_{L^2}$ and $\|\nabla v(t)\|_{L^2}$. Finally, we make use of useful properties of the Stokes operator to derive the L^∞-decay for the solution v.

The outline of this paper is as follows: In Section 2 we first introduce function spaces and notation, and then state some inequalities and fundamental properties of the Stokes operator. In Section 3 we apply fundamental properties of the Stokes operator to study the main linear operator of the system (1.3), and show the stability of the semigroup generated by our main linear operator. In Section 4 we give the proof of Theorem 1.2 and introduce our method to derive the decay rate with respect to time of energy solutions of (1.3) by applying an energy inequality and maximal L^p-in-time regularity for Hilbert space-valued functions.

2. Preliminaries.

In this section we prepare three tools to analyze the system (1.3). The first tool is some inequalities for functions in the Sobolev spaces and the Lorentz spaces. Using these inequalities, we derive an energy inequality for L^2-solutions of our system. The second one is the Stokes operator in a solenoidal L^2-space L^2_σ. Applying fundamental properties of the Stoke operator, we study basic properties of the main linear operator of (1.3). The third one is maximal L^p-in-time regularity for Hilbert space valued-functions. This is a key tool to derive the decay rate with respect to time of energy solutions to our system.

Let us first introduce function spaces. For $m \in \mathbb{N}$, $1 < p < \infty$, and $1 \leq q \leq \infty$, the symbols $L^q(\Omega)$, $W^{m,q}(\Omega)$, $L^{p,q}(\Omega)$ denote the usual Lebesgue space, Sobolev space, and Lorentz space with norms $\|\cdot\|_{L^q} (= \|\cdot\|_{L^q(\Omega)})$, $\|\cdot\|_{W^{m,q}} (= \|\cdot\|_{W^{m,q}(\Omega)})$, and $\|\cdot\|_{L^{p,q}} (= \|\cdot\|_{L^{p,q}(\Omega)})$, respectively. Furthermore,

$$\begin{aligned}
&C_{0,\sigma}^\infty(\Omega) := \{f = (f^1, f^2, f^3) \in [C_0^\infty(\Omega)]^3;\ \nabla \cdot f = 0\},\\
&L^p_\sigma = L^p_\sigma(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{L^p}},\\
&G_p(\Omega) := \{f = (f^1, f^2, f^3) \in L^p(\Omega);\ f = \nabla g \text{ for some } g \in L^p_{loc}(\overline{\Omega})\},\\
&\dot{W}^{1,p}(\Omega) := \{f \in L^p_{loc}(\Omega);\ \|\nabla f\|_{L^p} < +\infty\},\\
&\dot{W}_0^{1,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\nabla \cdot\|_{L^p}},\ \dot{W}_{0,\sigma}^{1,p}(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\nabla \cdot\|_{L^p}},\\
&H_0^1(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,2}}},\ H_{0,\sigma}^1(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{W^{1,2}}}.
\end{aligned}$$

Here

$$\|\nabla f\|_{L^p} = \|\nabla f\|_{L^p(\Omega)} := \left(\int_\Omega |\nabla f(x)|^p dx\right)^{1/p}.$$

See [**3**] and [**1**] for the Lebesgue spaces, Sobolev spaces, and Lorentz spaces. Throughout this paper, we always consider $L^q(\Omega)$, $W^{m,q}(\Omega)$, $L^{p,q}(\Omega)$, $C_0^\infty(\Omega)$, $C_{0,\sigma}^\infty(\Omega)$, $L_\sigma^p(U)$, $G_p(\Omega)$, $\dot{W}^{1,p}(\Omega)$, $\dot{W}_0^{1,p}(\Omega)$, $\dot{W}_{0,\sigma}^{1,p}(\Omega)$, $\dot{H}_0^1(\Omega)$, and $H_{0,\sigma}^1(\Omega)$ as real-valued function spaces. We write its complexification of a real Banach space X as $X \oplus \mathrm{i}X$. Here i is the imaginary unit, and $\oplus$ the direct sum. By $\langle \cdot, \cdot \rangle$, we define the usual L^2-inner product, that is, for all $f = (f^1, f^2, f^3)$, $g = (g^1, g^2, g^3) \in [L^2(\Omega) \oplus \mathrm{i}L^2(\Omega)]^3$

$$\langle f, g \rangle := \int_\Omega f(x) \cdot \overline{g}(x) dx = \int_\Omega \{ f^1(x)\overline{g}^1(x) + f^2(x)\overline{g}^2(x) + f^3(x)\overline{g}^3(x) \} dx,$$

where $\overline{g}$ is its complex conjugate of g.

Let X be a Banach space and $\mathscr{A}$ a linear operator on X. The symbols $D(\mathscr{A})$, $R(\mathscr{A})$, and $N(\mathscr{A})$ represent the domain of $\mathscr{A}$, the range of $\mathscr{A}$, and the null space of $\mathscr{A}$, respectively. When $\mathscr{A}$ generates a C_0-semigroup on X, we write the semigroup as $\mathrm{e}^{t\mathscr{A}}$.

We will use the symbol C to denote a positive constant. We write $C(\eta_1, \eta_2)$ if the constant C depends on certain quantities η_1, η_2.

Let us recall the weak Hölder and Sobolev inequalities.

Lemma 2.1 ([**27**, Proposition 2.1]). *Let $1 < p_1, p_2 < \infty$ and $1 \leq q_1, q_2 \leq \infty$. Set $1/p = 1/p_1 + 1/p_2$. Then there is $C = C(p_1, p_2, q_1, q_2) > 0$ such that for all $f \in L^{p_1,q_1}(\Omega)$ and $g \in L^{p_2,q_2}(\Omega)$*

$$\|fg\|_{L^{p,q}} \leq C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}, \text{ where } q := \min\{q_1, q_2\}.$$

Lemma 2.2 ([**20**, Lemma 2.1], [**26**, Lemma 2.1]). *Let $p \in [2, 3)$. Then there is $C = C(p) > 0$ such that for all $g \in \dot{W}_0^{1,p}(\Omega)$*

$$\|g\|_{L^{3p/(3-p),p}} \leq C\|\nabla g\|_{L^p}.$$

Combining Lemmas 2.1 and 2.2, we have a key inequality to construct an energy inequality.

Lemma 2.3. *There is $C > 0$ such that for all $f \in L^{3,\infty}(\Omega)$ and $g \in \dot{W}_0^{1,2}(\Omega)$*

$$\|fg\|_{L^2} \leq C\|f\|_{L^{3,\infty}}\|\nabla g\|_{L^2}. \tag{2.1}$$

Using the Extension theory and the Gagliardo–Nirenberg inequality, we obtain

Lemma 2.4. (i) *Let $2 \leq p \leq 6$. Then there is $C = C(p) > 0$ such that for all $f \in H_0^1(\Omega)$*

$$\|f\|_{L^p} \leq C\|f\|_{L^2}^{(6-p)/(2p)}\|\nabla f\|_{L^2}^{(3p-6)/(2p)}.$$

(ii) *Let $2 \leq p \leq 6$. Then there is $C = C(p, \Omega) > 0$ such that for all $f \in W^{2,2}(\Omega)$*

$$\|\nabla f\|_{L^p} \leq C\|\nabla f\|_{L^2}^{(6-p)/(2p)}\|f\|_{W^{2,2}}^{(3p-6)/(2p)}.$$

(iii) *Let* $6 \leq p \leq \infty$. *Then there is* $C = C(p, \Omega) > 0$ *such that for all* $f \in H^1_0(\Omega) \cap W^{2,2}(\Omega)$

$$\|f\|_{L^p} \leq C\|\nabla f\|_{L^2}^{(p+6)/(2p)} \|f\|_{W^{2,2}}^{(p-6)/(2p)},$$

where $(p+6)/(2p) := 1/2$ *and* $(p-6)/(2p) := 1/2$ *if* $p = \infty$.

Proof of Lemma 2.4. We only prove (iii). Fix $p \in [6, \infty]$. Applying the Extension theory, the Gagliardo–Nirenberg inequality, (i), and (ii), we check that for each $f \in H^1_0(\Omega) \cap W^{2,2}(\Omega)$

$$\begin{aligned}\|f\|_{L^p} &\leq C(p, \Omega)\|f\|_{L^6}^{(p+6)/(2p)} (\|f\|_{L^6} + \|\nabla f\|_{L^6})^{(p-6)/(2p)}\\ &\leq C(p, \Omega)\|\nabla f\|_{L^2}^{(p+6)/(2p)} \|f\|_{W^{2,2}}^{(p-6)/(2p)}.\end{aligned}$$

Note that the above constant $C(p, \Omega)$ is independent of f. Therefore we see (iii). □

Next we study the Stokes operator. Let $\nu > 0$. Let P be the Helmholtz projection such that $P : [L^2(\Omega)]^3 \to L^2_\sigma(\Omega)$ and $(I - P) : [L^2(\Omega)]^3 \to G_2(\Omega)$. See [**42**, 2.5.2 Lemma in Chapter II] for the Helmholtz projection. We define the *Stokes operator* A in $L^2_\sigma(\Omega) \oplus \mathrm{i}L^2_\sigma(\Omega)$ as follows:

$$\begin{cases} Af := P(-\nu\Delta)f,\\ D(A) := [L^2_\sigma(\Omega) \cap [W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)]^3] \oplus \mathrm{i}[L^2_\sigma \cap [W^{1,2}_0 \cap W^{2,2}]^3].\end{cases}$$

We recall some fundamental properties of the Stokes operator A. From [**42**, Chapter III] and [**11**, Chapter V], we have

Lemma 2.5 (Fundamental properties of the Stokes operator (I)).

(i) *The operator* $-A$ *generates a bounded analytic semigroup on* $L^2_\sigma(\Omega) \oplus \mathrm{i}L^2_\sigma(\Omega)$.

(ii) *There is* $C = C(\Omega, \nu) > 0$ *such that for all* $f \in D(A)$

$$\|f\|_{W^{2,2}} \leq C\|(A + 1)f\|_{L^2}. \tag{2.2}$$

(iii) $D(A^{1/2}) = H^1_{0,\sigma}(\Omega)$ *and for all* $f \in D(A^{1/2})$

$$\|A^{1/2}f\|_{L^2} = \nu^{1/2}\|\nabla f\|_{L^2}. \tag{2.3}$$

(iv) *For all* $f \in D(A)$

$$\|A^{1/2}f\|_{L^2} \leq \|f\|_{L^2}^{1/2}\|Af\|_{L^2}^{1/2}. \tag{2.4}$$

Since A is a non-negative selfadjoint operator, we easily see the property (i). Combining Lemma 2.4, (2.2), and (2.3), we derive properties of the Stokes operator.

Lemma 2.6 (Fundamental properties of the Stokes operator (II)).

(i) *Let* $2 \le p \le 6$. *Then there is* $C = C(\nu, p) > 0$ *such that for all* $f \in D(A^{1/2})$

$$\|f\|_{L^p} \le C\|f\|_{L^2}^{(6-p)/(2p)}\|A^{1/2}f\|_{L^2}^{(3p-6)/(2p)}. \tag{2.5}$$

(ii) *Let* $2 \le p \le 6$. *Then there is* $C = C(\Omega, \nu, p) > 0$ *such that for all* $f \in D(A)$

$$\|\nabla f\|_{L^p} \le C\|A^{1/2}f\|_{L^2}^{(6-p)/(2p)}(\|f\|_{L^2} + \|Af\|_{L^2})^{(3p-6)/(2p)}. \tag{2.6}$$

(iii) *Let* $6 \le p \le \infty$. *Then there is* $C = C(\Omega, \nu, p) > 0$ *such that for all* $f \in D(A)$

$$\|f\|_{L^p} \le C\|A^{1/2}f\|_{L^2}^{(p+6)/(2p)}(\|f\|_{L^2} + \|Af\|_{L^2})^{(p-6)/(2p)}, \tag{2.7}$$

where $(p+6)/(2p) := 1/2$ *and* $(p-6)/(2p) := 1/2$ *if* $p = \infty$.

Finally, we state maximal L^p-regularity (maximal L^p-in-time regularity for Hilbert space-valued functions). Since $-A$ generates a bounded analytic semigroup on $L^2_\sigma \oplus \mathrm{i}L^2_\sigma$, it follows from [**6**] that A has maximal L^p-regularity.

Lemma 2.7 (Maximal L^p-regularity). *Write* $H = L^2_\sigma(\Omega) \oplus \mathrm{i}L^2_\sigma(\Omega)$. *Let* $\nu > 0$ *and* $1 < p < \infty$. *Then for each* $T \in (0, \infty]$ *and* $(F, W_0) \in L^p(0,T;H) \times (H, D(A))_{1-1/p,p}$, *the system*

$$\begin{cases} W_t + AW = F, \ t \in (0,T), \\ W|_{t=0} = W_0, \end{cases}$$

has a unique solution W *satisfying*

$$\begin{aligned} &\|W_t\|_{L^p(0,T;L^2(\Omega))} + \|AW\|_{L^p(0,T;L^2(\Omega))} \\ &\quad \le C_p\left(\|F\|_{L^p(0,T;L^2(\Omega))} + \|W_0\|_{L^2(\Omega)} + \left(\int_0^1 \|A\mathrm{e}^{-tA}W_0\|^p_{L^2(\Omega)}dt\right)^{1/p}\right), \end{aligned}$$

where $C_p > 0$ *is independent of* (T, F, W_0). *Here* $(H, D(A))_{1-1/p,p}$ *is the real interpolation space between* H *and* $D(A)$.

See [**13**], [**7**], [**28**] for maximal L^p-regularity and [**30**, Proposition 6.2] for the real interpolation norms. See also [**36**] and [**21**, Appendix A].

3. Linear operators.

In this section, we study the main linear operator of the system (1.3). Let $\nu > 0$, $u_\infty \in \mathbb{R}^3$, and let w be as in the Assumption 1.1. Let P be the Helmholtz projection and A the Stokes operator defined by Section 2. Multiplying P into the system (1.3), we obtain the following abstract system.

$$\begin{cases} v_t + Lv = -P(v,\nabla)v, \quad t > 0, \\ v|_{t=0} = Pv_0. \end{cases} \tag{3.1}$$

Here $Lv = P(-\nu\Delta v + (u_\infty, \nabla)v + (w, \nabla)v + (v, \nabla)w)$. To study the main linear operator of the system (3.1), we define

$$\begin{cases} Bf := P\left((u_\infty, \nabla)f + (w, \nabla)f + (f, \nabla)w\right), \\ D(B) := \{f \in L^2_\sigma(\Omega) \oplus \mathrm{i}L^2_\sigma(\Omega);\ Bf \in L^2_\sigma(\Omega) \oplus \mathrm{i}L^2_\sigma(\Omega)\}, \end{cases}$$

$$\begin{cases} Lf := Af + Bf, \\ D(L) := D(A) \cap D(B). \end{cases}$$

From Lemmas 2.5 and 3.3, we observe that $D(L) = D(A)$ and that the operator $-L$ generates an analytic semigroup on $L^2_\sigma(\Omega) \oplus \mathrm{i}L^2_\sigma(\Omega)$. The aim of this section is to prove the following linear stability.

Lemma 3.1. *There is $\delta_0 = \delta_0(\nu) > 0$ such that if*

$$\|w\|_{L^{3,\infty}} < \delta_0$$

then for each $f \in L^2_\sigma(\Omega)$

$$\lim_{t\to\infty} \|\mathrm{e}^{-tL} f\|_{L^2} = 0.$$

Here δ_0 is the constant introduced in Lemma 3.5.

In order to prove Lemma 3.1, we shall apply the following abstract theory:

Lemma 3.2 ([**22**, Lemma 4.1]). *Let H be a Hilbert space and $\|\cdot\|_H$ its norm. Let $\mathscr{A} : D(\mathscr{A})(\subset H) \to H$ and $\mathscr{B} : D(\mathscr{B})(\subset H) \to H$ be two linear operators on H such that $D(\mathscr{A}) \subset D(\mathscr{B})$. Define $\mathscr{L}v := \mathscr{A}v + \mathscr{B}v$ and $D(\mathscr{L}) := D(\mathscr{A})$. Assume that $-\mathscr{A}$ generates a bounded analytic semigroup on H. Assume that there are $0 < \alpha \le 1$, $0 < \beta < 1$, and $C_1 > 0$ such that for all $\varphi \in D(\mathscr{A})$*

$$\|\mathscr{B}\varphi\|_H \le C_1 \|\mathscr{A}^\beta \varphi\|_H^\alpha (\|\varphi\|_H + \|\mathscr{A}\varphi\|_H)^{1-\alpha}. \tag{3.2}$$

Suppose that there is $C_2 > 0$ such that for all $\phi \in H$ and $s, t \ge 0 (s < t)$

$$\|\mathrm{e}^{-t\mathscr{L}}\phi\|_H^2 + C_2 \int_s^t \|\mathscr{A}^\beta \mathrm{e}^{-\tau\mathscr{L}}\phi\|_H^2 d\tau \le \|\mathrm{e}^{-s\mathscr{L}}\phi\|_H^2. \tag{3.3}$$

If the range $R(\mathscr{L})$ is dense in H, then for each $\psi \in H$

$$\lim_{t\to\infty} \|\mathrm{e}^{-t\mathscr{L}}\psi\|_H = 0. \tag{3.4}$$

To this end, we study the linear operator B.

Lemma 3.3 (Properties of the operator B).

(i) *Assume that $3 \le p_2 < \infty$. Then there is $C = C(\Omega, \nu, u_\infty, w, p_1, p_2) > 0$ such that for all $f \in D(A)$*

$$\|Bf\|_{L^2} \le C\{\|A^{1/2}f\|_{L^2} + \|A^{1/2}f\|_{L^2}^{(p_1-3)/p_1}(\|f\|_{L^2} + \|Af\|_{L^2})^{3/p_1} + \|A^{1/2}f\|_{L^2}^{3/p_2}(\|f\|_{L^2} + \|Af\|_{L^2})^{(p_2-3)/p_2}\}. \tag{3.5}$$

(ii) *Assume that* $2 \le p_2 \le 3$. *Then there is* $C = C(\Omega, \nu, u_\infty, w, p_1, p_2) > 0$ *such that for all* $f \in D(A)$

$$\|Bf\|_{L^2} \le C\{\|A^{1/2}f\|_{L^2} + \|A^{1/2}f\|_{L^2}^{(p_1-3)/p_1}(\|f\|_{L^2} + \|Af\|_{L^2})^{3/p_1} + \|A^{1/2}f\|_{L^2}^{(2p_2-3)/p_2}(\|f\|_{L^2} + \|Af\|_{L^2})^{(3-p_2)/p_2}\}. \tag{3.6}$$

(iii) *There are* $C = C(\Omega, \nu, u_\infty, w) > 0$ *and* $0 < \alpha \le 1$ *such that for all* $f \in D(A)$

$$\|Bf\|_{L^2} \le C\|A^{1/2}f\|_{L^2}^{\alpha}(\|f\|_{L^2} + \|Af\|_{L^2})^{1-\alpha}, \tag{3.7}$$

where $\alpha = \alpha(p_1, p_2)$.

(iv) *For each* $\epsilon > 0$ *there is* $C = C(\Omega, \nu, u_\infty, w, p_1, p_2, \epsilon) > 0$ *such that for all* $f \in D(A)$

$$\|Bf\|_{L^2} \le \epsilon\|Af\|_{L^2} + C\|f\|_{L^2}. \tag{3.8}$$

(v) *Assume that* $6 \le p_1 \le \infty$ *and that* $2 \le p_2 \le 6$. *Then there is* $C = C(\Omega, \nu, u_\infty, w, p_1, p_2) > 0$ *such that for all* $f \in D(A)$

$$\|Bf\|_{L^2} \le C\|A^{1/2}f\|_{L^2}^{1/2}(\|f\|_{L^2} + \|Af\|_{L^2})^{1/2}. \tag{3.9}$$

Proof of Lemma 3.3 . We first show (i) and (ii). Fix $f \in D(A)$. A calculation gives

$$\|Bf\|_{L^2} \le \|(u_\infty, \nabla)f\|_{L^2} + \|(w, \nabla)f\|_{L^2} + \|(f, \nabla)w\|_{L^2}.$$

From (2.3), we have

$$\|(u_\infty, \nabla)f\|_{L^2} \le C(\nu, u_\infty)\|A^{1/2}f\|_{L^2}.$$

Applying the Hölder inequality and (2.6), we check that

$$\begin{aligned}\|(w, \nabla)f\|_{L^2} &\le C\|w\|_{L^{p_1}}\|\nabla f\|_{L^{(2p_1)/(p_1-2)}} \\ &\le C(\Omega, \nu, w, p_1)\|A^{1/2}f\|_{L^2}^{(p_1-3)/p_1}(\|f\|_{L^2} + \|Af\|_{L^2})^{3/p_1}.\end{aligned}$$

The Hölder inequality shows that

$$\|(f, \nabla)w\|_{L^2} \le C(p_2)\|\nabla w\|_{L^{p_2}}\|f\|_{L^{(2p_2)/(p_2-2)}}.$$

Using (2.5) and (2.7), we see that

$\|f\|_{L^{(2p_2)/(p_2-2)}}$

$$\leq \begin{cases} C(\nu, p_2)\|A^{1/2}f\|_{L^2}^{3/p_2}(\|f\|_{L^2} + \|Af\|_{L^2})^{(p_2-3)/p_2} & \text{if } p_2 \in [3, \infty), \\ C(\Omega, \nu, p_2)\|A^{1/2}f\|_{L^2}^{(2p_2-3)/p_2}(\|f\|_{L^2} + \|Af\|_{L^2})^{(3-p_2)/p_2} & \text{if } p_2 \in [2, 3). \end{cases}$$

Consequently, we obtain (3.5) and (3.6).

Next we prove (iii), (iv), and (v). From (2.4), (3.5), and (3.6), we see (iii). Applying (2.4) and the Young inequality into (3.7), we observe that for each $\epsilon > 0$ there is $C = C(\Omega, \nu, u_\infty, w, p_1, p_2, \epsilon) > 0$ such that for each $f \in D(A)$

$$\begin{aligned} \|Bf\|_{L^2} &\leq C\|f\|_{L^2}^{\alpha/2}(\|f\|_{L^2} + \|Af\|_{L^2})^{1-\alpha/2} \\ &\leq \epsilon\|Af\|_{L^2} + C\|f\|_{L^2}, \end{aligned}$$

which is (iv). It is easy to check that

$$\frac{p_1 - 3}{p_1} \geq \frac{1}{2} \text{ if } 6 \leq p_1 \leq \infty, \quad \frac{3}{p_2} \geq \frac{1}{2} \text{ if } 3 \leq p_2 \leq 6,$$
$$\frac{2p_2 - 3}{p_2} \geq \frac{1}{2} \text{ if } 2 \leq p_2 < 3.$$

Therefore we have (3.9) from (2.4), (3.5), and (3.6). □

We shall study the linear operator B without a break.

LEMMA 3.4. *There is $C_* > 0$ such that for all $g \in L^2_\sigma(\Omega) \cap D(A)$ and $f \in D(A)$*

$$\begin{aligned} |\langle Bg, g\rangle| &\leq C_*\|w\|_{L^{3,\infty}}\|\nabla g\|_{L^2}^2, \\ |\langle Bf, f\rangle + \langle f, Bf\rangle| &\leq C_*\|w\|_{L^{3,\infty}}\|\nabla f\|_{L^2}^2. \end{aligned}$$

PROOF OF LEMMA 3.4. Fix $g \in L^2_\sigma(\Omega) \cap D(A)$. Integrating by parts, we observe that

$$\begin{aligned} \langle Bg, g\rangle &= \langle (u_\infty, \nabla)g, g\rangle + \langle (w, \nabla)g, g\rangle + \langle (g, \nabla)w, g\rangle \\ &= \langle \nabla \cdot (g \otimes w), g\rangle \\ &= -\langle g \otimes w, \nabla g\rangle. \end{aligned}$$

Using the Cauchy–Schwarz inequality and (2.1), we see that

$$\begin{aligned} |\langle Bg, g\rangle| &\leq C\|g \otimes w\|_{L^2}\|\nabla g\|_{L^2} \\ &\leq C\|w\|_{L^{3,\infty}}\|\nabla g\|_{L^2}^2. \end{aligned}$$

Let $f \in D(A)$. Similarly, we obtain

$$|\langle Bf, f\rangle + \langle f, Bf\rangle| \leq C\|w\|_{L^{3,\infty}}\|\nabla f\|_{L^2}^2.$$

Here we used the fact that

$$\begin{aligned} \langle (u_\infty, \nabla)f, f\rangle + \langle f, (u_\infty, \nabla)f\rangle &= 0, \\ \langle (w, \nabla)f, f\rangle + \langle f, (w, \nabla)f\rangle &= 0. \end{aligned}$$

Therefore the lemma follows. □

Lemma 3.4 gives one key lemma to show the stability of the semigroup e^{-tL}.

Lemma 3.5. *There is $\delta_0 = \delta_0(\nu) > 0$ such that if $\|w\|_{L^{3,\infty}} < \delta_0$ then for all $g \in L^2_\sigma(\Omega) \cap D(A)$ and $f \in D(A)$*

$$|\langle Bg, g\rangle| \leq \frac{\nu}{2}\|\nabla g\|^2_{L^2}, \tag{3.10}$$

$$|\langle Bf, f\rangle + \langle f, Bf\rangle| \leq \frac{\nu}{2}\|\nabla f\|^2_{L^2}. \tag{3.11}$$

Proof of Lemma 3.5. Let C_* be the constant appearing in Lemma 3.4. Set $\delta_0 = \nu/(2C_*)$. From Lemma 3.4, we easily obtain (3.10) and (3.11). □

From Lemma 3.3, we see that $D(L) = D(A) \cap D(B) = D(A)$. Since the Stokes operator $-A$ generates an analytic semigroup on $L^2_\sigma \oplus \mathrm{i}L^2_\sigma$, we apply a perturbation theory on analytic semigroups and the assertion (iv) of Lemma 3.3 to find that the operator $-L$ generates an analytic semigroup on $L^2_\sigma \oplus \mathrm{i}L^2_\sigma$.

Let us now discuss the adjoint operator L^* of the operator L.

Lemma 3.6. *Let L^* be the adjoint operator of the operator L in $L^2_\sigma(\Omega) \oplus \mathrm{i}L^2_\sigma(\Omega)$. Then*

$$\begin{cases} L^* f = P(-\nu\Delta f - (u_\infty, \nabla)f - (w, \nabla)f - (f, \nabla)w), \\ D(L^*) = D(A). \end{cases}$$

Moreover, the operator $-L^*$ generates an analytic semigroup on $L^2_\sigma(\Omega) \oplus \mathrm{i}L^2_\sigma(\Omega)$.

Proof of Lemma 3.6. Define

$$\begin{cases} B' f := P(-(u_\infty, \nabla)f - (w, \nabla)f - (f, \nabla)w), \\ D(B') := \{f \in L^2_\sigma(\Omega) \oplus \mathrm{i}L^2_\sigma(\Omega); B'f \in L^2_\sigma(\Omega) \oplus \mathrm{i}L^2_\sigma(\Omega)\}, \end{cases}$$

$$\begin{cases} L' f := P(-\nu\Delta f - (u_\infty, \nabla)f - (w, \nabla)f - (f, \nabla)w), \\ D(L') := D(A) \cap D(B'). \end{cases}$$

By the integration by parts, we check that for all $f, g \in D(A)$

$$\langle Lf, g\rangle = \langle f, L'g\rangle = \langle f, L^*g\rangle.$$

This shows that $L^* = L'$ on $D(A)$. By an argument similar to that in the proof of Lemma 3.3, we see that for each $\epsilon > 0$ there is $C = C(\Omega, \nu, u_\infty, w, p_1, p_2, \epsilon) > 0$ such that for all $f \in D(A)$

$$\|B'f\|_{L^2} \leq \epsilon\|Af\|_{L^2} + C\|f\|_{L^2}.$$

This implies that $D(L') = D(A) \cap D(B') = D(A)$. Since $-A$ generates an analytic semigroup on $L^2_\sigma \oplus \mathrm{i}L^2_\sigma$, we apply a perturbation theory on analytic semigroups to find that $-L'$ generates an analytic semigroup on $L^2_\sigma \oplus \mathrm{i}L^2_\sigma$. We make use of an argument in [**22**, the proof of Lemma 3.11] to conclude that $L^* = L'$ and $D(L^*) = D(L') = D(A)$. $\square$

Next we consider the two linear operators L and L^* when $\|w\|_{L^{3,\infty}}$ is sufficiently small.

LEMMA 3.7. *Let δ_0 be the constant appearing in Lemma 3.5. Assume that*

$$\|w\|_{L^{3,\infty}} < \delta_0.$$

Then the following three assertions hold:

(i) *For all $\varphi \in L^2_\sigma(\Omega) \oplus \mathrm{i}L^2_\sigma(\Omega)$ and $s, t \geq 0 (s < t)$*

$$\|\mathrm{e}^{-tL}\varphi\|^2_{L^2} + \nu \int_s^t \|\nabla \mathrm{e}^{-\tau L}\varphi\|^2_{L^2}\, d\tau \leq \|\mathrm{e}^{-tL}\varphi\|^2_{L^2}, \tag{3.12}$$

$$\|\mathrm{e}^{-tL^*}\varphi\|^2_{L^2} + \nu \int_s^t \|\nabla \mathrm{e}^{-\tau L^*}\varphi\|^2_{L^2}\, d\tau \leq \|\mathrm{e}^{-tL^*}\varphi\|^2_{L^2}. \tag{3.13}$$

(ii) *There is $C = C(\Omega, \nu, u_\infty, w) > 0$ such that for all $\phi \in D(A)$*

$$\|\phi\|_{W^{2,2}} \leq C\|(L+1)\phi\|_{L^2}, \tag{3.14}$$

$$\|\phi\|_{W^{2,2}} \leq C\|(L^*+1)\phi\|_{L^2}. \tag{3.15}$$

(iii) *The range $R(L)$ is dense in $L^2_\sigma(\Omega) \oplus \mathrm{i}L^2_\sigma(\Omega)$.*

PROOF OF LEMMA 3.7. We first show (i). Fix $\varphi \in L^2_\sigma(\Omega) \oplus \mathrm{i}L^2_\sigma(\Omega)$. Set $V(t) := \mathrm{e}^{-tL}\varphi$. A direct calculation gives

$$\begin{aligned}\frac{d}{dt}\|V(t)\|^2_{L^2} &= \langle V_t, V\rangle + \langle V, V_t\rangle \\ &= \langle -LV, V\rangle + \langle V, -LV\rangle \\ &= -\langle AV, V\rangle - \langle V, AV\rangle - \langle BV, V\rangle - \langle V, BV\rangle.\end{aligned}$$

By the integration by parts, we have

$$\frac{d}{dt}\|V(t)\|^2_{L^2} + 2\nu\|\nabla V(t)\|^2_{L^2} = -\langle BV, V\rangle - \langle V, BV\rangle.$$

Using (3.11) and integrating with respect to time, we observe that for $s, t \geq 0$ $(s < t)$

$$\|V(t)\|^2_{L^2} + \nu \int_s^t \|\nabla V(\tau)\|^2_{L^2}\, d\tau \leq \|V(s)\|^2_{L^2},$$

where $V(0) = \varphi$. This is (3.12). Similarly, we have (3.13).

Next we prove (ii). Let $f \in D(A)$. We use (3.8) with $\epsilon = 1/2$ to obtain

$$\|Af\|_{L^2} \leq \|(L+1)f\|_{L^2} + (1/2)\|Af\|_{L^2} + C\|f\|_{L^2}.$$

Since $\|f\|_{L^2} \leq C\|(L+1)f\|_{L^2}$, we have

$$\|Af\|_{L^2} \le C\|(L+1)f\|_{L^2}. \tag{3.16}$$

Combing the assertion (ii) of Lemma 2.5 and (3.16) gives (3.14). Similarly, we obtain (3.15).

Finally, we show (iii). Assume that $D(L) = D(L^*) = L^2_\sigma \cap D(A)$. Take $f \in D(L^*)$ such that $\langle L^* f, f\rangle = 0$. By the integration by parts, we have

$$0 = \langle L^* f, f\rangle = \langle f, Lf\rangle \ge \frac{\nu}{2}\|\nabla f\|^2_{L^2}.$$

This implies that $f = 0$. Therefore we see that $N(L^*) = \{0\}$, where $N(L^*)$ is the null set of L^*. Since L is a closed operator, we conclude that $R(L) \cap L^2_\sigma$ is dense in L^2_σ. Therefore we deduce (iii). □

Let us derive the stability of the semigroup e^{-tL}.

Proof of Lemma 3.1. Applying Lemmas 3.2 and 3.7, and the assertion (iii) of Lemma 3.3, we see that for each $f \in L^2_\sigma(\Omega)$

$$\lim_{t\to\infty} \|\mathrm{e}^{-tL} f\|_{L^2} = 0$$

when $\|w\|_{L^{3,\infty}} < \delta_0$, where δ_0 is the constant appearing in Lemma 3.5. □

4. Nonlinear stability and decay properties.

This section derives L^2-asymptotic stability and the decay rate with respect to time of energy solutions to the system (3.1). Let $\nu > 0$, $u_\infty \in \mathbb{R}^3$, and let w be as in Assumption 1.1. We first construct an energy inequality for solutions of (3.1).

Lemma 4.1. *Let δ_0 be the constant appearing in Lemma 3.5. Let $v_0 \in H^1_{0,\sigma}(\Omega)$, $T \in (0,\infty]$ and,*

$$v \in BC([0,T);\ H^1_{0,\sigma}(\Omega)) \cap C((0,T); D(A)) \cap C^1((0,T); L^2_\sigma(\Omega)).$$

Suppose that

$$\|w\|_{L^{3,\infty}} < \delta_0.$$

Assume that the function v is a strong solution of the system (3.1) on $(0,T)$. Then for all $s,t \ge 0$ with $s < t < T$

$$\|v(t)\|^2_{L^2} + \nu\int_s^t \|\nabla v(\tau)\|^2_{L^2} d\tau \le \|v(s)\|^2_{L^2}. \tag{4.1}$$

Proof of Lemma 4.1. Multiplying (3.1) by v, and then integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2} + \nu\|\nabla v(t)\|^2_{L^2} = -\langle Bv(t), v(t)\rangle.$$

Here we used the fact that $\langle (v,\nabla)v,v\rangle = 0$. Using Lemma 3.5 and integrating with respect to time, we have (4.1). □

Applying Lemmas 3.1, 3.7, 4.1, and arguments similar to those in [**21**, Chapters 7 and 8] or [**22**], we obtain the following proposition.

Proposition 4.2. *Let δ_0 be the constant appearing in Lemma 3.5. Assume that*

$$\|w\|_{L^{3,\infty}} < \delta_0.$$

Then there exists $c_0 = c_0(\Omega,\nu,u_\infty,w) > 0$ such that for every $v_0 \in H^1_{0,\sigma}(\Omega)$ with $\|v_0\|_{W^{1,2}} < c_0$ the system (3.1) has a unique global-in-time strong solution:

$$v \in BC([0,\infty);\ H^1_{0,\sigma}(\Omega)) \cap C((0,\infty);D(A)) \cap C^1((0,\infty);L^2_\sigma(\Omega)),$$

satisfying

$$\lim_{t\to\infty} \|v(t)\|_{L^2(\Omega)} = 0. \tag{4.2}$$

Next we discuss the uniqueness of the strong solutions of the system (3.1).

Lemma 4.3. *Let v be a global-in-time strong solution obtained by Proposition 4.2 of the system (3.1). Let $T_0, T_1 > 0$ such that $T_0 < T_1$. Suppose that*

$$\tilde v \in W^{1,p}(T_0,T_1;L^2_\sigma(\Omega)) \cap L^p(T_0,T_1;D(A)) \text{ for some } 1<p<\infty.$$

Assume that $\tilde v$ satisfies the following system:

$$\begin{cases} \tilde v_t + L\tilde v = -P(v,\nabla)v,\ T_0<t<T_1,\\ \tilde v|_{t=T_0} = v(T_0). \end{cases}$$

Then $v = \tilde v$ on $[T_0,T_1)$.

Proof of Lemma 4.3. Since $v \in C^1((0,\infty);L^2_\sigma(\Omega)) \cap C((0,\infty);D(A))$, we observe that

$$v \in W^{1,p}(T_0,T_1;L^2_\sigma(\Omega)) \cap L^p(T_0,T_1;D(A)).$$

The embedding theorem ([**30**, Corollary 1.14]) implies that

$$\begin{aligned} W^{1,p}(T_0,T_1;L^2_\sigma(\Omega)) \cap L^p(T_0,T_1;D(A)) &\subset BC([T_0,T_1);(L^2_\sigma,D(A))_{1-1/p,p})\\ &\subset BC([T_0,T_1);L^2_\sigma(\Omega)). \end{aligned}$$

Set $V = V(t) = v(t) - \tilde v(t)$. It is easy to check that V satisfies the following system:

$$\begin{cases} V_t + LV = 0,\ T_0<t<T_1,\\ V|_{t=T_0} = 0. \end{cases}$$

Since $-L$ generates analytic semigroup on L^2_σ, we see that $V \equiv 0$ on $[T_0, T_1)$. Therefore we conclude that $v = \tilde{v}$ on $[T_0, T_1)$. □

Finally, we derive the decay rate with respect to time of an energy solution of the system (3.1) to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Let δ_0, c_0 be the two constants appearing in Lemma 3.5 and Proposition 4.2, respectively. Assume that $\|w\|_{L^{3,\infty}} < \delta_0$. Let $v_0 \in H^1_{0,\sigma}(\Omega)$ with $\|v_0\|_{W^{1,2}} < c_0$. From Proposition 4.2, there exists a unique global-in-time strong solution v of the system (3.1) with the initial datum v_0, satisfying (4.2). By an argument similar to that in the proof of Lemma 4.1, we check that for all $s, t \geq 0$ with $s < t$

$$\|v(t)\|^2_{L^2} + \nu \int_s^t \|\nabla v(\tau)\|^2_{L^2} d\tau \leq \|v(s)\|^2_{L^2}. \tag{4.3}$$

To derive the decay rate with respect to time of v, we prepare the following lemma.

Lemma 4.4. *There is $T_0 > 0$ such that for all $t, s > 0$ such that $T_0 < s < t$*

$$\|v_t(t)\|_{L^2} \leq \frac{1}{t-s} \int_s^t \|v_t(\tau)\|_{L^2} d\tau.$$

Proof of Lemma 4.4. We first show that there is $T_0 > 0$ such that for all $s, t > 0$ and $h \in (-1, 1)$ with $T_0 \leq s \leq t$

$$\|v(t+h) - v(t)\|_{L^2} \leq \|v(s+h) - v(s)\|_{L^2}. \tag{4.4}$$

Fix $h \in (-1, 1)$. Set $Q(t) := v(t+h) - v(t)$. Since v is a strong solution of (3.1), we observe that

$$\begin{aligned} Q_t(t) &= v_t(t+h) - v_t(t) \\ &= -\{Lv(t+h) - P(v(t+h), \nabla)v(t+h)\} + \{Lv(t) + P(v(t), \nabla)v(t)\} \\ &= -LQ(t) - P(v(t), \nabla)Q(t) - P(Q(t), \nabla)v(t) - P(Q(t), \nabla)Q(t). \end{aligned}$$

A simple calculation gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|Q(t)\|^2_{L^2} &= \langle Q_t(t), Q(t)\rangle \\ &= \langle -LQ(t) - P(Q(t), \nabla)Q(t) - P(Q(t), \nabla)v(t) - P(Q(t), \nabla)Q(t), Q(t)\rangle. \end{aligned}$$

By the integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|Q(t)\|^2_{L^2} + \nu\|\nabla Q(t)\|^2_{L^2} \leq |\langle BQ(t), Q(t)\rangle| + |\langle (Q(t), \nabla)v(t), Q(t)\rangle|.$$

From Lemma 3.5, we obtain

$$|\langle BQ(t), Q(t)\rangle| \leq \frac{\nu}{2}\|\nabla Q(t)\|^2_{L^2}.$$

Using the integration by parts, the Hölder and the Gagliardo–Nirenberg inequalities, we check that

$$\begin{aligned}|\langle (Q(t),\nabla)v(t),Q(t)\rangle| &= |-\langle Q\otimes v,\nabla Q\rangle| \\ &\le C\|Q\|_{L^6}\|v\|_{L^3}\|\nabla Q\|_{L^2} \\ &\le C\|v(t)\|_{L^3}\|\nabla Q(t)\|_{L^2}^2.\end{aligned}$$

As a result, we have

$$\frac{1}{2}\frac{d}{dt}\|Q(t)\|_{L^2}^2+\frac{\nu}{2}\|\nabla Q(t)\|_{L^2}^2\le C\|v(t)\|_{L^3}\|\nabla Q(t)\|_{L^2}^2.$$

Integrating with respect to time, we obtain

$$\|Q(t)\|_{L^2}^2+\nu\int_s^t\|\nabla Q(\tau)\|_{L^2}^2d\tau\le\|Q(s)\|_{L^2}^2+\sup_{s\le\tau<\infty}\|v(t)\|_{L^3}\int_s^t\|\nabla Q(\tau)\|_{L^2}^2d\tau.$$

Since $v\in BC([0,\infty);H^1_{0,\sigma}(\Omega))$, we use the Gagliardo–Nirenberg inequality to see that

$$\begin{aligned}\sup_{s\le\tau<\infty}\|v(\tau)\|_{L^3} &\le C\sup_{s\le\tau<\infty}(\|v(\tau)\|_{L^2}^{1/2}\|\nabla v(\tau)\|_{L^2}^{1/2}) \\ &\le C\sup_{s\le\tau<\infty}\|v(\tau)\|_{L^2}^{1/2}.\end{aligned}$$

From (4.2), we find that there is $T_0>0$ such that for all $s\ge T_0$

$$\sup_{s\le\tau<\infty}\|v(t)\|_{L^3}<\frac{\nu}{2}.$$

Thus, if $s,t\ge T_0$ and $s<t$, then

$$\|Q(t)\|_{L^2}^2+\frac{\nu}{2}\int_s^t\|\nabla Q(\tau)\|_{L^2}^2d\tau\le\|Q(s)\|_{L^2}^2.$$

Therefore, there is $T_0>0$ such that for all $t>s>T_0$

$$\|Q(t)\|_{L^2}\le\|Q(s)\|_{L^2}.$$

This proves (4.4). From (4.4), we see that for $T_0<\tau<t$ and $h\ne 0$

$$\left\|\frac{v(t+h)-v(t)}{h}\right\|_{L^2}\le\left\|\frac{v(\tau+h)-v(\tau)}{h}\right\|_{L^2}.$$

Since $v\in C^1((0,\infty);L^2_\sigma(\Omega))$, we let $h\to 0$ to check that

$$\|v_t(t)\|_{L^2}\le\|v_t(\tau)\|_{L^2}.$$

Integrating with respect to τ, we find that for $T_0<s<t$

$$\|v_t(t)\|_{L^2} \leq \frac{1}{t-s}\int_s^t \|v_t(\tau)\|_{L^2} d\tau.$$

Therefore the lemma follows. □

Now we return to the proof of Theorem 1.2. We now assume that $p_1 \in [6,\infty]$ and $p_2 \in [2,6]$. Let T_0 be the constant appearing in Lemma 4.4. It is clear that for each $T > T_0$, v satisfies

$$\begin{cases} v_t + Av = -Bv - P(v,\nabla)v, \ t \in (T_0,T), \\ v|_{t=T_0} = v(T_0). \end{cases}$$

Fix $T > T_0$. Using maximal L^4-regularity (Lemma 2.7) and Lemma 4.3, we see that there is $C_4 > 0$ independent of T such that

$$\begin{aligned} &\int_{T_0}^T \|v_t\|_{L^2}^4 d\tau + \int_{T_0}^T \|Av\|_{L^2}^4 d\tau \\ &\leq C_4\left(\|v(T_0)\|_{W^{2,2}}^4 + \int_{T_0}^T \|-Bv - P(v,\nabla)v\|_{L^2}^4 d\tau\right). \end{aligned} \tag{4.5}$$

Here we used that fact that

$$\int_0^1 \|A\mathrm{e}^{-\tau A} v(T_0)\|_{L^2}^4 d\tau \leq \|Av(T_0)\|_{L^2}^4 \leq C(\nu)\|v(T_0)\|_{W^{2,2}}^4.$$

An easy computation gives

$$C_4\|-Bv - P(v,\nabla)v\|_{L^2}^4 \leq 16C_4(\|Bv\|_{L^2}^4 + \|P(v,\nabla)v\|_{L^2}^4). \tag{4.6}$$

By (3.9), we have

$$16C_4\|Bv\|_{L^2}^4 \leq C\|A^{1/2}v\|_{L^2}^2\|v\|_{L^2}^2 + C\|A^{1/2}v\|_{L^2}^2\|Av\|_{L^2}^2.$$

Since $ab \leq \varepsilon a^2 + b^2/(4\varepsilon)$ $(a,b \geq 0, \varepsilon > 0)$, we obtain

$$\begin{aligned} 16C_4\int_{T_0}^T \|Bv\|_{L^2}^4 d\tau &\leq C\int_{T_0}^T \|A^{1/2}v\|_{L^2}^2\|v\|_{L^2}^2 d\tau \\ &\quad + \frac{1}{4}\int_{T_0}^T \|Av\|_{L^2}^4 d\tau + C\int_{T_0}^T \|A^{1/2}v\|_{L^2}^4 d\tau. \end{aligned} \tag{4.7}$$

Using the Hölder inequality and Lemma 2.6, we check that

$$\begin{aligned} \|P(v,\nabla)v\|_{L^2} &\leq C\|v\|_{L^\infty}\|\nabla v\|_{L^2} \\ &\leq C\|A^{1/2}v\|_{L^2}^{3/2}(\|v\|_{L^2} + \|Av\|_{L^2})^{1/2}. \end{aligned} \tag{4.8}$$

The formula: $ab \leq \varepsilon a^2 + b^2/(4\varepsilon)$ $(a,b \geq 0, \varepsilon > 0)$ shows that

$$16C_4 \int_{T_0}^{T} \|P(v,\nabla)v\|_{L^2}^4 d\tau \le C \int_{T_0}^{T} \|A^{1/2}v\|_{L^2}^6 \|v\|_{L^2}^2 d\tau \\ + \frac{1}{4}\int_{T_0}^{T} \|Av\|_{L^2}^4 d\tau + C\int_{T_0}^{T} \|A^{1/2}v\|_{L^2}^{12} d\tau. \tag{4.9}$$

From (4.3) and (2.3), we have

$$\int_0^\infty \|A^{1/2}v(\tau)\|_{L^2}^2 d\tau < C < +\infty. \tag{4.10}$$

Since $v \in BC([0,\infty); H^1_{0,\sigma}(\Omega))$, we see that

$$\sup_{0\le\tau<\infty} (\|v(\tau)\|_{L^2} + \|\nabla v(\tau)\|_{L^2}) < C < +\infty. \tag{4.11}$$

Applying (4.6), (4.7), (4.9), (4.10), and (4.11), we have

$$C_4 \int_{T_0}^{T} \| - Bv - P(v,\nabla)v\|_{L^2}^4 d\tau \le \frac{1}{2}\int_{T_0}^{T} \|Av\|_{L^2}^4 dt + C_\sharp, \tag{4.12}$$

where $C_\sharp$ is independent of T. Combining (4.5) and (4.12) gives

$$\int_{T_0}^{T} \|v_t\|_{L^2}^4 d\tau + \frac{1}{2}\int_{T_0}^{T} \|Av\|_{L^2}^4 d\tau \le C\|v(T_0)\|_{W^{2,2}}^4 + C_\sharp.$$

Since the two constants C_4 and $C_\sharp$ do not depend on T, we can tend T to ∞ to have

$$\int_{T_0}^{\infty} \|v_t\|_{L^2}^4 d\tau + \frac{1}{2}\int_{T_0}^{\infty} \|Av\|_{L^2}^4 d\tau < C < +\infty. \tag{4.13}$$

Applying Lemma 4.4, the Hölder inequality, and (4.13), we check that for $T_0 < s < t$

$$\begin{aligned} \|v_t(t)\|_{L^2} &\le \frac{1}{t-s}\int_s^t \|v_t(\tau)\|_{L^2} d\tau \\ &\le \frac{1}{(t-s)^{1/4}} \left(\int_s^t \|v_t(\tau)\|_{L^2}^4 d\tau\right)^{1/4} \\ &\le C(t-s)^{-1/4}. \end{aligned}$$

Here we choose $s = t/2$ to see that

$$\|v_t(t)\|_{L^2} = O(t^{-1/4}) \text{ as } t \to \infty.$$

Next we show that

$$\begin{aligned} \|\nabla v(t)\|_{L^2(\Omega)} &= O(t^{-1/8}) && \text{as } t \to \infty, \\ \|v(t)\|_{L^\infty(\Omega)} &= O(t^{-1/16}) && \text{as } t \to \infty. \end{aligned}$$

Since v is a strong solution of (3.1), we use the integration by parts to observe that

$$
\begin{aligned}
\nu\|\nabla v(t)\|_{L^2}^2 &= \langle Av, v\rangle \\
&= \langle -v_t - Bv - P(v,\nabla)v, v\rangle \\
&= \langle -v_t, v\rangle - \langle Bv, v\rangle.
\end{aligned}
$$

By the Cauchy–Schwarz inequality and Lemma 3.5, we have

$$
\nu\|\nabla v(t)\|_{L^2}^2 \leq \|v_t(t)\|_{L^2}\|v(t)\|_{L^2} + \frac{\nu}{2}\|\nabla v(t)\|_{L^2}^2.
$$

This gives

$$
\|\nabla v(t)\|_{L^2} \leq C(\nu)\|v_0\|_{L^2}^{1/2}\|v_t(t)\|_{L^2}^{1/2} \leq Ct^{-1/8} \text{ as } t \to \infty. \tag{4.14}
$$

From the structure of the system (3.1), we check that

$$
\begin{aligned}
\|Av(t)\|_{L^2} &= \| - v_t - Bv - P(v,\nabla)v\|_{L^2} \\
&\leq \|v_t(t)\|_{L^2} + \|Bv(t)\|_{L^2} + \|P(v,\nabla)v(t)\|_{L^2}.
\end{aligned}
$$

Since $ab \leq \varepsilon a^2 + b^2/(4\varepsilon)$ $(a, b \geq 0, \varepsilon > 0)$, it follows from (3.9) and (4.8) to see that

$$
\|Bv(t)\|_{L^2} \leq \frac{1}{4}\|Av(t)\|_{L^2} + C\|A^{1/2}v(t)\|_{L^2}^{1/2}\|v(t)\|_{L^2}^{1/2} + C\|A^{1/2}v(t)\|_{L^2}
$$

and that

$$
\|P(v,\nabla)v(t)\|_{L^2} \leq \frac{1}{4}\|Av(t)\|_{L^2} + C\|A^{1/2}v(t)\|_{L^2}^{3/2}\|v(t)\|_{L^2}^{1/2} + C\|A^{1/2}v(t)\|_{L^2}^{3}.
$$

As a result, we have

$$
\begin{aligned}
\|Av(t)\|_{L^2} &\leq C(\|v_t(t)\|_{L^2} + \|A^{1/2}v(t)\|_{L^2}^{1/2} + \|A^{1/2}v(t)\|_{L^2} + \|A^{1/2}v(t)\|_{L^2}^{3}) \\
&\leq Ct^{-1/4} + Ct^{-1/16} + Ct^{-1/8} + Ct^{-3/16}.
\end{aligned}
$$

Therefore we find that

$$
\|Av(t)\|_{L^2} = O(t^{-1/16}) \text{ as } t \to \infty. \tag{4.15}
$$

By (2.3), (2.7), (4.14), and (4.15), we check that

$$
\|v(t)\|_{L^\infty} = O(t^{-1/16}) \text{ as } t \to \infty.
$$

Finally, we prove that for each $2 < q < 6$

$$
\|\nabla v(t)\|_{L^q(\Omega)} = O(t^{-(3/q-1/2)/8}) \ \text{ as } t \to \infty. \tag{4.16}
$$

Fix $2 < q < 6$. Using the Gagliardo–Nirenberg inequality and (2.2), we see that

$$
\begin{aligned}
\|\nabla v(t)\|_{L^q} &\leq C\|\nabla v(t)\|_{L^2}^{(3/q-1/2)}\|v(t)\|_{W^{2,2}}^{(3/2-3/q)} \\
&\leq C\|\nabla v(t)\|_{L^2}^{(3/q-1/2)}(\|v(t)\|_{L^2} + \|Av(t)\|_{L^2})^{(3/2-3/q)}.
\end{aligned}
$$

From (4.3), (4.14) and (4.15), we have (4.16). Therefore the theorem is proved. □

Remark that the proof of Lemma 4.4 is based on [**32**].

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