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The Chabauty and the Thurston topologies on the hyperspace of closed subsets

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Abstract. For a regularly locally compact topological space X of T_0 separation axiom but not necessarily Hausdorff, we consider a map σ from X to the hyperspace $C(X)$ of all closed subsets of X by taking the closure of each point of X. By providing the Thurston topology for $C(X)$, we see that σ is a topological embedding, and by taking the closure of $\sigma(X)$ with respect to the Chabauty topology, we have the Hausdorff compactification $\widehat{X}$ of X. In this paper, we investigate properties of $\widehat{X}$ and $C(\widehat{X})$ equipped with different topologies. In particular, we consider a condition under which a self-homeomorphism of a closed subspace of $C(X)$ with respect to the Chabauty topology is a self-homeomorphism in the Thurston topology.

1. Introduction.

The motivation of this work *was* to understand the topology of the space $\mathcal{GL}(X)$ of geodesic laminations on a hyperbolic surface X. A geodesic lamination $\alpha \in \mathcal{GL}(X)$ is a closed subset on X that is a disjoint union of simple closed or infinite geodesics. A natural topology on $\mathcal{GL}(X)$ is the one induced from the Hausdorff distance between closed subsets. We call this the *Chabauty topology* in this paper, which is also known as the *Fell topology*, mainly used for a topology on closed subgroups of a locally compact Lie group. See [**4**] for historical remarks on this topology. On the other hand, in connection with 3-dimensional hyperbolic geometry, Thurston [**8**, Section 8.10] introduced a useful topology on $\mathcal{GL}(X)$ when we realize a hyperbolic surface X in a hyperbolic 3-manifold bent along a geodesic lamination. This is called the *Thurston topology.* The precise definition will be given in Section 2.

The Thurston topology is a weaker topology than the Chabauty topology and it does not necessarily satisfy T_1 separation axiom. We are interested in the relationship between the Chabauty topology and the Thurston topology and in particular self-homeomorphisms with respect to each topology. Original questions on geodesic lamination spaces will be answered in Section 10 by applying the arguments for topological spaces in general. The purpose of this paper is now more than that, namely, to introduce another usage of the Thurston topology in the framework of general topology.

The Thurston topology on a family of closed subsets of X is not Hausdorff even if the base space X is regular enough. We embed a topological space X with T_0 separation

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axiom into the hyperspace $C(X)_{\mathrm{T}}$ of all closed subsets of X with the Thurston topology (represented by T), and apply this embedding for a certain subspace $\mathcal{Z}$ of $C(X)$ as a new space X. To perform this procedure, we cannot assume that our topological space X to be Hausdorff. However, a local compactness condition for X is necessary to obtain a preferable consequence, say, $C(X)$ is Hausdorff with the Chabauty topology. To this end, we assume that each point of X has a neighborhood system consisting of compact subsets of X. If X is Hausdorff, this is equivalent to the usual definition for local compactness, but since X is not necessarily Hausdorff, we especially call X *regularly locally compact* if this condition is satisfied. Now our fundamental result can be stated as follows, which summarizes the arguments developed in Section 2.

Theorem 1.1. *Suppose that a* T_0*-space* X *is regularly locally compact. Let* $\sigma = \sigma_X : X \to C(X)$ *be defined by taking the closure* $\sigma(x) = \overline{\{x\}}$ *for each point* $x \in X$*. Then* σ *is a topological embedding with respect to the Thurston topology on* $C(X)$ *and the closure* $\overline{\sigma(X)}^{\mathrm{CH}}$ *of the image* $\sigma(X)$ *with respect to the Chabauty topology is a compact Hausdorff space.*

This theorem implies that by changing the topology of X we have a compact Hausdorff space $\widehat{X} = \overline{\sigma(X)}^{\mathrm{CH}}$ that contains a dense subset homeomorphic to the changed X. We call $\widehat{X}$ the *Hausdorff compactification* of X. Note that, in the case where X is already Hausdorff and non-compact, $\widehat{X}$ coincides with the one-point compactification of X. See Section 4.

The Hausdorff compactification $\widehat{X}$ for a regularly locally topological compact space X was first introduced by Yoshino [**10**], [**11**] in a different manner from ours. He defined $\widehat{X}$ as the space of limit sets $\lim \mathcal{F} \in C(X)$ for all prime filters $\mathcal{F}$ on X and called this space the *topological blow-up* of X. The topology of $\widehat{X}$ is given so that each element of a closed basis of $\widehat{X}$ is defined by the family of $\lim \mathcal{F}$ for all prime filters $\mathcal{F}$ in an arbitrary subset $A \subset X$. This is an analogue to the way of defining Zariski closed sets in the hyperspace $\mathrm{Spec}(R)$ of all prime ideals of a commutative ring R. We will see in Section 3 that the topological blow-up $\widehat{X}$ due to Yoshino coincides with our Hausdorff compactification $\overline{\sigma(X)}^{\mathrm{CH}}$ respecting the topology. To see this, we utilize nets on X instead of filters. Necessary properties of complete nets in our setting are given in Section 2.

Next, by taking a certain subset of $C(X)$ as a topological space in question, we consider its topological embedding and the Hausdorff compactification. Let $\mathcal{Z} = \mathcal{Z}_{\mathrm{T}}$ be a subset of $C(X)$ equipped with the relative Thurston topology and closed with respect to the Chabauty topology. Note that such a $\mathcal{Z}$ is always a T_0-space. As before, we define a topological embedding $\widehat{\sigma}_{\mathcal{Z}} : \mathcal{Z} \to C(\mathcal{Z}_{\mathrm{T}})$ by $\alpha \mapsto \overline{\{\alpha\}}$. But, differently from before, $\widehat{\sigma}_{\mathcal{Z}}(\mathcal{Z})$ is already closed in the Chabauty topology of $C(\mathcal{Z}_{\mathrm{T}})$ in the present situation, and moreover, we have the following result. Section 5 is devoted to the arguments for this theorem.

Theorem 1.2. *Suppose that* X *is regularly locally compact. Let* $\mathcal{Z} \subset C(X)$ *be closed in the Chabauty topology and equipped with the relative Thurston topology. Then the Hausdorff compactification of* $\mathcal{Z}$ *coincides with* $\widehat{\sigma}_{\mathcal{Z}}(\mathcal{Z})$ *and* $\widehat{\sigma}_{\mathcal{Z}} : \mathcal{Z}_{\bullet} \to C(\mathcal{Z}_{\mathrm{T}})_{\bullet}$ *is a topological embedding both in the Thurston topology* $\bullet = \mathrm{T}$ *and in the Chabauty topology*

$\bullet = \mathrm{CH}$.

In Section 6, we introduce two topological embeddings of a regularly locally compact T_0-space X into hyperspaces of $\widehat{X}$ with different topologies. If we provide the Thurston topology for $\widehat{X}$, the map $\iota : \widehat{X}_{\mathrm{T}} \to C(\widehat{X}_{\mathrm{T}})_{\mathrm{T}}$ defined for $\widehat{X}_{\mathrm{T}}$ as $\sigma : X \to C(X)_{\mathrm{T}}$ for X in Theorem 1.1 is a topological embedding. Hence the composition $t = \iota \circ \sigma$ gives a topological embedding of X into $C(\widehat{X}_{\mathrm{T}})_{\mathrm{T}}$. On the other hand, the complemental topology to the Thurston topology, which we call the *dual Thurston topology* (T^*) and define in Section 2, can be also provided for $\widehat{X}$ and a topological embedding $\iota^* : \widehat{X}_{\mathrm{T}^*} \to C(\widehat{X}_{\mathrm{T}^*})_{\mathrm{T}}$ is similarly given. We also take the composition $\tau = \iota^* \circ \sigma$ and regard it as an injection of X into $C(\widehat{X})$, where we consider the Chabauty topology for $\widehat{X}$. Then τ coincides with the recovering map introduced by Yoshino [**11**] in order to obtain the information of X from the topological blow-up $\widehat{X}$. Providing the dual Thurston topology for $C(\widehat{X})$, we can explain his theorem by showing that $\tau : X \to C(\widehat{X})_{\mathrm{T}^*}$ is a topological embedding.

The remaining sections are devoted to the investigation of continuous maps $X \to Y$ between regularly locally compact topological spaces by using their Hausdorff compactifications. Any map $f : X \to Y$ defines the *closure map* $\widehat{f} : C(X) \to C(Y)$ by $\alpha \mapsto \overline{f(\alpha)}$ for $\alpha \in C(X)$. If f is a closed map, this is nothing but the restriction of the induced map $2^X \to 2^Y$ of f between their power sets, which we call the *power extension* of f. In Section 7, we consider the restriction of $\widehat{f}$ to the Hausdorff compactification $\widehat{X}$ for a proper continuous map $f : X \to Y$ and show that $\widehat{f} : \widehat{X} \to \widehat{Y}$ is (proper) continuous. This is also obtained for topological blow-ups in [**11**].

In Section 8, we further develop arguments for the power extension of any continuous map between compact Hausdorff spaces. This can be applied to a continuous map between the Hausdorff compactifications $\widehat{X}$ and $\widehat{Y}$. In particular, for the closure map $\widehat{f} : \widehat{X} \to \widehat{Y}$ as above, we obtain the following.

THEOREM 1.3. *Assume that $f : X \to Y$ is proper and continuous for regularly locally compact topological spaces X and Y. Then the closure map $\widehat{f} : \widehat{X} \to \widehat{Y}$ is proper and continuous and the power extension $F : C(\widehat{X})_\bullet \to C(\widehat{Y})_\bullet$ of $\widehat{f}$ is also proper and continuous for $\bullet = \mathrm{CH}, \mathrm{T}$.*

If $f : X \to Y$ is a homeomorphism, then $\widehat{f} : \widehat{X} \to \widehat{Y}$ is a homeomorphism. However, the converse question is more delicate, namely, under what condition a homeomorphism $\widetilde{f} : \widehat{X} \to \widehat{Y}$ is induced by a homeomorphism $f : X \to Y$ as a power extension. This problem is investigated in Section 9. A requirement for $\widetilde{f}$ to be induced by f is different according to the topology in which $\widetilde{f}$ is a homeomorphism, Chabauty or Thurston. Our result can be stated as follows.

THEOREM 1.4. *Assume that X and Y are regularly locally compact and satisfy* T_0 *separation axiom. Let $\widetilde{f} : \widehat{X} \to \widehat{Y}$ be a bijection such that $\widetilde{f}(\sigma_X(X)) = \sigma_Y(Y)$. Then the following conditions are equivalent*:

1. *$\widetilde{f} : \widehat{X} \to \widehat{Y}$ is a homeomorphism in the Chabauty topology and satisfies the condition that $\alpha \subset \beta$ if and only if $\widetilde{f}(\alpha) \subset \widetilde{f}(\beta)$ for any $\alpha, \beta \in \widehat{X}$;*
2. *$\widetilde{f} : \widehat{X}_{\mathrm{T}} \to \widehat{Y}_{\mathrm{T}}$ is a homeomorphism in the Thurston topology;*

3. *there exists a homeomorphism* $f : X \to Y$ *that induces* $\widetilde{f}$.

We apply this theorem to closed subsets $\mathcal{Z} \subset C(X)_{\mathrm{CH}}$ and $\mathcal{W} \subset C(Y)_{\mathrm{CH}}$. By Theorem 1.2 above, $\widehat{\sigma}_{\mathcal{Z}} : \mathcal{Z} \to \widehat{\sigma}_{\mathcal{Z}}(\mathcal{Z})$ and $\widehat{\sigma}_{\mathcal{W}} : \mathcal{W} \to \widehat{\sigma}_{\mathcal{W}}(\mathcal{W})$ are homeomorphisms both in the Chabauty and the Thurston topologies. Hence Theorem 1.4 turns out to be the following form in this special case.

Corollary 1.5. *Assume that* X *and* Y *are regularly locally compact. Let* $\mathcal{Z} \subset C(X)$ *and* $\mathcal{W} \subset C(Y)$ *be closed subsets with respect to the Chabauty topology. Then the following conditions are equivalent for a bijection* $f : \mathcal{Z} \to \mathcal{W}$:

1. f *is a homeomorphism with respect to the Chabauty topology such that* f *and* f^{-1} *preserve the inclusion relation*;

2. f *is a homeomorphism with respect to the Thurston topology.*

In Section 10, we give an answer to our original question on the space $\mathcal{GL}(X) \subset C(X)$ of geodesic laminations on a hyperbolic surface X. This is done by just setting $\mathcal{Z} = \mathcal{W} = \mathcal{GL}(X)$ in Corollary 1.5.

As is mentioned above, this paper has been influenced by the theory of topological blow-up by Yoshino [**10**], [**11**], though we do not try to show our theorems along his arguments. Rather than that, we intend to keep this paper independent and self-contained. Consequently, it can be read without reference of his work.

2. Chabauty topology and Thurston topology.

In this section, we show preliminary results on the Chabauty topology and the Thurston topology of the hyperspace of closed subsets of a regularly locally compact T_0-space.

Definition. We say that a topological space X is *regularly locally compact* if each point $x \in X$ has a neighborhood basis consisting of compact subsets.

Remark. There are several different definitions for local compactness. As a usual one, X is defined to be locally compact if each point $x \in X$ has a compact neighborhood. Regularly local compactness is stronger than local compactness in general but they are equivalent if X is Hausdorff. Actually, in Willard [**9**], local compactness as in the above definition is adopted for the concept of local compactness. In this paper, to distinguish these two definitions, we introduce the terminology “regularly locally compact”.

If X is compact, then X is locally compact in the usual sense but it is not necessarily regularly locally compact. For example, the trivial one-point compactification $\mathbb{Q} \cup \{\infty\}$ of the subspace of rational numbers in $\mathbb{R}$, where a neighborhood of ∞ is only the whole space, is compact but is not regularly locally compact (cf. Steen and Seebach [**7**, p.63]).

Definition. A topological space X satisfies T_0 separation axiom (X is a T_0-space or a Kolmogorov space) if distinct points in X have distinct systems of neighborhoods.

In other words, either of any two distinct points has an open neighborhood that does not contain the other.

Note that, when X is not a T_0-space, we can make it to be a T_0-space X' by taking the Kolmogorov quotient $X \to X'$, which is the quotient by an equivalence relation $x \sim y$ on X defined by the condition $\mathcal{V}(x) = \mathcal{V}(y)$. Here we denote by $\mathcal{V}(x)$ the system of neighborhoods of x in X.

For a topological space X, we denote by 2^X the power set of X, that is, the family of all subsets in X. We consider the subfamily $C(X)$ of 2^X, called the hyperspace, which consists of all closed subsets in X including the empty set $\emptyset$. For each point $x \in X$, let $\bar{x} \in C(X)$ denote the closure of the point set $\{x\}$. Remark that $\bar{x} = \{x\}$ for every $x \in X$ if and only if X satisfies T_1 separation axiom (X is a Fréchet space). We define a map $\sigma = \sigma_X : X \to C(X)$ by $x \mapsto \bar{x}$, which we call the *impression*.

Proposition 2.1. *If X is a T_0-space, then the impression $\sigma : X \to C(X)$ is injective.*

Proof. Take distinct points x and y in X. Since X is a T_0-space, one of x and y, say x, has an open neighborhood U that does not contain the other point y. Then $\bar{y}$ is included in the closed set $U^c = X - U$. On the other hand, $\bar{x} \cap U \neq \emptyset$. This shows that $\bar{x} \neq \bar{y}$. □

Remark. A non-empty closed subset $\alpha \subset X$ is called *irreducible* if α cannot be written as the union of two proper closed subsets of α. Since $\bar{x} = \sigma(x)$ is the smallest closed subset containing x, it is irreducible for every $x \in X$. We may ask a characterization of a topological space X for which the converse of this statement is also true, that is, $\sigma(X)$ is precisely the set of all irreducible closed subsets of X.

We provide two topologies for $C(X)$: the Chabauty topology and the Thurston topology (see Canary, Epstein and Green [**2**, Section I.3.1, Section I.4.1]). For any subset $A \subset X$, we define

$$O_1(A) = \{\alpha \in C(X) \mid \alpha \cap A \neq \emptyset\};$$
$$O_2(A) = \{\alpha \in C(X) \mid \alpha \cap A = \emptyset\}.$$

Definition. The *Chabauty topology* $\mathcal{O}_{\mathrm{CH}}$ on $C(X)$ is defined by giving $\{O_1(U)\}_U \cup \{O_2(K)\}_K$ as a sub-basis, where U runs over all open subsets of X and K runs over all compact subsets of X. On the other hand, the *Thurston topology* $\mathcal{O}_{\mathrm{T}}$ is given by only $\{O_1(U)\}_U$ as a sub-basis. Furthermore, the *dual Thurston topology* $\mathcal{O}_{\mathrm{T}^*}$ is given by $\{O_2(K)\}_K$ as a (sub)-basis.

For each $\bullet = \mathrm{CH}, \mathrm{T}, \mathrm{T}^*$, the symbol $C(X)_\bullet$ denotes the topological space $(C(X), \mathcal{O}_\bullet)$. For any subset $\mathcal{E} \subset C(X)$, the symbol $\mathcal{E}_\bullet$ denotes the subspace $(\mathcal{E}, \mathcal{O}_\bullet|_{\mathcal{E}})$ of $C(X)_\bullet$ with the relative topology (subspace topology) and $\overline{\mathcal{E}}^\bullet$ denotes the closure of $\mathcal{E}$ in $C(X)_\bullet$.

Since $\mathcal{O}_{\mathrm{CH}} \supset \mathcal{O}_{\mathrm{T}}, \mathcal{O}_{\mathrm{T}^*}$ from definition, the Chabauty topology is finer than the Thurston topology and the dual Thurston topology. Recall that $\emptyset \in C(X)$. It is known (see [**2**, Section I.3.1]) that $C(X)_{\mathrm{CH}}$ is compact, and hence so are $C(X)_{\mathrm{T}}$ and $C(X)_{\mathrm{T}^*}$, without any assumption on the topology of X. Moreover, for any compact subset $K \subset X$, the closed subset $O_1(K) = C(X) - O_2(K)$ of $C(X)_{\mathrm{CH}}$ is compact in $C(X)_{\mathrm{CH}}$ and hence $O_1(K)$ is also compact in $C(X)_{\mathrm{T}}$ and in $C(X)_{\mathrm{T}^*}$.

Proposition 2.2. *If X is regularly locally compact, then $C(X)_{\mathrm{CH}}$ is Hausdorff.*

Proof. Let α and β be any distinct elements in $C(X)$. Then there is a point $x \in X$, say in α, that does not belong to β. Since x is in an open set $X - \beta$, there is a compact neighborhood $K \subset X - \beta$ with $x \in U = \operatorname{Int} K$ due to regularly local compactness of X. Then by $\beta \cap K = \emptyset$, $O_2(K)$ is an open neighborhood of β in $C(X)_{\mathrm{CH}}$. On the other hand, by $\alpha \cap U \neq \emptyset$, $O_1(U)$ is an open neighborhood of α in $C(X)_{\mathrm{CH}}$. Moreover, the condition $U \subset K$ implies that $O_1(U)$ and $O_2(K)$ are disjoint. This proves that $C(X)_{\mathrm{CH}}$ is Hausdorff. □

This proposition has been shown in [**2**] under the assumption that X is locally compact and Hausdorff. If X is locally compact, Hausdorff and second countable, then $C(X)_{\mathrm{CH}}$ is metrizable and separable. Moreover, if X is compact and metrizable, then the Chabauty topology $\mathcal{O}_{\mathrm{CH}}$ is induced by the Hausdorff distance.

Remark. Concerning the Thurston topology, $C(X)_{\mathrm{T}}$ does not satisfy T_1 separation axiom in general, though it satisfies T_0 separation axiom without any assumption on X. The latter statement holds true since the open set $U = X - \beta$ gives the neighborhood $O_1(U)$ of α that does not contain β if $\alpha - \beta \neq \emptyset$.

Proposition 2.3. *If X is regularly locally compact, then so is $C(X)_{\mathrm{T}}$ and more generally, so is the subspace $\mathcal{Z}_{\mathrm{T}}$ for any closed subset $\mathcal{Z}$ of $C(X)_{\mathrm{CH}}$.*

Proof. A neighborhood basis of any $\alpha \in \mathcal{Z}_{\mathrm{T}}$ is given by

$$\left\{ \bigcap_{i=1}^{n} O_1(U_i) \cap \mathcal{Z} \right\}_{(n;U_1,\ldots,U_n)}$$

where $n \in \mathbb{N}$ is any positive integer and each U_i is taken over all open subsets $U_i \subset X$ with $\alpha \cap U_i \neq \emptyset$. In order to prove that $\mathcal{Z}_{\mathrm{T}}$ is regularly locally compact, we have only to show that, for every open subset U of X with $\alpha \cap U \neq \emptyset$, there is a neighborhood $\mathcal{E}$ of α in $C(X)_{\mathrm{T}}$ such that $\mathcal{E} \subset O_1(U)$ and $\mathcal{E}$ is closed in $C(X)_{\mathrm{CH}}$. This is because $\mathcal{Z}$ and $\mathcal{E}$ are also compact in the compact topological space $C(X)_{\mathrm{CH}}$ and hence in $C(X)_{\mathrm{T}}$.

Since X is regularly locally compact, for any $x \in \alpha \cap U$, there is a compact neighborhood K of x such that $K \subset U$. We set $\mathcal{E} = O_1(K) = C(X) - O_2(K)$ which is closed in $C(X)_{\mathrm{CH}}$ and contained in $O_1(U)$. Since the interior $\operatorname{Int} K$ of K contains $x \in \alpha$, it follows that $\alpha \in O_1(\operatorname{Int} K) \subset \mathcal{E}$ and so $\mathcal{E}$ is a neighborhood of α in $C(X)_{\mathrm{T}}$. □

A *directed set* $(N, \leq)$ is a (pseudo) ordered set satisfying that for any $\nu_1, \nu_2 \in N$ there is $\nu_3 \in N$ such that $\nu_1 \leq \nu_3$ and $\nu_2 \leq \nu_3$. A *net* (or a directed family of points) $\boldsymbol{D}$ in a set X is a family of points $\{x_\nu\}_{\nu \in N}$ indexed by a directed set N. We say that a net $\{x'_\mu\}_{\mu \in M}$ is a *subnet* of $\{x_\nu\}_{\nu \in N}$ if there is a map $i : M \to N$ satisfying that $x_{i(\mu)} = x'_\mu$ for every $\mu \in M$ and that for each $\nu \in N$ there is μ_0 such that $i(\mu) \geq \nu$ for every $\mu \geq \mu_0$.

A *complete net* (or a universal net) $\boldsymbol{C} = \{x_\nu\}_{\nu\in N}$ in X is a net satisfying that for every subset A in X either there is some $\nu_1 \in N$ such that $\{x_\nu\}_{\nu\geq\nu_1}$ is contained in A or there is some $\nu_2 \in N$ such that $\{x_\nu\}_{\nu\geq\nu_2}$ is contained in the complement $A^c = X - A$.

A net $\boldsymbol{D} = \{x_\nu\}_{\nu\in N}$ in a topological space X converges to a point $x \in X$ by definition if for every neighborhood U of x there is some $\nu_0 \in N$ such that $x_\nu \in U$ for every $\nu \geq \nu_0$. We define the *limit set* $\lim \boldsymbol{D}$ of a net $\boldsymbol{D}$ by the set of all points $x \in X$ to which $\boldsymbol{D}$ converges, which is a closed subset of X and hence an element of $C(X)$. If X is Hausdorff, then $\lim \boldsymbol{D}$ consists of at most one point. If $\boldsymbol{D}$ does not converge, then $\lim \boldsymbol{D} = \emptyset$. A topological space X is compact if and only if every complete net $\boldsymbol{C}$ in X converges, that is, $\lim \boldsymbol{C} \neq \emptyset$. Hence, if X is compact and Hausdorff, then every complete net $\boldsymbol{C}$ converges to a unique point. In this case, we denote the single limit point also by $\lim \boldsymbol{C}$. See Kelley [**6**, Chapter 2] for details.

The following proposition is clear from definition ([**6**, p. 81]).

Proposition 2.4. *Let X and Y be any sets and let $f : X \to Y$ be any map. Then, for a complete net $\boldsymbol{C} = \{x_\nu\}_{\nu\in N}$ in X, the image $f(\boldsymbol{C}) = \{f(x_\nu)\}_{\nu\in N}$ is also a complete net in Y.*

For a net $\boldsymbol{D} = \{x_\nu\}_{\nu\in N}$ in a topological space X, $\widehat{\boldsymbol{D}} = \{\bar{x}_\nu\}_{\nu\in N}$ is a net in the hyperspace $C(X)$. By proposition 2.4, if $\boldsymbol{C} = \{x_\nu\}_{\nu\in N}$ is a complete net in X, then $\widehat{\boldsymbol{C}} = \{\bar{x}_\nu\}_{\nu\in N}$ is a complete net in $C(X)$.

By Proposition 2.2, under the assumption that X is regularly locally compact, the limit set $\lim_{\mathrm{CH}} \widehat{\boldsymbol{C}}$ in $C(X)_{\mathrm{CH}}$ for any complete net $\boldsymbol{C}$ in X defines an unique element $\alpha \in C(X)$. This coincides with the limit set of $\boldsymbol{C}$ as the following theorem asserts.

Theorem 2.5. *Assume that a topological space X is regularly locally compact. Then, for any complete net $\boldsymbol{C}$ in X, the complete net $\widehat{\boldsymbol{C}}$ converges to $\lim \boldsymbol{C}$ in $C(X)_{\mathrm{CH}}$. Namely, $\lim \boldsymbol{C} = \lim_{\mathrm{CH}} \widehat{\boldsymbol{C}}$.*

Proof. First, we show the inclusion $\lim \boldsymbol{C} \subset \lim_{\mathrm{CH}} \widehat{\boldsymbol{C}}$. For $\boldsymbol{C} = \{x_\nu\}_{\nu\in N}$, take any $x \in \lim \boldsymbol{C}$; for every open neighborhood U of x, there is $\nu_0 \in N$ such that $x_\nu \in U$ for all $\nu \geq \nu_0$. We will show that $x \in \alpha$ for $\alpha = \lim_{\mathrm{CH}} \widehat{\boldsymbol{C}}$. Suppose to the contrary that $x \notin \alpha$. Since $X - \alpha$ is an open neighborhood of x and X is regularly locally compact, there is a compact neighborhood K of x with $K \cap \alpha = \emptyset$. We choose an open neighborhood U of x as $U = \operatorname{Int} K$. Then there is $\nu_0 \in N$ such that $x_\nu \in U$ for every $\nu \geq \nu_0$. On the other hand, $O_2(K)$ is an open neighborhood of $\alpha \in C(X)_{\mathrm{CH}}$. Since $\widehat{\boldsymbol{C}} = \{\bar{x}_\nu\}_{\nu\in N}$ converges to α in $C(X)_{\mathrm{CH}}$, there is $\nu_1 \in N$ such that $\bar{x}_\nu \in O_2(K)$, that is, $\bar{x}_\nu \cap K = \emptyset$ for every $\nu \geq \nu_1$. If we take $\nu \in N$ with $\nu \geq \nu_0, \nu_1$, then we have $\bar{x}_\nu \cap U \neq \emptyset$ and $\bar{x}_\nu \cap K = \emptyset$. This contradicts the inclusion $U \subset K$, which proves our claim.

Next, we show the other inclusion $\lim \boldsymbol{C} \supset \lim_{\mathrm{CH}} \widehat{\boldsymbol{C}}$. Take any $x \in \alpha = \lim_{\mathrm{CH}} \widehat{\boldsymbol{C}}$. For every open neighborhood U of x, $O_1(U)$ is an open neighborhood of $\alpha \in C(X)_{\mathrm{CH}}$. Since $\widehat{\boldsymbol{C}} = \{\bar{x}_\nu\}_{\nu\in N}$ converges to α, there is $\nu_2 \in N$ such that $\bar{x}_\nu \in O_1(U)$, that is, $\bar{x}_\nu \cap U \neq \emptyset$ for every $\nu \geq \nu_2$. Since $\bar{x}_\nu$ is the closure of $\{x_\nu\}$, we see that $x_\nu \in U$. Indeed, if $x_\nu \in X - U$, then $\bar{x}_\nu \subset X - U$. We have shown that for every open neighborhood U of x there is $\nu_2 \in N$ such that $x_\nu \in U$ for every $\nu \geq \nu_2$. Hence $x \in \lim \boldsymbol{C}$. □

Remark. In Theorem 2.5, we do not exclude the case where $\lim \boldsymbol{C} = \lim_{\mathrm{CH}} \widehat{\boldsymbol{C}}$ is satisfied by $\lim \boldsymbol{C} = \emptyset$ and $\lim_{\mathrm{CH}} \widehat{\boldsymbol{C}} = \{\emptyset\}$.

On the other hand, if X is a T_0-space, then the impression $\sigma : x \to C(X)_{\mathrm{T}}$ is injective by Proposition 2.1 and defines a topological embedding as in the following.

Theorem 2.6. *For any topological space X, the impression $\sigma : X \to C(X)_{\mathrm{T}}$ is continuous and $\sigma : X \to \sigma(X)_{\mathrm{T}}$ is open. In particular, if X satisfies T_0 separation axiom, then $\sigma : X \to \sigma(X)_{\mathrm{T}}$ is a homeomorphism, in other words, $\sigma : X \to C(X)_{\mathrm{T}}$ is a topological embedding.*

Proof. For the continuity, we show that, if a net $\boldsymbol{D} = \{x_\nu\}_{\nu \in N}$ converges to x in X, then the corresponding net $\widehat{\boldsymbol{D}} = \{\bar{x}_\nu\}_{\nu \in N}$ converges to $\bar{x}$ in $C(X)_{\mathrm{T}}$. A neighborhood sub-basis at $\bar{x} \in C(X)_{\mathrm{T}}$ is given by $\{O_1(U)\}_U$ where U runs over all open subsets U of X with $x \in U$. For any such U, there is $\nu_0 \in N$ such that $x_\nu \in U$ for every $\nu \geq \nu_0$. Then $\bar{x}_\nu \in O_1(U)$ for every $\nu \geq \nu_0$, which means that $\widehat{\boldsymbol{D}}$ converges to $\bar{x}$.

For being an open mapping, we consider $\sigma(U)$ in $\sigma(X)$ for any open subset $U \subset X$. Since $O_1(U)$ is open in $C(X)_{\mathrm{T}}$, we have only to show that

$$O_1(U) \cap \sigma(X) = \sigma(U).$$

It is clear that the former includes the latter. For the inverse inclusion, take any $\bar{x} \in \sigma(X)$ with $\bar{x} \in O_1(U)$. Then $\bar{x} \cap U \neq \emptyset$ implies $x \in U$. Thus $\bar{x}$ belongs to $\sigma(U)$. □

When σ is injective in the above theorem, we may alternatively prove the continuity of $\sigma^{-1} : \sigma(X) \to X$ as follows. Suppose that a net $\widehat{\boldsymbol{D}} = \{\bar{x}_\nu\}_{\nu \in N}$ converges to $\bar{x}$ in $\sigma(X)_{\mathrm{T}}$. Then we will show that the net $\boldsymbol{D} = \{x_\nu\}_{\nu \in N}$ converges to x in X. For any open neighborhood U of x, $O_1(U)$ is an open neighborhood of $\bar{x}$. Hence there is $\nu_1 \in N$ such that $\bar{x}_\nu \in O_1(U)$ for every $\nu \geq \nu_1$. This implies that $x_\nu \in U$ for every $\nu \geq \nu_1$, which yields that $\boldsymbol{D}$ converges to x.

Remark. By the continuity and the openness of σ and by the fact that $\sigma^{-1}(\sigma(U)) = U$ for any open subset U of X, we see that a net $\{x_\nu\}_{\nu \in N}$ converges to x in X if and only if $\{\bar{x}_\nu\}_{\nu \in N}$ converges to $\bar{x}$ in $C(X)_{\mathrm{T}}$. This statement is valid even if σ is not injective. Note that limit points of a net are not unique. In general, if a map $X \to Y$ is injective after taking the Kolmogorov quotient of X and it is continuous and open with respect to the relative topology on the image, then we see the same correspondence of convergence.

Concerning the Chabauty topology, we see that $\sigma : X \to \sigma(X)_{\mathrm{CH}}$ is open in general. Under the assumption that X is Hausdorff, we also have that σ is a topological embedding.

Proposition 2.7. *If X is Hausdorff, then the topologies of $\sigma(X)_{\mathrm{T}}$ and $\sigma(X)_{\mathrm{CH}}$ are the same. In particular, $\sigma : X \to C(X)_{\mathrm{CH}}$ is a topological embedding. On the other hand, if X is regularly locally compact and $\sigma : X \to C(X)_{\mathrm{CH}}$ is a topological embedding, then X is Hausdorff.*

Proof. Since X is Hausdorff, $\sigma(x) = \{x\}$ for every $x \in X$. Also, every compact subset $K \subset X$ is closed. Hence $O_2(K) \cap \sigma(X) = O_1(K^c) \cap \sigma(X)$, which is an open subset of $\sigma(X)_{\mathrm{T}}$. This shows the first statement. Conversely, if X is regularly locally compact, then $C(X)_{\mathrm{CH}}$ is Hausdorff by Proposition 2.2. Hence a topological embedding $\sigma : X \to C(X)_{\mathrm{CH}}$ implies that X is Hausdorff. □

Remark. If X is compact Hausdorff, then $\sigma(X)_{\mathrm{T}^*}$ is also the same as $\sigma(X)_{\mathrm{T}}$ and $\sigma(X)_{\mathrm{CH}}$, and $\sigma : X \to C(X)_{\mathrm{T}^*}$ is a topological embedding.

Finally in this section, we consider the relative version of the impression. Any non-empty subset A of X is regarded as a topological space with the relative topology. Then the impression $\sigma_A : A \to C(A)$ can be defined as before. On the other hand, the restriction of $\sigma : X \to C(X)$ to A gives $\sigma|_A : A \to C(X)$. We also have a map $r_A : C(X) \to C(A)$ sending a closed subset $\beta \in C(X)$ to the relative closed subset $\beta \cap A \in C(A)$. Theorem 2.6 implies that $\sigma_A : A \to C(A)_{\mathrm{T}}$ is continuous and $\sigma_A : A \to \sigma_A(A)_{\mathrm{T}}$ is open.

Proposition 2.8. *For any non-empty subset A of a topological space X, the impression $\sigma_A : A \to C(A)$ satisfies $\sigma_A = r_A \circ \sigma|_A$. If X satisfies T_0 separation axiom, then $\sigma_A : A \to C(A)_{\mathrm{T}}$ is a topological embedding.*

Proof. For any point $x \in A$, we have $\sigma_A(x) = \sigma(x) \cap A$. Then the first statement follows. If X is a T_0-space, then so is A. The second statement follows from Theorem 2.6. □

3. Topological blow-up.

In this section, we review the topological blow-up introduced by Yoshino [**10**], [**11**], which is defined by using the concept of filter. Then we show that the topological blow-up coincides with the Chabauty closure of the image of the impression σ by using the correspondence between filters and nets. Concerning filter, we refer to Bourbaki [**1**] and Dugundji [**5**].

Definition. A *filter* $\mathcal{F}$ in a set X is a family of subsets of X ($\mathcal{F} \subset 2^X$) that satisfies the following conditions:

- the empty set $\emptyset$ does not belong to $\mathcal{F}$;
- if $W_1, W_2 \in \mathcal{F}$, then $W_1 \cap W_2 \in \mathcal{F}$;
- if $W \subset W'$ and $W \in \mathcal{F}$, then $W' \in \mathcal{F}$.

An *ultrafilter* (or a maximal filter) $\mathcal{M}$ is a filter that has no larger filter containing it properly. For example, the principal filter $\mathcal{M}_x$ for $x \in X$, which consists of all subsets containing x, is an ultrafilter. Every filter $\mathcal{F}$ is contained in some ultrafilter $\mathcal{M}$.

Let X be a topological space. A filter $\mathcal{F} \subset 2^X$ converges to a point $x \in X$ by definition if the neighborhood system $\mathcal{V}(x) \subset 2^X$ of x is contained in $\mathcal{F}$. In this case, x is called a limit point of $\mathcal{F}$. The set of all such limit points of $\mathcal{F}$ is denoted by $\lim \mathcal{F} \subset X$, which is the *limit set* of $\mathcal{F}$. This is a closed subset of X; $\lim \mathcal{F} \in C(X)$. It is clear that if $\mathcal{F} \subset \mathcal{F}^*$ then $\lim \mathcal{F} \subset \lim \mathcal{F}^*$.

The topological space X is Hausdorff if and only if every filter $\mathcal{F}$ has at most one limit point. If X is compact, then every ultrafilter $\mathcal{M}$ converges, and vice versa. A filter $\mathcal{F}$ is called *prime* if any filter $\mathcal{F}^*$ with $\mathcal{F} \subset \mathcal{F}^*$ satisfies $\lim \mathcal{F} = \lim \mathcal{F}^*$. It is evident by definition that an ultrafilter is a prime filter.

The convergence of filter can be translated into the language of net. We can regard a filter $\mathcal{F}$ in X as a directed set so that the (partial) order $W' \leq W$ for elements in $\mathcal{F}$ is defined by the inclusion $W' \supset W$. Then a net $\boldsymbol{D}_{\mathcal{F}} = \{x_W\}_{W \in \mathcal{F}}$ with the index set $\mathcal{F}$ is given by choosing any point x_W from W. Conversely, for a net $\boldsymbol{D} = \{x_\nu\}_{\nu \in N}$ in X, we define a filter $\mathcal{F}_{\boldsymbol{D}}$ so that a subset W of X belongs to $\mathcal{F}_{\boldsymbol{D}}$ if there is some $\nu_0 \in N$ such that $\{x_\nu\}_{\nu \geq \nu_0}$ is contained in W. Then $\mathcal{F}$ converges to x if and only if any such $\boldsymbol{D}_{\mathcal{F}}$ converges to x, and $\boldsymbol{D}$ converges to x if and only if the $\mathcal{F}_{\boldsymbol{D}}$ converges to x. For every ultrafilter $\mathcal{M}$, the corresponding net $\boldsymbol{D}_{\mathcal{M}}$ is a complete net, and for every complete net $\boldsymbol{C}$, the corresponding filter $\mathcal{F}_{\boldsymbol{C}}$ is an ultrafilter. See Dugundji [**5**, p.213] and Kelley [**6**, p.83].

The topological blow-up is introduced for making a non-Hausdorff topological space to be Hausdorff.

Definition (Yoshino). For every prime filter $\mathcal{P}$ in a regularly locally compact topological space X, take its limit set $\lim \mathcal{P} \subset X$ and consider the set of all such limit sets $\widehat{X} = \{\lim \mathcal{P}\} \subset C(X)$. Provide a topology for $\widehat{X}$ by defining a closed basis $\{\widehat{A}_X\}_{A \subset X}$, where

$$\widehat{A}_X = \{\lim \mathcal{P} \in \widehat{X} \mid A \in \mathcal{P} : \text{a prime filter in } X\}$$

and A is taken over all subsets $A \subset X$. The *topological blow-up* of X is the space $\widehat{X}$ with this topology.

We can replace the prime filter $\mathcal{P}$ with an ultrafilter $\mathcal{M}$ containing $\mathcal{P}$ in the definition of $\widehat{X}$ since $\lim \mathcal{P} = \lim \mathcal{M}$. Then, by the correspondence between ultrafilters and complete nets, the topological blow-up $\widehat{X}$ is alternatively defined by the set of limit sets

$$\widehat{X} = \{\lim \boldsymbol{C} \mid \boldsymbol{C} : \text{a complete net in } X\} \subset C(X)$$

for all complete nets $\boldsymbol{C}$ in X. A closed basis of $\widehat{X}$ is given by $\{\widehat{A}_X\}_{A \subset X}$, where

$$\widehat{A}_X = \{\lim \boldsymbol{C} \in \widehat{X} \mid \boldsymbol{C} : \text{a complete net in } A\}$$

and A is taken over all subsets of X. We remark that the above $\lim \boldsymbol{C}$ is taken in X, which may be represented by $\lim_X \boldsymbol{C}$.

Remark. The principle filter $\mathcal{M}_x$ for $x \in X$ satisfies $\bar{x} = \sigma(x) = \lim \mathcal{M}_x$. Indeed, for every $x' \in \bar{x}$, any neighborhood of x' contains x. This means $\mathcal{V}(x') \subset \mathcal{M}_x$ and $x' \in \lim \mathcal{M}_x$. Conversely, every $x' \in \lim \mathcal{M}_x$ satisfies $\mathcal{V}(x') \subset \mathcal{M}_x$. Hence each $U \in \mathcal{V}(x')$ contains x. This implies that $x' \in \bar{x}$.

Yoshino [**11**] has proved that, if X is regularly locally compact, then the topological blow-up $\widehat{X}$ is a compact Hausdorff space. We will explain the topological blow-up $\widehat{X}$ in terms of the impression $\sigma : X \to C(X)$, and give another proof for this fact below.

LEMMA 3.1. *Assume that a topological space X is regularly locally compact. For any subset $A \subset X$, it holds that $\widehat{A}_X = \overline{\sigma(A)}^{\mathrm{CH}}$.*

PROOF. First, we show the inclusion $\widehat{A}_X \subset \overline{\sigma(A)}^{\mathrm{CH}}$. Take any element of $\widehat{A}_X$, which can be represented by the limit set $\lim \boldsymbol{C} = \lim_X \boldsymbol{C}$ of a complete net $\boldsymbol{C} = \{a_\nu\}_{\nu \in N}$ in A. Then, by Proposition 2.4, $\widehat{\boldsymbol{C}} = \{\bar{a}_\nu\}_{\nu \in N}$ is a complete net in $\sigma(A)$, and by Theorem 2.5, we have $\lim \boldsymbol{C} = \lim_{\mathrm{CH}} \widehat{\boldsymbol{C}} \in C(X)$. Hence $\lim \boldsymbol{C}$ belongs to $\overline{\sigma(A)}^{\mathrm{CH}}$.

Next, we show the inverse inclusion $\widehat{A}_X \supset \overline{\sigma(A)}^{\mathrm{CH}}$. Take any element $\alpha \in \overline{\sigma(A)}^{\mathrm{CH}}$. There is a net $\widehat{\boldsymbol{D}} = \{\bar{a}_\nu\}_{\nu \in N}$ in $\sigma(A)$ such that $\lim_{\mathrm{CH}} \widehat{\boldsymbol{D}} = \alpha$. From the net $\boldsymbol{D} = \{a_\nu\}_{\nu \in N}$ in A, we choose a complete sub-net $\boldsymbol{C}$, which is always possible; see Kelley [**6**, p.81]. Then the corresponding $\widehat{\boldsymbol{C}}$ is a complete sub-net of $\widehat{\boldsymbol{D}}$ by Proposition 2.4. Now Theorem 2.5 asserts that $\lim \boldsymbol{C} = \lim_{\mathrm{CH}} \widehat{\boldsymbol{C}}$, which coincides with $\lim_{\mathrm{CH}} \widehat{\boldsymbol{D}} = \alpha$. Hence $\alpha = \lim \boldsymbol{C}$ belongs to $\widehat{A}_X$. □

This lemma almost tells our description of $\widehat{X}$ by using the impression.

THEOREM 3.2. *Assume that a topological space X is regularly locally compact. Then $\widehat{X} = \overline{\sigma(X)}^{\mathrm{CH}}$ as a subset of $C(X)$. Moreover, the topology of $\widehat{X}$ as the topological blow-up of X coincides with the relative topology on $\overline{\sigma(X)}^{\mathrm{CH}} \subset C(X)_{\mathrm{CH}}$.*

PROOF. The coincidence $\widehat{X} = \overline{\sigma(X)}^{\mathrm{CH}}$ as a subset of $C(X)$ is just a consequence from Lemma 3.1 by taking $A = X$. By the definition of the topology of $\widehat{X}$, the closed basis is given by $\{\widehat{A}_X\}_{A \subset X}$. Lemma 3.1 says that this family coincides with $\{\overline{\sigma(A)}^{\mathrm{CH}}\}_{A \subset X}$, which can be taken as a closed basis of the subspace $\overline{\sigma(X)}^{\mathrm{CH}} \subset C(X)_{\mathrm{CH}}$. This is because $\sigma(X)$ is dense in the compact Hausdorff space, which is regular. Hence the topologies of $\widehat{X}$ and $\overline{\sigma(X)}^{\mathrm{CH}}$ are the same. □

COROLLARY 3.3 (Yoshino). *For a regularly locally compact topological space X, the topological blow-up $\widehat{X}$ is a compact Hausdorff space.*

PROOF. Since $C(X)_{\mathrm{CH}}$ is a compact Hausdorff space by Proposition 2.2, so is $\widehat{X}$. □

REMARK. We may ask a question about a condition under which $\sigma(X)$ is open in $\widehat{X}$. If this is the case, then $\widehat{X}$ is divided into the interior $\sigma(X)$ and the boundary $\partial\widehat{X} = \widehat{X} - \sigma(X)$. Another question goes somewhat to the opposite direction; when the closed basis $\{\widehat{A}_X\}_{A \subset X} = \{\overline{\sigma(A)}^{\mathrm{CH}}\}_{A \subset X}$ is precisely the family of all closed subsets of $\widehat{X}$.

4. Hausdorff compactification.

Theorem 2.6 says that a T_0-space X is homeomorphic to $\sigma(X)_{\mathrm{T}}$ in $C(X)_{\mathrm{T}}$ and Theorem 3.2 says that the topological blow-up $\widehat{X}$ of a regularly locally compact X coincides with the subspace $\overline{\sigma(X)}^{\mathrm{CH}}$ of $C(X)_{\mathrm{CH}}$.

Definition. For a regularly locally compact topological space X with T_0 separation axiom, we call hereafter $\widehat{X}$ the *Hausdorff compactification* of X.

Here we consider the situation when we regard $\widehat{X}$ as a subset in $C(X)_{\mathrm{T}}$. As defined in Section 2, for $\bullet = \mathrm{CH}, \mathrm{T}$, the notation $\widehat{X}_\bullet$ denotes the space $\widehat{X}$ as the subspace of $C(X)_\bullet$. The Hausdorff compactification assumes $\widehat{X} = \widehat{X}_{\mathrm{CH}}$. Since the Thurston topology is weaker than the Chabauty topology, we see that $\widehat{X}_{\mathrm{T}}$ is also compact in $C(X)_{\mathrm{T}}$ and $\sigma(X)_{\mathrm{T}}$ is dense in $\widehat{X}_{\mathrm{T}}$. These facts are summarized as follows.

Theorem 4.1. *Assume that a topological space X is regularly locally compact and satisfies T_0 separation axiom. Then $\widehat{X}_{\mathrm{T}}$ is compact and contains $X \cong \sigma(X)_{\mathrm{T}}$ as a dense subset. Hence $\widehat{X}_{\mathrm{T}}$ can be regarded as a compactification of X.*

Remark. We cannot conclude that $\widehat{X}$ is closed in $C(X)_{\mathrm{T}}$ against the fact that $\widehat{X}_{\mathrm{T}}$ is compact. This is due to the fact that $C(X)_{\mathrm{T}}$ is not necessarily Hausdorff. In general $\widehat{X}$ is a subset of the Thurston closure $\overline{\sigma(X)}^{\mathrm{T}}$. We may ask about what condition on X implies $\widehat{X} = \overline{\sigma(X)}^{\mathrm{T}}$. Another remark goes to the fact that $\sigma(X)$ is not necessarily open in $\widehat{X}_{\mathrm{T}}$. Indeed, if $\widehat{X} - \sigma(X)$ contains an element $\alpha \neq \emptyset$, then $\sigma(x) \in \sigma(X)$ is in the closure $\overline{\{\alpha\}}^{\mathrm{T}}$ of the point set $\{\alpha\}$ for any $x \in \alpha$.

Now suppose that X is Hausdorff. Then $\sigma(x) \in C(X)$ is the single point set $\{x\}$ for every $x \in X$, and the topology of $\sigma(X)$ is the same whichever we use the Thurston topology or the Chabauty topology (Proposition 2.7). For a locally compact Hausdorff space X, the Hausdorff compactification $\widehat{X} = \widehat{X}_{\mathrm{CH}}$ is a compact Hausdorff space containing $\sigma(X)_{\mathrm{CH}} \cong X$ as a dense subset. When X is not compact, $\widehat{X}_{\mathrm{CH}}$ turns out to be $\sigma(X) \cup \{\emptyset\}$, which is the one-point compactification of X and $\emptyset$ plays the role of the point at infinity. On the other hand, $\widehat{X}_{\mathrm{T}}$ is the trivial (one-point) compactification of $\sigma(X)_{\mathrm{T}} \cong X$. This means that $\widehat{X}_{\mathrm{T}} = \sigma(X) \cup \{\emptyset\}$ and the neighborhood of $\emptyset$ is only the whole set.

Proposition 4.2. *For a non-compact, locally compact Hausdorff space X, the Hausdorff compactification $\widehat{X} = \widehat{X}_{\mathrm{CH}}$ gives the one-point compactification of X, while $\widehat{X}_{\mathrm{T}}$ gives the trivial compactification of X.*

5. Topological embedding of a subset of $C(X)$.

In the previous sections, using the impression $\sigma : X \to C(X)$, we have characterized the topological blow-up $\widehat{X}$ of a regularly locally compact X as the Chabauty closure of the image $\sigma(X)$ and call it the Hausdorff compactification of X. This formulation can be applied to any subspace of $C(X)_{\mathrm{T}}$. In this section, we will discuss such an application.

First we consider the whole space $C(X)_{\mathrm{T}}$, which automatically satisfies T_0 separation axiom. As is shown in Proposition 2.3, this is also regularly locally compact if so is X. Let

$$\widehat{\sigma} = \sigma_{C(X)_{\mathrm{T}}} : C(X) \to C(C(X)_{\mathrm{T}})$$

be the impression given by the correspondence of $\alpha \in C(X)$ to the closure $\overline{\{\alpha\}}^{\mathrm{T}} \in C(C(X)_{\mathrm{T}})$ of the point set $\{\alpha\}$.

For any subset $\mathcal{E} \subset C(X)$, we obtain

$$\widehat{\mathcal{E}}_{C(X)_{\mathrm{T}}} = \{\lim\nolimits_{\mathrm{T}} \widehat{\boldsymbol{C}} \mid \widehat{\boldsymbol{C}}:\ \text{a complete net in } \mathcal{E}\} \subset C(C(X)_{\mathrm{T}}),$$

where $\lim_{\mathrm{T}} \widehat{\boldsymbol{C}} = \lim_{C(X)_{\mathrm{T}}} \widehat{\boldsymbol{C}}$ is the limit set of $\widehat{\boldsymbol{C}}$ taken in $C(X)_{\mathrm{T}}$. Lemma 3.1 asserts that this coincides with the closure $\overline{\widehat{\sigma}(\mathcal{E})}^{\mathrm{CH}}$ in $C(C(X)_{\mathrm{T}})_{\mathrm{CH}}$. In the case where $\mathcal{E}$ is the point set $\{\alpha\}$ for $\alpha \in C(X)$, we have $\widehat{\{\alpha\}}_{C(X)_{\mathrm{T}}} = \{\widehat{\sigma}(\alpha)\}$. Indeed, a complete net $\widehat{\boldsymbol{C}} = \{\alpha_\nu\}_{\nu \in N}$ in $\mathcal{E} = \{\alpha\}$ holds $\alpha_\nu = \alpha$ for all $\nu \in N$, and hence $\lim_{\mathrm{T}} \widehat{\boldsymbol{C}} = \overline{\{\alpha\}}^{\mathrm{T}} = \widehat{\sigma}(\alpha)$. Moreover:

Proposition 5.1. $\widehat{\sigma}(\alpha) = \{\alpha_* \in C(X) \mid \alpha_* \subset \alpha\} = C(\alpha)$ *for every* $\alpha \in C(X)$.

Proof. It is easy to see that $C(\alpha)$ is included in $\overline{\{\alpha\}}^{\mathrm{T}} = \widehat{\sigma}(\alpha)$. For the other inclusion, take $\beta \in C(X)$ with $\beta \not\subset \alpha$. Although $O_1(X - \alpha)$ is a neighborhood of β, α is not included in it, which means that β is not in the closure $\overline{\{\alpha\}}^{\mathrm{T}} = \widehat{\sigma}(\alpha)$. □

Remark. The injectivity of $\widehat{\sigma} : C(X) \to C(C(X)_{\mathrm{T}})$ follows also from Proposition 5.1. Indeed, if $\alpha \neq \beta$, then either $\alpha \notin \widehat{\sigma}(\beta)$ or $\beta \notin \widehat{\sigma}(\alpha)$ is satisfied.

The following lemma states the fundamental relation between the Chabauty topology and the Thurston topology.

Lemma 5.2. *Suppose that X is regularly locally compact. Let $\widehat{\boldsymbol{C}}$ be a complete net in $C(X)$ and $\alpha \in C(X)$ the unique limit point to which $\widehat{\boldsymbol{C}}$ converges in $C(X)_{\mathrm{CH}}$. Then $\lim_{\mathrm{T}} \widehat{\boldsymbol{C}} = \widehat{\sigma}(\alpha)$, which is a closed subset of $C(X)_{\mathrm{T}}$.*

Proof. Since the Chabauty topology is finer than the Thurston topology, $\widehat{\boldsymbol{C}}$ converges to α also in $C(X)_{\mathrm{T}}$. Since $\mathcal{V}_{\mathrm{T}}(\alpha_*) \subset \mathcal{V}_{\mathrm{T}}(\alpha)$ for any $\alpha_* \in C(X)$ with $\alpha_* \subset \alpha$, we see that $\widehat{\sigma}(\alpha) \subset \lim_{\mathrm{T}} \widehat{\boldsymbol{C}}$ by Proposition 5.1.

Suppose that $\beta \in C(X)$ with $\beta - \alpha \neq \emptyset$ is contained in $\lim_{\mathrm{T}} \widehat{\boldsymbol{C}}$. We choose some $x \in \beta - \alpha \subset X$. Since X is regularly locally compact, there is a compact neighborhood $K \subset X$ of x that is disjoint from α. Then $O_2(K)$ is a neighborhood of α in $C(X)_{\mathrm{CH}}$ and $O_1(\operatorname{Int} K)$ is a neighborhood of β in $C(X)_{\mathrm{T}}$.

The complete net $\widehat{\boldsymbol{C}} = \{\alpha_\nu\}_{\nu \in N}$ converges to α in $C(X)_{\mathrm{CH}}$ and also to β in $C(X)_{\mathrm{T}}$. The first convergence implies that there is some $\nu_1 \in N$ such that $\alpha_\nu \cap K = \emptyset$ for every $\nu \geq \nu_1$. The second convergence implies that there is some $\nu_2 \in N$ such that $\alpha_\nu \cap \operatorname{Int} K \neq \emptyset$ for every $\nu \geq \nu_2$. However, taking $\nu \geq \nu_1, \nu_2$ yields a contradiction.

Therefore, no such $\beta \in C(X)$ belongs to $\lim_{\mathrm{T}} \widehat{\boldsymbol{C}}$, which shows that $\lim_{\mathrm{T}} \widehat{\boldsymbol{C}}$ is precisely $\widehat{\sigma}(\alpha)$. □

The converse statement can be also seen from Lemma 5.2 itself. Namely, if $\lim_{\mathrm{T}} \widehat{\boldsymbol{C}} = \widehat{\sigma}(\alpha)$ for a complete net $\widehat{\boldsymbol{C}}$, then its Chabauty limit $\lim_{\mathrm{CH}} \widehat{\boldsymbol{C}}$ must be α. Moreover, Lemma 5.2 leads to the following claim used later, which gives the correspondence between the Chabauty limit point and the Thurston limit set for a complete net in a generalized situation.

Proposition 5.3. *For every complete net $\widehat{\boldsymbol{C}}$ in a subset $\mathcal{E} \subset C(X)$, there exists a unique element α in the closure $\overline{\mathcal{E}}^{\mathrm{CH}} \subset C(X)_{\mathrm{CH}}$ such that $\lim_{\mathrm{T}} \widehat{\boldsymbol{C}} = \widehat{\sigma}(\alpha)$. Conversely, for every $\alpha \in \overline{\mathcal{E}}^{\mathrm{CH}}$, there is a complete net $\widehat{\boldsymbol{C}}$ in $\mathcal{E}$ such that $\widehat{\sigma}(\alpha) = \lim_{\mathrm{T}} \widehat{\boldsymbol{C}}$.*

Proof. Since the subspace $\overline{\mathcal{E}}^{\mathrm{CH}} \subset C(X)_{\mathrm{CH}}$ is compact and Hausdorff, a complete net $\widehat{\boldsymbol{C}}$ in $\mathcal{E}$ converges to a unique limit point $\alpha \in \overline{\mathcal{E}}^{\mathrm{CH}}$. Then, by Lemma 5.2, we see that $\lim_{\mathrm{T}} \widehat{\boldsymbol{C}} = \widehat{\sigma}(\alpha)$. Conversely, $\widehat{\sigma}(\alpha)$ for each $\alpha \in \overline{\mathcal{E}}^{\mathrm{CH}}$ is given by $\lim_{\mathrm{T}} \widehat{\boldsymbol{C}}$ for some complete net $\widehat{\boldsymbol{C}}$ in $\mathcal{E}$. Indeed, we can take a net in $\mathcal{E}$ that converges to α in $C(X)_{\mathrm{CH}}$. Then it has a complete sub-net $\widehat{\boldsymbol{C}}$. □

The above arguments conclude the following result.

Theorem 5.4. *Suppose that X is regularly locally compact. Any subset $\mathcal{E} \subset C(X)$ satisfies $\widehat{\mathcal{E}}_{C(X)_{\mathrm{T}}} = \overline{\widehat{\sigma}(\mathcal{E})}^{\mathrm{CH}} = \widehat{\sigma}(\overline{\mathcal{E}}^{\mathrm{CH}})$.*

Proof. Since $C(X)_{\mathrm{T}}$ is regularly locally compact by Proposition 2.3, we can apply Lemma 3.1 to the subset $\mathcal{E} \subset C(X)_{\mathrm{T}}$. Then we have $\widehat{\mathcal{E}}_{C(X)_{\mathrm{T}}} = \overline{\widehat{\sigma}(\mathcal{E})}^{\mathrm{CH}}$. On the other hand, Proposition 5.3 proves that $\widehat{\mathcal{E}}_{C(X)_{\mathrm{T}}} = \widehat{\sigma}(\overline{\mathcal{E}}^{\mathrm{CH}})$. □

This theorem enables us to simplify the representation of the Hausdorff compactification $\widehat{C(X)_{\mathrm{T}}}$ of $C(X)_{\mathrm{T}}$.

Corollary 5.5. *The image $\widehat{\sigma}(C(X))$ is closed in $C(C(X)_{\mathrm{T}})_{\mathrm{CH}}$ and coincides with $\widehat{C(X)_{\mathrm{T}}}$.*

Proof. Apply Theorem 5.4 for $\mathcal{E} = C(X)$. Then $\widehat{C(X)_{\mathrm{T}}} = \overline{\widehat{\sigma}(C(X))}^{\mathrm{CH}} = \widehat{\sigma}(C(X))$. □

Besides the fact that $\widehat{\sigma}(C(X))$ is already closed in $C(C(X)_{\mathrm{T}})_{\mathrm{CH}}$, $\widehat{\sigma}$ is also a homeomorphism onto its image in the Chabauty topology. This is the feature of the Hausdorff compactification of the hyperspace $C(X)_{\mathrm{T}}$.

Theorem 5.6. *Let X be a regularly locally compact topological space. The impression*

$$\widehat{\sigma} : C(X)_{\bullet} \to C(C(X)_{\mathrm{T}})_{\bullet}$$

is a topological embedding both for $\bullet = \mathrm{T}$ *and for* $\bullet = \mathrm{CH}$.

PROOF. Theorem 2.6 implies this in the Thurston topology case. Concerning the Chabauty topology, Theorem 5.4 means that the injection $\widehat{\sigma}$ commutes with the closure operators in both sides. This implies that $\widehat{\sigma}$ is a closed embedding, that is, a topological embedding whose image is closed. □

COROLLARY 5.7. *For a regularly locally compact* T_0*-space* X *and its impression* $\sigma : X \to C(X)$*, the Hausdorff compactification* $\widehat{X} \subset C(X)$ *is homeomorphic to* $\widehat{\sigma(X)}_{C(X)_{\mathrm{T}}} \subset C(C(X)_{\mathrm{T}})_{\mathrm{CH}}$ *under* $\widehat{\sigma}$.

PROOF. We apply the above theorems to $\mathcal{E} = \sigma(X) \subset C(X)$; Theorem 5.6 implies that $\overline{\sigma(X)}^{\mathrm{CH}} = \widehat{X}$ is homeomorphic to the closed set $\widehat{\sigma}(\widehat{X})_{\mathrm{CH}}$, which coincides with $\widehat{\sigma(X)}_{C(X)_{\mathrm{T}}}$ by Theorem 5.4. □

REMARK. If we used the Thurston topology in Theorem 5.4, the inclusion $\overline{\widehat{\sigma}(\mathcal{E})}^{\mathrm{T}} \supset \widehat{\sigma}(\overline{\mathcal{E}}^{\mathrm{T}})$ would be satisfied. Indeed, since $\mathcal{E}$ is dense in $\overline{\mathcal{E}}^{\mathrm{T}}$ and $\widehat{\sigma} : C(X)_{\mathrm{T}} \to C(C(X)_{\mathrm{T}})_{\mathrm{T}}$ is continuous, $\widehat{\sigma}(\mathcal{E})$ is dense in $\widehat{\sigma}(\overline{\mathcal{E}}^{\mathrm{T}})$. Then the closure $\overline{\widehat{\sigma}(\mathcal{E})}^{\mathrm{T}}$ includes $\widehat{\sigma}(\overline{\mathcal{E}}^{\mathrm{T}})$. However, we do not know whether the equality holds or not. If $\widehat{\sigma}(\overline{\mathcal{E}}^{\mathrm{T}})$ were closed in $C(C(X)_{\mathrm{T}})_{\mathrm{T}}$, the equality would hold. We can only prove that $\widehat{\sigma}(\overline{\mathcal{E}}^{\mathrm{T}})$ is compact. Actually $\overline{\mathcal{E}}^{\mathrm{T}}$ is compact and $\widehat{\sigma}$ is continuous with respect to the Thurston topology. If $\widehat{\sigma}(C(X)) \subset C(C(X)_{\mathrm{T}})_{\mathrm{T}}$ is closed, then $\widehat{\sigma}(\overline{\mathcal{E}}^{\mathrm{T}})$ is also closed. A characterization of a regularly locally compact topological space X satisfying this property can be raised as a problem.

Next, we take a closed subset $\mathcal{Z}$ of $C(X)_{\mathrm{CH}}$ and provide it with the relative topology of $C(X)_{\mathrm{T}}$. Proposition 2.3 ensures that $\mathcal{Z} = \mathcal{Z}_{\mathrm{T}}$ is regularly locally compact when so is X. The topological blow-up $\widehat{\mathcal{Z}} \subset C(\mathcal{Z}_{\mathrm{T}})$ is defined as before:

$$\widehat{\mathcal{Z}} = \{\lim_{\mathcal{Z}_{\mathrm{T}}} \widehat{\boldsymbol{C}}' \in C(\mathcal{Z}_{\mathrm{T}}) \mid \widehat{\boldsymbol{C}}' : \text{ a complete net in } \mathcal{Z}\} \subset C(\mathcal{Z}_{\mathrm{T}}).$$

Note that $\lim_{\mathcal{Z}_{\mathrm{T}}} \widehat{\boldsymbol{C}}'$ is the limit set of $\widehat{\boldsymbol{C}}'$ taken in $\mathcal{Z}_{\mathrm{T}}$, not in $C(X)_{\mathrm{T}}$. But we can also regard $\widehat{\boldsymbol{C}}'$ as a complete net in $C(X)$. More precisely, by the inclusion map $\iota : \mathcal{Z} \to C(X)$, we define $\widehat{\boldsymbol{C}} = \iota(\widehat{\boldsymbol{C}}')$. Then $\lim_{\mathrm{T}} \widehat{\boldsymbol{C}} = \lim_{C(X)_{\mathrm{T}}} \widehat{\boldsymbol{C}} \in C(C(X)_{\mathrm{T}})$. We can verify that $\lim_{\mathcal{Z}_{\mathrm{T}}} \widehat{\boldsymbol{C}}' = \lim_{\mathrm{T}} \widehat{\boldsymbol{C}} \cap \mathcal{Z}$. The topology on $\widehat{\mathcal{Z}}$ is defined by the closed basis

$$\{\lim_{\mathrm{T}} \widehat{\boldsymbol{C}} \cap \mathcal{Z} \mid \widehat{\boldsymbol{C}} : \text{ a complete net in } \mathcal{E}\} \subset \widehat{\mathcal{Z}},$$

where $\mathcal{E}$ is taken over all subsets of $\mathcal{Z}$.

Let $\widehat{\sigma}_{\mathcal{Z}} = \sigma_{\mathcal{Z}_{\mathrm{T}}} : \mathcal{Z} \to C(\mathcal{Z}_{\mathrm{T}})$ denote the impression in this setting. By Proposition 2.8, we have $\widehat{\sigma}_{\mathcal{Z}} = r_{\mathcal{Z}} \circ \widehat{\sigma}|_{\mathcal{Z}}$, that is, $\widehat{\sigma}_{\mathcal{Z}}(\alpha) = \widehat{\sigma}(\alpha) \cap \mathcal{Z}$ for every $\alpha \in \mathcal{Z}$. By Theorem 3.2, the topological blow-up $\widehat{\mathcal{Z}}$ coincides with the Hausdorff compactification $\overline{\widehat{\sigma}_{\mathcal{Z}}(\mathcal{Z})}^{\mathrm{CH}}$ where the closure is taken in $C(\mathcal{Z}_{\mathrm{T}})_{\mathrm{CH}}$. However, we do not have to take the closure here. Namely, the relative versions of Theorems 5.4 and 5.6 are also satisfied, which can be stated as follows.

Theorem 5.8. *Let X be a regularly locally compact topological space. For a closed subset $\mathcal{Z}$ of $C(X)_{\mathrm{CH}}$, let $\mathcal{Z} = \mathcal{Z}_{\mathrm{T}}$ be equipped with the relative topology of $C(X)_{\mathrm{T}}$. Then the closed subset $\widehat{\mathcal{E}}_{\mathcal{Z}_T}$ in the Hausdorff compactification $\widehat{\mathcal{Z}}$ defined by an arbitrary subset $\mathcal{E} \subset \mathcal{Z}$ is represented by*

$$\widehat{\mathcal{E}}_{\mathcal{Z}_T} = \overline{\widehat{\sigma}_{\mathcal{Z}}(\mathcal{E})}^{\mathrm{CH}} = \widehat{\sigma}_{\mathcal{Z}}(\overline{\mathcal{E}}^{\mathrm{CH}}) \subset C(\mathcal{Z}_{\mathrm{T}}).$$

In particular, setting $\mathcal{E} = \mathcal{Z}$ yields

$$\widehat{\mathcal{Z}} = \overline{\widehat{\sigma}_{\mathcal{Z}}(\mathcal{Z})}^{\mathrm{CH}} = \widehat{\sigma}_{\mathcal{Z}}(\mathcal{Z}) \subset C(\mathcal{Z}_{\mathrm{T}}).$$

Hence the impression

$$\widehat{\sigma}_{\mathcal{Z}} : \mathcal{Z}_{\bullet} \to C(\mathcal{Z}_{\mathrm{T}})_{\bullet}$$

is a topological embedding both for $\bullet = \mathrm{T}$ and for $\bullet = \mathrm{CH}$.

Remark. It is easy to see that $\widehat{\mathcal{Z}}$ is included in $\widehat{C(X)_{\mathrm{T}}} \cap \mathcal{Z} = r_{\mathcal{Z}}(\widehat{C(X)_{\mathrm{T}}})$, but the converse inclusion is not necessarily true.

6. The recovering map and the duality of topology.

In this section, we give two topological embeddings of a regularly locally compact T_0-space X into $C(\widehat{X})$, where we equip different topologies on the Hausdorff compactification $\widehat{X}$ and the hyperspace $C(\widehat{X})$. The dual Thurston topology plays an important role here.

By Theorem 2.6, if X satisfies T_0 separation axiom, then the impression $\sigma : X \to C(X)_{\mathrm{T}}$ is a topological embedding. Moreover, if X is a Hausdorff space, then σ is a topological embedding for $C(X)_{\mathrm{T}}$ and $C(X)_{\mathrm{CH}}$ by Proposition 2.7.

First we consider the impression of $\widehat{X}$ endowed with the Thurston topology. This is denoted by $\widehat{X}_{\mathrm{T}} \subset C(X)_{\mathrm{T}}$ as before. The impression of $\widehat{X}_{\mathrm{T}}$ is especially written as

$$\iota = \sigma_{\widehat{X}_{\mathrm{T}}} : \widehat{X}_{\mathrm{T}} \to C(\widehat{X}_{\mathrm{T}}).$$

This is given in the correspondence

$$\alpha \mapsto \widehat{\sigma}(\alpha) \cap \widehat{X} = \{\alpha_* \in \widehat{X} \mid \alpha_* \subset \alpha\}$$

for $\widehat{\sigma} = \sigma_{C(X)_{\mathrm{T}}}$ by Propositions 2.8 and 5.1.

Since $\widehat{X}_{\mathrm{T}}$ clearly satisfies T_0 separation axiom, Theorem 2.6 shows that $\iota : \widehat{X}_{\mathrm{T}} \to C(\widehat{X}_{\mathrm{T}})_{\mathrm{T}}$ is a topological embedding. For the composition

$$t = t_X = \iota \circ \sigma : X \to C(\widehat{X}_{\mathrm{T}}),$$

we have the following.

Theorem 6.1. *Assume that X is regularly locally compact. Then $t : X \to C(\widehat{X}_{\mathrm{T}})_{\mathrm{T}}$ is continuous and open on the image $t(X)_{\mathrm{T}}$. If X satisfies T_0 separation axiom in* *addition, then t is a topological embedding.*

Proof. Since σ is continuous and open on the image by Theorem 2.6 and ι is a topological embedding, the composition t is also continuous and open on the image. If X is a T_0-space, then σ is a topological embedding by Theorem 2.6 again and so is the composition t. □

Next we consider the impression of $\widehat{X}$ endowed with the dual Thurston topology. This is denoted by $\widehat{X}_{\mathrm{T}^*} \subset C(X)_{\mathrm{T}^*}$. The impression of $\widehat{X}_{\mathrm{T}^*}$ is especially written as

$$\iota^* = \sigma_{\widehat{X}_{\mathrm{T}^*}} : \widehat{X}_{\mathrm{T}^*} \to C(\widehat{X}_{\mathrm{T}^*}).$$

This is given in the correspondence $\alpha \mapsto \overline{\{\alpha\}}^{\mathrm{T}^*} \cap \widehat{X}$, where $\overline{\{\alpha\}}^{\mathrm{T}^*}$ is the closure of the single point set $\{\alpha\}$ in $C(X)_{\mathrm{T}^*}$. By the following fact similar to Proposition 5.1, this can be explicitly represented, which reveals certain duality between ι and ι^*.

Proposition 6.2. $\overline{\{\alpha\}}^{\mathrm{T}^*} = \{\alpha^* \in C(X) \mid \alpha^* \supset \alpha\}$ *for every* $\alpha \in C(X)$.

Proof. Let $\alpha^* \in C(X)$ with $\alpha^* \supset \alpha$. Since any neighborhood of α^* in $C(X)_{\mathrm{T}^*}$ contains α by the definition of the dual Thurston topology, α^* belongs to $\overline{\{\alpha\}}^{\mathrm{T}^*}$. For the other inclusion, take $\beta \in C(X)$ with $\beta \not\supset \alpha$. Then, for any $x \in \alpha - \beta$, we have $\beta \in O_2(\{x\})$ but $\alpha \notin O_2(\{x\})$, which implies that $\beta \notin \overline{\{\alpha\}}^{\mathrm{T}^*}$. □

Since $\widehat{X}_{\mathrm{T}^*}$ satisfies T_0 separation axiom, ι^* is injective. In fact, Theorem 2.6 shows that $\iota^* : \widehat{X}_{\mathrm{T}^*} \to C(\widehat{X}_{\mathrm{T}^*})_{\mathrm{T}}$ is a topological embedding though this fact is not used later.

To investigate a regularly locally compact topological space X through its topological blow-up $\widehat{X}$, Yoshino [**11**] has introduced the recovering map $\tau = \tau_X : X \to 2^{\widehat{X}}$ defined by

$$\tau(x) = \{\alpha \in \widehat{X} \mid x \in \alpha\}$$

for each $x \in X$. We note some equivalent conditions in our notation to this:

$$x \in \alpha \Longleftrightarrow \bar{x} = \sigma(x) \subset \alpha \Longleftrightarrow \sigma(x) \in \widehat{\sigma}(\alpha) = C(\alpha).$$

Moreover, $\tau(x)$ belongs to $C(\widehat{X}_{\mathrm{T}^*})$, and hence $\tau : X \to C(\widehat{X}_{\mathrm{T}^*}) \subset C(\widehat{X})$. Indeed, since the point set $\{x\}$ is compact, $\tau(x) = O_2(\{x\})^c \cap \widehat{X}$ is a closed subset of $\widehat{X}_{\mathrm{T}^*}$.

We can relate the recovering map τ to the impression ι^* in the following way. This shows that τ is an analogue of t obtained by replacing the Thurston topology with the dual Thurston topology.

Proposition 6.3. *Assume that X is regularly locally compact. Then the composition $\iota^* \circ \sigma$ of $\sigma : X \to \sigma(X) \subset \widehat{X}$ and $\iota^* : \widehat{X} \to C(\widehat{X}_{\mathrm{T}^*})$ coincides with $\tau : X \to C(\widehat{X})$.*

Proof. By the definition of τ and Proposition 6.2, we have

$$\tau(x) = \{\alpha \in \widehat{X} \mid \sigma(x) \subset \alpha\} = \overline{\{\sigma(x)\}}^{\mathrm{T}^*} \cap \widehat{X}.$$

This shows that $\tau = \iota^* \circ \sigma$. □

Now we provide the dual Thurston topology for $C(\widehat{X})$ and consider $\tau : X \to C(\widehat{X})_{\mathrm{T}^*}$. We can obtain the following result, which explains the corresponding theorem by Yoshino [**11**] in a different language.

THEOREM 6.4. *If X is regularly locally compact, then $\tau : X \to C(\widehat{X})_{\mathrm{T}^*}$ is continuous and $\tau : X \to \tau(X)_{\mathrm{T}^*}$ is open.*

PROOF. For any subset $A \subset X$, we take $\widehat{A}_X = \overline{\sigma(A)}^{\mathrm{CH}}$, which is a closed and compact subset of $\widehat{X}$. Then the family $O_2(\widehat{A}_X) \subset C(\widehat{X})$ taken over all $A \subset X$ gives an open basis of $C(\widehat{X})_{\mathrm{T}^*}$. By definition, $O_2(\widehat{A}_X)$ is an open neighborhood of $\tau(x)$ precisely when $\tau(x) \cap \widehat{A}_X = \emptyset$. This condition is equivalent to the condition $\sigma(x) \notin \overline{\sigma(A)}^{\mathrm{T}}$. Indeed, $\tau(x) \cap \widehat{A}_X \neq \emptyset$ if and only if there is $\alpha \in \widehat{A}_X$ with $x \in \alpha$, and the last condition is equivalent to $\sigma(x) \in \overline{\sigma(A)}^{\mathrm{T}}$ by Lemma 6.5 below.

Consider the condition $\tau(x) \in O_2(\widehat{A}_X)$, which is equivalent to $\sigma(x) \notin \overline{\sigma(A)}^{\mathrm{T}}$ by the above claim. Since $\sigma : X \to C(X)_{\mathrm{T}}$ is continuous by Theorem 2.6, $\sigma^{-1}(C(X) - \overline{\sigma(A)}^{\mathrm{T}})$ is an open subset of X. The above condition is equivalent to the condition that x belongs to $\sigma^{-1}(C(X) - \overline{\sigma(A)}^{\mathrm{T}})$. Hence we see that the last open subset is the inverse image of $O_2(\widehat{A}_X)$ under τ. This shows that τ is continuous.

Consider any open subset $U \subset X$. By Theorem 2.6, $\sigma : X \to \sigma(X)_{\mathrm{T}}$ is open. Thus $\sigma(U)$ is open in $\sigma(X)_{\mathrm{T}}$. Set $A = X - U$. Note that $\sigma^{-1}(\sigma(U)) = U$. Then $\sigma(A) = \sigma(X) - \sigma(U)$ is closed in $\sigma(X)_{\mathrm{T}}$. Take the compact subset $\widehat{A}_X = \overline{\sigma(A)}^{\mathrm{CH}}$ in $\widehat{X}$ and consider the open subset $O_2(\widehat{A}_X) \subset C(\widehat{X})_{\mathrm{T}^*}$. As is shown above, we have

$$\tau^{-1}(O_2(\widehat{A}_X)) = \sigma^{-1}(C(X) - \overline{\sigma(A)}^{\mathrm{T}}) = \sigma^{-1}(\sigma(X) - \sigma(A)) = \sigma^{-1}(\sigma(U)) = U.$$

Thus we have $\tau(U) = O_2(\widehat{A}_X) \cap \tau(X)$ is open in $\tau(X)_{\mathrm{T}^*}$, which concludes that τ is open on the image $\tau(X)_{\mathrm{T}^*}$. □

LEMMA 6.5. *Suppose that X is regularly locally compact and take $x \in X$ and $\mathcal{E} \subset C(X)$ arbitrarily. Then there exists $\alpha \in \overline{\mathcal{E}}^{\mathrm{CH}}$ with $x \in \alpha$ if and only if $\sigma(x) \in \overline{\mathcal{E}}^{\mathrm{T}}$.*

PROOF. Assume that there is $\alpha \in \overline{\mathcal{E}}^{\mathrm{CH}}$ with $x \in \alpha$. We choose a complete net $\widehat{\boldsymbol{C}}$ in $\mathcal{E}$ that converges to α in $C(X)_{\mathrm{CH}}$; $\lim_{\mathrm{CH}} \widehat{\boldsymbol{C}} = \alpha$. Lemma 5.2 asserts that $\lim_{\mathrm{T}} \widehat{\boldsymbol{C}} = \widehat{\sigma}(\alpha)$. This implies that $\sigma(x) \in \lim_{\mathrm{T}} \widehat{\boldsymbol{C}}$, which shows that $\sigma(x) \in \overline{\mathcal{E}}^{\mathrm{T}}$.

Conversely, assume that $\sigma(x) \in \overline{\mathcal{E}}^{\mathrm{T}}$. Then we can choose a complete net $\widehat{\boldsymbol{C}}$ in $\mathcal{E}$ with $\sigma(x) \in \lim_{\mathrm{T}} \widehat{\boldsymbol{C}}$. Let $\alpha \in \overline{\mathcal{E}}^{\mathrm{CH}}$ be the limit of $\widehat{\boldsymbol{C}}$ in $C(X)_{\mathrm{CH}}$. Again by Lemma 5.2 we have $\sigma(x) \in \widehat{\sigma}(\alpha)$, or equivalently $x \in \alpha$. □

If X is a T_0-space, then σ is injective by Proposition 2.1. Since ι^* is always injective, the composition τ is injective in this case. Theorem 6.4 thus implies the following.

COROLLARY 6.6. *Assume that X is regularly locally compact and satisfies T_0 separation axiom. Then $\tau : X \to C(\widehat{X})_{\mathrm{T}^*}$ is a topological embedding.*

The recovering map τ can be used for understanding the topology of X. For instance, the next result is in Yoshino [**11**]. We remark that this does not hold true for $t : X \to C(\widehat{X})_{\mathrm{T}}$.

THEOREM 6.7. *Two distinct points x_1 and x_2 in a regularly locally compact topological space X are separable by disjoint open subsets in X if and only if $\tau(x_1) \cap \tau(x_2) = \emptyset$.*

PROOF. If $\alpha \in \tau(x_1) \cap \tau(x_2)$, then $\bar{x}_1 \subset \alpha$ and $\bar{x}_2 \subset \alpha$. Since $\alpha \in \widehat{X} = \overline{\sigma(X)}^{\mathrm{CH}} \subset \overline{\sigma(X)}^{\mathrm{T}}$, there is a net $\{\bar{x}_\nu\}_{\nu \in N}$ in $\sigma(X)$ that converges to α in $C(X)_{\mathrm{T}}$. Then it also converges to both $\bar{x}_1$ and $\bar{x}_2$. If x_1 and x_2 are separable by disjoint open subsets U_1 and U_2 of X, then $\bar{x}_1$ and $\bar{x}_2$ are also separable by disjoint open subsets $O_1(U_1) \cap \sigma(X)$ and $O_1(U_2) \cap \sigma(X)$ of $\sigma(X)_{\mathrm{T}}$. This is impossible and hence x_1 and x_2 are not separable.

Conversely, suppose that x_1 and x_2 are not separable by disjoint open subsets in X. Since X is regularly locally compact, there is a neighborhood basis $\mathcal{K} \subset \mathcal{V}(x_1)$ of x_1 consisting of compact subsets. Fix any $K \in \mathcal{K}$. For every neighborhood $U \in \mathcal{V}(x_2)$, there is a point $x_U \in K \cap U$. Then $\{x_U\}_{U \in \mathcal{V}(x_2)}$ defines a net in $K \subset X$ that converges to x_2. By taking a complete sub-net $\boldsymbol{C}$ of $\{x_U\}_{U \in \mathcal{V}(x_2)}$, we have $x_2 \in \lim \boldsymbol{C}$ and $K \cap \lim \boldsymbol{C} \neq \emptyset$. Set $\alpha_K = \lim \boldsymbol{C} \in \widehat{X}$ indexed by $K \in \mathcal{K}$, which belongs to $\tau(x_2)$ and satisfies $K \cap \alpha_K \neq \emptyset$. Then $\{\alpha_K\}_{K \in \mathcal{K}}$ gives a net in $\tau(x_2) \subset \widehat{X}$. We choose a complete sub-net $\widehat{\boldsymbol{C}}$ of $\{\alpha_K\}_{K \in \mathcal{K}}$. This takes the form $\{\alpha_{K_\lambda}\}_{\lambda \in \Lambda}$ indexed by a directed set Λ, where $\Lambda \to \mathcal{K} : \lambda \mapsto K_\lambda$ satisfies the condition that for every $K \in \mathcal{K}$ there is $\lambda_0 \in \Lambda$ such that $K_\lambda \subset K$ for every $\lambda \geq \lambda_0$. Take the unique Chabauty limit α of $\widehat{\boldsymbol{C}}$ in $\widehat{X}$. Since $\tau(x_2)$ is closed in $\widehat{X}$, we see that $\alpha \in \tau(x_2)$.

We will also see that $\alpha \in \tau(x_1)$, that is, $x_1 \in \alpha$. Indeed, if not, the regularly local compactness of X gives a compact neighborhood $K_1 \in \mathcal{K}$ of x_1 such that $K_1 \cap \alpha = \emptyset$, which means that $\alpha \in O_2(K_1)$. On the other hand, $\lim_{\mathrm{CH}} \widehat{\boldsymbol{C}} = \alpha$ implies that there is $\lambda_1 \in \Lambda$ such that $\alpha_{K_\lambda} \in O_2(K_1)$ for every $\lambda \geq \lambda_1$. However, if we choose some $\lambda \geq \lambda_1$ such that $K_\lambda \subset K_1$, we have a contradiction to the condition $K_\lambda \cap \alpha_{K_\lambda} \neq \emptyset$. Thus $\alpha \in \tau(x_1) \cap \tau(x_2)$ is proved. □

Finally in this section, we record the duality between the topologies T and T*. We introduce the following concept.

DEFINITION. For a subset A of an arbitrary topological space X, we call the intersection of all neighborhoods of A the *neighborhood core* of A and denote it by $[A]$.

PROPOSITION 6.8. *For an arbitrary subset $A \subset X$, it holds that*

$$[A] = \{x \in X \mid \bar{x} \cap A \neq \emptyset\}.$$

PROOF. Clearly $[A]$ contains the right side. Conversely, if $\bar{x} \cap A = \emptyset$, then $A \subset X - \bar{x}$ and $[A] \subset X - \bar{x}$, so $x \notin [A]$. □

REMARK. For a filter $\mathcal{F}$ in X, the *cluster set* $\operatorname{clus} \mathcal{F}$ is defined by the intersection of the closure $\overline{W}$ taken over all $W \in \mathcal{F}$ (see Bourbaki [**1**]). All the neighborhoods of a non-empty subset $A \subset X$ constitute a filter $\mathcal{F}_A$ and its cluster set satisfies $\operatorname{clus} \mathcal{F}_A \supset \overline{[A]}$.

It is clear from the definition that if $U \subset X$ is open then $[U] = U$. For a compact subset, we have the following.

Proposition 6.9. *If a subset $K \subset X$ is compact, then so is its neighborhood core $[K]$.*

Proof. Consider an open cover of $[K]$ in X. Since it is also an open cover of K in X and K is compact, there is a finite sub-cover $\{U_i\}_{i=1}^n$. Since $\bigcup_{i=1}^n U_i$ is an open neighborhood of K in X, this contains $[K]$. □

By Proposition 6.8, we have $\tau(x) = [\{\sigma(x)\}]_{\mathrm{T}} \cap \widehat{X}$, where $[\{\sigma(x)\}]_{\mathrm{T}}$ is the neighborhood core of the point set $\{\sigma(x)\}$ in $C(X)_{\mathrm{T}}$. This is compact by Proposition 6.9, but not necessarily closed. On the other hand, $\tau(x) = \overline{\{\sigma(x)\}}^{\mathrm{T}^*} \cap \widehat{X}$ is closed in $\widehat{X}_{\mathrm{T}^*}$.

Remark. For every $x \in X$, we consider a subset

$$A_x = \{a \in X \mid x \in \bar{a}\} \subset X,$$

which is the neighborhood core $[\{x\}]$ of the point set by Proposition 6.8 and is compact by Proposition 6.9. Then $\tau(x)$ contains $\sigma(A_x)$ and actually $\tau(x) \cap \sigma(X) = \sigma(A_x)$. Hence $\tau(x)$ contains $\overline{\sigma(A_x)}^{\mathrm{CH}}$. We may ask when they coincide.

We have seen a certain relationship between the closure and the neighborhood core for a point set, but this can be generalized as follows.

Proposition 6.10. *Let X be a regularly locally compact topological space and $\mathcal{E} \subset C(X)_{\mathrm{CH}}$ a closed subset. Then*

$$\overline{\mathcal{E}}^{\mathrm{T}} = \{\alpha_* \in C(X) \mid \alpha_* \subset \alpha,\ \exists\, \alpha \in \mathcal{E}\} = [\mathcal{E}]_{\mathrm{T}^*};$$
$$\overline{\mathcal{E}}^{\mathrm{T}^*} = \{\alpha^* \in C(X) \mid \alpha^* \supset \alpha,\ \exists\, \alpha \in \mathcal{E}\} = [\mathcal{E}]_{\mathrm{T}}.$$

Proof. We only show the first line. The proof for the second line is omitted, which is similarly given once we formulate the corresponding statement to Lemma 5.2, which is Lemma 6.11 below. Also, we only show the first equality, for the second equality is clear from Propositions 6.2 and 6.8. However, it is also evident from Proposition 5.1 that $\overline{\mathcal{E}}^{\mathrm{T}}$ includes the mid term. Hence we have only to prove the converse.

Take an arbitrary $\beta \in \overline{\mathcal{E}}^{\mathrm{T}}$. We can choose a complete net $\widehat{\boldsymbol{C}}$ in $\mathcal{E}$ that converges to β in $C(X)_{\mathrm{T}}$; $\beta \in \lim_{\mathrm{T}} \widehat{\boldsymbol{C}}$. On the other hand, Lemma 5.2 says that the unique Chabauty limit $\alpha = \lim_{\mathrm{CH}} \widehat{\boldsymbol{C}} \in \mathcal{E}$ satisfies $\widehat{\sigma}(\alpha) = \lim_{\mathrm{T}} \widehat{\boldsymbol{C}}$. Hence $\beta \in \widehat{\sigma}(\alpha)$, which is equivalent to $\beta \subset \alpha$. □

Lemma 6.11. *Suppose that X is regularly locally compact. Let $\widehat{\boldsymbol{C}}$ be a complete net in $C(X)$ and $\alpha \in C(X)$ the unique limit point to which $\widehat{\boldsymbol{C}}$ converges in $C(X)_{\mathrm{CH}}$. Then $\lim_{\mathrm{T}^*} \widehat{\boldsymbol{C}} = \overline{\{\alpha\}}^{\mathrm{T}^*}$, which is a closed subset of $C(X)_{\mathrm{T}^*}$.*

Proof. Since the Chabauty topology is finer than the dual Thurston topology,

$\widehat{\boldsymbol{C}}$ converges to α also in $C(X)_{\mathrm{T}^*}$. Since $\mathcal{V}_{\mathrm{T}^*}(\alpha^*) \subset \mathcal{V}_{\mathrm{T}^*}(\alpha)$ for any $\alpha^* \in C(X)$ with $\alpha^* \supset \alpha$, we see that $\overline{\{\alpha\}}^{\mathrm{T}^*} \subset \lim_{\mathrm{T}^*} \widehat{\boldsymbol{C}}$ by Proposition 6.2.

Suppose that $\beta \in C(X)$ with $\alpha - \beta \neq \emptyset$ is contained in $\lim_{\mathrm{T}^*} \widehat{\boldsymbol{C}}$. We choose some $x \in \alpha - \beta \subset X$. Since X is regularly locally compact, there is a compact neighborhood $K \subset X$ of x that is disjoint from β. Then $O_1(\mathrm{Int}\, K)$ is a neighborhood of α in $C(X)_{\mathrm{CH}}$ and $O_2(K)$ is a neighborhood of β in $C(X)_{\mathrm{T}^*}$.

The complete net $\widehat{\boldsymbol{C}} = \{\alpha_\nu\}_{\nu \in N}$ converges to α in $C(X)_{\mathrm{CH}}$ and also to β in $C(X)_{\mathrm{T}^*}$. The first convergence implies that there is some $\nu_1 \in N$ such that $\alpha_\nu \cap \mathrm{Int}\, K \neq \emptyset$ for every $\nu \geq \nu_1$. The second convergence implies that there is some $\nu_2 \in N$ such that $\alpha_\nu \cap K = \emptyset$ for every $\nu \geq \nu_2$. However, taking $\nu \geq \nu_1, \nu_2$ yields a contradiction. Therefore, no such $\beta \in C(X)$ belongs to $\lim_{\mathrm{T}^*} \widehat{\boldsymbol{C}}$, which shows that $\lim_{\mathrm{T}^*} \widehat{\boldsymbol{C}}$ is precisely $\overline{\{\alpha\}}^{\mathrm{T}^*}$. □

7. Induced maps between hyperspaces.

For a continuous map $f : X \to Y$ between topological spaces, we consider a certain map $\widehat{f} : C(X) \to C(Y)$ between the hyperspaces of their closed subsets induced by f. We will show that if f is proper in addition and if X and Y are regularly locally compact, then the restriction of $\widehat{f}$ to the Hausdorff compactification $\widehat{X}$ gives a continuous proper map to $\widehat{Y}$.

Definition. For any map $f : X \to Y$ between topological spaces X and Y, we define a map $\widehat{f} : C(X) \to C(Y)$ by $\widehat{f}(\alpha) = \overline{f(\alpha)}$ for each $\alpha \in C(X)$. We call this $\widehat{f}$ the *closure map* induced by f. The restriction of the closure map to any subset of $C(X)$ is also called a closure map.

The Thurston topology fits to the closure map as follows.

Proposition 7.1. *If $f : X \to Y$ is continuous, then $\widehat{f} : C(X)_{\mathrm{T}} \to C(Y)_{\mathrm{T}}$ is continuous.*

Proof. A sub-basis of $C(Y)_{\mathrm{T}}$ consists of sets of the form $O_1(U)$ for an open subset $U \subset Y$. A condition that $\widehat{f}(\alpha) \in O_1(U)$ for $\alpha \in C(X)$ is equivalent to that $f(\alpha) \cap U \neq \emptyset$ since $\widehat{f}(\alpha)$ is the closure of $f(\alpha)$. From this we have $\widehat{f}^{-1}(O_1(U)) = O_1(f^{-1}(U))$, where $f^{-1}(U)$ is open by the continuity of f. Thus we see that $\widehat{f}$ is continuous. □

We consider the restriction of the closure map $\widehat{f} : C(X) \to C(Y)$ to the Hausdorff compactification $\widehat{X}$. First we check it on $\sigma_X(X) \subset \widehat{X}$.

Proposition 7.2. *Suppose that $f : X \to Y$ is continuous. Then $\widehat{f} : C(X) \to C(Y)$ satisfies $\widehat{f} \circ \sigma_X = \sigma_Y \circ f$ for the impressions $\sigma_X : X \to C(X)$ and $\sigma_Y : Y \to C(Y)$. In particular, $\widehat{f}(\sigma_X(X)) \subset \sigma_Y(Y)$. If f is surjective in addition, then $\widehat{f}(\sigma_X(X)) = \sigma_Y(Y)$.*

Proof. By continuity of f, we see that $f(\sigma_X(x)) = f(\bar{x})$ is contained in $\overline{f(x)} = \sigma_Y(f(x))$ for every $x \in X$. Since $\overline{f(x)}$ is the smallest closed subset containing $f(x)$, the closure of $f(\sigma_X(x))$, which is $\widehat{f} \circ \sigma_X(x)$, coincides with $\sigma_Y \circ f(x)$. □

If $f : X \to Y$ is merely continuous, then the image $\widehat{f}(\widehat{X})$ is contained in $\overline{\sigma_Y(Y)}^{\mathrm{T}}$, which is larger than $\widehat{Y} = \overline{\sigma_Y(Y)}^{\mathrm{CH}}$ in general. Indeed, we have $\widehat{f}(\sigma_X(X)) \subset \sigma_Y(Y)$ by Proposition 7.2. Since $\widehat{X} = \overline{\sigma_X(X)}^{\mathrm{CH}} \subset \overline{\sigma_X(X)}^{\mathrm{T}}$, it follows that $\widehat{f}(\widehat{X}) \subset \widehat{f}(\overline{\sigma_X(X)}^{\mathrm{T}}) \subset \overline{\sigma_Y(Y)}^{\mathrm{T}}$ by Proposition 7.1.

Now, let X and Y be regularly locally compact. In addition to the continuity, we also assume that $f : X \to Y$ is proper. Then the closure map $\widehat{f}$ has the following favorable property.

LEMMA 7.3. *Assume that X and Y are regularly locally compact. If $f : X \to Y$ is proper and continuous, then $\overline{f(\lim \boldsymbol{C})} = \lim f(\boldsymbol{C})$ for every complete net $\boldsymbol{C}$ in X. In particular, $\widehat{f}(\alpha) = \overline{f(\alpha)} = \lim f(\boldsymbol{C}) \in \widehat{Y}$ for any $\alpha = \lim \boldsymbol{C} \in \widehat{X}$ and hence $\widehat{f}(\widehat{X}) \subset \widehat{Y}$.*

PROOF. Let $\boldsymbol{C} = \{x_\nu\}_{\nu \in N}$ be a complete net in X. By continuity of f, we have $f(\lim \boldsymbol{C}) \subset \lim f(\boldsymbol{C})$ and hence $\overline{f(\lim \boldsymbol{C})} \subset \lim f(\boldsymbol{C})$. We will show the other inclusion $\overline{f(\lim \boldsymbol{C})} \supset \lim f(\boldsymbol{C})$. Take any point $y \in \lim f(\boldsymbol{C})$. Since Y is regularly local compact, there is a neighborhood basis in $\mathcal{V}(y)$ consisting of compact subsets L. For any such $L \in \mathcal{V}(y)$, there is some $\nu_0 \in N$ such that $f(x_\nu) \in L$ for every $\nu \geq \nu_0$. Set $K = f^{-1}(L)$, which is a compact subset of X by properness of f. Since $\{x_\nu\}_{\nu \geq \nu_0}$ is a complete subnet of $\boldsymbol{C}$ in K, it has a limit point $x_L \in \lim \boldsymbol{C}$ in K. Then $y_L = f(x_L)$ belongs to $L \cap f(\lim \boldsymbol{C})$. Since this holds for every L in the compact neighborhood basis of y, we see that $y \in \overline{f(\lim \boldsymbol{C})}$. □

The following functorial property has been given in [**11**] by using the arguments of topological blow-up. Here we prove it relying on Theorem 3.2 and the claims in this section.

THEOREM 7.4 (Yoshino). *For regularly locally compact topological spaces X and Y, if $f : X \to Y$ is proper and continuous, then so is $\widehat{f} : \widehat{X} \to \widehat{Y}$.*

PROOF. We will show the continuity of $\widehat{f}$. A sub-basis of $\widehat{Y} = \overline{\sigma_Y(Y)}^{\mathrm{CH}} \subset C(Y)_{\mathrm{CH}}$ consists of sets of the form $O_1(U) \cap \widehat{Y}$ for an open subset $U \subset Y$ and of the form $O_2(K) \cap \widehat{Y}$ for a compact subset $K \subset Y$. Proposition 7.1 implies that

$$\widehat{f}^{-1}(O_1(U) \cap \widehat{Y}) = O_1(f^{-1}(U)) \cap \widehat{X}$$

is open in $\widehat{X}$ by the continuity of f. Hence it suffices to prove that $\widehat{f}^{-1}(O_2(K) \cap \widehat{Y})$ is open in $\widehat{X}$.

Let $\alpha \in \widehat{f}^{-1}(O_2(K) \cap \widehat{Y})$ be any element. It satisfies $\overline{f(\alpha)} \cap K = \emptyset$. Since Y is regularly locally compact, there is a compact neighborhood L_x of each $x \in K$ that is contained in the complement $\overline{f(\alpha)}^c$. By the compactness of K, we can choose finitely many $x \in K$ so that the finite unions of such $\operatorname{Int} L_x$ and L_x give an open set V and a compact set L with $K \subset V \subset L \subset \overline{f(\alpha)}^c$. In particular $f(\alpha) \cap L = \emptyset$ and hence α belongs to $O_2(f^{-1}(L)) \cap \widehat{X}$. This set is open by the properness of f and satisfies

$$O_2(f^{-1}(L)) \cap \widehat{X} \subset \widehat{f}^{-1}(O_2(K) \cap \widehat{Y}).$$

This shows that $\widehat{f}^{-1}(O_2(K) \cap \widehat{Y})$ is open.

The properness of $\widehat{f}$ follows from its continuity. Indeed, since $\widehat{X}$ and $\widehat{Y}$ are compact Hausdorff spaces by Corollary 3.3, compact subsets and closed subsets are the same in $\widehat{X}$ and $\widehat{Y}$. Since $\widehat{f}$ is continuous as above, the inverse image of a closed set is closed. Hence the inverse image of a compact set is compact. □

COROLLARY 7.5. *Under the same assumptions as in Theorem* 7.4, *if $f : X \to Y$ is surjective in addition, then $\widehat{f} : \widehat{X} \to \widehat{Y}$ is also surjective.*

PROOF. By Proposition 7.2, we see that $\sigma_Y(Y)$ is contained in the image $\widehat{f}(\widehat{X})$. Since $\widehat{f}(\widehat{X})$ is compact and hence closed by the continuity of $\widehat{f}$ proved in Theorem 7.4, we have $\widehat{Y} = \overline{\sigma_Y(Y)}^{\mathrm{CH}} \subset \widehat{f}(\widehat{X})$. □

Any map $f : X \to Y$ between any sets X and Y induces the map $2^X \to 2^Y$ between their power sets by $A \mapsto f(A)$ for $A \in 2^X$. We call this map the *power extension* of f and often denote it by the same symbol f. Its restriction to any subset $\mathcal{Z} \subset 2^X$ is also called a power extension. If $f : X \to Y$ is a closed map between topological spaces, then the closure map $\widehat{f} : C(X) \to C(Y)$ induced by f is clearly a power extension of f. In particular, for a homeomorphism $f : X \to Y$, we see that $\widehat{f} : C(X) \to C(Y)$ is the bijective power extension of f. Moreover in this case, $\widehat{f} : C(X)_\bullet \to C(Y)_\bullet$ is also a homeomorphism for any of $\bullet = \mathrm{CH}, \mathrm{T}, \mathrm{T}^*$.

8. Continuous maps between compact Hausdorff spaces.

We have seen in Proposition 7.1 that the closure map $\widehat{f} : C(X)_\mathrm{T} \to C(Y)_\mathrm{T}$ of a continuous map $f : X \to Y$ is also continuous. In this section, we will consider the closure map $C(\widetilde{X}) \to C(\widetilde{Y})$ of a continuous map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ between compact Hausdorff spaces $\widetilde{X}$ and $\widetilde{Y}$, in particular, the Hausdorff compactifications $\widehat{X}$ and $\widehat{Y}$ of regularly locally compact topological spaces X and Y. Note that the closure map is always the power extension F of $\widetilde{f}$ in this case. Indeed, since $\widetilde{X}$ is compact and $\widetilde{Y}$ is Hausdorff, the continuous map $\widetilde{f}$ is also closed. In the same reason, the continuous map $\widetilde{f}$ is always proper as is mentioned in the proof of Theorem 7.4. We provide three topologies for $C(\widetilde{X})$ and $C(\widetilde{Y})$: the Chabauty, the Thurston and the dual Thurston topologies.

PROPOSITION 8.1. *If $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is continuous for compact Hausdorff spaces $\widetilde{X}$ and $\widetilde{Y}$, then the power extension $F : C(\widetilde{X})_\mathrm{CH} \to C(\widetilde{Y})_\mathrm{CH}$ of $\widetilde{f}$ is proper and continuous. Moreover, $F : C(\widetilde{X})_\bullet \to C(\widetilde{Y})_\bullet$ is continuous for $\bullet = \mathrm{T}, \mathrm{T}^*$.*

PROOF. Since $C(\widetilde{X})_\mathrm{CH}$ is compact and $C(\widetilde{Y})_\mathrm{CH}$ is Hausdorff, the properness of F follows from its continuity. Hence we have only to show that F is continuous. For this purpose, take $O_1(U)$ and $O_2(K)$ where $U \subset \widetilde{Y}$ is open and $K \subset \widetilde{Y}$ is compact. A sub-basis of the open sets in $C(\widetilde{Y})_\mathrm{CH}$ consists of the sets of these forms. Then consider the inverse images of these sets by F:

$$F^{-1}(O_1(U)) = O_1(\widetilde{f}^{-1}(U)); \quad F^{-1}(O_2(K)) = O_2(\widetilde{f}^{-1}(K)).$$

Here $\widetilde{f}^{-1}(U)$ is open and $\widetilde{f}^{-1}(K)$ is compact since $\widetilde{f}$ is continuous and proper. Hence these are open subsets of $C(\widetilde{X})_{\mathrm{CH}}$, which shows that $F : C(\widetilde{X})_{\mathrm{CH}} \to C(\widetilde{Y})_{\mathrm{CH}}$ is continuous. This also shows that $F : C(\widetilde{X})_\bullet \to C(\widetilde{Y})_\bullet$ is continuous for $\bullet = \mathrm{T}, \mathrm{T}^*$. □

Remark. The continuity of $F : C(\widetilde{X})_{\mathrm{T}} \to C(\widetilde{Y})_{\mathrm{T}}$ also follows from Proposition 7.1.

Next we prove the properness of F for the (dual) Thurston topology. To this end, we need a characterization of compact subsets in $C(\widetilde{X})_{\mathrm{T}}$ (in $C(\widetilde{X})_{\mathrm{T}^*}$). For a subset $\mathcal{E} \subset C(X)$, we denote by $\mathcal{E}_0$ (by $\mathcal{E}^0$) the set of all minimal (maximal, respectively) elements in $\mathcal{E}$ concerning the inclusion relation of closed subsets in a topological space X.

Lemma 8.2. *Let X be a regularly locally compact topological space. Then a subset $\mathcal{E} \subset C(X)$ is compact in $C(X)_{\mathrm{T}}$ if and only if $\mathcal{E}_0 = (\overline{\mathcal{E}}^{\mathrm{CH}})_0$, that is, $\mathcal{E}_0$ is also the set of all minimal elements in $\overline{\mathcal{E}}^{\mathrm{CH}}$.*

Proof. Suppose that $\mathcal{E}_0 = (\overline{\mathcal{E}}^{\mathrm{CH}})_0$. Take any open cover $\{\mathcal{O}_i\}_{i\in I}$ of $\mathcal{E}$ in $C(X)_{\mathrm{T}}$. Since $\{\mathcal{O}_i\}_{i\in I}$ covers $(\overline{\mathcal{E}}^{\mathrm{CH}})_0$, it also covers $\overline{\mathcal{E}}^{\mathrm{CH}}$. Since each $\mathcal{O}_i$ is open in $C(X)_{\mathrm{CH}}$, $\{\mathcal{O}_i\}_{i\in I}$ is an open cover of $\overline{\mathcal{E}}^{\mathrm{CH}}$ in $C(X)_{\mathrm{CH}}$. The compactness of $\overline{\mathcal{E}}^{\mathrm{CH}}$ in $C(X)_{\mathrm{CH}}$ implies that $\mathcal{E}$ is compact in $C(X)_{\mathrm{T}}$.

Conversely, suppose that $\mathcal{E}_0 \neq (\overline{\mathcal{E}}^{\mathrm{CH}})_0$. Then there is a minimal element α_0 in $\overline{\mathcal{E}}^{\mathrm{CH}}$ that does not belong to $\mathcal{E}$. Take a complete net $\widehat{\boldsymbol{C}}$ in $\mathcal{E}$ that converges to α_0 in $C(X)_{\mathrm{CH}}$. By Lemma 5.2, we have $\lim_{\mathrm{T}} \widehat{\boldsymbol{C}} = \widehat{\sigma}(\alpha_0)$. However, since $\alpha_0 \notin \mathcal{E}$ and hence there is no element of $\mathcal{E}$ contained in α_0, it follows that $\lim_{\mathrm{T}} \widehat{\boldsymbol{C}} \cap \mathcal{E} = \emptyset$. This shows that $\mathcal{E}$ is not compact in $C(X)_{\mathrm{T}}$. □

Lemma 8.3. *Let X be a regularly locally compact topological space. Then a subset $\mathcal{E} \subset C(X)$ is compact in $C(X)_{\mathrm{T}^*}$ if and only if $\mathcal{E}^0 = (\overline{\mathcal{E}}^{\mathrm{CH}})^0$, that is, $\mathcal{E}^0$ is also the set of all maximal elements in $\overline{\mathcal{E}}^{\mathrm{CH}}$.*

Proof. We apply Lemma 6.11 instead of Lemma 5.2 in the proof of Lemma 8.2. Then the same arguments work except for replacing minimal with maximal. □

The properness of F for the Thurston topology follows from Lemma 8.2.

Proposition 8.4. *If $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is continuous for compact Hausdorff spaces $\widetilde{X}$ and $\widetilde{Y}$, then the power extension $F : C(\widetilde{X})_{\mathrm{T}} \to C(\widetilde{Y})_{\mathrm{T}}$ is proper.*

Proof. Taking a compact subset $\mathcal{H}$ of $C(\widetilde{Y})_{\mathrm{T}}$, we show that $F^{-1}(\mathcal{H})$ is a compact subset of $C(\widetilde{X})_{\mathrm{T}}$. Since $F : C(\widetilde{X})_{\mathrm{CH}} \to C(\widetilde{Y})_{\mathrm{CH}}$ is proper by Proposition 8.1, $F^{-1}(\overline{\mathcal{H}}^{\mathrm{CH}})$ is a compact closed subset of $C(\widetilde{X})_{\mathrm{CH}}$. Hence $F^{-1}(\overline{\mathcal{H}}^{\mathrm{CH}})$ contains $\overline{F^{-1}(\mathcal{H})}^{\mathrm{CH}}$. According to Lemma 8.2, to see that $F^{-1}(\mathcal{H})$ is compact in $C(\widetilde{X})_{\mathrm{T}}$, it suffices to show that the set of minimal elements in $F^{-1}(\mathcal{H})$ coincides with that in $F^{-1}(\overline{\mathcal{H}}^{\mathrm{CH}})$.

Take any minimal element ξ of $F^{-1}(\overline{\mathcal{H}}^{\mathrm{CH}})$. Then, for $\eta = F(\xi)$, we have $\eta \in \overline{\mathcal{H}}^{\mathrm{CH}}$.

However, since $\mathcal{H}$ and $\overline{\mathcal{H}}^{\mathrm{CH}}$ have the same set of minimal elements by Lemma 8.2, there is a minimal element $\eta_0 \in C(\widetilde{Y})$ of both $\mathcal{H}$ and $\overline{\mathcal{H}}^{\mathrm{CH}}$ that is contained in η. Set $\xi_0 = \widetilde{f}^{-1}(\eta_0) \cap \xi$, which is an element of $C(\widetilde{X})$ because $\widetilde{f}^{-1}(\eta_0)$ is closed by the continuity of $\widetilde{f}$. Since the power extension F preserves the inclusion relation, we see that $F(\xi_0) = \eta_0$. Then ξ_0 belongs to both $F^{-1}(\mathcal{H})$ and $F^{-1}(\overline{\mathcal{H}}^{\mathrm{CH}})$. However, the minimality of ξ in $F^{-1}(\overline{\mathcal{H}}^{\mathrm{CH}})$ implies that $\xi_0 = \xi \in F^{-1}(\mathcal{H})$. □

To obtain the properness of F for the dual Thurston topology, we need an extra assumption that $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is surjective.

PROPOSITION 8.5. *If $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is continuous and surjective for compact Hausdorff spaces $\widetilde{X}$ and $\widetilde{Y}$, then the power extension $F : C(\widetilde{X})_{\mathrm{T}^*} \to C(\widetilde{Y})_{\mathrm{T}^*}$ is proper.*

PROOF. This is similarly obtained by using Lemma 8.3; to see that $F^{-1}(\mathcal{H})$ is a compact subset of $C(\widetilde{X})_{\mathrm{T}^*}$ for any compact subset $\mathcal{H}$ of $C(\widetilde{Y})_{\mathrm{T}^*}$, we have only to check that the set of maximal elements in $F^{-1}(\mathcal{H})$ coincides with that in $F^{-1}(\overline{\mathcal{H}}^{\mathrm{CH}})$.

Take any maximal element ξ of $F^{-1}(\overline{\mathcal{H}}^{\mathrm{CH}})$ and consider $\eta = F(\xi) \in \overline{\mathcal{H}}^{\mathrm{CH}}$. Since $\mathcal{H}$ and $\overline{\mathcal{H}}^{\mathrm{CH}}$ have the same set of maximal elements by Lemma 8.3, there is a maximal element $\eta^0 \in C(\widetilde{Y})$ of both $\mathcal{H}$ and $\overline{\mathcal{H}}^{\mathrm{CH}}$ that contains η. Set $\xi^0 = \widetilde{f}^{-1}(\eta^0) \in C(\widetilde{X})$, which contains ξ. Since $\widetilde{f}$ is surjective, $F(\xi^0) = \widetilde{f}(\xi^0) = \widetilde{f}\widetilde{f}^{-1}(\eta^0) = \eta^0$. Then ξ^0 belongs to both $F^{-1}(\mathcal{H})$ and $F^{-1}(\overline{\mathcal{H}}^{\mathrm{CH}})$. However, the maximality of ξ in $F^{-1}(\overline{\mathcal{H}}^{\mathrm{CH}})$ implies that $\xi^0 = \xi \in F^{-1}(\mathcal{H})$. □

Finally, we apply the above results to the closure map $\widehat{f} : \widehat{X} \to \widehat{Y}$ of a proper continuous map $f : X \to Y$.

THEOREM 8.6. *For regularly locally compact topological spaces X and Y, suppose that $f : X \to Y$ is proper and continuous. Then the power extension $F : C(\widehat{X})_\bullet \to C(\widehat{Y})_\bullet$ of the closure map $\widehat{f} : \widehat{X} \to \widehat{Y}$ is proper and continuous for $\bullet = \mathrm{CH}, \mathrm{T}, \mathrm{T}^*$, where for the properness in the case of the dual Thurston topology T^*, f is further assumed to be surjective.*

PROOF. By Theorem 7.4, the closure map $\widehat{f} : \widehat{X} \to \widehat{Y}$ is continuous. If $f : X \to Y$ is surjective in addition then so is $\widehat{f}$ by Corollary 7.5. Then by Propositions 8.1, 8.4 and 8.5, we obtain the assertion. □

In the above situation, we consider the proper continuous map $F : C(\widehat{X})_{\mathrm{T}} \to C(\widehat{Y})_{\mathrm{T}}$ in the Thurston topology for instance. By setting $f_1 = F$, $X_1 = C(\widehat{X})_{\mathrm{T}}$ and $Y_1 = C(\widehat{Y})_{\mathrm{T}}$, we can apply Theorem 8.6 to $f_1 : X_1 \to Y_1$. Then we have that the power extension $F_1 : C(\widehat{X}_1)_{\mathrm{T}} \to C(\widehat{Y}_1)_{\mathrm{T}}$ of $\widehat{f}_1$ is proper and continuous. This process can be repeated.

9. Reproduction of homeomorphisms between base spaces.

It is obvious that a homeomorphism $f : X \to Y$ induces a homeomorphism $\widetilde{f} : \widehat{X} \to \widehat{Y}$ between the Hausdorff compactifications by the power extension, which coincides with the closure map $\widehat{f}$ in this case. In this section, we conversely consider whether a given homeomorphism $\widetilde{f}:\widehat{X}\to\widehat{Y}$ is induced by $f:X\to Y$.

Suppose that a map $\widetilde{f}:\mathcal{Z}\subset 2^X\to 2^Y$ is given. We consider necessary conditions for the existence of a map $f:X\to Y$ whose power extension $2^X\to 2^Y$ restricted to $\mathcal{Z}$ coincides with $\widetilde{f}$. As an obvious condition, we introduce the following.

Definition. We say that a map $\widetilde{f}:\mathcal{Z}\subset 2^X\to 2^Y$ preserves the *inclusion relation* if $\alpha,\beta\in\mathcal{Z}$ satisfy $\alpha\subset\beta\subset X$ then $\widetilde{f}(\alpha)\subset\widetilde{f}(\beta)\subset Y$.

The Thurston topology (and also the dual Thurston topology, though we do not use this later) on $C(X)$ and $C(Y)$ is suitable for this property.

Proposition 9.1. *If $\widetilde{f}:\mathcal{Z}_{\mathrm{T}}\subset C(X)_{\mathrm{T}}\to C(Y)_{\mathrm{T}}$ is continuous, then $\widetilde{f}$ preserves the inclusion relation.*

Proof. In the Thurston topology, $\alpha,\beta\in\mathcal{Z}$ satisfy $\alpha\subset\beta$ if and only if $\beta\in O$ for every $O\in\mathcal{V}(\alpha)$. Take any $O'\in\mathcal{V}(\widetilde{f}(\alpha))$. By the continuity, $\widetilde{f}^{-1}(O')$ is in $\mathcal{V}(\alpha)$, and hence $\beta\in\widetilde{f}^{-1}(O')$, which implies that $\widetilde{f}(\beta)\in O'$. This proves that $\widetilde{f}(\alpha)\subset\widetilde{f}(\beta)$. □

By Proposition 7.2, we have another necessary condition for $\widetilde{f}:\widehat{X}\to\widehat{Y}$ to be the power extension of a continuous map $f:X\to Y$.

Definition. We say that a map $\widetilde{f}:\widehat{X}\to\widehat{Y}$ preserves the *point structure* if $\widetilde{f}(\sigma_X(X))\subset\sigma_Y(Y)$.

Although any continuous map $\widetilde{f}:\widehat{X}_{\mathrm{T}}\to\widehat{Y}_{\mathrm{T}}$ preserves the inclusion relation by Proposition 9.1, we do no know whether such an $\widetilde{f}$ always preserves the point structure.

Given a bijection between the Hausdorff compactifications preserving the inclusion relation and the point structure, we consider bijections both in the upper and the lower levels that are induced by impressions.

Lemma 9.2. *Assume that X and Y are regularly locally compact and satisfy T_0 separation axiom. Let $\widetilde{f}:\widehat{X}\to\widehat{Y}$ be a bijection such that $\widetilde{f}$ and $\widetilde{f}^{-1}$ preserve the inclusion relation and the point structure. Then the following are satisfied*:

1. *$f=\sigma_Y^{-1}\circ\widetilde{f}\circ\sigma_X:X\to Y$ is bijective and $\widetilde{f}$ is a power extension of f;*
2. *the bijection $F=\iota_Y\circ\widetilde{f}\circ\iota_X^{-1}:\iota_X(\widehat{X})\to\iota_Y(\widehat{Y})$ for $\iota_X=\sigma_{\widehat{X}_{\mathrm{T}}}$ and $\iota_Y=\sigma_{\widehat{Y}_{\mathrm{T}}}$ is a power extension of $\widetilde{f}$;*
3. *the bijection $F^*=\iota_Y^*\circ\widetilde{f}\circ\iota_X^{*-1}:\iota_X^*(\widehat{X})\to\iota_Y^*(\widehat{Y})$ for $\iota_X^*=\sigma_{\widehat{X}_{\mathrm{T}^*}}$ and $\iota_Y^*=\sigma_{\widehat{Y}_{\mathrm{T}^*}}$ is a power extension of $\widetilde{f}$.*

Proof. (1) Since $\widetilde{f}$ and $\widetilde{f}^{-1}$ preserve the point structure, the restriction of $\widetilde{f}$ to $\sigma_X(X)$ gives the bijection $\widetilde{f}:\sigma_X(X)\to\sigma_Y(Y)$. Then $f=\sigma_Y^{-1}\circ\widetilde{f}\circ\sigma_X:X\to Y$ is bijective. For each $\alpha\in\widehat{X}$, set $\beta=\widetilde{f}(\alpha)\in\widehat{Y}$. To prove that $\widetilde{f}:\widehat{X}\to\widehat{Y}$ is the power extension of f, it is enough to show that

$$f(\alpha) = \{f(x) \in Y \mid x \in \alpha\} \subset \beta; \quad f^{-1}(\beta) = \{f^{-1}(y) \in X \mid y \in \beta\} \subset \alpha.$$

For every $x \in \alpha$, we have $\sigma_X(x) \subset \alpha$, and since $\widetilde{f}$ preserves the inclusion relation, we have $\widetilde{f}(\sigma_X(x)) \subset \widetilde{f}(\alpha) = \beta$. Then $y = f(x) = \sigma_Y^{-1} \circ \widetilde{f} \circ \sigma_X(x)$ satisfies $\sigma_Y(y) \subset \beta$, and thus $f(x) \in \beta$. This implies $f(\alpha) \subset \beta$. The other inclusion can be proved similarly by using $\widetilde{f}^{-1}$.

(2) For each $\widehat{\alpha} = \iota_X(\alpha) \in \iota_X(\widehat{X})$, set $\widehat{\beta} = F(\widehat{\alpha}) \in \iota_Y(\widehat{Y})$ ($\widehat{\beta} = \iota_Y(\beta)$, $\beta = \widetilde{f}(\alpha) \in \widehat{Y}$). We prove that F is the power extension of $\widetilde{f}$ by showing

$$\widetilde{f}(\widehat{\alpha}) = \{\widetilde{f}(\alpha') \in \widehat{Y} \mid \alpha' \in \widehat{\alpha}\} \subset \widehat{\beta}; \quad \widetilde{f}^{-1}(\widehat{\beta}) = \{\widetilde{f}^{-1}(\beta') \in \widehat{X} \mid \beta' \in \widehat{\beta}\} \subset \widehat{\alpha}.$$

For every $\alpha' \in \widehat{\alpha}$, we have $\alpha' \subset \alpha$ by Proposition 5.1. Since $\widetilde{f}$ preserves the inclusion relation, we have $\widetilde{f}(\alpha') \subset \widetilde{f}(\alpha) = \beta$. Then $\widetilde{f}(\alpha') \in \iota_Y(\beta) = \widehat{\beta}$ again by Proposition 5.1. This implies $\widetilde{f}(\widehat{\alpha}) \subset \widehat{\beta}$. The other inclusion can be proved similarly by using $\widetilde{f}^{-1}$.

(3) For each $\widehat{\alpha}^* = \iota_X^*(\alpha) \in \iota_X^*(\widehat{X})$, set $\widehat{\beta}^* = F^*(\widehat{\alpha}^*) \in \iota_Y^*(\widehat{Y})$ ($\widehat{\beta}^* = \iota_Y^*(\beta)$, $\beta = \widetilde{f}(\alpha) \in \widehat{Y}$). We prove that F^* is the power extension of $\widetilde{f}$ by showing

$$\widetilde{f}(\widehat{\alpha}^*) = \{\widetilde{f}(\alpha') \in \widehat{Y} \mid \alpha' \in \widehat{\alpha}^*\} \subset \widehat{\beta}^*; \quad \widetilde{f}^{-1}(\widehat{\beta}^*) = \{\widetilde{f}^{-1}(\beta') \in \widehat{X} \mid \beta' \in \widehat{\beta}^*\} \subset \widehat{\alpha}^*.$$

For every $\alpha' \in \widehat{\alpha}^*$, we have $\alpha' \supset \alpha$ (inverse inclusion) by Proposition 6.2. Since $\widetilde{f}$ preserves the inclusion relation, we have $\widetilde{f}(\alpha') \supset \widetilde{f}(\alpha) = \beta$. Then $\widetilde{f}(\alpha') \in \iota_Y^*(\beta) = \widehat{\beta}^*$ again by Proposition 6.2 (whose twice applications cancel the reverse of the inclusion relation). This implies $\widetilde{f}(\widehat{\alpha}^*) \subset \widehat{\beta}^*$. The other inclusion can be proved similarly by using $\widetilde{f}^{-1}$. □

Now we are ready to show the main theorem in this section, which gives conditions for a homeomorphism between the Hausdorff compactifications preserving the point structure to be induced from a homeomorphism of base spaces.

THEOREM 9.3. *Assume that X and Y are regularly locally compact and satisfy T_0 separation axiom. Let $\widetilde{f} : \widehat{X} \to \widehat{Y}$ be a bijection such that $\widetilde{f}$ and $\widetilde{f}^{-1}$ preserve the point structure. Then the following conditions are equivalent*:

1. *$\widetilde{f} : \widehat{X} \to \widehat{Y}$ is a homeomorphism such that $\widetilde{f}$ and $\widetilde{f}^{-1}$ preserve the inclusion relation*;
2. *$\widetilde{f} : \widehat{X}_{\mathrm{T}} \to \widehat{Y}_{\mathrm{T}}$ is a homeomorphism*;
3. *there exists a homeomorphism $f : X \to Y$ whose power extension coincides with $\widetilde{f}$.*

PROOF. Assume condition (1). The power extension $C(\widehat{X})_\bullet \to C(\widehat{Y})_\bullet$ induced by the homeomorphism $\widetilde{f} : \widehat{X} \to \widehat{Y}$ is a homeomorphism in particular for $\bullet = \mathrm{T}^*$. By Lemma 9.2 (3), the bijection $F^* = \iota_Y^* \circ \widetilde{f} \circ \iota_X^{*-1} : \iota_X^*(\widehat{X}) \to \iota_Y^*(\widehat{Y})$ is a power extension of $\widetilde{f}$. Hence $F^* : \iota_X^*(\widehat{X})_{\mathrm{T}^*} \subset C(\widehat{X})_{\mathrm{T}^*} \to \iota_Y^*(\widehat{Y})_{\mathrm{T}^*} \subset C(\widehat{Y})_{\mathrm{T}^*}$ is a homeomorphism. On the other hand, by Lemma 9.2 (1), we have the bijection $f = \sigma_Y^{-1} \circ \widetilde{f} \circ \sigma_X : X \to Y$ that induces $\widetilde{f}$ as a power extension. Since the recovering map is given as $\tau = \iota^* \circ \sigma$ by Proposition 6.3 and $\tau_X : X \to C(\widehat{X})_{\mathrm{T}^*}$ and $\tau_Y : Y \to C(\widehat{Y})_{\mathrm{T}^*}$ are topological embeddings by Corollary 6.6, we see that $f = \tau_Y^{-1} \circ F^* \circ \tau_X$ is a homeomorphism. This verifies condition (3).

Assume condition (2). By Proposition 9.1, $\widetilde{f}$ and $\widetilde{f}^{-1}$ preserve the inclusion relation. By Lemma 9.2 (1), we have the bijection $f = \sigma_Y^{-1} \circ \widetilde{f} \circ \sigma_X : X \to Y$ that induces $\widetilde{f}$ as a power extension. Since $\sigma_X : X \to \widehat{X}_{\mathrm{T}}$ and $\sigma_Y : Y \to \widehat{Y}_{\mathrm{T}}$ are topological embeddings, we see that f is a homeomorphism, which gives condition (3).

Finally we show that condition (3) implies both conditions (1) and (2). The power extension $C(X)_\bullet \to C(Y)_\bullet$ of f is a homeomorphism for $\bullet = \mathrm{CH}, \mathrm{T}$. Since $\widetilde{f}$ is its restriction to $\widehat{X}$, we see that $\widetilde{f} : \widehat{X}_\bullet \to \widehat{Y}_\bullet$ is a homeomorphism for $\bullet = \mathrm{CH}, \mathrm{T}$. Clearly $\widetilde{f}$ and $\widetilde{f}^{-1}$ preserve the inclusion relation. □

Finally in this section, we examine the special case of Theorem 9.3 where $\widetilde{f} : \widehat{\mathcal{Z}} \to \widehat{\mathcal{W}}$ is a bijection between the Hausdorff compactifications of $\mathcal{Z} = \mathcal{Z}_{\mathrm{T}}$ and $\mathcal{W} = \mathcal{W}_{\mathrm{T}}$. Here $\mathcal{Z}$ and $\mathcal{W}$ are closed subsets of $C(X)_{\mathrm{CH}}$ and $C(Y)_{\mathrm{CH}}$, respectively. By Theorem 5.8, the impressions define topological embeddings $\widehat{\sigma}_{\mathcal{Z}} : \mathcal{Z} \to C(\mathcal{Z}_{\mathrm{T}})$ and $\widehat{\sigma}_{\mathcal{W}} : \mathcal{W} \to C(\mathcal{W}_{\mathrm{T}})$ both in the Thurston topology and in the Chabauty topology. In this case, $\widetilde{f} : \widehat{\mathcal{Z}} = \widehat{\sigma}_{\mathcal{Z}}(\mathcal{Z}) \to \widehat{\mathcal{W}} = \widehat{\sigma}_{\mathcal{W}}(\mathcal{W})$ of course preserves the point structure. Condition (3) of Theorem 9.3 concerns the existence of a map $f : \mathcal{Z} \to \mathcal{W}$ that induces $\widetilde{f}$, but in the present situation we can define it just by $f = \widehat{\sigma}_{\mathcal{W}}^{-1} \circ \widetilde{f} \circ \widehat{\sigma}_{\mathcal{Z}}$. Hence Theorem 9.3 turns out to be the following statement for the map $f : \mathcal{Z} \to \mathcal{W}$.

Corollary 9.4. *Assume that X and Y are regularly locally compact. Let $\mathcal{Z}$ and $\mathcal{W}$ be closed subsets of $C(X)_{\mathrm{CH}}$ and $C(Y)_{\mathrm{CH}}$, respectively. Then the following conditions are equivalent for a bijection $f : \mathcal{Z} \to \mathcal{W}$:*

1. *$f : \mathcal{Z}_{\mathrm{CH}} \to \mathcal{W}_{\mathrm{CH}}$ is a homeomorphism such that f and f^{-1} preserve the inclusion relation;*
2. *$f : \mathcal{Z}_{\mathrm{T}} \to \mathcal{W}_{\mathrm{T}}$ is a homeomorphism.*

Proof. Consider $\widetilde{f} = \widehat{\sigma}_{\mathcal{W}} \circ f \circ \widehat{\sigma}_{\mathcal{Z}}^{-1}$, which is a bijection $\widetilde{f} : \widehat{\mathcal{Z}} \to \widehat{\mathcal{W}}$. Clearly $\widetilde{f}$ and $\widetilde{f}^{-1}$ preserve the point structure. Note that $\mathcal{Z} = \mathcal{Z}_{\mathrm{T}}$ and $\mathcal{W} = \mathcal{W}_{\mathrm{T}}$ are regularly locally compact by Proposition 2.3 and satisfy T_0 separation axiom. Also, since $\widehat{\sigma}_{\mathcal{Z}}(\alpha) = \widehat{\sigma}(\alpha) \cap \mathcal{Z}$ and $\widehat{\sigma}_{\mathcal{W}}(\beta) = \widehat{\sigma}(\beta) \cap \mathcal{W}$ for $\alpha \in \mathcal{Z}$ and $\beta \in \mathcal{W}$ by Proposition 2.8, $\widetilde{f}$ and $\widetilde{f}^{-1}$ preserve the inclusion relation if and only if f and f^{-1} preserve the inclusion relation. Then Theorem 9.3 asserts that conditions (1) and (2) for $\widetilde{f}$ instead of f are equivalent. Since $\widehat{\sigma}_{\mathcal{Z}} : \mathcal{Z} \to \widehat{\mathcal{Z}}$ and $\widehat{\sigma}_{\mathcal{W}} : \mathcal{W} \to \widehat{\mathcal{W}}$ are homeomorphisms by Theorem 5.8 in both topologies, these two conditions are equivalent for f itself. □

10. An application to the space of geodesic laminations.

We consider the space of geodesic laminations on a complete hyperbolic surface X not necessarily of finite area. Here a geodesic lamination $\lambda \in C(X)$ is a closed subset of X consisting of a disjoint union of simple closed or infinite geodesics. We denote the set of all geodesic laminations on X by $\mathcal{GL}(X)$.

Since X is locally compact, Hausdorff and second countable, $C(X)_{\mathrm{CH}}$ is metrizable and separable. The Chabauty topology coincides with the topology induced by the Hausdorff distance between closed subsets. The space $\mathcal{GL}(X)$ of geodesic laminations is a closed subset of $C(X)_{\mathrm{CH}}$. In fact, this is also closed in $C(X)_{\mathrm{T}}$.

We apply Corollary 9.4 to a bijection $f : \mathcal{Z} \to \mathcal{Z}$ for $\mathcal{Z} = \mathcal{GL}(X) \subset C(X)$ and obtain the following.

THEOREM 10.1. *Let X be a complete hyperbolic surface and $\mathcal{GL}(X) \subset C(X)$ the space of all geodesic laminations on X. For a bijection $f : \mathcal{GL}(X) \to \mathcal{GL}(X)$, the following conditions are equivalent*:

1. *$f : \mathcal{GL}(X)_{\mathrm{CH}} \to \mathcal{GL}(X)_{\mathrm{CH}}$ is a homeomorphism such that f and f^{-1} preserve the inclusion relation (i.e. for any $\lambda, \lambda^* \in \mathcal{GL}(X)$, $\lambda \subset \lambda^* \subset X$ if and only if $f(\lambda) \subset f(\lambda^*) \subset X$);*
2. *$f : \mathcal{GL}(X)_{\mathrm{T}} \to \mathcal{GL}(X)_{\mathrm{T}}$ is a homeomorphism.*

When a complete hyperbolic surface X is of finite area, a problem about whether a self-homeomorphism of $\mathcal{GL}(X)$ is induced geometrically by an automorphism of X has been studied in Charitos, Papadoperakis and Papadopoulos [**3**].

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References

[1] N. Bourbaki, Topologie générale, Diffusion C.C.L.S., Paris, 1971.
[2] R. Canary, D. Epstein and P. Green, Notes on notes of Thurston, Fundamentals of Hyperbolic Manifolds, LMS Lecture Note Series, **328**, Cambridge Univ. Press, 2006, pp. 1–115.
[3] C. Charitos, I. Papadoperakis and A. Papadopoulos, On the homeomorphisms of the space of geodesic laminations on a hyperbolic surface, Proc. Amer. Math. Soc., **142** (2014), 2179–2191.
[4] P. de la Harpe, Spaces of closed subgroups of locally compact groups, arXiv:0807.2030v2.
[5] J. Dugundji, Topology, Allyn and Bacon, 1966.
[6] J. Kelley, General Topology, GTM, **27**, Springer, 1975.
[7] L. A. Steen and J. A. Seebach, Counterexamples in Topology, 2nd ed., Springer, 1978.
[8] W. P. Thurston, The Geometry and Topology of Three-Manifolds, lecture note at Princeton Univ., 1980. http://www.msri.org/publications/books/gt3m/
[9] S. Willard, General Topology, Addison-Wesley, 1970.
[10] T. Yoshino, Topological blow-up and discontinuous groups, Representation Theory and Harmonic Analysis, Oberwolfach Report 52/2010, pp. 3076–3080.
[11] T. Yoshino, On topological blow-up.

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