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Deformations of Killing spinors on Sasakian and 3-Sasakian manifolds

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Abstract. We consider some natural infinitesimal Einstein deformations on Sasakian and 3-Sasakian manifolds. Some of these are infinitesimal deformations of Killing spinors and further some integrate to actual Killing spinor deformations. In particular, on 3-Sasakian 7 manifolds these yield infinitesimal Einstein deformations preserving 2, 1, or none of the 3 independent Killing spinors. Toric 3-Sasakian manifolds provide non-trivial examples with integrable deformation preserving precisely 2 Killing spinors. Thus in contrast to the case of parallel spinors the dimension of Killing spinors is not preserved under Einstein deformations but is only upper semi-continuous.

Introduction.

Let M be an n-dimensional Riemannian spin manifold with spinor bundle Σ. A Killing spinor is a non-trivial section $\psi \in \Gamma(\Sigma)$ with

$$\nabla_X \psi = cX \cdot \psi, \tag{1}$$

for some constant c, where ∇ is the Levi-Civita connection, X any tangent vector, and $X \cdot \psi$ denotes Clifford multiplication. An easy computation shows that $\mathrm{Ric}_g = 4(n-1)c^2 g$. Thus c must be either purely imaginary in which case M is non-compact, $c = 0$ with ψ a parallel spinor and M is Ricci-flat, or c is real and M is positive Einstein and compact assuming completeness. In the latter case ψ is a *real Killing spinor*. We will only consider real Killing spinors with $c \neq 0$. Since c is rescaled by homotheties of the metric, only its sign is of significance. We denote by N_+ (respectively N_-) the dimension of the space of Killing spinors with $c > 0$ (respectively $c < 0$).

Killing spinors are of interest in physics in supergravity and string theories [**11**]. But they are also of interest purely mathematically. See [**3**] for a survey. Much work has been done in classifying manifolds admitting a Killing spinor. Bär [**2**] classified simply connected manifolds admitting a real Killing spinor in terms of the underlying geometry of (M, g). The classification is given in terms of the holonomy of the metric cone $(C(M), \bar{g})$, $C(M) = \mathbb{R}_+ \times M$, $\bar{g} = dr^2 + r^2 g$. The argument in [**2**] is essentially that the connection $\nabla_X - cX$ on Σ is identified with the Levi-Civita connection $\overline{\nabla}$ of $\bar{g}$ on $\overline{\Sigma}$ (the spin bundle of $C(M)$ when n is even, and half-spin bundle when n is odd). Then the classification is in terms of irreducible holonomies admitting a parallel spinors [**41**]. See Table 1 for the classification. Therefore, just as for the irreducible reduced Ricci-flat

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holonomies there are two cases occurring in infinitely many dimensions, the Sasaki–Einstein and 3-Sasakian manifolds, and two exceptional cases, nearly Kähler and weak G_2 in dimensions 6 and 7 respectively.

Nearly Kähler structures, introduced by Gray in the context of weak holonomy, are almost Hermitian structures (g, J, ω) with $\nabla_X J(X) = 0$ for any $X \in TM$. Note that for a proper nearly Kähler structure, i.e. not Kähler, the almost complex structure J is not integrable and $d\omega \neq 0$. When $n = 6$ the torsion of the $SU(3)$-structure is contained in a 1-dimensional subbundle. In [**31**] it is shown that every nearly Kähler manifold is locally the Riemannian product of Kähler manifolds, 3-symmetric spaces, twistor spaces over positive quaternion-Kähler manifolds and 6-dimensional nearly Kähler manifolds. Thus most questions about nearly Kähler manifolds reduces to proper 6-dimensional nearly Kähler manifolds.

A weak G_2 manifold is a 7-manifold with a vector cross product coming from the imaginary octonians, or equivalently a *stable* 3-form $\sigma \in \Omega^3$ with $d\sigma = -\lambda \star \sigma$ with $\lambda \neq 0$ a constant. The form σ defines a reduction of the structure group of M to G_2 and thus a metric g, as $G_2 \subset SO(7)$, which is Einstein with scalar curvature $s = (21/8)\lambda^2$. Again, the torsion of the G-structure lies in a 1-dimensional subbundle. See [**16**] for results on weak G_2 manifolds including a classification of homogeneous examples.

Most interesting is perhaps $n = 7$ for which, when M is simply connected and not of constant curvature, $N_+ = 1, 2$, or 3, in which case (M, g) is said to be of type 1, 2, or 3 respectively. Recall that the spinor representation $\mathbb{S}$ of $Spin(7)$ is real, $\mathbb{S} = \mathbb{S}_{\mathbb{R}} \otimes \mathbb{C}$. Thus M has a real spinor bundle $\Sigma_{\mathbb{R}}$, and the space of solutions to (1) is the complexification of solutions in $\Gamma(\Sigma_{\mathbb{R}})$. Each section $\psi \in \Gamma(\Sigma_{\mathbb{R}})$ defines a G_2-structure on M with stable 3-form σ_ψ, and there is a bijective correspondence between sections of $\mathbb{P}(\Sigma_{\mathbb{R}})$ and G_2-structures with metric g and given orientation. If ψ is a representative of such a section with $|\psi| = 1$, then σ_ψ defines a weak G_2-structure, $d\sigma_\psi = -\lambda \star \sigma_\psi$, if and only if ψ satisfies (1), with $\lambda = 8c$. If (M, g) is type 1, then there is a unique 3-form inducing the given metric and orientation. If it is of type 2, then (M, g) is Sasaki–Einstein but not 3-Sasakian and there is a space of compatible 3-forms parameterized by $\mathbb{RP}^1$. And if it is of type 3, then (M, g) is 3-Sasakian and has a space of compatible 3-forms parameterized by $\mathbb{RP}^2$. See [**16**].

Note that an easy computation of the curvature of the warped product shows that $(C(M), \bar{g})$ is Ricci-flat if and only if (M, g) is Einstein with $\mathrm{Ric}_g = (n-1)g$. Thus the classification as in Table 1 gives a natural scaling in which $c = \pm 1/2$ in (1) and $s = n(n-1)$.

We consider deformations of the Killing spinor Equation (1) under deformations of g, both infinitesimal and genuine. As solutions to (1) imply that (M, g) is Einstein we consider Einstein deformations. The beginnings of a general theory of deformations of Killing spinors was developed by Wang [**42**], making use of the work of Bourguignon and Gauduchon [**6**] on the variations of spinors under metric variations.

More recently there has been some work on the two exceptional cases in Table 1. In [**28**] and [**30**] it is shown that the space of infinitesimal Einstein deformations of a proper nearly Kähler 6-manifold consists of eigenspaces of the Laplace operator Δ restricted to the space E of co-closed primitive $(1,1)$-forms. If $E(\lambda)$ denotes the λ-eigenspace of Δ restricted to E, then the space of essential infinitesimal Einstein defor-

Table 1. real Killing spinors.

$\dim M$	N_+	N_-	$\mathrm{Hol}(C(M))$	geometry
n	$2^{\lfloor n/2 \rfloor}$	$2^{\lfloor n/2 \rfloor}$	Id	n-sphere
$4m-1$	2	0	$SU(2m)$	Sasaki–Einstein
$4m+1$	1	1	$SU(2m+1)$	Sasaki–Einstein
$4m-1$	$m+1$	0	$Sp(m)$	3-Sasakian
6	1	1	G_2	nearly Kähler
7	1	0	$Spin(7)$	weak G_2

mations is $E(2) \oplus E(6) \oplus E(12)$. The space of infinitesimal deformations of nearly Kähler structures is $E(12)$. Besides S^6, which has no Einstein deformations the only examples of proper nearly Kähler 6-manifolds are 3-symmetric spaces, $\mathbb{CP}^3 = SO(5)/U(2)$, $F(1,2) = SU(3)/U(1) \times U(1)$, and $S^3 \times S^3 = SU(2) \times SU(2) \times SU(2)/\Delta$. In [**29**] it is shown that the nearly Kähler structures on $\mathbb{CP}^3$ and $S^3 \times S^3$ have no infinitesimal Einstein deformations, and on $F(1,2)$ $E(2)$ and $E(6)$ vanish while $E(12)$ is an 8-dimensional space.

Similar results are known for weak G_2 manifolds. In [**1**] a similar decomposition of the infinitesimal Einstein deformations on a weak G_2 manifold are given. First recall that a G_2-structure induces a decomposition of the 3-forms into irreducible G_2-representations $\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$. And there is a map $\iota : S^2_0(T^*) \to \Lambda^3$, which on a decomposable element $\alpha \odot \beta$ is $\iota(\alpha \odot \beta) = \alpha \wedge (\beta \lrcorner \sigma) + \beta \wedge (\alpha \lrcorner \sigma)$, which is an isomorphism onto Λ^3_{27}. It is proved in [**1**] that the essential infinitesimal Einstein deformations is given by the direct sum

$$E(16) \oplus E(4) \oplus E(8),$$

where $E(16) = \{\gamma \in \Omega^3_{27} | \star d\gamma = -4\gamma\}$, $E(4) = \{\gamma \in \Omega^3_{27} | \star d\gamma = 2\gamma\}$, and $E(8) = \{\gamma \in \Omega^3_{27} | dd^*\gamma = 8\gamma\}$. The notation $E(\lambda)$ indicates that these are subspaces of the λ-eigenspace of Δ. The space $E(16)$ is the subspace of infinitesimal deformations of weak G_2-structures, or more precisely, those not fixing the metric and deforming the Killing spinor. This space is computed on the normal homogeneous examples: the isotropy irreducible space $SO(5)/SO(3)$, the pinched metric on S^7, and the second Einstein metric on the Aloff–Wallach space $N(1,1) = SU(3)/U(1)$. The first two cases have no infinitesimal Einstein deformations, while for the third the infinitesimal Einstein deformations correspond to $E(16)$ which is 8-dimensional.

These results might lead one to suspect that there might be some stability for Killing spinors under Einstein deformations, either infinitesimal or integrable. Furthermore, for the case $c = 0$ in (1), i.e. parallel spinors, there are strong stability results [**33**], [**42**]. Recall that a simply-connected, spin, irreducible Riemannian manifold (M, g) admits a parallel spinor if and only if the holonomy $\mathrm{Hol}(g) = G$ where $G = SU(m), Sp(m), G_2$, or $Spin(7)$. Define a G-manifold to be a connected oriented manifold of dimension $2m, 4m, 7$ or 8 respectively with a torsion-free G-structure with G from this list. This means $\mathrm{Hol}(g) \subseteq G$. Thus a G-manifold M is Ricci-flat, and we define $\mathcal{W}_G$ to be the moduli space of torsion-free G-structures on M, $\mathcal{M}_G$ the moduli space of G-metrics, i.e. metrics induced by a torsion-free G-structure, and $\mathcal{M}_0$ the moduli space of Ricci-flat metrics on M. Here the moduli spaces are defined by quotienting by diffeomorphisms isotopic to the identity. We have the following result of J. Nordström extending similar results of Wang [**42**].

Theorem 1 ([**33**]). *Let M be a compact G-manifold with $G = SU(m), Sp(m), G_2$, or $Spin(7)$. Then $\mathcal{M}_G$ is open in $\mathcal{M}_0$, actually a union of connected components. Furthermore, $\mathcal{M}_G$ is a smooth manifold and the natural map*

$$m : \mathcal{W}_G \to \mathcal{M}_G$$

that sends a torsion-free G-structure to the metric it defines is a submersion.

This article will show that there is no analogous result for Killing spinors. Under Einstein deformations N_+, N_- are merely upper semi-continuous and can drop under infinitesimal and integrable Einstein deformations. In particular, the *toric* 3-Sasakian 7-manifolds of [**9**] have interesting infinitesimal Einstein deformations. Let $H^1(\mathcal{A}^\bullet)$ be the first cohomology of the complex (30), that is the first order deformations of the complex structure of the Reeb foliation $\mathscr{F}_\xi$. We know that $\dim_{\mathbb{C}} H^1(\mathcal{A}^\bullet) = b_2(M) - 1$ if (M, g) is a toric 3-Sasakian 7-manifold [**38**].

Theorem 2. *Let (M, g) be a 3-Sasakian 7-manifold with $\dim_{\mathbb{C}} H^1(\mathcal{A}^\bullet) > 0$, e.g. a toric 3-Sasakian 7-manifold with $b_2(M) \geq 2$. Thus (M, g) has three linearly independent Killing spinors. Then there exist infinitesimal Einstein deformations of g preserving two, one, and zero dimensional subspaces of the Killing spinors.*

It is unknown whether the infinitesimal Einstein deformations preserving only 1-dimensional subspaces of Killing spinors or none are integrable. But in Section 3 some infinitesimal Einstein deformations are proved to be integrable. For example the infinitesimal deformations of a toric 3-Sasakian 7-manifold in the theorem preserving a 2-dimensional subspace of Killing spinors can be shown to be integrable.

Theorem 3. *Let (M, g) be a* toric *3-Sasakian 7-manifold, so $N_+ = 3$. There exists an effective space $\mathcal{U} \subset \mathbb{C}^{b_2(M)-1}$ of Einstein deformations of $g = g_0$. For $t \in \mathcal{U}$ and $t \neq 0$, g_t is Sasaki–Einstein but not 3-Sasakian. Thus $g_t, t \neq 0$, admits only a two dimensional space of Killing spinors ($N_+ = 2, N_- = 0$).*

We also prove in Theorem 3.3 that certain infinitesimal Einstein deformations on a general 3-Sasakian manifold are integrable. In Section 4.1 we see that this has implications for the local premoduli space of Einstein metrics.

Corollary 4. *Suppose (M, g) is 3-Sasakian with $\dim_{\mathbb{C}} H^1(\mathcal{A}^\bullet) > 0$, e.g. a toric 3-Sasakian 7-manifold with $b_2(M) \geq 2$. Then either there exist Einstein deformations of g preserving no Killing spinors, or the Einstein premoduli space is singular.*

In Section 1 we review necessary background on the deformations of Einstein metrics, the variation of spin structures, and deformations of Killing spinors. In Section 2 we show that infinitesimal deformations of the transversal complex structure of a Sasaki–Einstein manifold give infinitesimal Einstein deformations. We then give the basic results on these deformations regarding the behavior of Killing spinors, on Sasaki–Einstein and 3-Sasakian manifolds. In Section 3 we give some results on when these infinitesimal Einstein deformations integrate to genuine Einstein deformations. In Section 4.1 we study the space of these infinitesimal Einstein deformations on a 3-Sasakian manifold more closely, and we prove Theorem 2, Theorem 3 and Corollary 4. In Section 4.2 the examples of toric 3-Sasakian 7-manifolds from [**9**] provide non-trivial examples of the above results.

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1. Preliminaries.

1.1. Spinors.

We review the explicit construction of the spin representations via explicit representations of the Clifford algebras $\mathrm{Cl}(n)$. For more details see [**23**] and [**3**]. These representation will give the complex representations of the complex Clifford algebras $\mathbb{C}\mathrm{l}(n) = \mathrm{Cl}(n) \otimes \mathbb{C}$. Suppose V is a real vector space of dimension $n = 2m$ with a metric g and compatible almost complex structure $I : V \to V$. We have the decomposition $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$, and the spinor space is

$$\mathbb{S}(V) := \Lambda^{*,0} V = \Lambda^* V^{1,0}.$$

The representation $c : \mathbb{C}\mathrm{l}(V) \to \mathrm{End}(\mathbb{S}(V))$ is defined by its action on $V \otimes \mathbb{C}$. For $v \in V^{1,0}$ define $c(v) := \sqrt{2} v \wedge \cdot$, and for $w \in V^{0,1}$ define $c(w) := -\sqrt{2} w \lrcorner \cdot$, where the contraction is induced by the metric g on V extended complex bilinearly.

Recall we have the splitting $\mathrm{Cl}(V) = \mathrm{Cl}_0(V) \oplus \mathrm{Cl}_1(V)$ into even and odd elements making $\mathrm{Cl}(V)$ into a *superalgebra*, that is

$$\mathrm{Cl}_r(V) \cdot \mathrm{Cl}_s(V) \subseteq \mathrm{Cl}_t(V) \text{ with } t = r + s \mod 2.$$

We have $\mathrm{Pin}(n) \subset \mathrm{Cl}(n)$, where $\mathrm{Pin}(n)$ is the universal cover of $O(n)$, and $Spin(n) \subset \mathrm{Cl}_0(n)$ is the universal cover of $SO(n)$.

The representation has a splitting preserved by the superalgebra structure of $\mathbb{C}\mathrm{l}(V)$

$$\mathbb{S}(V) = \mathbb{S}_{2m} = \mathbb{S}^+_{2m} \oplus \mathbb{S}^-_{2m}, \tag{2}$$

that is $\mathbb{C}\mathrm{l}_0(V) \cdot \mathbb{S}^{\pm}_{2m} \subseteq \mathbb{S}^{\pm}_{2m}$ while $\mathbb{C}\mathrm{l}_1(V) \cdot \mathbb{S}^{\pm}_{2m} \subseteq \mathbb{S}^{\mp}_{2m}$. The restriction of $\mathbb{S}(V)$ to $Spin(2m)$ is the *spin representation*, which splits into components in (2) which are irreducible.

As in [**41**], we define $\mathbb{S}^+_{2m}$ to be the half-spin representation with highest weight $(x_1 + \cdots + x_m)/2$, while $\mathbb{S}^-_{2m}$ has highest weight $(x_1 + \cdots + x_{m-1} - x_m)/2$, with the usual choice of fundamental weights. If $\{e_1, \ldots, e_{2m}\}$ is an orthonormal basis of V, then

$\mathbb{S}^{\pm}_{2m}$ are the $+1$ and -1 eigenspaces of $\omega_{\mathbb{C}} = \left(\sqrt{-1}\right)^{m^2+2m} e_1 \cdots e_{2m}$.

Remark 1.1. Note that this differs from the convention in [**23**], where $\mathbb{S}^{\pm}_{2m}$ are defined as the $+1$ and -1 eigenspaces of $\omega_{\mathbb{C}} = \left(\sqrt{-1}\right)^{m} e_1 \cdots e_{2m}$, by a factor of $(-1)^{m(m+1)/2}$.

Explicitly, we have

$$\begin{aligned} \mathbb{S}^{+}_{2m} &= \Lambda^{m,0}V \oplus \Lambda^{m-2,0}V \oplus \cdots, \\ \mathbb{S}^{-}_{2m} &= \Lambda^{m-1,0}V \oplus \Lambda^{m-3,0}V \oplus \cdots. \end{aligned} \tag{3}$$

For the odd dimensional case, $n = 2m+1$, let $\{e_1, \dots, e_{2m}\}$ be an orthonormal basis of V and define $V' = V \oplus \mathbb{R}e_{2m+1}$, with e_{2m+1} unit length and orthogonal to V. We define $c' : \mathrm{Cl}(V') \to \mathrm{End}(\mathbf{S}(V))$ as follows. If $v \in V$ we let $c'(v) := c(v) \in \mathrm{End}(\mathbf{S}(V))$ as above, and we define $c'(e_{2m+1}) := -(-1)^{(m+1)/2} c(e_1 \cdots e_{2m}) \in \mathrm{End}(\mathbf{S}(V))$. Note that $\mathbb{C}\mathrm{l}(V') = \mathbb{C}\mathrm{l}(2m+1)$ has two irreducible complex representations, each of dimension 2^m, and changing the sign of $c'(e_{2m+1})$ gives the other representation of $\mathbb{C}\mathrm{l}(V')$.

Alternatively, let $V = V_0 \oplus \mathbb{R}e_{2m}$ be an orthogonal sum. Then

$$\mathrm{Cl}(V_0) \overset{\gamma}{\cong} \mathrm{Cl}_0(V) \xrightarrow{c} \mathrm{End}(\mathbb{S}^{\pm}(V)), \tag{4}$$

where the isomorphism $\gamma : \mathrm{Cl}(V_0) \overset{\gamma}{\cong} \mathrm{Cl}_0(V)$ is given by $e_i \mapsto e_i \cdot e_{2m}$. The choice of half-spin representations $\mathbb{S}^{\pm}(V)$ gives the two representations of $\mathbb{C}\mathrm{l}(V_0)$ denoted by $\mathbb{S}^{\pm}_{2m-1}$. The restrictions of $\mathbb{S}^{\pm}_{2m-1}$ to $\mathbb{C}\mathrm{l}_0(V_0)$ are identical, thus restricting to $Spin(2m-1) \subset \mathbb{C}\mathrm{l}_0(V_0)$ gives the complex spin representation $\mathbb{S}_{2m-1}$, without a superscript.

Let (M, g) be an oriented Riemannian manifold with a spin structure. We have the principal bundle of orthonormal frames $L_{SO(n)}$ with the spin structure a $Spin(n)$ principal bundle $L_{Spin(n)}$ with 2-fold cover $\theta : L_{Spin(n)} \to L_{SO(n)}$, restricting to the 2-fold cover $Spin(n) \to SO(n)$ on each fiber. The *spin bundle* is $\Sigma = L_{Spin(n)} \times_{Spin(n)} \mathbb{S}_n$. If $n = 2m$ then $\Sigma = \Sigma^+ \oplus \Sigma^-$, where $\Sigma^{\pm} = L_{Spin(n)} \times_{Spin(n)} \mathbb{S}^{\pm}_n$. When n is odd there is a unique spinor bundle Σ, although there are two choices as a bundle of Clifford modules over $\mathbb{C}\mathrm{l}(TM)$.

Since Killing spinors correspond to a holonomy reduction we will make use of the decomposition of some restrictions of the spinor representation $\mathbb{S}_n$. Let μ_m be the usual representation of $SU(m) \subset SO(2m)$ on $\mathbb{C}^m$. Since $SU(m)$ is simply connected, $SU(m) \subset SO(2m)$ lifts to an embedding $SU(m) \subset Spin(2m)$ under $\theta : Spin(2m) \to SO(2m)$. We have from our conventions

$$\begin{aligned} \mathbb{S}^{+}_{2m}|_{SU(m)} &= \Lambda^{m}\mu_m \oplus \Lambda^{m-2}\mu_m \oplus \cdots \\ \mathbb{S}^{-}_{2m}|_{SU(m)} &= \Lambda^{m-1}\mu_m \oplus \Lambda^{m-3}\mu_m \oplus \cdots \end{aligned} \tag{5}$$

We will need to consider the spin representation restricted to $\mathfrak{sp}(m) \oplus \mathfrak{sp}(1) \subset SO(4m)$. Let ν_{2m} be the complex representation of $Sp(m)$ given by $Sp(m) \subset SU(2m)$. Contraction by the symplectic form gives $\Lambda^k \nu_{2m} = \Lambda_k \oplus \Lambda^{k-2}\nu_{2m}$, for $2 \geq k \geq m$, as $Sp(m)$-representations where Λ_k is the irreducible representation of $Sp(m)$ with highest weight $x_1 + \cdots + x_k$. It is an elementary result (see [**10**, Proposition 4.14]) that an irreducible representation of $Sp(m) \times Sp(1)$ is of the form $V\hat{\otimes}W$ where V and W are irreducible representations of $Sp(m)$ and $Sp(1)$ respectively. A little more work shows that

$$\begin{aligned} \mathbb{S}^{+}_{4m}|_{\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)} &= \Lambda_0\hat{\otimes}\gamma_m \oplus \Lambda_2\hat{\otimes}\gamma_{m-2} \oplus \cdots \\ \mathbb{S}^{-}_{4m}|_{\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)} &= \Lambda_1\hat{\otimes}\gamma_{m-1} \oplus \Lambda_3\hat{\otimes}\gamma_{m-3} \oplus \cdots \end{aligned} \tag{6}$$

where $\gamma_k = S^k(\mu_2)$ is the irreducible representation of $SU(2) = Sp(1)$ of dimension $k+1$. It follows from (6) that for m even the inclusion $Sp(m)\cdot Sp(1) = Sp(m)\times Sp(1)/\mathbb{Z}_2 \subset SO(4m)$ lifts under $\theta : Spin(4m) \to SO(4m)$ to $Sp(m)\times Sp(1)/\mathbb{Z}_2 \subset Spin(4m)$. While when m is odd $\theta^{-1}(Sp(m)\cdot Sp(1)) = Sp(m)\times Sp(1) \subset Spin(4m)$, which contains $(-I,-1) = -1 \in Spin(4m)$.

1.2. Deformation of Einstein metrics and Killing spinors.

1.2.1. Deformation of Einstein metrics.

We describe what we will need from the theory of deformations of Einstein metrics and deformations of Killing spinors. For more on the deformation theory of Einstein metrics see [**5**, Chapter 12] or [**20**]. See [**6**] for the apparatus for working with spinors under metric variations, and see [**42**] for this applied to the Killing spinor equation. In this article M denotes a compact connected n-dimensional manifold.

DEFINITION 1.2. Let g be an Einstein metric on M. A family g_t of Einstein metrics on M of fixed volume with $g_0 = g$ depending smoothly on $t \in U \subset \mathbb{R}^k$ is an *Einstein deformation* of g.

Because Einstein metrics are critical points of the total scalar curvature functional $g \mapsto \int_M s_g\,\mu_g$ restricted to metrics of a fixed volume, a deformation of Einstein metrics has fixed scalar curvature $s = s_{g_t}$. Thus

$$\mathrm{Ric}_{g_t} = \lambda g_t, \tag{7}$$

where $\lambda = s/n$. We will consider positive scalar curvature Einstein metrics, and it will be convenient for us to assume $\lambda = n-1$.

Let $\mathscr{M}_c$ be the space of Riemannian metrics on M of fixed volume c. This is acted upon by the diffeomorphism group $\mathscr{D}$. A local description of the quotient $\mathscr{M}_c/\mathscr{D}$ is given by Ebin's Slice Theorem [**12**]. The tangent space to $\mathscr{M}_c$ at g denoted by $T_g\mathscr{M}_c$ consists of symmetric 2 tensors $h \in \Gamma\big(\mathrm{S}^2\,T^*M\big)$ with $\int_M \mathrm{tr}\,h\,\mu_g = 0$. The tangent space to the orbit $\mathscr{D}^*g$ consists of all Lie derivatives $\mathcal{L}_X g = 2\delta_g^* X^\flat$, where $X^\flat$ is the 1-form dual to the vector field X and

$$(\delta_g^* X^\flat)_{ij} = \frac{1}{2}\big(\nabla_i X^\flat_j + \nabla_j X^\flat_i\big), \tag{8}$$

with ∇ the Levi-Civita connection. One can show that $\mathrm{Im}\,\delta_g^* \subset T_g\mathscr{M}_c$ is closed, and

$$T_g\mathscr{M}_c = \mathrm{Im}\,\delta_g^* \oplus \big(T_g\mathscr{M}_c \cap \ker\delta\big), \tag{9}$$

where $(\delta_g h)_i = -\nabla^j h_{ji}$ is adjoint to δ_g^*.

Let $h = dg_t/dt|_{t=0}$, then differentiating (7) gives

$$2E'_g(h) = \big(\overline{\Delta} + 2L - \delta_g^*\delta_g - \nabla d\,\mathrm{tr}_g\big)h = 0, \tag{10}$$

where $(Lh)_{ij} = R_i{}^k{}_j{}^l h_{kl}$ and $\overline{\Delta} = \nabla^*\nabla$ is the rough Laplacian.

Definition 1.3. Let (M,g) be an Einstein manifold. A symmetric 2-tensor $h \in \Gamma\big(\mathrm{S}^2 T^*M\big)$ is an *infinitesimal Einstein deformation* of g if h satisfies (10) and $\int_M \mathrm{tr}_g h\,\mu_g = 0$. The space of infinitesimal Einstein deformations is denoted by $\mathcal{ED}(g)$.

An infinitesimal Einstein deformations of the form $\mathcal{L}_X g$ is said to be *trivial*. The space of trivial infinitesimal Einstein deformations is denoted by $\mathcal{TED}(g)$. An infinitesimal Einstein deformation h is said to be *essential* if it is orthogonal to $\mathcal{TED}(g)$. The space of essential infinitesimal Einstein deformations is denoted by $\mathcal{EED}(g)$. We can use the following lemma due to Berger and Ebin as the definition of $\mathcal{EED}(g)$.

Lemma 1.4 ([**4**]). *Let (M,g) be an Einstein manifold. An $h \in \Gamma\big(\mathrm{S}^2 T^*M\big)$ is an element of $\mathcal{EED}(g)$ if and only if h satisfies*

$$\big(\overline{\Delta} + 2L\big)h = 0, \quad \delta_g h = 0, \quad \mathrm{tr}_g h = 0. \tag{11}$$

We have the decomposition of closed spaces

$$\mathcal{ED}(g) = \mathcal{EED}(g) \oplus \mathcal{TED}(g), \tag{12}$$

with $\mathcal{EED}(g)$ finite dimensional.

Definition 1.5. Let (M,g) be an Einstein manifold. The subset of Einstein metrics in the Ebin slice $\mathscr{S}_g$ (cf. [**12**]) at g is called the *local premoduli space of Einstein structures* and denoted by $\mathscr{PM}(g)$.

The local moduli space is $\mathscr{PM}(g)/\operatorname{Isom}(g)$, but it will be more convenient to work with the local premoduli space.

1.2.2. Deformation of spinors.

We will need the machinery due to Bourguignon and Gauduchon [**6**] for describing variations of spinor bundles and spinors under metric variations and applied by Wang [**42**] to study Killing spinor variations.

Let $P = L_{SO(n)}$ be the bundle of oriented orthonormal frames on (M,g). A spin structure is a double cover $\tilde{P}$. Given a symmetric, with respect to g, automorphism $\alpha : TM \to TM$ we have a new metric

$$g^\alpha(X,Y) = g(\alpha^{-1}X, \alpha^{-1}Y).$$

If P^α is the bundle of g^α-orthonormal oriented frames, $\alpha : P \to P^\alpha$ is $SO(n)$-equivariant, and gives an isomorphism

$$\Sigma = \tilde{P} \times_{Spin(n)} \mathbb{S}_n \xrightarrow{\tilde{\alpha}} \Sigma^\alpha = \tilde{P}^\alpha \times_{Spin(n)} \mathbb{S}_n.$$

Let $\alpha(t)$ be a smooth path of symmetric automorphisms with $\alpha(0) = \mathbb{1}_{TM}$, and $\hat{\sigma}_t$ Killing spinors for $g^{\alpha(t)}$,

$$\nabla_X^{\alpha(t)} \hat{\sigma}_t = cX \cdot_t \hat{\sigma}_t.$$

Set $\sigma_t = \tilde{\alpha}(t)^{-1}(\hat{\sigma}_t)$, then in terms of the original spin bundle

$$\overline{\nabla}_X^{\alpha(t)} \sigma_t = c\alpha(t)^{-1}(X) \cdot \sigma_t, \tag{13}$$

where $\overline{\nabla}_X^{\alpha(t)} = \tilde{\alpha}(t)^{-1} \circ \nabla_X^{\alpha(t)} \circ \tilde{\alpha}(t)$.

A *deformation of the Killing spinor* σ_0 is a path $(\alpha(t), \sigma_t)$ satisfying

$$\mathcal{L}^c(\alpha(t), \sigma_t)(X) := \overline{\nabla}_X^{\alpha(t)} \sigma_t - c\alpha(t)^{-1}(X) \cdot \sigma_t = 0. \tag{14}$$

We will make use of the twisted Dirac operator

$$\mathcal{D} : \Gamma(TM_{\mathbb{C}}^* \otimes \Sigma) \to \Gamma(TM_{\mathbb{C}}^* \otimes \Sigma). \tag{15}$$

Decomposing into irreducible representations of *Spin*(n)

$$TM_{\mathbb{C}}^* \otimes \Sigma = \Sigma \oplus \Sigma_{3/2},$$

where $\Sigma_{3/2}$ is the bundle coming from the kernel of Clifford multiplication $p : T \otimes \mathbb{S}_n \to \mathbb{S}_n$. The component of $\mathcal{D}$ on $\Sigma_{3/2}$ is the *Rarita–Schwinger operator*

$$\mathcal{Q} : \Gamma(\Sigma_{3/2}) \to \Gamma(\Sigma_{3/2}). \tag{16}$$

If $\Psi \in \Gamma(\Sigma_{3/2})$ then $\mathcal{D}\Psi = \mathcal{Q}\Psi$ if and only if $\delta_g \Psi = 0$.

We define tensors $\Psi^{(\beta,\sigma_0)}, \Theta^{(\beta,\sigma_0)} \in \Gamma(T^*M_{\mathbb{C}} \otimes \Sigma)$ for $\beta : TM \to TM$ and $\sigma_0 \in \Gamma(\Sigma)$:

$$\Psi^{(\beta,\sigma_0)}(X) = \beta(X) \cdot \sigma_0 \tag{17}$$

$$\Theta^{(\beta,\sigma_0)}(X) = \sum_i e_i(\nabla_i \beta)(X) \cdot \sigma_0, \tag{18}$$

where $X \in TM$ and $\{e_i\}$ is a local orthonormal frame. If β is symmetric, $\operatorname{tr}_g \beta = 0$, and $\delta_g \beta = 0$ then $\Psi^{(\beta,\sigma_0)}, \Theta^{(\beta,\sigma_0)} \in \Gamma(\Sigma_{3/2})$. And if σ_0 is a Killing spinor, then $\delta_g \Psi^{(\beta,\sigma_0)} = \delta_g \Theta^{(\beta,\sigma_0)} = 0$.

Differentiating (14) at $(\mathbb{1}_{TM}, \sigma_0)$:

Proposition 1.6 ([**42**]).

$$d\mathcal{L}^c(\dot{\alpha}, \dot{\sigma})(X) = \nabla \dot{\sigma}_X - cX\dot{\sigma} + c\dot{\alpha}(X)\sigma_0 - \frac{1}{2}\sum_i e_i(\nabla_i \dot{\alpha})(X)\sigma_0 + \frac{1}{2} g(\delta\dot{\alpha}, X)\sigma_0.$$

If $\operatorname{tr}_g(\dot{\alpha}) = \delta\dot{\alpha} = 0$*, then* $d\mathcal{L}^c(\dot{\alpha}, \dot{\sigma}) = 0$ *if and only if* $\nabla_X \dot{\sigma} = cX\dot{\sigma}$ *and* $\mathcal{D}\Psi^{(\dot{\alpha},\sigma_0)} = nc\Psi^{(\dot{\alpha},\sigma_0)}$.

For $\beta : TM \to TM$ g-symmetric, define $h(X, Y) = -2g(\beta(X), Y)$.

Proposition 1.7 ([**42**]). *If* $\operatorname{tr}_g \beta = \delta\beta = 0$ *and* $\mathcal{D}\Psi^{(\beta,\sigma_0)} = cn\Psi^{(\beta,\sigma_0)}$*, then* $\big(\overline{\Delta} + 2L\big)h = 0$ *where* $(Lh)_{ij} = R_i{}^k{}_j{}^l h_{kl}$.

So $h \in \Gamma\big(\mathrm{S}^2 T^*M\big)$ is an infinitesimal Einstein deformation.

Definition 1.8. An *infinitesimal deformation of the Killing spinor* σ_0 is a pair (β, σ), $\beta : TM \to TM$ symmetric and $\sigma \in \Gamma(\Sigma)$, satisfying:

(i) σ is a Killing spinor with constant c,

(ii) $\operatorname{tr}_g \beta = \delta\beta = 0$,

(iii) $\mathcal{D}\Psi^{(\beta,\sigma_0)} = nc\Psi^{(\beta,\sigma_0)}$.

The following result will have applications for the existence of eigenvectors of $\mathcal{Q}$.

Proposition 1.9 ([**42**]). *Let* (M, g) *be spin with nonzero Killing spinor* σ_0*. Let* $h \in \mathcal{EED}(g)$*, and define* $\beta : TM \to TM$ *by* $h(X, Y) = -2g(\beta(X), Y)$*. Then we have an eigenvector of* $\mathcal{Q}$ *of either eigenvalue* cn *or* $c(2-n)$*, that is*

(i) $\mathcal{D}\Psi^{(\beta,\sigma_0)} = nc\Psi^{(\beta,\sigma_0)}$ *and* β *is an infinitesimal deformation of* σ_0*, or*

(ii) $\Theta^{(\beta,\sigma_0)} - 2c\Psi^{(\beta,\sigma_0)} \neq 0$ *and*

$$\mathcal{D}\big(\Theta^{(\beta,\sigma_0)} - 2c\Psi^{(\beta,\sigma_0)}\big) = c(2-n)\big(\Theta^{(\beta,\sigma_0)} - 2c\Psi^{(\beta,\sigma_0)}\big).$$

Let (M, g) be Einstein, then the Einstein premoduli space $\mathscr{PM}(g) \subseteq Z$, where Z is a finite dimensional real analytic submanifold of the slice $\mathscr{S}_g$ [**21**]. The bundles $\Sigma_{g'}$ and Equation (1) depend real analytically on $g' \in Z$. Define $\mathcal{N}^+_{g'}$ (resp. $\mathcal{N}^-_{g'}$) to be space of solutions of (1) for $g' \in Z$ and $c = 1/2$ (resp. $c = -1/2$). Since (1) has injective symbol $\dim_{\mathbb{C}} \mathcal{N}^{\pm}_{g'}$ is upper semi-continuous. See for example [**20**, Lemma 4.3]. We will see by example that it is not locally constant as in the case of parallel spinors.

1.3. Sasakian manifolds.

1.3.1. Sasakian structures.

The Killing spinor deformations we consider are of the non-exceptional cases of Sasakian and 3-Sasakian manifolds in Table 1. See [**7**] or the monograph [**8**] for more on Sasakian geometry.

Definition 1.10. A Riemannian manifold (M, g) is Sasakian if the metric cone $(C(M), \bar{g})$, $C(M) := \mathbb{R}_+ \times M$ and $\bar{g} = dr^2 + r^2 g$, is Kähler, that is $\bar{g}$ admits a compatible almost complex structure J so that $(C(M), \bar{g}, J)$ is a Kähler structure. Equivalently, $\operatorname{Hol}(C(M), \bar{g}) \subseteq U(m)$, where $\dim M = n = 2m - 1$.

It is convenient to identify M with $\{r = 1\} = \{1\} \times M \subset C(M)$. A Sasaki structure is a special type of metric contact structure. Traditionally the Sasakian structure on M was defined as a metric contact structure (g, η, ξ, Φ) satisfying an additional condition called *normality*, which is an integrability condition, where η is a contact form with Reeb vector field ξ and Φ is a $(1, 1)$ tensor. Here ξ and η are restrictions to M of

$$\xi = Jr\partial r, \quad \eta(X) = \frac{1}{r^2}\xi \lrcorner \bar{g}, \tag{19}$$

on $C(M)$, which are given the same notation. It follows from the latter formula that

$$\eta = d^c \log r, \tag{20}$$

where $d^c = \sqrt{-1}(\bar{\partial} - \partial)$. One can show from the warped product structure of $(C(M), \bar{g})$ that ξ is Killing and real holomorphic. If ω is the Kähler form of $\bar{g}$, then

$$\omega = \frac{1}{2}d(r^2\eta) = \frac{1}{4}dd^c r^2.$$

We also have

$$\omega = \frac{1}{2}d(r^2\eta) = rdr \wedge \eta + \frac{1}{2}r^2 d\eta. \tag{21}$$

Let $D \subset TM$ be the contact distribution which is defined by

$$D_x = \ker \eta_x \tag{22}$$

for $x \in M$. There is a splitting of the tangent bundle TM

$$TM = D \oplus L_\xi, \tag{23}$$

where L_ξ is the trivial subbundle generated by ξ. The tensor $\Phi \in \mathrm{End}(TM)$ is defined by $\Phi|_D = J$ and $\Phi(\xi) = 0$. Since ξ is Killing one can show that $\Phi = \nabla\xi$. We denote the Sasakian structure by (g, η, ξ, Φ).

The vector field $\xi + \sqrt{-1}r\partial_r$ is holomorphic on $C(M)$, thus it defines a holomorphic action of $\tilde{\mathbb{C}}^*$, the universal cover of $\mathbb{C}^*$. The intersection of each orbit with $M \subset C(M)$ is an orbit of the action of ξ on M. Thus the orbits define a transversely holomorphic foliation $\mathscr{F}_\xi$ on M called the *Reeb foliation*. If ξ generates a free $U(1)$-action, then the Sasakian structure is *regular*. The Sasakian structure is *quasi-regular* if it generates a locally free $U(1)$-action, and *irregular* if not all the orbits are compact.

The foliation $\mathscr{F}_\xi$ together with its transverse holomorphic structure is given by an open covering $\{U_\alpha\}_{\alpha\in A}$ and submersions $\pi_\alpha : U_\alpha \to W_\alpha \subset \mathbb{C}^{m-1}$ such that when $U_\alpha \cap U_\beta \neq \emptyset$ the map

$$\phi_{\beta\alpha} = \pi_\beta \circ \pi_\alpha^{-1} : \pi_\alpha(U_\alpha \cap U_\beta) \to \pi_\beta(U_\alpha \cap U_\beta)$$

is a biholomorphism.

Note that on U_α the differential $d\pi_\alpha : D_x \to T_{\pi_\alpha(x)}W_\alpha$ at $x \in U_\alpha$ is an isomorphism taking the almost complex structure J_x to that on $T_{\pi_\alpha(x)}W_\alpha$. Since $\xi \lrcorner d\eta = 0$ the 2-form $(1/2)d\eta$ descends to a form ω_α^T on W_α. Similarly, $g^T = (1/2)d\eta(\cdot, \Phi\cdot)$ satisfies $\mathcal{L}_\xi g^T = 0$ and vanishes on vectors tangent to the leaves, so it descends to an Hermitian metric g_α^T on W_α with Kähler form ω_α^T. The Kähler metrics $\{g_\alpha^T\}$ and Kähler forms $\{\omega_\alpha^T\}$ on $\{W_\alpha\}$ by construction are isomorphic on the overlaps

$$\phi_{\beta\alpha} : \pi_\alpha(U_\alpha \cap U_\beta) \to \pi_\beta(U_\alpha \cap U_\beta).$$

We will use g^T, respectively ω^T, to denote both the Kähler metric, respectively Kähler form, on the local charts and the globally defined pull-back on M.

If we define $\nu(\mathscr{F}_\xi) = TM/L_\xi$ to be the normal bundle to the leaves, then we can generalize the above concept.

Definition 1.11. A tensor $\Psi \in \Gamma\big((\nu(\mathscr{F}_\xi)^*)^{\otimes p} \bigotimes \nu(\mathscr{F}_\xi)^{\otimes q}\big)$ is *basic* if $\mathcal{L}_V \Psi = 0$ for any vector field $V \in \Gamma(L_\xi)$.

Note that it is sufficient to check the above property for $V = \xi$. Then g^T and ω^T are such tensors on $\nu(\mathscr{F}_\xi)$. We will also make use of the bundle isomorphism $\pi : D \to \nu(\mathscr{F}_\xi)$, which induces an almost complex structure $\overline{J}$ on $\nu(\mathscr{F}_\xi)$ so that $(D, J) \cong (\nu(\mathscr{F}_\xi), \overline{J})$ as complex vector bundles. Clearly, $\overline{J}$ is basic and is mapped to the natural almost complex structure on W_α by the local chart $d\pi_\alpha : D_x \to T_{\pi_\alpha(x)} W_\alpha$.

To work on the Kähler leaf space we define the Levi-Civita connection of g^T by

$$\nabla^T_X Y = \begin{cases} \pi_\xi(\nabla_X Y) & \text{if } X, Y \text{ are smooth sections of } D, \\ \pi_\xi([V, Y]) & \text{if } X = V \text{ is a smooth section of } L_\xi, \end{cases} \tag{24}$$

where $\pi_\xi : TM \to D$ is the orthogonal projection onto D. Then ∇^T is the unique torsion free connection on $D \cong \nu(\mathscr{F}_\xi)$ so that $\nabla^T g^T = 0$. Then for $X, Y \in \Gamma(TM)$ and $Z \in \Gamma(D)$ we have the curvature of the transverse Kähler structure

$$R^T(X, Y)Z = \nabla^T_X \nabla^T_Y Z - \nabla^T_Y \nabla^T_X Z - \nabla^T_{[X,Y]} Z, \tag{25}$$

and similarly we have the transverse Ricci curvature Ric^T and scalar curvature s^T. We will denote the transverse Ricci form by ρ^T. From O'Neill's tensors computation for Riemannian submersions [**34**] and elementary properties of Sasakian structures we have the following.

Proposition 1.12. *Let (M, g, η, ξ, Φ) be a Sasakian manifold of dimension $n = 2m - 1$, then*

(i) $\mathrm{Ric}_g(X, \xi) = (2m - 2)\eta(X)$, *for* $X \in \Gamma(TM)$,
(ii) $\mathrm{Ric}^T(X, Y) = \mathrm{Ric}_g(X, Y) + 2g^T(X, Y)$, *for* $X, Y \in \Gamma(D)$.

In particular, if (M, g, η, ξ, Φ) is *Sasaki–Einstein*, then by 1.12 i it has Einstein constant $n - 1$, that is

$$\mathrm{Ric}_g = (n - 1)g. \tag{26}$$

Note that (26) is equivalent to $(C(M), \bar{g})$ being Ricci-flat, since

$$\mathrm{Ric}_{\bar{g}} = \mathrm{Ric}_g - (n - 1)g.$$

1.3.2. 3-Sasakian structures.

Recall that a hyperkähler structure on a $4m$-dimensional manifold consists of a metric g which is Kähler with respect to three complex structures J_1, J_2, J_3 satisfying the quaternionic relations $J_1 J_2 = -J_2 J_1 = J_3$ etc.

Definition 1.13. A Riemannian manifold (M, g) is 3-Sasakian if the metric cone $(C(M), \bar{g})$ is hyperkähler, that is $\bar{g}$ admits compatible almost complex structures $J_\alpha,\ \alpha = 1,2,3$ such that $(C(M), \bar{g}, J_1, J_2, J_3)$ is a hyperkähler structure. Equivalently, $\mathrm{Hol}(C(M)) \subseteq Sp(m)$.

A consequence of the definition is that (M, g) is equipped with three Sasakian structures $(g, \eta_i, \xi_i, \Phi_i),\ i = 1,2,3$. The Reeb vector fields $\xi_i = J_i(r\partial_r),\ i = 1,2,3$ are orthogonal and satisfy $[\xi_i, \xi_j] = -2\varepsilon^{ijk}\xi_k$, where ε^{ijk} is anti-symmetric in the indices $i, j, k \in \{1,2,3\}$ and $\varepsilon^{123} = 1$. The tensors $\Phi_i,\ i = 1,2,3$ satisfy the identities

$$\Phi_i(\xi_j) = \varepsilon^{ijk}\xi_k \tag{27}$$

$$\Phi_i \circ \Phi_j = -\delta_{ij}\mathbb{1} + \varepsilon^{ijk}\Phi_k + \eta_j \otimes \xi_i. \tag{28}$$

It is easy to see that there is an S^2 of Sasakian structures with Reeb vector field $\xi_\tau = \tau_1\xi_1 + \tau_2\xi_2 + \tau_3\xi_3$ with $\tau \in \mathrm{S}^2$.

The Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ generate a Lie algebra $\mathfrak{sp}(1)$, so there is an effective isometric action of either $SO(3)$ or $Sp(1)$ on (M, g). Both cases occur in the examples in this article. This action generates a foliation $\mathscr{F}_{\xi_1,\xi_2,\xi_3}$ with generic leaves either $SO(3)$ or $Sp(1)$.

If we set $D_i = \ker \eta_i \subset TM,\ i = 1,2,3$ to be the contact subbundles, then the complex structures $J_i,\ i = 1,2,3$ are recovered by

$$J_i(r\partial_r) = \xi_i, \quad J_i|_{D_i} = \Phi_i. \tag{29}$$

Because a hyperkähler manifold is always Ricci-flat we have the following.

Proposition 1.14. *A 3-Sasakian manifold (M, g) of dimension $4m-1$ is Einstein with Einstein constant $\lambda = 4m-2$.*

We choose a Reeb vector field ξ_1, fixing a quasi-regular Sasakian structure, then the leaf space $\mathscr{F}_{\xi_1}$ is a Kähler orbifold Z with respect to the transversal complex structure $\overline{J} = \Phi_1$. But it has in addition a complex contact structure and a fibering by rational curves which we now describe. The 1-form $\eta^c = \eta_2 + \sqrt{-1}\eta_3$ is a $(1,0)$-form with respect to $\overline{J}$. But it is not invariant under the $U(1)$ group generated by $\exp(t\xi_1)$. We have $\exp(t\xi_1)^*\eta^c = e^{2\sqrt{-1}\,t}\eta^c$. Let $\boldsymbol{L} = M \times_{U(1)} \mathbb{C}$, with $U(1)$ acting on $\mathbb{C}$ by $e^{-2\sqrt{-1}\,t}$. This is a holomorphic orbifold line bundle; in fact $C(M)$ is either $\boldsymbol{L}^{-1}$ or $\boldsymbol{L}^{-1/2}$ minus the zero section. It is easy to see that each of these cases occur precisely where the Reeb vector fields generate an effective action of $SO(3)$ and $Sp(1)$ respectively. Then η^c descends to an $\boldsymbol{L}$ valued holomorphic 1-form $\theta \in \Gamma\big(\Omega^{1,0}(\boldsymbol{L})\big)$. It follows easily from (28) that $d\eta^c$ restricted to $D_1 \cap \ker \eta^c$ is a non-degenerate type $(2,0)$ form. Thus θ is a complex contact form on Z, and $\theta \wedge (d\theta)^{m-1} \in \Gamma\big(\boldsymbol{K}_Z \otimes \boldsymbol{L}^m\big)$ is a non-vanishing section. Thus $\boldsymbol{L} \cong \boldsymbol{K}_Z^{-1/m}$.

Each leaf of $\mathscr{F}_{\xi_1,\xi_2,\xi_3}$ descends to a rational curve in Z. Each curve is a $\mathbb{CP}^1$ but may have orbifold singularities for non-generic leaves. It is also well-known that restricted to a leaf $\boldsymbol{L}|_{\mathbb{CP}^1} = \mathcal{O}(2)$, the degree 2 line bundle on a generic smooth leaf, while $\mathcal{O}(2)$ is interpreted as an orbifold line bundle when the leaf has orbifold singularities. The element $\exp((\pi/2)\xi_2)$ acts on M taking ξ_1 to $-\xi_1$, thus it descends to an anti-holomorphic involution $\varsigma : Z \to Z$. This *real structure* is crucial to the twistor approach. Note that $\varsigma^*\theta = \bar{\theta}$. This all depends on the choice $\xi_1 \in \mathrm{S}^2$ of the Reeb vector field. But taking a different Reeb vector field gives an isomorphic twistor space under the transitive action of $Sp(1)$.

2. Killing spinor deformations on Sasaki–Einstein manifolds.

2.1. Deformations of transversal complex structures.

Let (M, g, η, ξ, Φ) be a Sasakian manifold. Then the Reeb foliation $(\mathscr{F}_\xi, \bar{J})$ has a transversely holomorphic structure. The existence of a versal deformation space for $(\mathscr{F}_\xi, \bar{J})$, fixing the smooth structure of $\mathscr{F}$, was proved in [**14**] and [**18**] using arguments similar to those in [**22**].

Let $\mathcal{A}^k = \Gamma(\Lambda_b^{0,k} \otimes \nu(\mathscr{F})^{1,0})$ be the space of smooth basic forms of type $(0,k)$ with values in $\nu(\mathscr{F})^{1,0}$. We have the Dolbeault complex

$$0 \to \mathcal{A}^0 \xrightarrow{\bar{\partial}_b} \mathcal{A}^1 \xrightarrow{\bar{\partial}_b} \mathcal{A}^2 \to \cdots . \tag{30}$$

Here (30) is the basic version of the complex used by Kuranishi [**22**] whose degree one cohomology is the space of first order deformation of the complex structure modulo diffeomorphisms. Likewise, the first order deformations of $(\mathscr{F}_\xi, \bar{J})$ modulo foliate diffeomorphisms are given by $H^1(\mathcal{A}^\bullet)$. As in [**22**] there is an open set $\mathcal{U} \subset H^1(\mathcal{A}^\bullet)$ and the versal deformation space $\mathcal{V} \subset \mathcal{U}$ is the germ of $\theta^{-1}(0)$ where θ is an analytic map

$$H^1(\mathcal{A}^\bullet) \xrightarrow{\theta} H^2(\mathcal{A}^\bullet).$$

Proposition 2.1. *Suppose (M, g, η, ξ, Φ) is Sasaki–Einstein (just $\mathrm{Ric}^T > 0$ is sufficient). We have $H^2(\mathcal{A}^\bullet) = \{0\}$, so the versal deformation space is smooth, $\mathcal{U} \subset H^1(\mathcal{A}^\bullet)$.*

Proof. The basic version of Serre duality gives

$$H^2(\mathcal{A}^\bullet) = H^{m-3}_{\bar{\partial}_b}(\Gamma(\Lambda_b^{1,\bullet} \otimes \Lambda_b^{m-1,0})) = 0,$$

where the second equality is given by Kodaira–Nakano vanishing, since $\Lambda_b^{m-1,0} < 0$ and $(m-3)+1 = m-2 < m-1$. The proof of Kodaira–Nakano vanishing in [**19**] goes through in transversally Kähler case using the transversal harmonic theory of [**15**]. □

Since $\mathrm{Ric}^T > 0$, the obstruction to lifting a deformation $\bar{J}_t,\ t \in \mathcal{U}$, to a deformation of Sasakian structures vanish.

Proposition 2.2. *Let (M, g, η, ξ, Φ) be Sasaki–Einstein (or just $\mathrm{Ric}^T > 0$ is sufficient), then after possibly shrinking $\mathcal{U}$, the deformation $\bar{J}_t,\ t \in \mathcal{U}$, lifts to a smooth family $(g_t, \eta_t, \xi, \Phi_t),\ t \in \mathcal{U}$, where Φ_t induces the transversal complex structure $\bar{J}_t$.*

Proof. We first show that the basic Dolbeault cohomology $H^{0,k}_b = H^k_{\bar\partial_b}(\Lambda^{0,\bullet}_b) = \{0\}$. This can be proved using Kodaira vanishing as above or from the Weitzenböck formula on $\psi \in \Omega^{0,k}_b$

$$2\Delta_{\bar\partial_b}\psi_{\bar\alpha_1\ldots\bar\alpha_k} = \overline{\Delta}^T\psi_{\bar\alpha_1\ldots\bar\alpha_k} + \sum_{j=1}^{k}(g^T)^{\beta\bar\gamma}\,\mathrm{Ric}^T_{\bar\alpha_j\beta}\,\psi_{\bar\alpha_1\ldots\bar\alpha_{j-1}\bar\gamma\bar\alpha_{j+1}\ldots\bar\alpha_k}, \tag{31}$$

where $\overline{\Delta}^T = (\nabla^T)^*\nabla^T$ is the transversal rough Laplacian. Then if ψ is harmonic and $\mathrm{Ric}^T \geq \lambda g^T$ then integrating (31) gives

$$0 \geq \int_M \big(\langle\nabla^T\psi, \nabla^T\psi\rangle + k\lambda\langle\psi,\psi\rangle\big)\,\mu_g,$$

where $\langle\cdot,\cdot\rangle$ is the Hermitian product and $\mu_g = (1/(m-1)!)\eta\wedge((1/2)d\eta)^{m-1}$. Therefore $\psi = 0$.

By [**13**] there is a family of transversal Kähler metrics with Kähler forms ω^T_t on $(\mathscr{F}_\xi, \overline{J}_t)$ depending smoothly on $t\in\mathcal{U}$ with $\omega^T_0 = \omega^T$. The above argument shows that after shrinking $\mathcal{U}$ the Dolbeault groups on $(\mathscr{F}_\xi, \overline{J}_t)$ also satisfy $H^{0,k}_{b,t} = \{0\}$. Since the harmonic space $\mathcal{H}^2_{\Delta_{\bar\partial_{b,t}}}$, of the transverse Laplacian $\Delta_{\bar\partial_{b,t}}$ with respect to ω^T_t, has constant dimension, by for example [**20**, Lemma 4.3] there are isomorphisms $R_t : \mathcal{H}^2_{\Delta_{\bar\partial_b}} \to \mathcal{H}^2_{\Delta_{\bar\partial_{b,t}}}$ depending smoothly on t. There exists smoothly varying $\alpha_t \in \mathcal{H}^2_{\Delta_{\bar\partial_b}}$ so that $R_t(\alpha_t) = [\omega^T - \omega^T_t]_h$, the harmonic component. Let G be the Green's operator for $\Delta_{\bar\partial_b}$. Let $\beta_t = d^*G(\omega^T_t + R_t(\alpha_t) - \omega^T)$, and define $\eta_t = \eta + \beta_t$. Then $(1/2)d\eta_t = \omega^T_t + R_t(\alpha_t)$ which is of type $(1,1)$ and is positive definite for small enough t.

The family of Sasakian structures $(g_t, \eta_t, \xi, \Phi_t)$ is defined by lifting $\overline{J}_t$ to $\ker\eta_t$ to get Φ_t, while

$$g_t = \frac{1}{2}d\eta_t(\cdot, \Phi_t\cdot) + \eta_t\otimes\eta_t. \tag{32}$$

□

Remark 2.3. With the assumption $c_1(\mathscr{F}_\xi, \overline{J}_t) > 0$ made in this article, the deformations in Proposition 2.2 along with transversal Kähler deformations

$$\tilde\eta = \eta + d^c\varphi, \quad \tilde\Phi = \Phi - \xi\otimes\tilde\eta\circ\Phi,$$

for $\varphi \in C^\infty_b(M)$ basic, give all local deformations of the Sasakian structure fixing the Reeb vector field. See [**40**] for details.

Since a Sasaki–Einstein structure is transversally Kähler–Einstein by Proposition 1.12.ii, a necessary condition for a compatible Sasaki–Einstein structure is that

$$\pi c_1(\mathscr{F}_\xi, \overline{J}) = m\omega^T. \tag{33}$$

It follows from the proof of Proposition 2.2 that if (33) holds for (M, g, η, ξ, Φ), then the family $(g_t, \eta_t, \xi, \Phi_t)$, $t\in\mathcal{U}$, also satisfies

$$\pi c_1(\mathscr{F}_\xi, \overline{J}_t) = m\omega_t^T.$$

We consider some properties of a first order deformation through Sasakian metrics which will be used later. We differentiate (32) and use the notation

$$\dot{\overline{J}}_t = I, \quad \dot{\omega}^T = \phi, \text{ and } \dot{g}^T = h$$

where we have

$$d\dot{\eta} = 2\phi. \tag{34}$$

Since $\omega_t^T(X,Y) = g_t^T(\overline{J}_t X, Y)$, we have

$$\phi_{\alpha\beta} = \sqrt{-1}h_{\alpha\beta} + I_{\alpha\beta} \tag{35}$$

$$\phi_{\alpha\overline{\beta}} = \sqrt{-1}h_{\alpha\overline{\beta}}. \tag{36}$$

Note that since I anti-commutes with $\overline{J}_0$, it only has components $I_\alpha{}^{\overline{\beta}}$ and $I_{\overline{\alpha}}{}^{\beta}$.

In addition differentiating

$$g_t^T(\overline{J}_t X, Y) + g_t^T(X, \overline{J}_t Y) = 0 \tag{37}$$

gives

$$2\sqrt{-1}h_{\alpha\beta} + (I_{\alpha\beta} + I_{\beta\alpha}) = 0. \tag{38}$$

Finally (35) and (38) give

$$\phi_{\alpha\beta} = \frac{1}{2}(I_{\alpha\beta} - I_{\beta\alpha}). \tag{39}$$

2.2. Skew–Hermitian Einstein deformations.

By Proposition 1.12.ii if (M, g, η, ξ, Φ) is Sasaki–Einstein then the transversal Kähler metric g^T on $\mathscr{F}_\xi$ is Einstein

$$\mathrm{Ric}_{g^T} = 2mg^T.$$

We define the space $\mathcal{EED}(g^T)$ just as in Section 1.2.1 using the transversal Levi-Civita connection defined in (24), that is

$$\mathcal{EED}(g^T) = \{h \in \Gamma\big(\mathrm{S}^2\, T_b^* M\big) \mid \mathrm{tr}_{g^T} h = \delta_{g^T} h = 0,\ \big(\overline{\Delta}^T + 2L^T\big)h = 0\},$$

where L^T is defined as in (10) but with the transverse curvature R^T.

Given $h \in \Gamma\big(\mathrm{S}^2\, T_b^* M\big)$ we decompose h into its Hermitian h_H and anti-Hermitian h_A parts with respect to the transversal complex structure $\overline{J}$ on $\nu(\mathscr{F}_\xi)$, i.e.

$$h_H(\overline{J}X, \overline{J}Y) = h_H(X,Y), \quad h_A(\overline{J}X, \overline{J}Y) = -h_A(X,Y).$$

We denote by $\mathcal{EED}_H(g^T)$ (resp. $\mathcal{EED}_A(g^T)$) the space of Hermitian (resp. anti-Hermitian) essential infinitesimal Einstein deformations. The following is an adaptation of results of Koiso [**20**] to the current situation.

PROPOSITION 2.4. *Suppose (M, g, η, ξ, Φ) is Sasaki–Einstein. Then we have the decomposition*

$$\mathcal{EED}(g^T) = \mathcal{EED}_H(g^T) \oplus \mathcal{EED}_A(g^T), \tag{40}$$

and $h \in \Gamma\big(\mathrm{S}^2\, \Lambda_b^{0,1}\big)$ is an element of $\mathcal{EED}_A(g^T)$ if and only if

$$\nabla^T_{\bar\alpha} h_{\bar\beta\bar\gamma} - \nabla^T_{\bar\beta} h_{\bar\alpha\bar\gamma} = 0 \tag{41}$$

$$(\nabla^T)^{\bar\alpha} h_{\bar\alpha\bar\beta} = 0. \tag{42}$$

PROOF. Suppose $h \in \Gamma\big(\mathrm{S}^2\, \Lambda_b^{0,1}\big)$. If $h^\sharp$ denotes raising the second index, then $h^\sharp \in \mathcal{A}^1$. We have the Weitzenböck formula

$$\bar\partial_b \bar\partial_b^* h^\sharp + \bar\partial_b^* \bar\partial_b h^\sharp = \frac{1}{2}\big(\overline{\Delta}^T + 2L^T\big) h^\sharp. \tag{43}$$

Suppose $h \in \mathcal{EED}(g^T)$. Then $\big(\overline{\Delta}^T + 2L^T\big)h_A = 0$ and (43) implies $\delta_{g^T} h_A = 0$. Trivially, $\mathrm{tr}_{g^T} h_A = 0$. Thus $h_A \in \mathcal{EED}(g^T)$ and (40) follows.

It follows from (43) that $h \in \Gamma\big(\mathrm{S}^2\, \Lambda_b^{0,1}\big)$ is in $\mathcal{EED}_A(g^T)$ if and only if (41) and (42) hold. $\square$

Let $\mathcal{H}^k_{\mathcal{A}}$ denote the k-th harmonic space of the complex (30).

COROLLARY 2.5. *Let (M, g, η, ξ, Φ) be Sasaki–Einstein. Then there is a canonical isomorphism*

$$\begin{aligned} \mathcal{H}^1_{\mathcal{A}} &\xrightarrow{\ \sim\ } \mathcal{EED}_A(g^T) \\ h_{\bar\alpha\bar\beta} &\longmapsto -\sqrt{-1}\, h_{\bar\alpha\bar\beta}. \end{aligned} \tag{44}$$

PROOF. First note that from Proposition 2.4 and formula (43) we have a decomposition

$$\mathcal{H}^1_{\mathcal{A}} = \mathcal{H}^1_{\mathcal{A},S} \oplus \mathcal{H}^1_{\mathcal{A},A}, \tag{45}$$

into symmetric and anti-symmetric parts. If $\phi \in \mathcal{H}^1_{\mathcal{A},A}$ then $L\phi = 0$. Thus (43) shows that $\overline{\Delta}^T \phi = 0$, and we have $\nabla^T \phi = 0$. Lowering an index gives an harmonic $\phi_{\bar\alpha\bar\beta} \in \Omega_b^{0,2}$. Since M is Sasaki–Einstein (31) becomes

$$2\Delta_{\bar\partial_b} \phi_{\bar\alpha\bar\beta} = \overline{\Delta}^T \phi_{\bar\alpha\bar\beta} + 4m\phi_{\bar\alpha\bar\beta}.$$

Since all but the last term are zero, $\phi_{\bar\alpha\bar\beta} = 0$. $\square$

LEMMA 2.6. *Let (M, g, η, ξ, Φ) be Sasaki–Einstein and $h^T \in \Gamma\big(\mathrm{S}^2\, T_b^* M\big)$ an element of $\mathcal{EED}_A(g^T)$. If $h = \pi^* h^T$ is the pull-back of the basic tensor h^T to M then $h \in \mathcal{EED}(g)$.*

Proof. First note that the O'Neill tensor of the local projection π onto the leaf space of the foliation $\mathscr{F}_\xi$ is

$$A_X Y = g(\xi, \nabla_X Y)\xi = -g(\Phi X, Y)\xi, \quad X, Y \in \Gamma(D). \tag{46}$$

We will use the formulae of O'Neill on the curvature of a Riemannian submersion. See [**5**, Chapter 9] for more details.

If $X, Y, Z, W \in \Gamma(D)$ are basic vector fields, then we have

$$\begin{aligned} g(R(X,Y)Z,W) &= g^T(R^T(X,Y)Z,W) + 2g(\Phi X,Y)g(\Phi Z,W) + g(\Phi X,Z)g(\Phi Y,W) \\ &\quad - g(\Phi Y,Z)g(\Phi X,W), \quad (47) \\ g(R(X,Y)\xi,W) &= g(X,W)g(Y,\xi) - g(X,\xi)g(Y,W). \quad (48) \end{aligned}$$

A routine calculation shows that

$$\begin{aligned} \overline{\Delta} h(X,Y) &= \pi^*\big(\overline{\Delta}^T h^T\big)(X,Y) + 4h(X,Y) - 2h(\Phi X, \Phi Y), \\ \overline{\Delta} h(\xi, X) &= -2\delta h^T(\Phi X), \\ \overline{\Delta} h(\xi,\xi) &= -2 \operatorname{tr} h^T. \end{aligned}$$

We compute from (47) using an orthonormal frame $\{e_1, \dots, e_{2m-2}, \xi\}$ that

$$\begin{aligned} Lh(X,Y) &= \pi^*\big(L^T h^T\big)(X,Y) + \sum_{i,j}\Big[2g(\Phi X, e_i)g(\Phi Y, e_j) + g(\Phi X, Y)g(\Phi e_i, e_j) \\ &\qquad - g(\Phi e_i, Y)g(\Phi X, e_j)\Big] h(e_i, e_j) \\ &= \pi^*\big(L^T h^T\big)(X,Y) + 2h(\Phi X, \Phi Y) + h(\Phi Y, \Phi X) \\ &= \pi^*\big(L^T h^T\big)(X,Y) - 3h(X,Y). \end{aligned} \tag{49}$$

And (48) easily gives

$$Lh(X,\xi) = -g(\xi, X) \operatorname{tr} h + h(\xi, X) = 0. \tag{50}$$

It follows from the above equations that

$$\big(\overline{\Delta} + 2L\big)h = \pi^*\big(\overline{\Delta}^T h^T\big) + 2\pi^*\big(L^T h^T\big), \tag{51}$$

and $\delta h = 0$, $\operatorname{tr} h = 0$ are trivial. □

Remark 2.7. It is clear from the proof that a non-zero $h = \pi^* h^T$ is not an infinitesimal Einstein deformation if h^T is not anti-Hermitian.

2.3. Infinitesimal deformations on Sasaki–Einstein manifolds.

From Proposition 2.4 and Lemma 2.6 for any $\beta \in \mathcal{H}^1_{\mathcal{A}}$ we have $h^\beta \in \mathcal{EED}(g)$, where $h^\beta(X,Y) = g^T(\bar{J}\beta X, Y)$. We define as in Section 1.2.2 $\Psi^{\beta,\sigma_0}(X) = \alpha(X)\sigma_0$, where $\alpha = -(h^\beta)^\sharp/2$ and σ_0 is Killing spinor.

Proposition 2.8. *Let (M,g) be a spin Sasaki–Einstein manifold admitting the 2 defining Killing spinors σ_j, $j=0,1$. If $\beta\in\mathcal{H}^1_{\mathcal{A}}$ then the corresponding basic anti-Hermitian symmetric tensor h^β is an infinitesimal Einstein deformation of g, and $(\alpha,0)$, $\alpha=-(h^\beta)^\sharp/2$ is an infinitesimal deformation of the Killing spinors σ_j for $j=0,1$.*

Remark 2.9. The definitions of h^β, Ψ^{β,σ_0} and α are made to agree with the identifications made in Corollary 2.5 and Section 1.2.2.

Proof. That h^β is an infinitesimal Einstein deformation follows from Lemma 2.6.

In the proof we denote $(h^\beta)^\sharp$ by h which can be considered to be a basic tensor with values in $D=\ker\eta$ and $\Phi h=-h\Phi$. By Proposition 1.6 it is sufficient to prove

$$\sum_i e_i\cdot\big(\nabla_i h\big)(X)\sigma_j=2ch(X)\sigma_j,\quad\text{for all } X\in TM,\ j=0,1,\tag{52}$$

for a local orthonormal frame $\{e_1,\dots,e_{2m-1}\}$ for which we may choose $e_i\in\Gamma\big(D\big)$ for $i=1,\dots,2m-2$, $e_{m-1+i}=\Phi e_i$ for $i=1,\dots,m-1$ and $e_{2m-1}=\xi$. We extend to an orthonormal frame on $C(M)$ by setting $e_{2m}=\partial_r$.

Define an Hermitian frame by $\varepsilon_\alpha=(e_\alpha-\sqrt{-1}Je_\alpha)/\sqrt{2}$, $\alpha=1,\dots,m-1$ and $\varepsilon_m=(e_{2m-1}-\sqrt{-1}Je_{2m-1})/\sqrt{2}=(\xi+\sqrt{-1}\partial_r)/\sqrt{2}$. Denote their duals by $\varepsilon^\alpha=(e_\alpha+\sqrt{-1}Je_\alpha)/\sqrt{2}$ and define $\varepsilon_{\bar\alpha}=\bar\varepsilon_\alpha$. Note that $\varepsilon_{\bar\alpha}=\varepsilon^\alpha$.

Since $\mathrm{Hol}(\bar g)\subseteq SU(m)$ the spinor bundle Σ of M can be identified, on the neighborhood of the frame, with $\Lambda^{ev}\operatorname{Span}_{\mathbb{C}}\{\varepsilon_\alpha|\alpha=1,\dots,m\}=\Lambda^{ev}T^{1,0}C(M)|_M$, or $\Lambda^{odd}\operatorname{Span}_{\mathbb{C}}\{\varepsilon_\alpha|\alpha=1,\dots,m\}$. Clifford multiplication is given by $e_i\mapsto e_ie_{2m}$, $1\le i\le 2m-1$ (or $e_i\mapsto -e_ie_{2m}$ giving the other Clifford module structure on Σ).

If m is even we take $\Sigma=\Lambda^{ev}\operatorname{Span}_{\mathbb{C}}\{\varepsilon_\alpha|\alpha=1,\dots,m\}$. If m is odd, then we take $\Sigma=\Lambda^{odd}\operatorname{Span}_{\mathbb{C}}\{\varepsilon_\alpha|\alpha=1,\dots,m\}$ when considering $\sigma_1\in\Gamma(\Sigma)$, and $\Sigma=\Lambda^{ev}\operatorname{Span}_{\mathbb{C}}\{\varepsilon_\alpha|\alpha=1,\dots,m\}$ when considering $\sigma_0\in\Gamma(\Sigma)$. In the latter case we take Clifford multiplication to act through $e_i\mapsto -e_ie_{2m}$ in order to obtain the same Clifford module structure on Σ (in this case $c=-1/2$).

The Killing spinors are locally $\sigma_0=a(x)\in\Gamma\big(\Lambda^0\big)$ and $\sigma_1=b(x)\varepsilon_1\wedge\dots\wedge\varepsilon_m\in\Gamma\big(\Lambda^m\big)$, where a,b are smooth functions.

Note that for $X,Y\in\Gamma\big(D\big)$ basic

$$\begin{aligned}\nabla_Y h(X)&=\nabla^T_Y h(X)+g(\nabla_Y(hX),\xi)\xi\\&=\nabla^T_Y h(X)-g(h(X),\Phi Y)\xi.\end{aligned}\tag{53}$$

Thus

$$\begin{aligned}\sum_{i=1}^{2m-1}e_i\big(\nabla_i h\big)(X)\sigma_j&=\sum_{i=1}^{2m-2}e_i\big(\nabla^T_i h\big)(X)\sigma_j+\xi\big(\nabla_\xi h\big)(X)\sigma_j+\sum_i e_ig(\Phi h(X),e_i)\xi\sigma_j\\&=\sum_{i=1}^{2m-2}e_i\big(\nabla^T_i h\big)(X)\sigma_j+2\xi\Phi h(X)\sigma_j+\Phi h(X)\xi\sigma_j\end{aligned}$$

$$= \sum_{i=1}^{2m-2} e_i\big(\nabla^T_i h\big)(X)\sigma_j - \Phi h(X)\xi\sigma_j. \tag{54}$$

We will show that the first term on the right of (54) vanishes. First suppose $X = \varepsilon_\gamma$, then

$$\begin{aligned}\sum_{i=1}^{2m-2} e_i\big(\nabla^T_i h\big)(X)\sigma_j &= \sum_{\alpha=1}^{m-1} \varepsilon^\alpha\big(\nabla^T_{\varepsilon_\alpha} h\big)(\varepsilon_\gamma)\sigma_j + \sum_{\alpha=1}^{m-1} \varepsilon^{\bar\alpha}\big(\nabla^T_{\varepsilon_{\bar\alpha}} h\big)(\varepsilon_\gamma)\sigma_j \\ &= \varepsilon^\alpha \nabla^T_\alpha h_\gamma{}^{\bar\beta}\varepsilon_{\bar\beta}\sigma_j + \varepsilon^{\bar\alpha}\nabla^T_{\bar\alpha} h_\gamma{}^{\bar\beta}\varepsilon_{\bar\beta}\sigma_j.\end{aligned} \tag{55}$$

If $j = 0$, then this vanishes since $\varepsilon_{\bar\beta}\sigma_0 = 0$. Suppose $j = 1$, then the first term on the right of (55) is

$$\begin{aligned}\varepsilon^\alpha \nabla^T_\alpha h_\gamma{}^{\bar\beta}\varepsilon_{\bar\beta}\sigma_1 &= \nabla^T_\alpha h_{\beta\gamma}\varepsilon^\alpha\varepsilon^\beta\sigma_1 \\ &= \sum_{\alpha<\beta}\big(\nabla^T_\alpha h_{\beta\gamma} - \nabla^T_\beta h_{\alpha\gamma}\big)\varepsilon^\alpha\varepsilon^\beta\sigma_1 = 0,\end{aligned} \tag{56}$$

because of (41). And the second term on the right of (55) is

$$\begin{aligned}\varepsilon^{\bar\alpha}\nabla^T_{\bar\alpha} h_\gamma{}^{\bar\beta}\varepsilon_{\bar\beta}\sigma_1 &= \nabla^T_{\bar\alpha} h_{\beta\gamma}\varepsilon^{\bar\alpha}\varepsilon^\beta\sigma_1 \\ &= \nabla^T_{\bar\alpha} h_{\beta\gamma}\big(-\varepsilon^\beta\varepsilon^{\bar\alpha} - 2g(\varepsilon^{\bar\alpha},\varepsilon^\beta)\big)\sigma_1 \\ &= -2\big(\nabla^T\big)^\alpha h_{\alpha\gamma}\sigma_1 = 0,\end{aligned} \tag{57}$$

because of (42). The case of $X = \varepsilon_{\bar\gamma}$ is completely analogous.

We have

$$\begin{aligned}\sum_{i=1}^{2m-1} e_i\big(\nabla_i h\big)(\xi)\sigma_j &= -\sum_{i=1}^{2m-2} e_i h(\Phi e_i)\sigma_j \\ &= -\sum_{i,k=1}^{2m-2} e_i h(\Phi e_i, e_k)e_k\sigma_j \\ &= \sum_{i=1}^{2m-2} h(\Phi e_i, e_i)\sigma_j = 0,\end{aligned} \tag{58}$$

for $j = 0, 1$. The last two equalities follow because $h(\Phi\cdot,\cdot)$ is symmetric and anti-Hermitian.

We have that

$$\sum_{i=1}^{2m-1} e_i\big(\nabla_i h\big)(X)\sigma_j = -\Phi h(X)\xi\sigma_j, \quad \text{for } X \in TM. \tag{59}$$

Recall that Clifford multiplication is $X\cdot\sigma_j = X\partial_r\sigma_j$, for $X \in TM$ with our representation space, unless σ_j has $c = -1/2$ in which case we must take $X\cdot\sigma_j = -X\partial_r\sigma_j$. It is easy to check that

$$-\Phi h(X)\xi\sigma_j = h(X)\partial_r\sigma_j, \quad j = 1, 2. \tag{60}$$

Then (52) follows from (59) and (60) and the Proposition follows. □

2.4. Infinitesimal deformations on 3-Sasakian manifolds.

Recall the important result of Pedersen and Poon that 3-Sasakian structures are rigid.

THEOREM 2.10 ([**35**]). *Let (M, g), $\dim M = 4m-1$, be a 3-Sasakian manifold with Killing spinors σ_i, $i = 0, \ldots, m$. Then any Einstein deformation (M, g_t) of g with compatible 3-Sasakian structures, i.e. preserving the existence of the σ_i, $i = 0, \ldots, m$, is trivial. That is, there exists a family ϕ_t of diffeomorphisms of M with $\phi_t^* g_t = g$.*

The transversal space $\mathscr{F}_\xi$, for any fixed Reeb vector field $\xi \in \mathrm{S}^2$, is an orbifold Z with a complex contact structure. Recall that the twistor spaces for any two $\xi \in \mathrm{S}^2$ are isomorphic via the transitive action of $Sp(1)$ on the S^2 of Reeb vector fields. We denote by $\mathcal{H}^1_{\mathcal{A}}(\xi)$ the harmonic space of the particular $\xi \in \mathrm{S}^2$. Although, the $\mathcal{H}^1_{\mathcal{A}}(\xi)$, $\xi \in \mathrm{S}^2$, are isomorphic they give different deformations $h^\beta \in \mathcal{EED}(g)$, $\beta \in \mathcal{H}^1_{\mathcal{A}}(\xi)$.

The proof of Theorem 2.10, and the earlier similar result [**24**] of LeBrun, follow mainly from the vanishing of $H^1(Z, \mathcal{O}(\boldsymbol{L}))$. We have

$$H^1(Z, \mathcal{O}(\boldsymbol{L})) = H^1(Z, \Omega^{2m-1}(\boldsymbol{K}_Z^{-1} \otimes \boldsymbol{L})) = \{0\}$$

by Kodaira vanishing, since $\boldsymbol{K}_Z^{-1} \otimes \boldsymbol{L} > 0$.

The following provides a spinor version of this vanishing result.

PROPOSITION 2.11. *Let (M, g), $\dim M = 4m-1$, be a 3-Sasakian manifold with Killing spinors σ_j, $j = 0, \ldots, m$. If $\beta \in \mathcal{H}^1_{\mathcal{A}}(\xi)$ is nonzero, then the corresponding basic anti-Hermitian symmetric tensor h^β is an infinitesimal Einstein deformation of g, and $(\alpha, 0)$, $\alpha = -(h^\beta)^\sharp/2$ is an infinitesimal deformation of the Killing spinors σ_j for $j = 0, m$, but never for any nonzero $\sigma \in \mathrm{Span}_{\mathbb{C}}\{\sigma_j | j = 1, \ldots, m-1\}$.*

It will be convenient to introduce some notation. Given $\sigma \in \mathcal{N}_g$ we change notation and write the formula in Proposition 1.6 as

$$\mathcal{L}(\alpha, \sigma)(X) = -\frac{1}{2}\sum_i e_i \cdot \big(\nabla_i \alpha\big)(X)\sigma + \frac{1}{2}\alpha(X)\sigma, \quad \text{for all } X \in TM. \tag{61}$$

Then the proposition asserts that $\mathcal{L}(\alpha, \sigma) = 0$ for $\sigma = \sigma_j$ $j = 0, m$ and $\mathcal{L}(\alpha, \sigma) \neq 0$ for nonzero $\sigma \in \mathrm{Span}_{\mathbb{C}}\{\sigma_j | j = 1, \ldots, m-1\}$.

PROOF. We consider a local orthonormal frame which is in the $Sp(m)$-structure of $C(M)$

$$(e_1, e_2, \ldots, e_{4m}) = (f_1, J_1 f_1, J_2 f_1, J_3 f_1; f_2, J_1 f_2 \ldots; f_m, J_1 f_m, J_2 f_m, J_3 f_m),$$

where $e_1, \ldots, e_{4m-4} \in \bigcap_{i=1,2,3} D_i = \mathcal{D}$, $f_m = -\xi_3, J_1 f_m = \xi_2, J_2 f_m = -\xi_1$ and $J_3 f_m = \partial_r$.

We define an Hermitian frame by $\varepsilon_\alpha = (e_{2\alpha-1} - \sqrt{-1}J_1 e_{2\alpha-1})/\sqrt{2} = (e_{2\alpha-1} - \sqrt{-1}e_{2\alpha})/\sqrt{2}$, $\alpha = 1,\dots,2m$, and their duals $\varepsilon^\alpha = \varepsilon_{\overline{\alpha}} = \bar{\varepsilon}_\alpha$. In particular, we have $\varepsilon_{2m-1} = (-\xi_3 - \sqrt{-1}\xi_2)/\sqrt{2}$ and $\varepsilon_{2m} = (-\xi_1 - \sqrt{-1}\partial_r)/\sqrt{2}$. As in the proof of Proposition 2.8 the spinor bundle of (M,g) can be identified with $\Sigma = \Lambda^{ev}T^{1,0}C(M)|_M = \Lambda^{ev}\operatorname{Span}_{\mathbb{C}}\{\varepsilon_\alpha | \alpha = 1,\dots,2m\}$.

Define the "symplectic form"

$$\varpi = \sum_{\alpha=1}^{m} \varepsilon_{2\alpha-1} \wedge \varepsilon_{2\alpha}. \tag{62}$$

The Killing spinors on (M,g) can be identified with

$$\sigma_k = \frac{1}{k!}\varpi^k \in \Gamma\big(\Lambda^{ev}T^{1,0}C(M)\big),\ k = 0,\dots,m.$$

From the proof of Proposition 2.8 a Killing spinor σ_k is preserved to first order by the Einstein deformation h if and only if

$$\sum_{\alpha=1}^{2m-1} \varepsilon^\alpha \nabla^T_\alpha h(X)\sigma_k + \sum_{\alpha=1}^{2m-1} \varepsilon^{\overline{\alpha}} \nabla^T_{\overline{\alpha}} h(X)\sigma_k + \xi_1 \Phi_1 h(X)\sigma_k = 2ch(X)\sigma_k, \tag{63}$$

holds for all $X \in D_1$. Here $c = 1/2$.

Define $\psi \in \Omega^{0,1}(\boldsymbol{L})$ by $\psi_{\overline{\beta}} = h^{\gamma}_{\overline{\beta}}\theta_\gamma$. Since θ is holomorphic $\bar{\partial}\psi = 0$. The line bundle $\boldsymbol{L}$ has a natural Hermitian metric by the identification $\boldsymbol{L} = \boldsymbol{K}_Z^{-1/m}$, so there is a natural connection on $\boldsymbol{L}$. Then

$$\begin{aligned}\bar{\partial}^*\psi &= -\nabla^{T\overline{\beta}}\psi_{\overline{\beta}} \\ &= -h_{\overline{\beta}}{}^{\gamma}\nabla^{\overline{\beta}}\theta_\gamma = 0,\end{aligned} \tag{64}$$

where the second equality holds from $\nabla^{T\overline{\beta}} h_{\overline{\beta}}{}^{\gamma} = 0$. For the third equality observe that $\nabla_\beta\theta_\gamma$ lifts to the form $d\eta^c = g(\Phi_2\cdot,\cdot) + \sqrt{-1}g(\Phi_3\cdot,\cdot)$ restricted to D_1, but $h^{\beta\gamma}$ is symmetric and so the contraction is zero.

Therefore $\psi \in \Omega^{0,1}(\boldsymbol{L})$ is harmonic. But as we observed, $H^1(Z,\mathcal{O}(\boldsymbol{L})) = 0$, so $\psi = 0$. It follows that $h(X) \in \mathcal{D}$ for all $X \in TM$. This fact will be used repeatedly in the rest of the proof.

Substituting $\xi_1 = (-1/\sqrt{2})(\varepsilon_{2m} + \varepsilon_{\overline{2m}})$ and $\partial_r = (\sqrt{-1}/\sqrt{2})(\varepsilon_{2m} - \varepsilon_{\overline{2m}})$ into (63) and canceling terms gives

$$\sum_{\alpha=1}^{2m-1} \varepsilon^\alpha \nabla^T_\alpha h(X)\sigma_k + \sum_{\alpha=1}^{2m-1} \varepsilon^{\overline{\alpha}} \nabla^T_{\overline{\alpha}} h(X)\sigma_k - \sqrt{-1}\sqrt{2}h(X)^{0,1}\varepsilon_{2m} + \sqrt{-1}\sqrt{2}h(X)^{1,0}\varepsilon_{\overline{2m}} = 0. \tag{65}$$

We saw in the proof of Proposition 2.8 that

$$\varepsilon^\alpha \nabla^T_\alpha h_{\beta\gamma}\varepsilon^\gamma = \varepsilon^{\overline{\alpha}} \nabla^T_{\overline{\alpha}} h_{\overline{\beta}\overline{\gamma}}\varepsilon^{\overline{\gamma}} = 0,$$

so (65) becomes

$$\sum_{\alpha=1}^{2m-1}\sum_{\gamma=1}^{2m-1}\nabla^T_{\bar\alpha}h_{\beta\gamma}\varepsilon^{\bar\alpha}\varepsilon^{\gamma}\sigma_k-\sqrt{-1}\sqrt{2}h(\varepsilon_\beta)\varepsilon_{2m}\sigma_k=0,\ \text{for } X=\varepsilon_\beta, \tag{66}$$

$$\sum_{\alpha=1}^{2m-1}\sum_{\gamma=1}^{2m-1}\nabla^T_{\alpha}h_{\bar\beta\bar\gamma}\varepsilon^{\alpha}\varepsilon^{\bar\gamma}\sigma_k+\sqrt{-1}\sqrt{2}h(\varepsilon_{\bar\beta})\varepsilon_{\overline{2m}}\sigma_k=0,\ \text{for } X=\varepsilon_{\bar\beta}. \tag{67}$$

Define $\vartheta=\sum_{\alpha=1}^{m-1}\varepsilon_{2\alpha-1}\wedge\varepsilon_{2\alpha}$, then we have

$$\sigma_k=\frac{1}{k!}\vartheta^k+\frac{1}{(k-1)!}\vartheta^{k-1}\wedge\varepsilon_{2m-1}\wedge\varepsilon_{2m}. \tag{68}$$

The second term of (66) is

$$\begin{aligned}-\sqrt{-1}\sqrt{2}h(\varepsilon_\beta)\varepsilon_{2m}\sigma_k&=-\frac{\sqrt{-1}2\sqrt{2}}{k!}\varepsilon_{2m}\wedge\big(h(\varepsilon_\beta)\lrcorner\,\vartheta^k\big)\\&=-\frac{\sqrt{-1}2\sqrt{2}}{(k-1)!}\varepsilon_{2m}\wedge\big(h(\varepsilon_\beta)\lrcorner\,\vartheta\big)\wedge\vartheta^{k-1}\\&=-\frac{\sqrt{-1}2\sqrt{2}}{(k-1)!}\varepsilon_{2m}\wedge\Phi_2h(\varepsilon_\beta)\wedge\vartheta^{k-1}.\end{aligned} \tag{69}$$

Note that every term of (69) contains ε_{2m} but does not contain ε_{2m-1}. The terms of the first component of (66) which also contain ε_{2m} but not ε_{2m-1} are

$$\sum_{\alpha=1}^{2m-1}\nabla^T_{\bar\alpha}h_{\beta 2m-1}\varepsilon^{\bar\alpha}\varepsilon^{2m-1}\sigma_k. \tag{70}$$

We simplify (70) to get

$$\begin{aligned}\sum_{\alpha=1}^{2m-1}\nabla^T_{\bar\alpha}h_{\beta 2m-1}\varepsilon^{\bar\alpha}\varepsilon^{2m-1}\sigma_k&=\sum_{\alpha=1}^{2m-1}-h(\varepsilon_\beta,\nabla^T_{\varepsilon_{\bar\alpha}}\varepsilon_{2m-1})\varepsilon^{\bar\alpha}\varepsilon^{2m-1}\sigma_k\\&=\sqrt{-1}\sqrt{2}\sum_{\alpha=1}^{2m-1}h(\varepsilon_\beta,\Phi_2\varepsilon_{\bar\alpha})\varepsilon^{\bar\alpha}\varepsilon^{2m-1}\sigma_k\\&=-\sqrt{-1}\sqrt{2}\Phi_2h(\varepsilon_\beta)\varepsilon^{2m-1}\sigma_k\\&=\frac{\sqrt{-1}2\sqrt{2}}{(k-1)!}\Phi_2h(\varepsilon_\beta)\wedge\vartheta^{k-1}\wedge\varepsilon_{2m}.\end{aligned} \tag{71}$$

Together the terms of (66) which contain ε_{2m} but not ε_{2m-1} are

$$-\frac{\sqrt{-1}4\sqrt{2}}{(k-1)!}\varepsilon_{2m}\wedge\Phi_2h(\varepsilon_\beta)\wedge\vartheta^{k-1}. \tag{72}$$

We claim that (72) is non-zero for $1\le k\le m-1$ when $\Phi_2h(\varepsilon_\beta)$ is non-zero. But this follows because ϑ is a complex symplectic form on $\mathcal{D}$. Thus $h(\varepsilon_\beta)=0$.

A similar argument will be carried out with (67). The second term of (67) is

$$\sqrt{-1}\sqrt{2}h(\varepsilon_{\overline{\beta}})\varepsilon_{\overline{2m}}\sigma_k = \frac{\sqrt{-1}2}{(k-1)!}h(\varepsilon_{\overline{\beta}})\big(\vartheta^{k-1}\wedge\varepsilon_{2m-1}\big)$$
$$= \frac{\sqrt{-1}2\sqrt{2}}{(k-1)!}h(\varepsilon_{\overline{\beta}})\wedge\vartheta^{k-1}\wedge\varepsilon_{2m-1}. \tag{73}$$

The terms of the first component of (67) which contain ε_{2m-1} but not ε_{2m} are

$$\sum_{\alpha=1}^{2m-1}\nabla^T_\alpha h_{\overline{\beta}\,\overline{2m-1}}\varepsilon^\alpha\varepsilon^{\overline{2m-1}}\sigma_k. \tag{74}$$

We compute

$$\sum_{\alpha=1}^{2m-1}\nabla^T_\alpha h_{\overline{\beta}\,\overline{2m-1}}\varepsilon^\alpha\varepsilon^{\overline{2m-1}}\sigma_k = \sum_{\alpha=1}^{2m-1}-h(\varepsilon_{\overline{\beta}},\nabla^T_\alpha\varepsilon_{\overline{2m-1}})\varepsilon^\alpha\varepsilon^{\overline{2m-1}}\sigma_k$$
$$= -\sqrt{-1}\sqrt{2}\sum_{\alpha=1}^{2m-1}g(h(\varepsilon_{\overline{\beta}}),\Phi_2\varepsilon_\alpha)\varepsilon^\alpha\varepsilon^{\overline{2m-1}}\sigma_k$$
$$= \sqrt{-1}\sqrt{2}\Phi_2 h(\varepsilon_{\overline{\beta}})\varepsilon_{2m-1}\sigma_k$$
$$= \frac{\sqrt{-1}2\sqrt{2}}{k!}\varepsilon_{2m-1}\wedge\big(\Phi_2 h(\varepsilon_{\overline{\beta}})\lrcorner\vartheta^k\big)$$
$$= \frac{\sqrt{-1}2\sqrt{2}}{(k-1)!}\varepsilon_{2m-1}\wedge\big(\Phi_2 h(\varepsilon_{\overline{\beta}})\lrcorner\vartheta\big)\wedge\vartheta^{k-1}$$
$$= -\frac{\sqrt{-1}2\sqrt{2}}{(k-1)!}\varepsilon_{2m-1}\wedge h(\varepsilon_{\overline{\beta}})\wedge\vartheta^{k-1}. \tag{75}$$

Combining (73) and (75) give

$$-\frac{\sqrt{-1}4\sqrt{2}}{(k-1)!}\varepsilon_{2m-1}\wedge h(\varepsilon_{\overline{\beta}})\wedge\vartheta^{k-1}. \tag{76}$$

We have for $X\in\Gamma(D^{1,0})$ that the component of $\mathcal{L}(\alpha,\sigma_k)(X)$ containing ε_{2m} but not ε_{2m-1} is $-1/2$ of (72). Since these terms are linearly independent, for $\sigma=\sum_{k=1}^{m-1}a_k\sigma_k$, $\mathcal{L}(\alpha,\sigma_k)(X)=0$ for all X implies $h=0$. □

The proof involved determining the component of (61) with the spinor component containing precisely one vector in $\mathrm{Span}_{\mathbb{C}}\{\varepsilon_{2m-1},\varepsilon_{2m}\}$. This is given in (72) and (76). This component is preserved under changes of the frame used in the calculation. This will be used later in Section 4.1 where more details will be given. It will be useful that this component is

$$-\Phi_1 h(X)\xi_1\cdot\sigma - h(X)\partial_r\cdot\sigma. \tag{77}$$

3. Integrable deformations of Killing spinors.

We consider the integrability of the infinitesimal Einstein deformations $h^\beta\in\mathcal{EED}(g)$ for $\beta\in\mathcal{H}^1_{\mathcal{A}}$ from the last section. We will also consider the integrability of infinitesimal Killing spinor deformations. This is essentially the problem of deforming Sasaki–Einstein metrics. We give some sufficient conditions for integrating these infinitesimal deformations. A deeper sufficient condition for deforming Sasaki–Einstein metrics is K-polystability (see [**40**]), but here we merely give some sufficiency results using analytic methods.

3.1. Integrability on Sasaki–Einstein manifolds.

We state a result from [**39**] giving a sufficient condition for deforming Sasaki–Einstein structures. Let (M, g, η, ξ, Φ) be a Sasaki–Einstein structure, and let $G \subseteq G' = \mathrm{Aut}(g, \eta, \xi, \Phi)$ be a compact subgroup. We consider G-equivariant deformations of the foliation $(\mathscr{F}_\xi, \overline{J})$. We have the G-equivariant Dolbeault complex

$$0 \to \mathcal{A}_G^0 \xrightarrow{\bar{\partial}_b} \mathcal{A}_G^1 \xrightarrow{\bar{\partial}_b} \mathcal{A}_G^2 \to \cdots, \tag{78}$$

with $\mathcal{A}_G^k = \Gamma(\Lambda_b^{0,k} \otimes \nu(\mathscr{F})^{1,0})^G$ the subspace of G-invariant sections. Then $H^1(\mathcal{A}_G^\bullet)$ gives the first order deformations of $(\mathscr{F}_\xi, \overline{J})$ preserving the action of G. We saw in Proposition 2.1 that the versal deformation space $\mathcal{U}$ is smooth. The space of G-equivariant deformations $\mathcal{U}^G \subseteq \mathcal{U}$ is a submanifold with tangent space $H^1(\mathcal{A}_G^\bullet) \subseteq H^1(\mathcal{A})$. With respect to a fixed transversal Kähler structure we have the G-invariant harmonic space $\mathcal{H}^1_{\mathcal{A},G}$ and $H^1(\mathcal{A}_G^\bullet) \cong \mathcal{H}^1_{\mathcal{A},G}$.

If $(\mathscr{F}_\xi, \overline{J}_t)_{t\in\mathcal{V}}$ is a G-equivariant deformation, then one can show as in Proposition 2.2 that there is a family of Sasakian structures $(g_t, \eta_t, \xi, \Phi_t)$, $t \in \mathcal{V}$, with $G \subseteq \mathrm{Aut}(g_t, \eta_t, \xi, \Phi_t)$ where Φ_t induces the transversal complex structure $\overline{J}_t$. Arguments using the implicit function theorem can show the following.

Theorem 3.1 ([**39**]). *Suppose (M, g, η, ξ, Φ) is a Sasaki–Einstein manifold. Let $G \subseteq \mathrm{Aut}(g, \eta, \xi, \Phi)$ be a maximal torus, and let $(\mathscr{F}_\xi, \overline{J}_t)_{t\in\mathcal{V}}$ be a G-equivariant deformation with $\mathcal{V}$ smooth. Then after possibly shrinking $\mathcal{V}$, there is a family $(g_t, \eta_t, \xi, \Phi_t)$, $t \in \mathcal{V}$ of Sasaki–Einstein structures with $(g_0, \eta_0, \xi, \Phi_0) = (g, \eta, \xi, \Phi)$ and with Φ_t inducing the transversal complex structure $\overline{J}_t$.*

This implies the following in terms of Killing spinors.

Corollary 3.2. *Let (M, g) be a spin Sasaki–Einstein manifold admitting the two defining Killing spinors σ_j, $j = 0, 1$, e.g. M is simply connected. Then the infinitesimal Einstein deformations h^β, for $\beta \in \mathcal{H}^1_{\mathcal{A},G}$, integrate to a family g_t, $t \in \mathcal{V} \subset \mathbb{C}^d$, $d = \dim_{\mathbb{C}} \mathcal{H}^1_{\mathcal{A},G}$, of Einstein deformations preserving σ_j, $j = 0, 1$.*

The components in $\mathcal{EED}(g)$ of $\{v(g_t) \mid v \in T_0\mathcal{V}\}$ are precisely the original infinitesimal Einstein deformations $\{h^\beta \mid \beta \in \mathcal{H}^1_{\mathcal{A},G}\}$.

Proof. Just the last statement remains to be proved. Consider the family $(g_t, \eta_t, \xi, \Phi_t)$, $t \in \mathcal{V}$ of Proposition 2.2. Using the notation of Section 2.1 and differentiating in the direction of some $v \in T_0\mathcal{V}$ we have

$$\phi_{\alpha\beta} = 0 \tag{79}$$

$$\phi_{\alpha\bar{\beta}} = \sqrt{-1}h_{\alpha\bar{\beta}} \tag{80}$$

$$h_{\alpha\beta} = \sqrt{-1} I_{\alpha\beta}, \tag{81}$$

which follow from (39), (36) and (35) respectively. In the proof of Proposition 2.2 the basic cohomology class $[\omega_t^T]$ is constant. Thus ϕ is an exact $(1,1)$-form. We may replace η_t with $\eta_t + d^c\psi_t$, so that using the same notation we have $(1/2)d\dot{\eta}_t = \phi = 0$.

The possible contact forms for a fixed Reeb vector field ξ and transversal complex structure $\overline{J}_t$ are $\eta_t + d^c\psi_t + d\theta_t$ for basic functions $\psi_t, \theta_t \in C_b^\infty(M)$. See [**39**, Lemma 2.2.3], where we also use that $\mathrm{Ric}^T > 0$, which implies that the basic cohomology $H_b^1 = H^1(M,\mathbb{R}) = \{0\}$. And $d\theta_t$ is given by a gauge transformation $\exp(\theta_t\xi)^*\eta_t$, which fixes basic tensors. Therefore, by adding a factor of $d\theta_t$ to η_t, we may arrange that $\dot{\eta}_t = 0$. We assume that the family $(g_t, \eta_t, \xi, \Phi_t),\ t \in \mathcal{V}$ is chosen so that $\dot{\eta}_t = 0$ at $t = 0$. Thus the only component of $\dot{g}_t$ at $t = 0$ is $h_{\alpha\beta} = \sqrt{-1}I_{\alpha\beta} \in \mathcal{EED}(g)$.

Recall that if $\psi \in C_b^\infty(M)$ is sufficiently small there is a Sasakian structure $(g_{t,\psi}, \eta_{t,\psi}, \xi, \Phi_{t,\psi})$ with contact form $\eta_{t,\psi} = \eta_t + d^c\psi$ and transversal complex structure $\overline{J}_t$. The metric is

$$g_{t,\psi} = \frac{1}{2} d\eta_{t,\psi}(\cdot, \overline{J}_t \cdot) + \eta_{t,\psi} \otimes \eta_{t,\psi},$$

and $\Phi_{t,\psi}$ is the lift of $\overline{J}_t$ to $\ker \eta_{t,\psi}$.

Theorem 3.1 is proved by using the implicit function theorem to find $\psi_t \in C_b^\infty(M),\ t \in \mathcal{V}$, so that the Sasakian structure $(g_{t,\psi}, \eta_{t,\psi}, \xi, \Phi_{t,\psi})$ has scalar curvature $s_{t,\psi_t} = 0$. We review enough of the proof of Theorem 3.1 to prove the corollary. For more details see [**39**].

We consider the G-invariant Sobolev space $L^2_{k+4,G}(M),\ k > m$, of $k+4$ times weakly differentiable functions. For $\psi \in L^2_{k+4,G}(M)$ small we have the Sasakian structure with metric $g_{t,\psi}$ as above. We have the space of holomorphy potentials $\mathcal{H}^{\mathfrak{g}}_{t,\psi}$ for this metric where $\mathfrak{g}$ is the Lie algebra of G (cf. [**39**]). Using the metric $g_{t,\psi}$ to define the L^2 inner product on $L^2_{k,G}(M)$ we have the orthogonal decomposition

$$L^2_{k,G}(M) = \sqrt{-1}\mathcal{H}^{\mathfrak{g}}_{t,\psi} \oplus W_{k,t,\psi},$$

and the projections

$$\pi^G_{t,\psi} : L^2_{k,G}(M) \to \sqrt{-1}\mathcal{H}^{\mathfrak{g}}_{t,\psi}, \text{ and } \pi^W_{t,\psi} : L^2_{k,G}(M) \to W_{k,t,\psi}.$$

The reduced scalar curvature of $g_{t,\psi}$ is given by

$$s^G_{t,\psi} = \pi^W_{t,\psi}(s_{t,\psi}) = (\mathbb{1} - \pi^G_{t,\psi})(s_{t,\psi}). \tag{82}$$

Let $U \subset \mathcal{V} \times L^2_{k+4,G}(M)$ be a neighborhood of $(0,0)$ so that for $(t,\psi) \in U$, $(g_{t,\psi}, \eta_{t,\psi}, \xi, \Phi_{t,\psi})$ is well defined. For $\mathcal{U} = U \cap \big(\mathcal{V} \times W_{k+4,0}\big)$ we define a map

$$\begin{aligned} \mathcal{S} :\ \mathcal{U} &\to W_{k,0} \\ (t,\psi) &\mapsto \pi^W_0(s^G_{t,\psi}). \end{aligned} \tag{83}$$

The derivative of (83) is

$$d\mathcal{S} : W_{k+4,0} \to W_{k,0}, \tag{84}$$

with $d\mathcal{S}(\dot\psi) = -2\mathbb{L}_g\dot\psi$. Here $\mathbb{L}_g$ is the self-adjoint operator

$$\mathbb{L}_g\psi = \frac{1}{2}\Delta_b^2\psi + \frac{1}{2}(\mathrm{Ric}^T, dd^c\psi) + \frac{1}{2}(d\psi, ds_g).$$

As proved in [**39**, Corollary 4.2.5] there is a family $\psi_t,\ t \in \mathcal{U}$, with

$$\mathcal{S}(t, \psi_t) = \pi_0^W(s^G_{t,\psi_t}) = 0. \tag{85}$$

Since $\dot g_t \in \mathcal{EED}(g)$ it is easy to check that $(d/dt)s^G_{t,0} = 0$ at $t = 0$. Then differentiating (85) at $t = 0$ gives $-2\mathbb{L}_g\dot\psi_t = 0$. But (84) is an isomorphism, so $\dot\psi_t = 0$ at $t = 0$. Therefore at $t = 0$ we have $\dot g_{t,\psi_t} = \dot g_t$ which is $h_{\alpha\beta} = \sqrt{-1}I_{\alpha\beta} \in \mathcal{EED}(g)$. □

We will give an application of Theorem 3.2 in Section 4.2.

3.2. Integrability on 3-Sasakian manifolds.

We can prove integrability of many of the transversal infinitesimal deformations on a 3-Sasakian manifold. The infinitesimal deformations of the real subspace $\mathrm{Re}\,\mathcal{H}^1_{\mathcal{A}}(\xi) \subset \mathcal{H}^1_{\mathcal{A}}(\xi)$ with respect to the real structure $\varsigma : \mathcal{H}^1_{\mathcal{A}}(\xi) \to \mathcal{H}^1_{\mathcal{A}}(\xi)$ induced by the anti-holomorphic real structure $\varsigma : Z \to Z$ integrate to Einstein deformations preserving the existence of precisely two Killing spinors.

Theorem 3.3. *Let (M, g), $\dim M = 4m-1$, be a 3-Sasakian manifold, and denote by $\sigma_i,\ i = 0, \dots, m$ the Killing spinors associated to the 3-Sasakian structure. Then the infinitesimal Einstein deformations h^β of g for $\beta \in \mathrm{Re}\,\mathcal{H}^1_{\mathcal{A}}(\xi)$ in Proposition 2.8 integrate to a family $g_t,\ t \in \mathcal{N} \subset \mathbb{R}^d,\ d = \dim_{\mathbb{C}} \mathcal{H}^1_{\mathcal{A}}$, of Einstein deformations of g preserving σ_0 and σ_m but not the remaining. The components in $\mathcal{EED}(g)$ of $\{v(g_t) \mid v \in T_0\mathcal{N}\}$ are precisely the original infinitesimal Einstein deformations $\{h^\beta \mid \beta \in \mathcal{H}^1_{\mathcal{A}}(\xi)\}$.*

Corollary 3.4. *Let (M, g), $\dim M = 4m-1$, be a 3-Sasakian manifold with $d = \dim_{\mathbb{C}} H^1(\mathcal{A}^\bullet)$. Then g has a d-dimensional family of non-trivial deformations, $\{g_t \mid t \in \mathcal{N} \subset \mathbb{R}^d\}$, where $g_t,\ t \neq 0$, has a compatible Sasaki–Einstein structure but no 3-Sasakian structure.*

Recall that the quotient of M, $\dim M = 4m + 3$, by the action of $Sp(1)$-action generated by $\{\xi_1, \xi_2, \xi_3\}$ is a quaternion-Kähler orbifold $(\hat M, \hat g)$, $\dim \hat M = 4m$. If $m \geq 2$, this means there is a three dimensional bundle $\mathcal{J} \subset \mathrm{End}(TM)$ which is locally spanned by almost complex structures $\hat J_i,\ i = 1, 2, 3$ satisfying the quaternionic identities which is preserved by the Levi-Civita connection of $\hat g$. This is equivalent to the existence of a 1-integrable $Sp(m)\,Sp(1)$-structure on $\hat M$. The O'Neill formulas of the submersion $\pi : M \to \hat M$ show that $(\hat M, \hat g)$ is Einstein with constant $\lambda = 4m + 8$. If $m = 1$, every oriented manifold satisfies this with $\mathcal{J} = \Lambda^2_+$. A 4-dimensional quaternion-Kähler orbifold $(\hat M, \hat g)$ is defined to be oriented and satisfy $W_g^+ \equiv 0$ and $\mathrm{Ric}_g = \lambda g$.

We will consider a weaker condition, that of a quaternionic structure (cf. [**37**]).

DEFINITION 3.5. A *quaternionic structure* on $\hat{M}$, of dimension $4m$, $m \geq 2$, is a three dimensional subbundle $\mathcal{J} \subset \mathrm{End}(T\hat{M})$ which is locally spanned by almost complex structures $\hat{J}_i$, $i = 1,2,3$ satisfying the quaternionic identities and preserved by a torsion-free connection on $T\hat{M}$. This is equivalent to the existence of a 1-integrable $GL(m,\mathbb{H})\,Sp(1)$-structure.

If $m = 1$, then a quaternionic structure is defined to be a conformal class $[g]$ with an orientation on $\hat{M}$ satisfying $W^+_{[g]} \equiv 0$.

Part of the interest in quaternionic manifolds is due to an attractive twistor correspondence [**36**]. If $(\hat{M},\mathcal{J})$ is a $4m$-dimensional quaternionic manifold, then the *twistor space* is $Z = \mathbb{P}(\boldsymbol{E})$ where $\boldsymbol{E}$ is the locally defined complex 2-dimensional bundle associated to the complex 2-dimensional representation of the $Sp(1)$-factor of $GL(m,\mathbb{H})\,Sp(1)$. Then Z is a $2m+1$-dimensional complex manifold with a family of twistor lines $\mathbb{CP}^1$ with normal bundle $\mathcal{O}_{\mathbb{CP}^1}(1)^{\oplus 2m}$ and an anti-holomorphic involution $\varsigma : Z \to Z$ preserving the *real* twistor lines. Conversely, if Z is a $2m+1$-dimensional complex manifold with a family of twistor lines $\mathbb{CP}^1$ with normal bundle $\mathcal{O}_{\mathbb{CP}^1}(1)^{\oplus 2m}$ and an anti-holomorphic involution $\sigma : Z \to Z$, then a connected component of real twistor lines is a $4m$-dimensional manifold with a quaternionic structure. Since the twistor correspondence is natural, if $(M,\mathcal{J})$ is a quaternionic orbifold we may define the twistor space over each uniformizing chart as for manifolds and quotient by the orbifold group.

We say that a diffeomorphism of a quaternionic manifold(orbifold) $F : \hat{M} \to \hat{M}$ is a quaternionic automorphism if the derivative of F preserves the bundle $\mathcal{J}$, or equivalently preserves the $GL(m,\mathbb{H})\,Sp(1)$-structure. The following is essentially different proof of a result of LeBrun [**26**, Corollary C], but we need to consider the case in which $(\hat{M},\hat{g})$ is an orbifold.

LEMMA 3.6. *Let $(\hat{M},\hat{g})$ be a quaternion-Kähler manifold or orbifold whose associated 3-Sasakian space M is smooth. If $(\hat{M},\hat{g})$ admits a quaternionic automorphism which is not an isometry, then $(\hat{M},\hat{g})$ is locally isometric to $\mathbb{HP}^m$ with the symmetric metric. Thus $(\hat{M},\hat{g}) \overset{\mathrm{isom}}{\cong} \Gamma\backslash\mathbb{HP}^m$, $\Gamma \subset Sp(m+1)$.*

PROOF. Let $M \to \hat{M}$ be the $Sp(1)$ or $SO(3)$ orbifold bundle with M the 3-Sasakian space associated to $\hat{M}$. Suppose there is such a quaternionic automorphism $F : \hat{M} \to \hat{M}$, then F lifts to a diffeomorphism $\overline{F} : M \to M$ which maps each ξ_i, $i = 1,2,3$ to itself and preserves the complex structure on the transverse space Z. The complex contact form θ of Z lifts to $\eta^c = \eta_2 + \sqrt{-1}\eta_3$. Since $F : \hat{M} \to \hat{M}$ is an isometry if and only if the biholomorphism induced on Z is complex contact [**25**], [**32**], $\hat{\eta} = \overline{F}^*\eta^c \neq \eta^c$. And $C(M)$ has two holomorphic symplectic forms $\varpi = d(r^2\eta^c)$ and $\hat{\varpi} = d(r^2\hat{\eta})$. If $\overline{\nabla}$ is the Levi-Civita connection of $(C(M),\bar{g})$, then $\overline{\nabla}\varpi = 0$. Note that both ϖ and $\hat{\varpi}$ are of order 2 with respect to the Euler vector field $r\partial_r$. Since $\overline{\nabla}_{\partial_r}\partial_r = 0$ and $\overline{\nabla}_{r\partial_r}X = X$ for a vector field X on M viewed as a vector field on $C(M)$, it is easy to check that $\overline{\nabla}_{\partial_r}\hat{\varpi} = 0$.

We have the following formula on a Kähler–Einstein manifold with Einstein constant λ:

$$\overline{\nabla}^{\beta}\overline{\nabla}_{\beta}\hat{\varpi}_{\alpha_1\alpha_2} = \overline{\nabla}^{\bar{\beta}}\overline{\nabla}_{\bar{\beta}}\hat{\varpi}_{\alpha_1\alpha_2} + 2\lambda\hat{\varpi}_{\alpha_1\alpha_2}. \tag{86}$$

Since $\lambda = 0$ and $\hat{\varpi}$ is holomorphic, we have $\overline{\nabla}^{\beta}\overline{\nabla}_{\beta}\hat{\varpi}_{\alpha_1\alpha_2} = \overline{\nabla}^{\bar{\beta}}\overline{\nabla}_{\bar{\beta}}\hat{\varpi}_{\alpha_1\alpha_2} = 0$. Consider $TC(M)|_M$ as an Hermitian vector bundle on M and denote by ∇ the connection $\overline{\nabla}$ restricted to M. Then $\nabla^*\nabla\hat{\varpi} = \overline{\nabla}^*\overline{\nabla}\hat{\varpi} = 0$ and

$$\begin{aligned} 0 &= \int_M \langle \nabla^*\nabla\hat{\varpi}, \hat{\varpi}\rangle\, \mu_g \\ &= \int_M \langle \nabla\hat{\varpi}, \nabla\hat{\varpi}\rangle\, \mu_g. \end{aligned}$$

Therefore $\overline{\nabla}\hat{\varpi} = 0$. So the holonomy of $(C(M), \bar{g})$ stabilizes two linearly independent $(2,0)$-forms of maximal rank, and the holonomy of the universal cover $\widetilde{C(M)}$ is reducible. It follows from [**17**, Proposition 3.1] that $\widetilde{C(M)}$ is flat. Thus M is isometric to a space form $\Gamma\backslash \mathrm{S}^{4m+3}$. □

Proof of Theorem. Fixing a $\xi \in \mathrm{S}^2$ we have the foliation $(\mathscr{F}_\xi, \overline{J})$ whose transversal space is the twistor space Z. There is a subspace $\mathcal{N} \subset \mathcal{U} \subset H^1(\mathcal{A}^\bullet)$ of the versal deformation space of $(\mathscr{F}_\xi, \overline{J})$ of *real deformations.* These are the deformations $\overline{J}_t$ for which $\varsigma(\overline{J}_t) = -\overline{J}_t$. By straightforward averaging one can choose the family of compatible Sasakian structures in Proposition 2.2 $(g_t, \eta_t, \xi, \Phi_t)$ to satisfy

$$\varsigma^* g_t = g_t, \quad \varsigma^*\eta_t = -\eta_t, \quad \varsigma_*\xi = -\xi, \quad \varsigma^*\Phi_t = -\Phi_t, \tag{87}$$

for $t \in \mathcal{N}$. In particular, we also have $\varsigma^*\omega^T = -\omega^T$. For $t \in \mathcal{N}$ with respect to $(g_t, \eta_t, \xi, \Phi_t)$ we have $\operatorname{Re} H^1(\mathcal{A}^\bullet) = \operatorname{Re}\mathcal{H}^1_{\mathcal{A}}(\xi)$ for the tangent space to $\mathcal{N}$ at 0. Therefore $(\mathscr{F}_\xi, \overline{J}_t) = (Z, \overline{J}_t)$ has a Kähler structure ω_t^T, with $\omega_t^T \in (\pi/2m)c_1(\mathscr{F}_\xi, \overline{J}_0)$ depending smoothly on $t \in \mathcal{N}$ and $\mathrm{Ricci}(\omega_0^T) = 4m\omega_0^T$. Since the leaf space is an orbifold we will denote the transversal Kähler space by $(Z, \overline{J}_t, \omega_t)$.

Let $\mathfrak{g}$ be the Lie algebra of quaternionic automorphisms of $(\hat{M}, \hat{g})$. By the twistor correspondence, $\mathfrak{g} \cong \{X \in \mathfrak{hol}(Z, J_0) | \varsigma_* X = X\}$. Since $\mathfrak{g}$ is a real form of $\mathfrak{hol}(Z, \overline{J}_0)$, $\mathfrak{g}\otimes\mathbb{C} = \mathfrak{hol}(Z, \overline{J}_0)$. By Lemma 3.6 $\mathfrak{g} \subseteq \mathfrak{isom}(\hat{M}, \hat{g}, \mathcal{J})$. Thus $\mathfrak{g} \subseteq \mathfrak{isom}(Z, \omega_0, \overline{J}_0)$. Since $(Z, \omega_0, \overline{J}_0)$ is Kähler–Einstein the results of Matsushima [**27**] show that $\mathfrak{isom}(Z, \omega_0, \overline{J}_0) \subset \mathfrak{g}\otimes\mathbb{C}$ is a real form, so $\mathfrak{g} = \mathfrak{isom}(Z, \omega_0, \overline{J}_0)$.

Recall that $f \in C^\infty(Z, \mathbb{C})$ is a *holomorphy potential* if

$$\partial^\# f := (\bar{\partial} f)^\# = \sum_{i,j} \frac{\partial f}{\partial \bar{z}^{\bar{k}}} g^{\bar{k}j} \frac{\partial}{\partial z^j}$$

is holomorphic. We define the space of normalized holomorphy potential functions,

$$\mathcal{H}_g := \left\{ f \in C^\infty(Z, \mathbb{C}) \;\middle|\; f \text{ is Hamiltonian and } \int f\, \mu_g = 0 \right\}. \tag{88}$$

Suppose $W \in \Gamma(T^{1,0}Z)$ is holomorphic with $\operatorname{Re} W = X \in \mathfrak{g} = \{Y \in \mathfrak{hol}(Z, J_0) \mid \varsigma_* Y = Y\}$, so $\mathcal{L}_X\omega = 0$. And let $f_W \in C^\infty(Z)$ be a symplectic Hamiltonian, with $\int_Z f_W \mu_g = 0$, that is

$$X \lrcorner\, \omega = df_W. \tag{89}$$

Then

$$\partial^{\#} f_W = \frac{1}{2}(df_W + \sqrt{-1}J^* df_W)^{\#} = \frac{1}{2}(JX + \sqrt{-1}X) = \frac{\sqrt{-1}}{2}W.$$

From (87) and (89) we have $\varsigma^* df_W = -df_W$, and $\int_Z f_W \, \mu_g = 0$ implies that $\varsigma^* f_W = -f_W$. Since $\mathcal{H}_g$ is the complexification of the real functions f_W considered, we have that $\varsigma^* f = -f$ for all $f \in \mathcal{H}_g$.

There are $F_t \in C^\infty(Z)$ depending smoothly on $t \in \mathcal{N}$ with

$$\sqrt{-1}\partial_t \bar{\partial}_t F_t = \mathrm{Ricci}(\omega_t) - 4m\omega_t. \tag{90}$$

Since F_t is defined up to a constant, $\varsigma^* F_t = F_t + c_t$ for $t \in \mathrm{Re}\,\mathcal{N}$. But $\int (F_t - \varsigma^* F_t)\mu_{g_t} = 0$, so $\varsigma^* F_t = F_t$.

Define $C^{k,\alpha}(Z)_{\mathrm{sym}}$ to be the Hölder space of functions f with $\varsigma^* f = f$. The Monge-Ampère equation

$$\Psi(\varphi_t, t) = \log\Big(\frac{(\omega_t + \sqrt{-1}\partial_t \bar{\partial}_t \varphi_t)^{2m-1}}{\omega_t^{2m-1}}\Big) + 4m\varphi_t = F_t, \tag{91}$$

is ς-invariant for $t \in \mathrm{Re}\,\mathcal{N}$, and Ψ defines a smooth map

$$\Psi : C^{k+2,\alpha}(Z)_{\mathrm{sym}} \times \mathrm{Re}\,\mathcal{N} \to C^{k,\alpha}(Z)_{\mathrm{sym}}. \tag{92}$$

The differential of (92) is

$$D_\varphi \Psi(\dot{\varphi}) = \big(-\Delta_{\bar{\partial}} + 4m\big)\dot{\varphi}. \tag{93}$$

But it is a result of Matsushima [**27**] that $\mathcal{H}_g = \ker(\Delta_{\bar{\partial}} - \lambda)$, where $\lambda = 4m$ is the Einstein constant. Thus $D_\varphi \Psi : C^{k+2,\alpha}(Z)_{\mathrm{sym}} \to C^{k,\alpha}(Z)_{\mathrm{sym}}$ is an isomorphism. By the implicit function theorem, after possibly replacing $\mathcal{N}$ by a smaller neighborhood of 0, for $t \in \mathcal{N}$ there is a $\varphi_t \in C^{k+2,\alpha}(Z)_{\mathrm{sym}}$ with $\Psi(\varphi_t) = F_t$, and

$$\omega_t' = \omega_t + \sqrt{-1}\partial_t \bar{\partial}_t \varphi_t \tag{94}$$

is Kähler–Einstein. The well-known regularity results show that $\varphi_t \in C^\infty(Z)_{\mathrm{sym}}$.

Let $\pi : M_t \to Z_t$ be the $U(1)$-bundle associated to either $\boldsymbol{K}_{Z_t}^{1/m}$ or $\boldsymbol{K}_{Z_t}^{-1/2m}$, depending on whether (M, g) fibers over $(\hat{M}, \hat{g})$ with generic $SO(3)$ or $Sp(1)$ fibers. Choose the connection form on M_t to be $\eta_t' = \eta_t + d_t^c \varphi_t$. Then from (94) one has $(1/2)d\eta_t = \omega_t'$. We get a Sasaki–Einstein structure $(g_t', \eta_t', \xi, \Phi_t')$ on M_t where

$$\tilde{g}_t' = \omega_t'(\cdot, \Phi_t' \cdot) + \eta_t' \otimes \eta_t', \tag{95}$$

and Φ_t' is the lift of $\bar{J}_t$ to $\ker \eta_t'$.

By Theorem 2.10 for small $t \in \mathcal{N}$, (M, g_t') has no compatible 3-Sasakian structure.

It remains to prove that the components in $\mathcal{EED}(g)$ of $\{v(g_t) \mid v \in T_0\mathcal{N}\}$ are precisely the original infinitesimal Einstein deformations $\{h^\beta \mid \beta \in \mathcal{H}^1_{\mathcal{A}}(\xi)\}$. Consider the family $(g_t, \eta_t, \xi, \Phi_t)$, $t \in \mathcal{N}$ of Proposition 2.2. Using the notation of Section 2.1 and differentiating in the direction of some $v \in T_0\mathcal{N}$ we have

$$\phi_{\alpha\beta} = 0 \tag{96}$$

$$\phi_{\alpha\bar{\beta}} = \sqrt{-1}h_{\alpha\bar{\beta}} \tag{97}$$

$$h_{\alpha\beta} = \sqrt{-1}I_{\alpha\beta}, \tag{98}$$

which follow from (39), (36) and (35) respectively. In the proof of Proposition 2.2 the basic cohomology class $[\omega_t^T]$ is constant. Thus ϕ is an exact $(1,1)$-form. We may replace η_t with $\eta_t + d^c\psi_t$, so that using the same notation we have $(1/2)d\dot{\eta}_t = \phi = 0$.

The possible contact forms for a fixed Reeb vector field ξ and transversal complex structure $\bar{J}_t$ are $\eta_t + d^c\psi_t + d\theta_t$ for basic functions $\psi_t, \theta_t \in C_b^\infty(M)$. See [**39**, Lemma 2.2.3], where we also use that $\mathrm{Ric}^T > 0$, which implies that the basic cohomology $H_b^1 = H^1(M,\mathbb{R}) = \{0\}$. And $d\theta_t$ is given by a gauge transformation $\exp(\theta_t\xi)^*\eta_t$, which fixes basic tensors. Therefore, by adding a factor of $d\theta_t$ to η_t, we may arrange that $\dot{\eta}_t = 0$.

We suppose now that we have chosen $(g_t, \eta_t, \xi, \Phi_t)$, $t \in \mathcal{N}$ as such. Thus the only component of h is $h_{\alpha\beta} = \sqrt{-1}I_{\alpha\beta}$, which is a transversal infinitesimal Einstein deformation. Differentiating (90) gives

$$\sqrt{-1}\partial_b\bar{\partial}_b\dot{F}_t = 0.$$

Then differentiating (91) with respect to t gives

$$\left(-\Delta_{\bar{\partial}} + 4m\right)\dot{\varphi}_t = 0,$$

and it follows that $\dot{\varphi}_t = 0$ at $t = 0$. Therefore $(g_t', \eta_t', \xi, \Phi_t')$ gives the same first order Einstein deformation at $t = 0$ as $(g_t, \eta_t, \xi, \Phi_t)$ which is $h_{\alpha\beta} = \sqrt{-1}I_{\alpha\beta}$. □

4. Deformations on a 3-Sasakian manifold.

4.1. Space of deformations on a 3-Sasakian manifold.

The space of Einstein deformations on a 3-Sasakian manifold constructed in Section 2 has an interesting structure. Suppose (M,g) has a 3-Sasakian structure with Reeb vector fields ξ_1, ξ_2, ξ_3 satisfying $[\xi_i, \xi_j] = -2\varepsilon^{ijk}\xi_k$ and space of Reeb fields S^2.

For $\xi \in \mathrm{S}^2$ and $\beta \in \mathcal{H}^1_{\mathcal{A}}(\xi)$ we define $h^{\beta,\xi} \in \mathcal{EED}(g)$, where $h^{\beta,\xi}(X,Y) = g^T(\bar{J}\beta X, Y)$ where we distinguish the particular Reeb vector field. We have the following space of infinitesimal Einstein deformations

$$\mathscr{E}\mathscr{D}(g) := \sum_{\xi \in \mathrm{S}^2} \{h^{\beta,\xi} \mid \beta \in \mathcal{H}^1_{\mathcal{A}}(\xi)\} \subseteq \mathcal{EED}(g). \tag{99}$$

We have a left action of $Sp(1)$ on (M,g) generated by ξ_1, ξ_2, ξ_3. Since $Sp(1)$ acts by isometries and on the space of Sasakian structures S^2, it acts on $\mathscr{E}\mathscr{D}(g)$, and all the subspaces $\mathcal{H}^1_{\mathcal{A}}(\xi), \xi \in \mathrm{S}^2$, are isomorphic. The subspace $\mathcal{H}^1_{\mathcal{A}}(\xi_1)$ is preserved by ξ_1, so by elementary representation theory

$$\dim_{\mathbb{R}} \mathscr{E}\mathscr{D}(g) = 2\dim_{\mathbb{C}} \mathscr{E}\mathscr{D}(g) \geq 6\dim_{\mathbb{C}} \mathcal{H}^1_{\mathcal{A}}(\xi_1).$$

This $Sp(1)$-action acts on $(C(M), J_1, J_2, J_3)$ by quaternionic automorphisms. That is, it preserves the bundle of quaternionic frames $L_{Sp(m)Sp(1)}(C(M))$. This lifts, via the spin structure to an action on $\tilde{L}_{Sp(m)Sp(1)}(C(M)) \subset L_{Spin(4m)}(C(M))$ if m is even or $\tilde{L}_{Sp(m)\times Sp(1)}(C(M)) \subset L_{Spin(4m)}(C(M))$ if m is odd. The Killing spinors are contained in the γ_m factor of $\mathbb{S}^+_{4m}$ of (6). Thus $Sp(1)$ acts on the Killing spinors via the representation of $Sp(1) = SU(2)$ on $\gamma_m = S^2(\mu_2)$.

We will consider a principal subbundle $E \subset L_{Sp(m)Sp(1)}(C(M))$ with structure group $\big(Sp(m-1)\times Sp(1)\big)\, Sp(1)$ generated by all the local frames considered in the proof of Proposition 2.11. This subbundle is invariant under the isometric $Sp(1)$-action. In order to determine the $Sp(1)$ action on spinors we consider the spin bundle

$$\Sigma = \tilde{E} \times_{\big(Sp(m-1)\times Sp(1)\big)\, Sp(1)} \mathbb{S}^+_{4m}.$$

Importantly, the subspace of spinors, considered in the proof of Proposition 2.11, with precisely one vector in $\operatorname{Span}_{\mathbb{C}}\{\varepsilon_{2m-1}, \varepsilon_{2m}\}$ is preserved by $\big(Sp(m-1)\times Sp(1)\big)\, Sp(1)$.

The $Sp(1)$ action on E is easily computed. Given $a \in Sp(1)$ and $u \in E$, write $a_* u = u\psi(a)$, then

$$\psi(a) = \big((\cdot, kak^{-1}), a\big) \in \big(Sp(m-1)\times Sp(1)\big)\, Sp(1)$$

is the factor acting non-trivially on the component of spinors with one vector in $\operatorname{Span}_{\mathbb{C}}\{\varepsilon_{2m-1}, \varepsilon_{2m}\}$. It will be useful that the spin bundle has the decomposition (6) with the $Sp(1)$-action acting on the $\gamma_m, \gamma_{m-2}, \ldots$ factors in the usual way with γ_m being the space of Killing spinors.

We will need a lemma in the proofs of the main theorems.

Lemma 4.1. *Suppose $\xi, \xi' \in \mathrm{S}^2$. If $\xi \neq \xi'$ and $\xi \neq -\xi'$, then*

$$\{h^{\beta,\xi} \mid \beta \in \mathcal{H}^1_{\mathcal{A}}(\xi)\} \cap \{h^{\beta,\xi'} \mid \beta \in \mathcal{H}^1_{\mathcal{A}}(\xi')\} = \{0\}.$$

Suppose that $m = 2$, $\xi_1, \xi_2, \xi_3 \in \mathrm{S}^2$ are linearly independent, and $\beta_i \in \mathcal{H}^1_{\mathcal{A}}(\xi_i), i = 1, 2, 3$ are non-zero. Then

$$h^{\beta_1,\xi_1} + h^{\beta_2,\xi_2} + h^{\beta_3,\xi_3} \neq 0.$$

Proof. Let $\sigma_k, k = 0, \ldots, m$ be the Killing spinors as in the proof of Proposition 2.11 which span the representation γ_m of $Sp(1)$. More precisely, $\gamma_m \cong S^2(\mathbb{C}^2)$ where we identify $Sp(1) \cong SU(2)$. And under this identification each σ_k is identified with $\binom{m}{k} e_1^k e_2^{m-k}$ where e_1, e_2 are the standard basis of $\mathbb{C}^2$. By acting by $Sp(1)$ we may suppose that ξ is ξ_1.

By Proposition 2.11 the elements $h^{\beta,\xi}$ preserve the spinors corresponding to the span of e_1^m and e_2^m but not the remaining. Let $g \in SU(2)$ be such that $g\xi = \xi'$. Then the elements $h^{\beta,\xi'}$ preserve precisely the spinors $g(e_1^m)$ and $g(e_2^m)$. This is the same subspace as that spanned by e_1^m and e_2^m if and only if g is in the subgroup generated by the elements

$$\begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}, \text{ such that } |u| = 1, \quad \text{and} \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

This is precisely the subgroup fixing $\xi_1 \in \mathbb{RP}^2$.

For the second part recall that γ_2 is a real representation, with real Killing spinors

$$\begin{aligned} \varsigma_0 &= 1 + \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4 \\ \varsigma_1 &= i\varepsilon_1 \wedge \varepsilon_2 + i\varepsilon_3 \wedge \varepsilon_4 \\ \varsigma_2 &= i - i\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4. \end{aligned} \tag{100}$$

Again we may assume that ξ_1 is the standard Reeb vector field, thus h^{β_1,ξ_1} preserves σ_0 and σ_2. Suppose $\xi_2 = a\xi_1$ and $\xi_3 = b\xi_1$ where $a, b \in Sp(1)$. By assumption $\mathrm{Span}_{\mathbb{R}}\{\sigma_0, \sigma_2\} \cap \mathrm{Span}_{\mathbb{R}}\{a\sigma_0, a\sigma_2\}$ is 1-dimensional, and let σ be a non-zero element. Then $\sigma \notin \mathrm{Span}_{\mathbb{R}}\{b\sigma_0, b\sigma_2\}$. Then

$$\mathcal{L}(h^{\beta_1,\xi_1} + h^{\beta_2,\xi_2} + h^{\beta_3,\xi_3}, \sigma) = \mathcal{L}(h^{\beta_3,\xi_3}, \sigma) \neq 0$$

by Proposition 2.11. □

Proposition 4.2. *Let (M, g) be 3-Sasakian with $\dim M = 4m - 1$. Suppose $\xi, \xi' \in$ S^2 with $\xi \neq \xi'$ and $\xi \neq -\xi'$. And suppose $\beta \in \mathcal{H}^1_{\mathcal{A}}(\xi)$ and $\beta' \in \mathcal{H}^1_{\mathcal{A}}(\xi')$ are non-zero, then*

$$h^{\beta,\xi} + h^{\beta',\xi'} \in \mathscr{E}\mathscr{D}(g)$$

is non-zero and preserves a 1-dimensional subspace of Killing spinors if $m = 2$ and no Killing spinors if $m > 2$.

Proof. We may suppose that $\xi = \xi_1$ and $\xi' = \cos(t)\xi_1 + \sin(t)\xi_2$, $0 < t < \pi$, after possibly transforming by $Sp(1)$. Then $\xi' = \exp((t/2)\pi)_*\xi_1$. Set $a = \exp((t/2)\pi) \in Sp(1)$. By Lemma 4.1 $h^{\beta,\xi} + h^{\beta',\xi'} \neq 0$. Set $h_1 = h^{\beta,\xi}$ and $h^{\beta',\xi'} = ah_2$ with $h_2 \in \mathcal{H}^1_{\mathcal{A}}(\xi_1)$. Suppose

$$\begin{aligned} 0 = \mathcal{L}(h^{\beta,\xi} + h^{\beta',\xi'}, \sigma)(X) &= \mathcal{L}(h_1 + ah_2, \sigma)(X) \\ &= \mathcal{L}(h, \sigma)(X) + a\mathcal{L}(h_2, a^{-1}\sigma)(a^{-1}\sigma X). \end{aligned} \tag{101}$$

The component of interest in this is given by (77) which is

$$-\Phi_1 h(X)\xi_1 \cdot \sigma - h(X)\partial_r \cdot \sigma - \Phi' h'(X)\xi' \cdot \sigma - h'(X)\partial_r \cdot \sigma, \tag{102}$$

where for shorthand $h = h_1$, $h' = h^{\beta',\xi'}$ and $\Phi' = \cos(t)\Phi_1 + \sin(t)\Phi_2$. Here $\sigma = c_0\sigma_0 + \cdots + c_m\sigma_m$ is an arbitrary Killing spinor.

We consider the case $m > 2$ first. We compute (102) using the notation in the proof of Proposition 2.11. In particular,

$$\sigma_k = \frac{1}{k!}\vartheta^k + \frac{1}{(k-1)!}\vartheta^{k-1} \wedge \varepsilon_{2m-1} \wedge \varepsilon_{2m}.$$

The first two terms of (102) with $\sigma = \sigma_k$ are

$$\frac{2\sqrt{2}\sqrt{-1}}{(k-1)!}\big(\vartheta^{k-1}\wedge\Phi_2 h(X)^{0,1}\wedge\varepsilon_{2m}+\vartheta^{k-1}\wedge h(X)^{1,0}\wedge\varepsilon_{2m-1}\big). \tag{103}$$

The second two terms are

$$\begin{aligned}&-\big(\cos(t)\Phi_1 h'(X)+\sin(t)\Phi_2 h'(X)\big)\\ &\times\left(\frac{-\cos(t)}{\sqrt{2}}(\varepsilon 2m+\varepsilon\overline{2m})+\frac{\sqrt{-1}\sin(t)}{\sqrt{2}}(\varepsilon_{2m-1}-\varepsilon_{\overline{2m-1}})\right)\sigma_k\\ &-h'(X)\frac{\sqrt{-1}}{\sqrt{2}}(\varepsilon_{2m}-\varepsilon_{\overline{2m}})\sigma_k.\end{aligned} \tag{104}$$

After a routine computation we get that (102) with $\sigma=\sigma_k$ is

$$\begin{aligned}&-\frac{\sqrt{-1}\sqrt{2}\sin^2(t)}{k!}h'(X)^{1,0}\wedge\varepsilon_{2m}\vartheta^k+\frac{\sqrt{-1}2\sqrt{2}\cos^2(t)}{(k-1)!}\wedge\Phi_2 h'(X)^{1,0}\wedge\varepsilon_{2m}\wedge\vartheta^{k-1}\\ &+\frac{\sqrt{-1}2\sqrt{2}\cos^2(t)}{(k-1)!}h'(X)^{1,0}\wedge\varepsilon_{2m-1}\vartheta^{k-1}-\frac{\sqrt{-1}\sqrt{2}\sin^2(t)}{(k-2)!}\Phi_2 h'(X)^{1,0}\wedge\varepsilon_{2m-1}\wedge\vartheta^{k-2}\\ &+\frac{\sqrt{2}\sin(t)\cos(t)}{k!}h'(X)^{1,0}\varepsilon_{2m-1}\vartheta^k+\frac{2\sqrt{2}\sin(t)\cos(t)}{(k-1)!}\Phi_2 h'(X)^{1,0}\wedge\varepsilon_{2m-1}\wedge\vartheta^{k-1}\\ &+\frac{2\sqrt{2}\sin(t)\cos(t)}{(k-1)!}h'(X)^{1,0}\wedge\varepsilon_{2m}\wedge\vartheta^{k-1}+\frac{\sqrt{2}\sin(t)\cos(t)}{(k-2)!}\Phi_2 h'(X)^{1,0}\wedge\varepsilon_{2m}\wedge\vartheta^{k-2}\\ &+\frac{\sqrt{2}\sin(t)\cos(t)}{k!}\Phi_2 h'(X)^{1,0}\wedge\varepsilon_{2m}\wedge\vartheta^k+\frac{\sqrt{2}\sin(t)\cos(t)}{(k-2)!}h'(X)^{1,0}\wedge\vartheta_{2m-1}\wedge\vartheta^{k-2}\\ &-\frac{\sqrt{-1}\sqrt{2}\sin^2(t)}{k!}\Phi_2 h'(X)^{1,0}\wedge\varepsilon_{2m-1}\wedge\vartheta^k-\frac{\sqrt{-1}\sqrt{2}\sin^2(t)}{(k-2)!}h'(X)^{1,0}\wedge\varepsilon_{2m}\wedge\vartheta^{k-2}\\ &+\frac{\sqrt{-1}2\sqrt{2}}{(k-1)!}\Phi_2 h(X)^{1,0}\wedge\varepsilon_{2m}\wedge\vartheta^{k-1}+\frac{\sqrt{-1}2\sqrt{2}}{(k-1)!}h(X)^{1,0}\wedge\varepsilon_{2m-1}\wedge\vartheta^{k-1}.\end{aligned}$$

Consider the image of a general Killing spinor $\sigma=c_0\sigma_0+\cdots+c_m\sigma_m$ under (102). In particular, consider its component of degree $2k+2$ given by this formula for $0\le k\le m-2$. From the ε_{2m} and ε_{2m-1} components we get the following equations after some manipulation:

$$\begin{aligned}0&=c_k(\sqrt{2}\sin(t)\Phi' h'(X))\\ &\quad+c_{k+1}(2\sqrt{2}\cos^2(t)h'(X)-2\sqrt{2}\sin(t)\cos(t)\Phi_3 h'(X)+2\sqrt{2}h(X))\\ &\quad+c_{k+2}(\sqrt{2}\sin(t)\Phi' h'(X))\end{aligned}$$

and

$$\begin{aligned}0&=c_k(\sqrt{2}\sin(t)\Phi' h'(X))\\ &\quad+c_{k+1}(-2\sqrt{2}\cos^2(t)h'(X)+2\sqrt{2}\sin(t)\cos(t)\Phi_3 h'(X)-2\sqrt{2}h(X))\\ &\quad+c_{k+2}(\sqrt{2}\sin(t)\Phi' h'(X)).\end{aligned}$$

From these we get $c_k + c_{k+2} = 0$ and $c_{k+1}(\cos(t)\Phi' h'(X) + \Phi_1 h(X)) = 0$, which implies $c_{k+1} = 0$ from Lemma 4.1. This implies $\sigma = 0$ when $m > 2$.

If $m = 2$ then we have $c_1 = 0$ and $c_0 + c_2 = 0$. So the only possible Killing spinors preserved by $h^{\beta,\xi} + h^{\beta',\xi'}$ are spanned by the real spinor ς_2. And one easily sees that $\mathcal{L}(h^{\beta',\xi'}, \varsigma_2) = 0$ since $\exp(tk)\varsigma_2 = \varsigma_2$. □

Recall that γ_2 is the real representation of $Sp(1)$, and easy calculation shows that the standard basis of $\mathfrak{sp}(1)$ acts as follows in the basis $\varsigma_0, \varsigma_1, \varsigma_2$

$$i = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}, j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}, k = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

PROPOSITION 4.3. *Let (M, g) be a 7-dimensional 3-Sasakian manifold. Suppose $\xi_1, \xi_2, \xi_3 \in \mathrm{S}^2$ are linearly independent and $\beta_k \in \mathcal{H}^1_{\mathcal{A}}(\xi_k)$ $k = 1, 2, 3$ are each non-zero. Then*

$$h^{\beta_1,\xi_1} + h^{\beta_2,\xi_2} + h^{\beta_3,\xi_3} \in \mathscr{E}\mathscr{D}(g)$$

is non-zero and preserves no Killing spinors.

PROOF. By Lemma 4.1 $h^{\beta_1,\xi_1} + h^{\beta_2,\xi_2} + h^{\beta_3,\xi_3}$ is non-zero, so we need to show it preserves no Killing spinors.

For simplicity we assume that $\xi_k, k = 1, 2, 3$ is an orthonormal basis, which we may assume to be the standard basis after a possibly acting by $Sp(1)$. By considering the $Sp(1)$-action on γ_2, we see that $\mathcal{H}^1_{\mathcal{A}}(\xi_2)$ preserves ς_1, ς_2 and $\mathcal{H}^1_{\mathcal{A}}(\xi_3)$ preserves ς_0, ς_1. Let $\sigma = c_0\varsigma_0 + c_1\varsigma_1 + c_2\varsigma_2$, and denote $h^{\xi_k} = h^{\beta_k,\xi_k}$. Then suppose

$$\begin{aligned} 0 &= \mathcal{L}(h^{\beta_1,\xi_1} + h^{\beta_2,\xi_2} + h^{\beta_3,\xi_3}, \sigma) \\ &= c_1\mathcal{L}(h^{\beta_1,\xi_1}, \varsigma_1) + c_0\mathcal{L}(h^{\beta_2,\xi_2}, \varsigma_0) + c_2\mathcal{L}(h^{\beta_3,\xi_3}, \varsigma_2) \\ &= -c_1(\Phi_1 h^{\xi_1}(X)\xi_1 \cdot \varsigma_1 + h^{\xi_1}(X)\partial_r \cdot \varsigma_1) \\ &\quad - c_0(\Phi_2 h^{\xi_2}(X)\xi_2 \cdot \varsigma_0 + h^{\xi_2}(X)\partial_r \cdot \varsigma_0) \\ &\quad - c_2(\Phi_3 h^{\xi_3}(X)\xi_3 \cdot \varsigma_2 + h^{\xi_3}(X)\partial_r \cdot \varsigma_2). \end{aligned} \tag{105}$$

Routine calculation gives

$$\begin{aligned} &\Phi_1 h^{\xi_1}(X)\xi_1 \cdot \varsigma_1 + h^{\xi_1}(X)\partial_r \cdot \varsigma_1 = 2\sqrt{2}(\Phi_2 h^{\xi_1}(X)^{1,0} \wedge \varepsilon_4 + h^{\xi_1}(X)^{1,0} \wedge \varepsilon_3) \\ &\Phi_2 h^{\xi_2}(X)\xi_2 \cdot \varsigma_0 + h^{\xi_2}(X)\partial_r \cdot \varsigma_0 = 2\sqrt{2}\sqrt{-1}(\Phi_2 h^{\xi_2}(X)^{1,0} \wedge \varepsilon_3 + h^{\xi_2}(X)^{1,0} \wedge \varepsilon_4) \\ &\Phi_3 h^{\xi_3}(X)\xi_3 \cdot \varsigma_2 + h^{\xi_3}(X)\partial_r \cdot \varsigma_2 = 2\sqrt{2}(\Phi_2 h^{\xi_3}(X)^{1,0} \wedge \varepsilon_3 - h^{\xi_3}(X)^{1,0} \wedge \varepsilon_4). \end{aligned}$$

Thus we have

$$\begin{aligned} 0 = &- c_1 2\sqrt{2}(\Phi_2 h^{\xi_1}(X)^{1,0} \wedge \varepsilon_4 + h^{\xi_1}(X)^{1,0} \wedge \varepsilon_3) \\ &- c_0 2\sqrt{2}\sqrt{-1}(\Phi_2 h^{\xi_2}(X)^{1,0} \wedge \varepsilon_3 + h^{\xi_2}(X)^{1,0} \wedge \varepsilon_4) \end{aligned}$$

$$- c_2 2\sqrt{2}(\Phi_2 h^{\xi_3}(X)^{1,0} \wedge \varepsilon_3 - h^{\xi_3}(X)^{1,0} \wedge \varepsilon_4).$$

The ε_3 component gives

$$c_1\Phi_1 h^{\xi_1}(X) - c_0\Phi_2 h^{\xi_2}(X) + c_2\Phi_3 h^{\xi_3}(X) = 0.$$

Lemma 4.1 now implies that $c_0 = c_1 = c_2 = 0$. □

This proves Corollary 4. By Theorem 3.3 for any $\beta \in \operatorname{Re}\mathcal{H}^1_{\mathcal{A}}(\xi)$ the deformation $h^{\beta,\xi}$ is integrable. By Proposition 4.2 for $m > 2$, and Proposition 4.3 for $m = 2$ there are elements in the span of these elements preserving no Killing spinors.

4.2. Toric 3-Sasakian manifolds.

The examples of toric 3-Sasakian 7-manifolds from [**9**] provide interesting examples of Einstein deformations, integrable and infinitesimal, preserving various numbers of Killing spinors. This will give non-trivial examples of the theorems of the previous sections.

Definition 4.4. A 3-Sasakian manifold (M, g), $\dim M = 4m-1$, is *toric* if there is a $T^m \subseteq \operatorname{Aut}(M, g, \xi_1, \xi_2, \xi_3)$.

Remark 4.5. Note that a toric 3-Sasakian manifold is generally not toric as a Sasakian manifold.

The isometry group of a 3-Sasakian manifold is

$$\operatorname{Aut}(M, g, \xi_1, \xi_2, \xi_3) \times Sp(1) \quad \text{or} \quad \operatorname{Aut}(M, g, \xi_1, \xi_2, \xi_3) \times SO(3),$$

where the $Sp(1)$ or $SO(3)$ factor is generated by the Reeb vector fields.

Toric 3-Sasakian manifolds have been constructed from 3-Sasakian quotients by torus actions on S^{4n-1} [**7**], [**9**], with the 3-Sasakian structure given by right multiplication by $Sp(1)$. A subtorus $T^k \subset T^n$ is determined by a weight matrix $\Omega_{k,n} \in \operatorname{Mat}(k, n, \mathbb{Z})$. There are conditions on Ω, Boyer, Galicki, Mann, Rees, 1998 [**9**], that imply the moment map

$$\mu : S^{4n-1} \to (\mathfrak{t}^k)^* \otimes \mathbb{R}^3$$

is a submersion, and further that the quotient

$$M_{\Omega_{k,n}} = S^{4n-1} /\!\!/ T^k = \mu^{-1}(0)/T^k$$

is smooth. When $n = k+2$ the above authors showed there are infinitely many weight matrices in $\operatorname{Mat}(k, n, \mathbb{Z})$ for $k \geq 1$ giving infinitely many 7-manifolds $M_{\Omega_{k,n}}$ for each $b_2 = k \geq 1$.

Lemma 4.6 ([**38**]). *Let Z be the twistor space of a toric 3-Sasakian 7-manifold M, then $H^1(Z, \Theta_Z) = H^1(Z, \Theta_Z)^{T^2}$ and*

$$\dim_{\mathbb{C}} H^1(Z, \Theta_Z) = b_2(M) - 1 = k - 1.$$

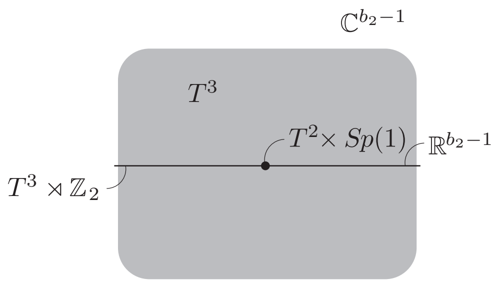

Figure 1. Space of Sasaki–Einstein metrics.

Thus Z has a local $b_2(M)-1$-dimensional space of deformations.

If $b_2(M) \geq 1$, then the maximal torus of Sasakian automorphisms, $T^3 \subset \mathrm{Aut}(M, \xi_1)$, is 3-dimensional. Theorem 3.1 implies the following.

THEOREM 4.7. *Let (M, g) be a toric 3-Sasakian 7-manifold. Then (M, g) has a 3-dimensional space of Killing spinors spanned by $\sigma_0, \sigma_1, \sigma_2$. Then g is in an effective complex $b_2(M)-1$-dimensional family $\{g_t\}_{t\in\mathcal{U}}$, $\mathcal{U} \subset \mathbb{C}^{b_2(M)-1}$ with $g_0 = g$, of Sasaki–Einstein metrics where g_t is not 3-Sasakian for $t \neq 0$.*

Therefore the deformations preserve a two dimensional subspace of Killing spinors spanned by σ_0, σ_2.

The deformation space of Sasaki–Einstein metrics with their isometry groups is illustrated in Figure 1.

For a given $\xi \in \mathrm{S}^2$, the space of infinitesimal Einstein deformations $\{h^{\beta,\xi} \mid \beta \in \mathcal{H}^1_{\mathcal{A}}(\xi)\} \subseteq \mathcal{EED}(g)$ integrate to Einstein deformations preserving Killing spinors σ_0 and σ_2 but not σ_1. Note that the space $\mathscr{E}\mathscr{D}(g)$ defined in (99) is spanned by integrable Einstein deformations. Theorem 2 now follows from Proposition 4.2 and Proposition 4.3.

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