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Locally standard torus actions and h'-numbers of simplicial posets

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Abstract. We consider the orbit type filtration on a manifold with a locally standard torus action and study the corresponding spectral sequence in homology. When all proper faces of the orbit space are acyclic and the free part of the action is trivial, this spectral sequence can be described in full. The ranks of diagonal terms of its second page are equal to h'-numbers of a simplicial poset dual to the orbit space. Betti numbers of a manifold with a locally standard torus action are computed: they depend on the combinatorics and topology of the orbit space but not on the characteristic function.

A toric space whose orbit space is the cone over a Buchsbaum simplicial poset is studied by the same homological method. In this case the ranks of the diagonal terms of the spectral sequence at infinity are the h''-numbers of the simplicial poset. This fact provides a topological evidence for the nonnegativity of h''-numbers of Buchsbaum simplicial posets and links toric topology to some recent developments in enumerative combinatorics.

1. Introduction.

An action of a compact torus T^n on a smooth compact manifold M of dimension $2n$ is called locally standard if it is locally modeled on the standard representation of T^n on $\mathbb{C}^n$. The orbit space $Q = M/T^n$ is a manifold with corners. Every manifold with a locally standard torus action is equivariantly homeomorphic to the quotient construction $X = Y/\sim$, where Y is a principal T^n-bundle over Q and $\sim$ is the equivalence relation determined by the characteristic function on Q [**13**].

In the case when all faces of the orbit space (including Q itself) are acyclic, Masuda and Panov [**9**] proved that $H^*_T(M;\mathbb{Z}) \cong \mathbb{Z}[S_Q]$ and $H^*(M;\mathbb{Z}) \cong \mathbb{Z}[S_Q]/(l.s.o.p)$, where S_Q is a simplicial poset dual to Q; $\mathbb{Z}[S_Q]$ is the face ring with even grading; and $(l.s.o.p)$ is a linear system of parameters determined by the characteristic function. In this situation S_Q is a Cohen–Macaulay simplicial poset, so $(l.s.o.p)$ is actually a regular sequence in the ring $\mathbb{Z}[S_Q]$. In particular this implies $H^{2j+1}(M) = 0$ and $\dim H^{2j}(M) = h_j(S_Q)$, where the h-numbers are determined by the combinatorics of Q.

These considerations generalize similar results for quasitoric manifolds, complete smooth toric varieties, and symplectic toric manifolds which were known before. One can see that there are many examples of manifolds M whose orbit spaces are acyclic. Nevertheless, several constructions have appeared in the last years providing natural

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examples of manifolds with torus actions whose orbit spaces have nontrivial topology. Among others, these constructions include toric origami manifolds [**5**] and toric log symplectic manifolds [**7**].

It seems that the most reasonable assumption which is weaker than acyclicity of all faces but still allows for explicit calculations is as follows. We assume that every *proper* face of Q is acyclic, and Y is a trivial T^n-bundle: $Y = Q \times T^n$ (this condition may be replaced by homological triviality: $H_*(Y) \cong H_*(Q) \otimes H_*(T^n)$). This is the second paper in the series of works, where we study homological structure of M under these assumptions by using the orbit type filtration. In [**1**] we proved several technical results, which will be used in this work.

It is convenient to work with the quotient construction $X = (Q \times T^n)/\sim$ instead of M. The orbit type filtration $X_0 \subset X_1 \subset \cdots \subset X_n$ covers the natural filtration $Q_0 \subset Q_1 \subset \cdots \subset Q_n$ on Q, and is covered by a filtration $Y_0 \subset Y_1 \subset \cdots \subset Y_n$ on Y, where $Y_i = Q_i \times T^n$. Previously we proved that homological spectral sequences associated with filtrations on Y and X are closely related. More precisely, there is an isomorphism of the second pages $f_*^2\colon (E_Y)^2_{p,q} \to (E_Y)^2_{p,q}$ for $p > q$, when Q has acyclic proper faces.

In this paper we compute the ranks of groups in the spectral sequence and the Betti numbers of X. Since Y is just the direct product $Q \times T^n$, the spectral sequence $(E_Y)^*_{*,*}$ is isomorphic to $(E_Q)^*_{*,*} \otimes H_*(T^n)$. The structure of $(E_Q)^*_{*,*}$ can be explicitly described. This is done in Section 3. As a technical tool, we introduce the modified spectral sequence $(\dot{E}_Q)^*_{*,*}$ which coincides with $(E_Q)^*_{*,*}$ from the second page, and whose first page $(\dot{E}_Q)^1_{*,*}$, in certain sense, lies between $(E_Q)^1_{*,*}$ and $(E_Q)^2_{*,*}$. Similar constructions of modified spectral sequences $(\dot{E}_Y)^*_{*,*}$ and $(\dot{E}_X)^*_{*,*}$ are introduced for the spaces Y and X in Section 4.

The induced map $\dot{f}_*^1\colon (\dot{E}_Y)^1_{p,q} \to (\dot{E}_X)^1_{p,q}$ is an isomorphism for $p > q$, as follows essentially from the result of [**1**]. This gives a description of all differentials and all non-diagonal terms of $(\dot{E}_X)^1_{*,*}$, which is stated in detail in Theorem 4.3. The diagonal terms of the spectral sequence are considered separately. We prove, in particular, that $\dim(E_X)^2_{q,q} = \dim(\dot{E}_X)^2_{q,q} = h'_{n-q}(S_Q)$, the h'-number of the dual simplicial poset (Theorem 4.6). The proof is purely combinatorial and is placed in a separate Section 5, where we give all necessary definitions from the combinatorial theory of simplicial posets. The appearance of h'-numbers here is quite natural. If Q has acyclic proper faces, the dual simplicial poset S_Q is Buchsbaum. Recall that h'- and h''-numbers are combinatorial notions which were specially devised to study the combinatorics of Buchsbaum simplicial complexes.

In Section 6 we introduce the bigraded structure on $H_*(X)$ and compute the bigraded Betti numbers (Theorem 6.2). Bigraded Poincare duality easily follows from this computation.

Most of the arguments used for manifolds with locally standard actions work equally well for the space $X = (P \times T^n)/\sim$ where P is the cone over Buchsbaum simplicial poset equipped with the dual face structure. In this case we have $\dim(E_X)^\infty_{q,q} = h''_q(S)$ (Theorem 4.7). Every simplicial poset admits a characteristic function over rational numbers. Hence our theorem implies that $h''_q(S) \geq 0$ for any Buchsbaum simplicial poset S. This result was proved by Novik and Swartz in [**11**] by a different method.

Both manifolds with acyclic proper faces and cones over Buchsbaum posets are unified in the notion of Buchsbaum pseudo-cell complex introduced in Section 2. The technique developed in the paper can be applied to any Buchsbaum pseudo-cell complex.

In the last section we analyze a simple example which shows that, when the assumption of proper face acyclicity is dropped, the problem of computing Betti numbers of X is more complicated. In general, Betti numbers of X may depend not only on the orbit space Q, but also on the characteristic function.

2. Preliminaries.

2.1. Coskeleton filtrations and manifolds with corners.

DEFINITION 2.1. A finite partially ordered set (poset in the following) is called *simplicial* if there is a minimal element $\hat{0} \in S$ and, for any $I \in S$, the lower interval $\{J \in S \mid J \leq I\}$ is isomorphic to the poset of faces of a $(k-1)$-simplex, for some number $k \geq 0$.

The elements of S are called *simplices*. The number k in the definition is denoted by $|I|$ and is called the rank of I. Also set $\dim I = |I| - 1$. A simplex of rank 1 is called a *vertex*; the set of all vertices is denoted $\mathrm{Vert}(S)$. The *link* of a simplex $I \in S$ is the set $\mathrm{lk}_S I = \{J \in S \mid J \geq I\}$. This set inherits the order relation from S, and $\mathrm{lk}_S I$ is a simplicial poset on its own, with I being the minimal element. Let S' denote the barycentric subdivision of S. By definition, S' is a simplicial complex on the vertex set $S \setminus \{\hat{0}\}$ whose simplices are the ordered chains in $S \setminus \{\hat{0}\}$. The geometric realization of S is the geometric realization of its barycentric subdivision $|S| \overset{\text{def}}{=} |S'|$. One can also think of $|S|$ as a CW-complex with simplicial cells (such complexes were called *simplicial cell complexes* in [**3**]). A poset S is called *pure* if all its maximal elements have equal dimensions. A poset S is pure whenever S' is pure.

Let $\Bbbk$ denote the ground ring, which may be either $\mathbb{Z}$ or a field. The term "(co)homology of simplicial poset" means the (co)homology of its geometrical realization. If the coefficient ring in the notation of (co)homology is omitted, it is supposed to be $\Bbbk$. The rank of a $\Bbbk$-module A is denoted by $\dim A$.

DEFINITION 2.2. A simplicial poset S of dimension $n-1$ is called *Buchsbaum* (over $\Bbbk$) if $\widetilde{H}_i(\mathrm{lk}_S I; \Bbbk) = 0$ for all $\hat{0} \neq I \in S$ and $i \neq n-1-|I|$. If S is Buchsbaum and, moreover, $\widetilde{H}_i(S; \Bbbk) = 0$ for $i \neq n-1$, then S is called *Cohen–Macaulay* (over $\Bbbk$).

By abuse of terminology we call S a *homology manifold* of dimension $n-1$ if its geometric realization $|S|$ is a homology $(n-1)$-manifold. Simplicial poset S is a homology manifold if and only if it is Buchsbaum and, moreover, its local homology stack of highest degree is isomorphic to a constant sheaf (the details are discussed in [**1**]).

If S is Buchsbaum and connected, then S is pure. In the following we consider only pure posets, and assume $\dim S = n-1$.

CONSTRUCTION 2.3. For any pure simplicial poset S, there is an associated space $P(S) = \mathrm{Cone}|S|$ endowed with the dual face structure (also called coskeleton structure), defined as follows. The complex $P(S)$ is a simplicial complex on the set S and k-simplices of $P(S)$ have the form $(I_0 < I_1 < \cdots < I_k)$, where $I_j \in S$. For each $I \in S$ consider the subsets:

$$G_I = |\{(I_0 < I_1 < \cdots) \in S' \text{ such that } I_0 \geq I\}| \subset P(S),$$

$$\partial G_I = |\{(I_0 < I_1 < \cdots) \in S' \text{ such that } I_0 > I\}| \subset P(S).$$

and the subset $G_I^\circ = G_I \setminus \partial G_I$. We have $G_{\hat{0}} = P(S)$; $G_I \subset G_J$ whenever $J < I$, and $\dim G_I = n - |I|$, since S is pure. The subset G_I is called the *dual face* of a simplex $I \in S$. A subset ∂G_I is a union of faces of smaller dimensions.

Recall several facts about manifolds with corners. A smooth connected manifold with corners Q is called *nice* (or a *manifold with faces*) if every codimension k face lies in exactly k distinct facets. In the following we consider only nice compact orientable manifolds with corners. Any such Q determines a simplicial poset S_Q whose elements are the faces of Q ordered by reversed inclusion. The whole Q is the maximal face of itself, thus represents the minimal element of S_Q.

DEFINITION 2.4. A nice manifold with corners Q is called Buchsbaum if Q is orientable and every proper face of Q is acyclic. If, moreover, Q is acyclic itself, it is called Cohen–Macaulay.

If Q is a Buchsbaum manifold with corners, then its underlying simplicial poset S_Q is Buchsbaum (moreover, S_Q is a homology manifold), and when Q is Cohen–Macaulay, then so is S_Q (moreover, S_Q is a homology sphere) by [**1**, Lemma 6.2].

2.2. Buchsbaum pseudo-cell complexes.

It is convenient to introduce a notion generalizing both manifolds with corners and cones over pure simplicial posets.

CONSTRUCTION 2.5 (Pseudo-cell complex). A CW-pair $(F, \partial F)$ will be called k-dimensional pseudo-cell, if F is compact and connected, $\dim F = k$, $\dim \partial F \leq k - 1$. A (regular finite) *pseudo-cell complex* Q is a space which is a union of an expanding sequence of subspaces Q_k such that Q_{-1} is empty and Q_k is the pushout obtained from Q_{k-1} by attaching finite number of k-dimensional pseudo-cells $(F, \partial F)$ along injective attaching CW-maps $\partial F \to Q_{k-1}$. We assume that the boundary of each pseudo-cell is a union of lower dimensional pseudo-cells. The poset of pseudo-cells, ordered by the reversed inclusion is denoted by S_Q. The abstract elements of S_Q are denoted by I, J, etc. and the corresponding pseudo-cells considered as subsets of Q are denoted by F_I, F_J, etc.

A pseudo-cell complex Q, of dimension n is called *simple* if S_Q is a simplicial poset of dimension $n-1$ and $\dim F_I = n - |I|$ for any $I \in S_Q$. In particular, the space Q itself represents the maximal pseudo-cell, $Q = F_{\hat{0}}$. Pseudo-cells of a simple pseudo-cell complex Q will be called *faces*, faces different from Q—*proper faces*, and maximal proper faces—*facets*. Facets correspond to vertices of S_Q.

Nice manifolds with corners and cones over simplicial posets stratified by dual faces are examples of simple pseudo-cell complexes. Simple polytopes are the examples which may be considered as both cones over simplicial poset and manifolds with corners.

Every regular finite CW-complex Q is a pseudo-cell complex. If Q has more than one maximal cell, then Q is an example of non-simple pseudo-cell complex.

DEFINITION 2.6. A simple pseudo-cell complex Q is called Buchsbaum (over $\Bbbk$) if, for any proper face $F_I \subset Q$, $I \neq \hat{0}$, the following conditions hold:

1. F_I is acyclic, $\widetilde{H}_*(F_I;\Bbbk) = 0$;
2. $H_j(F_I, \partial F_I;\Bbbk) = 0$ for each $j \neq \dim F_I$.

Buchsbaum complex Q is called Cohen–Macaulay (over $\Bbbk$) if these two conditions also hold for the maximal face $F_{\hat{0}} = Q$.

Both Buchsbaum manifolds and cones over Buchsbaum posets are examples of Buchsbaum pseudo-cell complexes (and the same for Cohen–Macaulay property). Indeed, for Buchsbaum manifolds with corners $\widetilde{H}_*(F_I)$ vanishes by definition, and $H_*(F_I, \partial F_I)$ vanishes in the required degrees by the Poincare–Lefschetz duality, since every face F_I is an orientable manifold with boundary. In the cone case we have $G_I = \operatorname{Cone}(\partial G_I)$ and $\partial G_I \cong |\operatorname{lk}_S I|$, so the conditions of Definition 2.6 follow from the isomorphism $H_*(G_I, \partial G_I) \cong H_{*-1}(\partial G_I)$ and Definition 2.2.

On the other hand, the converse holds. If Q is a manifold with corners, having non-acyclic proper faces, or a cone over non-Buchsbaum simplicial poset, then Q is a non-Buchsbaum simple pseudo-cell complex.

For a general simple pseudo-cell complex we have the filtration

$$Q_0 \subset Q_1 \subset \cdots \subset Q_{n-1} \subset Q_n = Q;$$

and the truncated filtration

$$Q_0 \subset Q_1 \subset \cdots \subset Q_{n-1} = \partial Q,$$

where Q_j is the union of j-dimensional faces. The homological spectral sequences associated with these filtrations are denoted by $(E_Q)^r_{p,q}$ and $(E_{\partial Q})^r_{p,q}$ respectively. The same argument as in [**1**, Lemma 6.2] proves the following

PROPOSITION 2.7. (1) *Let Q be a Buchsbaum pseudo-cell complex, S_Q be its underlying poset, and $P = P(S_Q)$ be the cone complex. Then there exists a face-preserving map $\varphi\colon Q \to P$ which induces the identity isomorphism of posets of faces and an isomorphism of the truncated spectral sequences $\varphi_*\colon (E_{\partial Q})^r_{*,*} \xrightarrow{\cong} (E_{\partial P})^r_{*,*}$ for $r \geq 1$. In particular, it follows that S_Q is a Buchsbaum simplicial poset.*

(2) *If Q is a Cohen–Macaulay pseudo-cell complex of dimension n, then φ induces an isomorphism of non-truncated spectral sequences $\varphi_*\colon (E_Q)^r_{*,*} \xrightarrow{\cong} (E_P)^r_{*,*}$ for $r \geq 1$. In particular, it follows that S_Q is a Cohen–Macaulay simplicial poset.*

Thus all homological information about Buchsbaum pseudo-cell complex Q away from its maximal cell is encoded in the underlying poset S_Q. This makes Buchsbaum pseudo-cell complexes and in particular Buchsbaum manifolds with corners natural candidates for study.

3. Spectral sequence of Q.

3.1. Truncated and non-truncated spectral sequences.

In Buchsbaum case the spectral sequence $(E_Q)^r_{p,q}$ can be described explicitly. We have $(E_Q)^r_{p,q} \Rightarrow H_{p+q}(Q)$, the differentials act as $(d_Q)^r\colon (E_Q)^r_{p,q} \to (E_Q)^r_{p-r,q+r-1}$, and

$$(E_Q)^1_{p,q} \cong H_{p+q}(Q_p, Q_{p-1}) \cong \bigoplus_{I,\dim F_I = p} H_{p+q}(F_I, \partial F_I).$$

By the definition of Buchsbaum pseudo-cell complex we have $(E_Q)^1_{p,q} = 0$ unless $q = 0$ or $p = n$. Such form of the spectral sequence will be referred to as $\urcorner$-shaped.

By forgetting the last term of the filtration we get the spectral sequence $(E_{\partial Q})^r_{p,q} \Rightarrow H_{p+q}(\partial Q)$, whose terms vanish unless $q = 0$. Thus $(E_{\partial Q})^r_{p,q}$ collapses at a second page, giving the isomorphism $(E_{\partial Q})^2_{p,0} \cong H_p(\partial Q)$.

In the non-truncated case we have $(E_Q)^2_{p,0} \cong (E_{\partial Q})^2_{p,0}$ for $p \neq n, n-1$. The terms $(E_Q)^2_{n,q}$ coincide with $(E_Q)^1_{n,q} \cong H_{n+q}(Q, \partial Q)$ when $q \neq 0$. The term $(E_Q)^2_{n-1,0}$ differs from $(E_{\partial Q})^2_{n-1,0} \cong H_{n-1}(\partial Q)$ by the image of the first differential $(d_Q)^1$ which hit it at the previous step. Similarly, the term $(E_Q)^2_{n,0}$ is the kernel of the same differential. To avoid mentioning these two exceptional cases every time in the following, we introduce the formalism of modified spectral sequence.

3.2. Modified spectral sequence.

Let $(\dot{E}_Q)^1_{*,*}$ be the collection of $\Bbbk$-modules defined by

$$(\dot{E}_Q)^1_{p,q} \stackrel{\text{def}}{=} \begin{cases} (E_{\partial Q})^2_{p,q}, & \text{if } p \leq n-1, \\ (E_Q)^1_{p,q}, & \text{if } p = n, \\ 0, & \text{otherwise.} \end{cases}$$

Let d_Q^- be the differential of degree $(-1,0)$ acting on $\bigoplus (E_Q)^1_{p,q}$ by:

$$d_Q^- = \begin{cases} (d_Q)^1\colon (E_Q)^1_{p,q} \to (E_Q)^1_{p-1,q}, & \text{if } p \leq n-1, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily seen that the bigraded homology module $H((E_Q)^1_{*,*}, d_Q^-)$ is isomorphic to $(\dot{E}_Q)^1_{*,*}$. Now consider the differential $(\dot{d}_Q)^1$ of degree $(-1,0)$ acting on $\bigoplus (\dot{E}_Q)^1_{p,q}$:

$$(\dot{d}_Q)^1 = \begin{cases} 0, & \text{if } p \leq n-1, \\ (E_Q)^1_{n,q} \xrightarrow{(d_Q)^1} (E_Q)^1_{n-1,q}, & \text{if } p = n. \end{cases}$$

In the latter case, the image of the differential lies in $(\dot{E}_Q)^1_{n-1,q} \subseteq (E_Q)^1_{n-1,q}$ since $(\dot{E}_Q)^1_{n-1,q}$ is just the kernel of $(d_Q)^1$. We have $(E_Q)^2_{*,*} \cong H((\dot{E}_Q)^1_{*,*}, (\dot{d}_Q)^1)$. These considerations are shown on the diagram:

$$\begin{tikzcd}[column sep=small]
 & (\dot{E}_Q)^1 \arrow[dr, dotted, "(\dot{d}_Q)^1"] & & & \\
(E_Q)^1 \arrow[ur, dotted, "d_Q^-"] \arrow[rr, dotted, "(d_Q)^1"] & & (E_Q)^2 \arrow[r, dotted, "(d_Q)^2"] & (E_Q)^3 \arrow[r, dotted, "(d_Q)^3"] & \cdots
\end{tikzcd}$$

in which the dotted arrows represent passing to homology. To summarize:

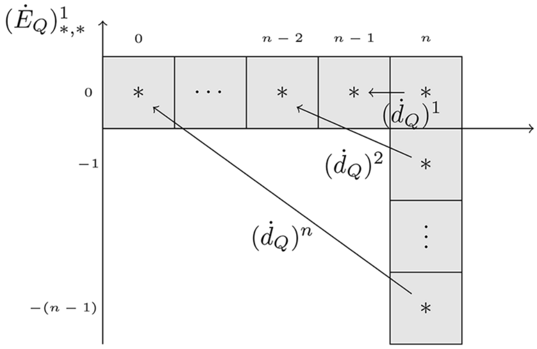

Figure 1. The shape of the spectral sequence.

Claim 3.1. *There is a homological spectral sequence $(\dot{E}_Q)^r_{p,q} \Rightarrow H_{p+q}(Q)$ such that $(\dot{E}_Q)^1_{*,*} = H((E_Q)^1_{*,*}, d_Q^-)$, and $(\dot{E}_Q)^r_{*,*} = (E_Q)^r_{*,*}$ for $r \geq 2$. The only nontrivial differentials of this sequence have the form*

$$(\dot{d}_Q)^r \colon (\dot{E}_Q)^r_{n,1-r} \to (\dot{E}_Q)^r_{n-r,0}$$

for $r \geq 1$ (see Figure 1).

The differentials have distinct domains and targets. Thus the whole spectral sequence $(E_Q)^r_{p,q} \Rightarrow H_{p+q}(Q)$ folds into a single long exact sequence, which is isomorphic to the long exact sequence of the pair $(Q, \partial Q)$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1-r}(Q) \arrow[r] \arrow[dd, equal] & (\dot{E}_Q)^r_{n,1-r} \arrow[r, "(\dot{d}_Q)^r"] \arrow[d, equal] & (\dot{E}_Q)^r_{n-r,0} \arrow[r] \arrow[d, equal] & H_{n-r}(Q) \arrow[r] \arrow[dd, equal] & \cdots \\
 & & (\dot{E}_Q)^1_{n,1-r} \arrow[d, equal] & (\dot{E}_Q)^1_{n-r,0} \arrow[d, "\cong"] & & \\
\cdots \arrow[r] & H_{n+1-r}(Q) \arrow[r] & H_{n+1-r}(Q, \partial Q) \arrow[r, "\delta_{n+1-r}"] & H_{n-r}(\partial Q) \arrow[r] & H_{n-r}(Q) \arrow[r] & \cdots
\end{tikzcd} \tag{3.1}$$

In particular, the differentials $(d_Q)^r \colon (E_Q)^r_{n,1-r} \to (E_Q)^r_{n-r,0}$ coincide up to isomorphism with the connecting homomorphisms

$$\delta_{n+1-r} \colon H_{n+1-r}(Q, \partial Q) \to H_{n-r}(\partial Q).$$

This proves

PROPOSITION 3.2. *Up to isomorphism, the spectral sequence* $(\dot{E}_Q)^r_{*,*} \Rightarrow H_*(Q)$ *has the form shown on Figure* 2.

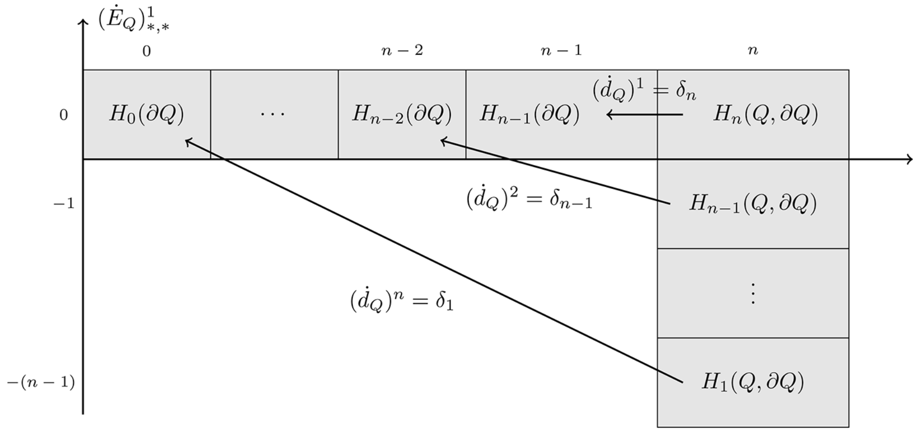

Figure 2. Modified spectral sequence for Q.

4. Quotient construction and its spectral sequence.

4.1. Quotient construction.

Let T^n denote the compact torus, and Λ_* be its homology algebra, $\Lambda_* = \bigoplus_{j=0}^n \Lambda_j$, $\Lambda_j = H_j(T^n;\Bbbk)$. Let Q be a simple pseudo-cell complex of dimension n, and S_Q be its dual simplicial poset. The map

$$\lambda\colon \mathrm{Vert}(S_Q) \to \{\text{1-dimensional toric subgroups of } T^n\}$$

is called a *characteristic function* if the following condition (so called $(*)$-condition) holds: whenever $i_1,\ldots,i_k$ are the vertices of some simplex in S_Q, the map

$$\lambda(i_1)\times\cdots\times\lambda(i_k)\to T^n \tag{4.1}$$

induced by inclusions $\lambda(i_j)\hookrightarrow T^n$ is injective and splits. Note that $i_1,\ldots,i_k$ are the vertices of a simplex if and only if $F_{i_1}\cap\cdots\cap F_{i_k}\neq\emptyset$. Denote the image of the map (4.1) by T_I, where I is a simplex with the vertices $i_1,\ldots,i_k$.

It follows from the $(*)$-condition that the map

$$H_1(\lambda(F_1)\times\cdots\times\lambda(F_k);\Bbbk)\to H_1(T^n;\Bbbk) \tag{4.2}$$

is injective and splits for every $\Bbbk$. If the map (4.2) splits for a specific ground ring $\Bbbk$, we say that λ satisfies $(*_\Bbbk)$-condition and call it a $\Bbbk$-characteristic function. It is easy to see that the topological $(*)$-condition is equivalent to $(*_\mathbb{Z})$, and that $(*_\mathbb{Z})$ implies $(*_\Bbbk)$ for any $\Bbbk$.

For a simple pseudo-cell complex Q of dimension n, consider the space $Y = Q\times T^n$.

CONSTRUCTION 4.1. For any $\Bbbk$-characteristic function λ over Q consider the quotient construction

$$X = Y/\sim = (Q \times T^n)/\sim,$$

where $(q_1, t_1) \sim (q_2, t_2)$ if and only if $q_1 = q_2 \in F_I^\circ$ for some $I \in S_Q$ and $t_1 t_2^{-1} \in T_I$. The action of T^n on the second coordinate of Y descends to the action on X. The orbit space of this action is Q and the stabilizer of the point $q \in F_I^\circ \subset Q$ is T_I. Let f denote the canonical quotient map, $f\colon Y \to X$.

The filtration on Q induces filtrations on Y and X:

$$Y_i = Q_i \times T^n, \quad X_i = Y_i/\sim, \quad i = 0, \ldots, n.$$

The filtration $X_0 \subset X_1 \subset \cdots \subset X_n = X$ is the orbit type filtration on X. This means that X_i is the union of all torus orbits of dimension at most i. We have $\dim X_i = 2i$. We will use the following notation

$$\begin{aligned} Y_I &= F_I \times T^n, & \partial Y_I &= (\partial F_I) \times T^n, \\ X_I &= Y_I/\sim, & \partial X_I &= \partial Y_I/\sim \end{aligned}$$

for $I \in S_Q$. Note that ∂X_I does not have the meaning of topological boundary of X_I, this is just a notational convention. Since $\partial Q = Q_{n-1}$, we have $\partial Y = (\partial Q) \times T^n = Q_{n-1} \times T^n = Y_{n-1}$ and $\partial X = \partial Y/\sim = Y_{n-1}/\sim = X_{n-1}$.

Consider the homological spectral sequences

$$\begin{aligned} (E_Y)^r_{p,q} &\Rightarrow H_{p+q}(Y), & (E_X)^r_{p,q} &\Rightarrow H_{p+q}(X), \\ (E_{\partial Y})^r_{p,q} &\Rightarrow H_{p+q}(\partial Y), & (E_{\partial X})^r_{p,q} &\Rightarrow H_{p+q}(\partial X), \end{aligned}$$

associated with these filtrations. The canonical map $f\colon Y \to X$ induces the morphisms $f_*^r\colon (E_Y)^r_{*,*} \to (E_X)^r_{*,*}$ and $f_*^r\colon (E_{\partial Y})^r_{*,*} \to (E_{\partial X})^r_{*,*}$.

Since homology groups of the torus are torsion free, we have

$$(E_Y)^r_{p,q} \cong \bigoplus_{q_1+q_2=q} (E_Q)^r_{p,q_1} \otimes \Lambda_{q_2}, \tag{4.3}$$

for $r \geq 1$ by Kunneth's formula. Similarly,

$$(E_{\partial Y})^r_{p,q} \cong \bigoplus_{q_1+q_2=q} (E_{\partial Q})^r_{p,q_1} \otimes \Lambda_{q_2}.$$

4.2. Modified spectral sequences.

Absolutely similar to the case of Q, we introduce the modified spectral sequences $(\dot{E}_Y)_{*,*}$ and $(\dot{E}_X)_{*,*}$. Consider the bigraded module:

$$(\dot{E}_Y)^1_{p,q} = \begin{cases} (E_{\partial Y})^2_{p,q}, & \text{if } p < n, \\ (E_Y)^1_{n,q}, & \text{if } p = n. \end{cases}$$

and define the differentials $d_Y^-\colon (E_Y)^1_{p,q} \to (E_Y)^1_{p-1,q}$ and $(\dot{d}_Y)^1\colon (\dot{E}_Y)^1_{p,q} \to (\dot{E}_Y)^1_{p-1,q}$ by

$$d_Y^- = \begin{cases} (d_Y)^1, & \text{if } p < n, \\ 0, & \text{if } p = n. \end{cases} \qquad (\dot{d}_Y)^1 = \begin{cases} 0, & \text{if } p < n, \\ (E_Y)^1_{n,q} \xrightarrow{(d_Y)^1} (E_Y)^1_{n-1,q} & \text{if } p = n. \end{cases}$$

It is easy to see that $(\dot{E}_Y)^1_{*,*} \cong H((E_Y)^1_{*,*}, d_Y^-)$ and $(E_Y)^2_{*,*} \cong H((\dot{E}_Y)^1_{*,*}, (\dot{d}_Y)^1)$. Let $(\dot{E}_Y)^r_{*,*} = (E_Y)^r_{*,*}$ for $r \geq 2$. Thus there is a modified spectral sequence $(\dot{E}_Y)^r_{*,*} \Rightarrow H_*(Y)$ and for $r \geq 1$ there holds

$$(\dot{E}_Y)^r_{p,q} \cong \bigoplus_{q_1+q_2=q} (\dot{E}_Q)^r_{p,q_1} \otimes \Lambda_{q_2}. \tag{4.4}$$

The same construction applies to X and gives the spectral sequence $(\dot{E}_X)^r_{*,*} \Rightarrow H_*(X)$ such that $(\dot{E}_X)^1_{*,*} \cong H((E_X)^1_{*,*}, d_X^-)$, and $(\dot{E}_X)^r_{*,*} = (E_X)^r_{*,*}$ for $r \geq 2$. The map f induces the map of modified spectral sequences:

$$\dot{f}^r_*\colon (\dot{E}_Y)^r_{*,*} \to (\dot{E}_X)^r_{*,*}.$$

By dimensional reasons the homological spectral sequence $(E_X)^r_{p,q} \Rightarrow H_{p+q}(X)$ (and hence its modified version) has an obvious vanishing property:

$$(E_X)^1_{p,q} = H_{p+q}(X_p, X_{p-1}) = 0 \quad \text{if } p < q.$$

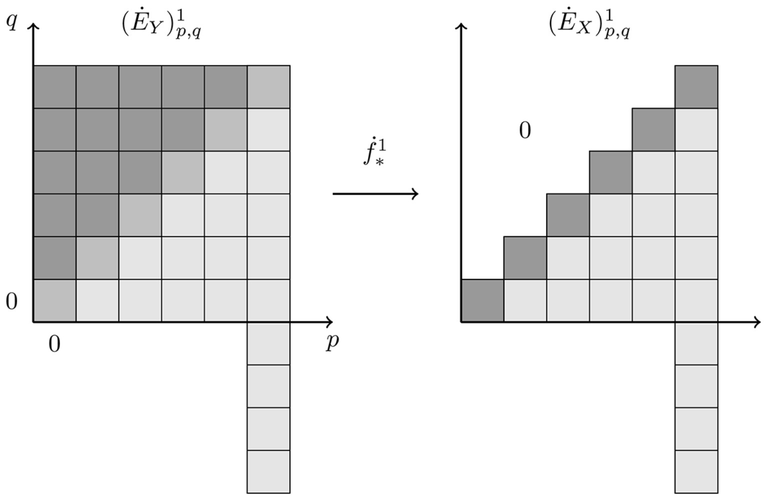

Figure 3. The induced map of spectral sequences.

Proposition 4.2. *The map $\dot{f}^1_*\colon (\dot{E}_Y)^1_{p,q} \to (\dot{E}_X)^1_{p,q}$ is an isomorphism if $p > q$ or $p = q = n$. It is injective if $p = q$. In the remaining cases, that is $p < q$, the modules $(\dot{E}_X)^1_{p,q}$ vanish.*

Proof. The map $f^2_*\colon (E_{\partial Y})^2_{p,q} \to (E_{\partial X})^2_{p,q}$ is an isomorphism for $p > q$ and injective for $p = q$ (see [**1**, Theorem 5.2 and Remark 6.6]). Thus $\dot{f}^1_*\colon (\dot{E}_Y)^1_{p,q} \to (\dot{E}_X)^1_{p,q}$ is an isomorphism for $q < p < n$ and injective for $q = p < n$. Note that in [**1**] we considered manifolds with corners, but the argument used there can be applied to any Buchsbaum pseudo-cell complex without significant changes.

As for the case $p = n$, the map $f_*\colon (E_Y)^1_{n,q} \to (E_X)^1_{n,q}$ is an isomorphism since the identification $\sim$ does not touch the interior of Y, and therefore,

$$X_n/X_{n-1} = X/\partial X \cong Y/\partial Y = Y_n/Y_{n-1}.$$

Thus $\dot{f}^1_*\colon (\dot{E}_Y)^1_{n,q} = H_{n+q}(Y, \partial Y) \to (\dot{E}_X)^1_{n,q} = H_{n+q}(X, \partial X)$ is an isomorphism by excision. ☐

This proposition together with (4.4) and Proposition 3.2 provides a complete description of differentials and non-diagonal terms of $(\dot{E}_X)^r_{*,*}$.

Theorem 4.3. *Let Q be a Buchsbaum (over $\Bbbk$) pseudo-cell complex, and let $X = (Q \times T^n)/\sim$ be the quotient construction determined by some $\Bbbk$-characteristic function on Q. There exists a homological spectral sequence $(\dot{E}_X)^r_{*,*}$ converging to $H_*(X)$. Starting from the second page this spectral sequence coincides with $(E_X)^*_{*,*}$ (the spectral sequence associated with the orbit type filtration). The first page, $(\dot{E}_X)^1$ is the homology module of $(E_X)^1$, with respect to the differential d^-_X of degree $(-1, 0)$. The following properties hold for $(\dot{E}_X)^*_{*,*}$:*

(1) *Non-diagonal terms of the first page have the form*

$$(\dot{E}_X)^1_{p,q} \cong \begin{cases} H_p(\partial Q) \otimes \Lambda_q, & \text{if } q < p < n, \\ \bigoplus\limits_{q_1+q_2=q+n} H_{q_1}(Q, \partial Q) \otimes \Lambda_{q_2}, & \text{if } p = n, \\ 0, & \text{if } q > p. \end{cases}$$

(2) *There exist injective maps $\dot{f}^1_*\colon H_q(\partial Q) \otimes \Lambda_q \hookrightarrow (\dot{E}_X)^1_{q,q}$ for $q \leq n$.*
(3) *Nontrivial differentials for $r \geq 1$ have the form*

$$(\dot{d}_X)^r \cong \begin{cases} \delta_{q_1} \otimes \mathrm{id}_\Lambda\colon \underset{\substack{\cup \\ (\dot{E}_X)^1_{n,q_1+q_2-n}}}{H_{q_1}(Q, \partial Q) \otimes \Lambda_{q_2}} \to \underset{\substack{\cup \\ (\dot{E}_X)^1_{q_1-1,q_2}}}{H_{q_1-1}(\partial Q) \otimes \Lambda_{q_2}}, & \\ & \text{if } r = n - q_1 + 1, q_1 - 1 > q_2, \\ \dot{f}^1_* \circ (\delta_{q_1} \otimes \mathrm{id}_\Lambda)\colon H_{q_1}(Q, \partial Q) \otimes \Lambda_{q_2} & \\ \quad \to H_{q_1-1}(\partial Q) \otimes \Lambda_{q_2} \hookrightarrow (\dot{E}_X)^*_{q_1-1,q_1-1}, & \text{if } r = n - q_1 + 1, q_1 - 1 = q_2, \\ 0, & \text{otherwise.} \end{cases}$$

REMARK 4.4. We can weaken the condition of triviality $Y \cong Q \times T^n$, and require only the homological triviality, namely that (4.3) holds for Y. All results of the paper hold in this generality. In [**1**] we proved that $f_*^2\colon (E_Y)^2_{*,*} \to (E_X)^2_{*,*}$ and $f_*^2\colon (E_{\partial Y})^2_{*,*} \to (E_{\partial X})^2_{*,*}$ are isomorphisms if $q < p$ and injective for $q = p$ for any principal T^n-bundle Y over Q. So one can describe the structure of $(E_X)^*_{*,*}$ if the structure of $(E_Y)^*_{*,*}$ is known. The problem is that there is no uniform description of $(E_Y)^*_{*,*}$ when Y is a general T^n-bundle. We have such description when Y is homologically trivial, and this was the main reason to introduce this assumption.

Note that if Q is an orientable manifold with boundary, the condition $H_*(Y) \cong H_*(Q) \otimes H_*(T^n)$ implies $H_*(Y, \partial Y) \cong H_*(Q, \partial Q) \otimes H_*(T^n)$ by Poincare–Lefschetz duality. In this case the homological triviality condition (4.3) holds for Y.

4.3. Diagonal terms of the spectral sequence.

Our next goal is to compute the diagonal terms $(\dot{E}_X)^1_{q,q}$ since Theorem 4.3 does not describe them explicitly. In this subsection we state the results about their dimensions. The proofs are given in the next section.

Let $\widetilde{\beta}_p(S)$ denote the rank of $\widetilde{H}_p(S)$ for each $p < n$. If Q is a Buchsbaum pseudo-cell complex, we have $\dim \widetilde{H}_p(\partial Q) = \widetilde{\beta}_p(S_Q)$ since S_Q is homologous to ∂Q by Proposition 2.7. Let $h_q(S)$, $h'_q(S)$ and $h''_q(S)$ be the h-, h'- and h''-numbers of a simplicial poset S (see definitions in Section 5).

THEOREM 4.5. *In the notation and under conditions of Theorem* 4.3 *there holds*

$$\dim(\dot{E}_X)^1_{q,q} = h_q(S_Q) + \binom{n}{q} \sum_{p=0}^{q} (-1)^{p+q} \widetilde{\beta}_p(S_Q) \quad \textit{for} \quad q \le n-1.$$

THEOREM 4.6. *Let Q be a Buchsbaum manifold with corners and $X = (Q \times T^n)/\sim$. Then*:

(1) $\dim(\dot{E}_X)^1_{q,q} = h'_{n-q}(S_Q)$ *for* $q \le n-2$, *and* $\dim(\dot{E}_X)^1_{n-1,n-1} = h'_1(S_Q) + n$.
(2) $\dim(E_X)^2_{q,q} = \dim(\dot{E}_X)^2_{q,q} = h'_{n-q}(S_Q)$ *for* $0 \le q \le n$.

For the cone over Buchsbaum simplicial poset, the diagonal components of the ∞-page also have a clear combinatorial meaning.

THEOREM 4.7. *Let S be a Buchsbaum simplicial poset, $P = P(S)$ be the cone over its geometric realization, and $X = (P \times T^n)/\sim$. Then*

$$\dim(E_X)^\infty_{q,q} = \dim(\dot{E}_X)^\infty_{q,q} = h''_q(S)$$

for $0 \le q \le n$.

COROLLARY 4.8. *If S is Buchsbaum, then $h''_i(S) \ge 0$.*

PROOF. For any simplicial poset S there exists a characteristic function on $P = P(S)$ over $\mathbb{Q}$. Thus we can consider the space $X = (P \times T^n)/\sim$ and apply Theorem 4.7. $\square$

5. Face vectors and ranks of diagonal components.

In this section we prove Theorems 4.5, 4.6 and 4.7.

5.1. Preliminaries on face numbers.

First recall several standard definitions from combinatorial theory of simplicial posets.

CONSTRUCTION 5.1. Let S be a pure simplicial poset, $\dim S = n-1$. Let $f_i(S)$ be the the number of i-dimensional simplices in S and, in particular, $f_{-1}(S) = 1$ (the element $\hat{0} \in S$ has dimension -1). The array $(f_{-1}, f_0, \dots, f_{n-1})$ is called the *f-vector of S*. We write f_i instead of $f_i(S)$ since the poset is always clear from the context. Let $f_S(t)$ be the generating polynomial: $f_S(t) = \sum_{i\geq 0} f_{i-1}t^i$.

The *h-numbers* are determined by the formula:

$$\sum_{i=0}^{n} h_i t^i = \sum_{i=0}^{n} f_{i-1} t^i (1-t)^{n-i} = (1-t)^n f_S\left(\frac{t}{1-t}\right), \tag{5.1}$$

where t is a formal variable. Let $\beta_i(S) = \dim H_i(S)$, $\widetilde{\beta}_i(S) = \dim \widetilde{H}_i(S)$, and

$$\chi(S) = \sum_{i=0}^{n-1} (-1)^i \beta_i(S) = \sum_{i=0}^{n-1} (-1)^i f_i(S), \qquad \widetilde{\chi}(S) = \sum_{i=0}^{n-1} \widetilde{\beta}_i(S) = \chi(S) - 1.$$

Thus $f_S(-1) = 1 - \chi(S)$. Note that

$$h_n = (-1)^{n-1} \widetilde{\chi}(S). \tag{5.2}$$

The *h'- and h''-numbers* of S are defined by the formulas

$$h'_i = h_i + \binom{n}{i}\left(\sum_{j=1}^{i-1} (-1)^{i-j-1} \widetilde{\beta}_{j-1}(S)\right) \quad \text{for } 0 \leq i \leq n,$$

$$h''_i = h'_i - \binom{n}{i}\widetilde{\beta}_{i-1}(S) = h_i + \binom{n}{i}\left(\sum_{j=1}^{i} (-1)^{i-j-1} \widetilde{\beta}_{j-1}(S)\right) \quad \text{for } 0 \leq i \leq n-1,$$

and $h''_n = h'_n$. The sum over an empty set is assumed zero. It follows from (5.2) that

$$h'_n = h_n + \sum_{j=0}^{n-1} (-1)^{n-j-1} \widetilde{\beta}_{j-1}(S) = \widetilde{\beta}_{n-1}(S). \tag{5.3}$$

PROPOSITION 5.2 (Dehn–Sommerville relations). *For a homology manifold S there holds*

$$h_i = h_{n-i} + (-1)^i \binom{n}{i} (1 - (-1)^n - \chi(S)), \tag{5.4}$$

or, equivalently:

$$h_i = h_{n-i} + (-1)^i \binom{n}{i} (1 + (-1)^n \widetilde{\chi}(S)). \tag{5.5}$$

Moreover, $h''_i = h''_{n-i}$.

PROOF. The first statement can be found e.g. in [**12**] or [**4**, Theorem 3.8.2]. Also see Remark 5.5 below. The last statement follows from the definition of h''-numbers and Poincare duality $\beta_i(S) = \beta_{n-1-i}(S)$ (see [**10**, Lemma 7.3]). ☐

Now we introduce an auxiliary numerical characteristic of a simplicial poset S.

DEFINITION 5.3. Let S be a Buchsbaum simplicial poset. For $i \geq 0$ consider the number

$$\widehat{f}_i(S) = \sum_{I \in S, \dim I = i} \dim \widetilde{H}_{n-1-|I|}(\mathrm{lk}_S I).$$

For a homology manifold S there holds $\widehat{f}_i = f_i$ since all proper links are homology spheres. In general, there is another formula connecting these quantities.

PROPOSITION 5.4. *For Buchsbaum simplicial poset S there holds*

$$f_S(t) = (1 - \chi(S)) + (-1)^n \sum_{k \geq 0} \widehat{f}_k(S) \cdot (-t-1)^{k+1}.$$

PROOF. This follows from the general statement [**8**, Theorem 9.1], [**4**, Theorem 3.8.1], but we provide an independent proof for completeness. For simplicial posets we have:

$$\frac{d}{dt} f_S(t) = \sum_{v \in \mathrm{Vert}(S)} f_{\mathrm{lk}\, v}(t),$$

(see [**2**, Lemmas 3.7 and 3.8]) and, more generally,

$$\left(\frac{d}{dt}\right)^k f_S(t) = k! \sum_{I \in S, |I| = k} f_{\mathrm{lk}\, I}(t).$$

Thus for $k \geq 1$:

$$f_S^{(k)}(-1) = k! \sum_{I \in S, |I| = k} (1 - \chi(\mathrm{lk}_S I))$$

$$= k! \sum_{I\in S, |I|=k} (-1)^{n-|I|} \dim \widetilde{H}_{n-|I|-1}(\mathrm{lk}\, I) = (-1)^{n-k} k! \widehat{f}_{k-1}.$$

The Taylor expansion of $f_S(t)$ at -1 has the form:

$$f_S(t) = f_S(-1) + \sum_{k\geq 1} \frac{1}{k!} f_S^{(k)}(-1)(t+1)^k = (1-\chi(S)) + \sum_{k\geq 0} (-1)^{n-k-1} \widehat{f}_k \cdot (t+1)^{k+1},$$

which completes the proof. □

Remark 5.5. If S is a homology manifold, then Proposition 5.4 implies

$$f_S(t) = (1-(-1)^n - \chi(S)) + (-1)^n f_S(-t-1),$$

which is yet another equivalent form of Dehn–Sommerville relations (5.4).

Lemma 5.6. *For Buchsbaum poset S there holds*

$$\sum_{i=0}^{n} h_i t^i = (1-t)^n (1-\chi(S)) + \sum_{k\geq 0} \widehat{f}_k \cdot (t-1)^{n-k-1}.$$

Proof. Substitute $t/(1-t)$ in Proposition 5.4 and apply (5.1). □

Comparing the coefficients of t^i in the identity of Lemma 5.6 we get:

$$h_i(S) = (1-\chi(S))(-1)^i \binom{n}{i} + \sum_{k\geq 0} (-1)^{n-k-i-1} \binom{n-k-1}{i} \widehat{f}_k(S). \tag{5.6}$$

5.2. Ranks of $(E_X)^1_{*,*}$.

To prove Theorem 4.5 we use the following straightforward idea. The module $(\dot{E}_X)^1_{*,*}$ is the homology of $(E_X)^1_{*,*}$ with respect to the differential d_X^- of degree $(-1,0)$. Theorem 4.3 describes the ranks of all groups $(\dot{E}_X)^1_{p,q}$ except for $p=q$; the terms $(E_X)^1_{p,q}$ are known as well. Thus the ranks of the remaining terms $\dim(\dot{E}_X)^1_{q,q}$ can be found by equating Euler characteristics of $(E_X)^1_{*,*}$ and $(\dot{E}_X)^1_{*,*}$.

When we pass from $(E_X)^1_{*,*}$ to $(\dot{E}_X)^1_{*,*}$, the terms with $p=n$ do not change. The other groups are the same as if we passed from $(E_{\partial X})^1_{*,*}$ to $(E_{\partial X})^2_{*,*}$. Thus it is sufficient to perform calculations with the truncated sequence $(E_{\partial X})^*_{*,*}$.

Let χ^1_q be the Euler characteristic of the q-th row of $(E_{\partial X})^1_{*,*}$:

$$\chi^1_q = \sum_{p\leq n-1} (-1)^p \dim (E_{\partial X})^1_{p,q}. \tag{5.7}$$

Lemma 5.7. *For $q \leq n-1$ there holds $\chi^1_q = (\chi(S_Q)-1)\binom{n}{q} + (-1)^q h_q(S_Q)$.*

Proof. By Proposition 2.7 there is an isomorphism of spectral sequences

$(E_{\partial Q})^* \to (E_{\partial P(S_Q)})^*$. Thus, in particular, for any $I \in S_Q \setminus \{\hat{0}\}$, $|I| = n - p$ we have an isomorphism

$$H_p(F_I, \partial F_I) \cong H_p(G_I, \partial G_I) \cong H_{p-1}(\mathrm{lk}_{S_Q} I) \tag{5.8}$$

where G_I is the face of $P(S_Q)$ dual to I. The last isomorphism in (5.8) is due to the long exact sequence of the pair $(G_I, \partial G_I)$, since $G_I = \mathrm{Cone}(\partial G_I)$ and $\partial G_I \cong \mathrm{lk}_{S_Q} I$.

For $p < n$ we have

$$\dim(E_{\partial X})^1_{p,q} = \sum_{I, \dim F_I = p} \dim H_{p+q}(X_I, \partial X_I).$$

We have $X_I \cong F_I \times (T^n/T_I)/\sim_{F_I}$, where $\sim_{F_I}$ identifies points over ∂F_I. Thus $H_{p+q}(X_I, \partial X_I) \cong H_p(F_I, \partial F_I) \otimes H_q(T^n/T_I)$. Therefore,

$$\dim(E_{\partial X})^1_{p,q} = \sum_{I, |I| = n-p} \dim(H_p(F_I, \partial F_I) \otimes H_q(T^n/T_I)) = \binom{p}{q} \cdot \widehat{f}_{n-p-1}(S_Q).$$

In the last equality we applied (5.8) and the definition of $\widehat{f}$-numbers. Therefore,

$$\chi^1_q = \sum_{p \leq n-1} (-1)^p \dim(E_{\partial X})^1_{p,q} = \sum_{p \leq n-1} (-1)^p \binom{p}{q} \widehat{f}_{n-p-1}(S_Q). \tag{5.9}$$

Now substitute $i = q$ and $k = n - p - 1$ in (5.6) and combine with (5.9). $\square$

5.3. Ranks of $(\dot{E}_X)^1_{*,*}$.

By the definition of modified spectral sequence we have $(\dot{E}_X)^1_{p,q} \cong (E_{\partial X})^2_{p,q}$ for $p \leq n-1$. Let χ^2_q be the Euler characteristic of q-th row of $(E_{\partial X})^2_{*,*}$:

$$\chi^2_q = \sum_{p \leq n-1} (-1)^p \dim(E_{\partial X})^2_{p,q}. \tag{5.10}$$

Euler characteristics of the first and the second pages coincide: $\chi^2_q = \chi^1_q$. By Theorem 4.3, for $q < p < n$ we have

$$\dim(\dot{E}_X)^1_{p,q} = \binom{n}{q} \beta_p(S_Q).$$

Lemma 5.7 yields

$$(-1)^q \dim(\dot{E}_X)^1_{q,q} + \sum_{p=q+1}^{n-1} (-1)^p \binom{n}{q} \beta_p(S_Q) = (\chi(S_Q) - 1)\binom{n}{q} + (-1)^q h_q(S_Q).$$

This relation combined with the equality $\chi(S_Q) = \sum_{p=0}^{n-1} \beta_p(S_Q)$ and the obvious relation between reduced and non-reduced Betti numbers, completes the proof of Theorem 4.5.

5.4. Manifold case.

Let us prove Theorem 4.6. If Q is a Buchsbaum manifold with corners, then S_Q is a homology manifold. Then Poincare duality $\beta_i(S_Q) = \beta_{n-1-i}(S_Q)$ and the Dehn–Sommerville relations (5.5) imply

$$\begin{aligned}
\dim(\dot{E}_X)^1_{q,q} &= h_q + \binom{n}{q}\sum_{p=0}^{q}(-1)^{p+q}\widetilde{\beta}_p \\
&= h_q - (-1)^q\binom{n}{q} + \binom{n}{q}\sum_{p=0}^{q}(-1)^{p+q}\beta_p \\
&= h_q - (-1)^q\binom{n}{q} + \binom{n}{q}\sum_{p=n-1-q}^{n-1}(-1)^{n-1-p+q}\beta_p \\
&= h_{n-q} + (-1)^q\binom{n}{q}\Big[(-1)^n\widetilde{\chi} + \sum_{p=n-1-q}^{n-1}(-1)^{n-1-p}\beta_p\Big] \\
&= h_{n-q} + (-1)^q\binom{n}{q}\Big[-(-1)^n + \sum_{p=0}^{n-q-2}(-1)^{p+n}\beta_p\Big].
\end{aligned}$$

The last expression in brackets coincides with $\sum_{p=-1}^{n-q-2}(-1)^{p+n}\widetilde{\beta}_p$ whenever the summation is taken over nonempty set, that is for $q \leq n-2$. Thus $\dim(\dot{E}_X)^1_{q,q} = h'_{n-q}$ for $q \leq n-2$. In the case $q = n-1$ we have $\dim(\dot{E}_X)^1_{n-1,n-1} = h_1 + \binom{n}{n-1} = h'_1 + n$. This proves part (1) of Theorem 4.6.

Part (2) follows easily. Indeed, for $q = n$ we have

$$\dim(\dot{E}_X)^2_{n,n} = \dim(\dot{E}_X)^1_{n,n} = \binom{n}{n}\dim H_n(Q, \partial Q) = 1 = h'_0(S_Q).$$

For $q = n-1$:

$$\dim(\dot{E}_X)^2_{n-1,n-1} = \dim(\dot{E}_X)^1_{n-1,n-1} - \binom{n}{n-1}\dim \operatorname{Im}\delta_n = h'_1(S_Q),$$

since the map $\delta_n\colon H_n(Q, \partial Q) \to H_{n-1}(\partial Q)$ is injective and $\dim H_n(Q, \partial Q) = 1$.

If $q \leq n-2$, then $(\dot{E}_X)^2_{q,q} = (\dot{E}_X)^1_{q,q}$, and the statement follows from part (1).

5.5. Cone case.

If $P = P(S) \cong \operatorname{Cone}|S|$, then the map $\delta_i\colon H_i(P, \partial P) \to \widetilde{H}_{i-1}(\partial P)$ is an isomorphism as follows from the long exact sequence of the pair $(P, \partial P)$. Thus for $q \leq n-1$ Theorem 4.3 implies

$$\dim(\dot{E}_X)^\infty_{q,q} = \dim(\dot{E}_X)^1_{q,q} - \binom{n}{q}\dim H_{q+1}(P, \partial P) = \dim(\dot{E}_X)^1_{q,q} - \binom{n}{q}\widetilde{\beta}_q(S).$$

By Theorem 4.5, this expression is equal to

$$h_q + \binom{n}{q}\left[\sum_{p=0}^{q}(-1)^{p+q}\widetilde{\beta}_p(S)\right] - \binom{n}{q}\widetilde{\beta}_q(S) = h_q(S) + \binom{n}{q}\sum_{p=0}^{q-1}(-1)^{p+q}\widetilde{\beta}_p(S) = h''_q.$$

The case $q = n$ follows from (5.3). Indeed, the term $(\dot{E}_X)^1_{n,n}$ survives in the spectral sequence, thus we have

$$\dim(\dot{E}_X)^{\infty}_{n,n} = \binom{n}{n}\dim H_n(P,\partial P) = \beta_{n-1}(S) = h'_n(S) = h''_n(S).$$

This proves Theorem 4.7.

6. Homology of X.

In this section we assume that $\Bbbk$ is a field. Theorem 4.3 gives an additional grading on $H_*(X)$, namely the one induced by degrees of exterior forms, as described below. In the following Q is an arbitrary Buchsbaum pseudo-cell complex of dimension n.

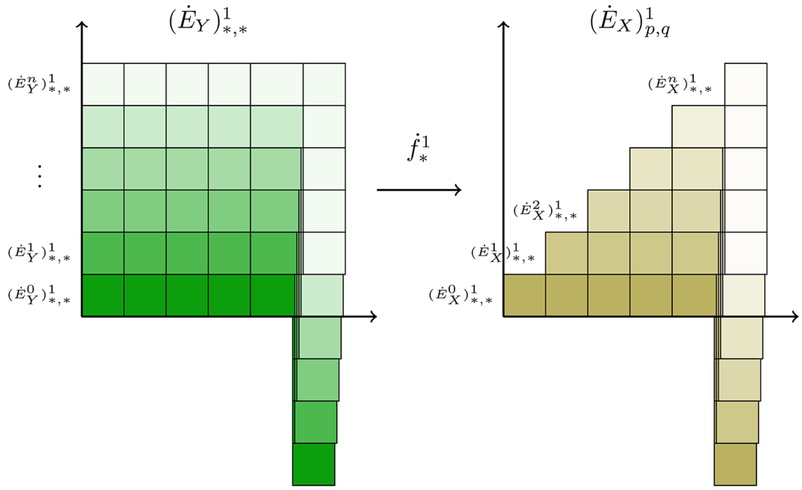

Figure 4. Decomposition of spectral sequences into graded components.

CONSTRUCTION 6.1. The spectral sequence $(\dot{E}_Y)^*$ splits as the direct sum of spectral subsequences, indexed by degrees of exterior forms. For $0 \leq j \leq n$ consider the ⌉-shaped spectral sequence

$$(\dot{E}^j_Y)^r_{p,q} = (\dot{E}_Q)^r_{p,q-j} \otimes \Lambda_j.$$

Clearly, $(\dot{E}_Y)^r_{*,*} = \bigoplus_{j=0}^{n}(\dot{E}^j_Y)^r_{*,*}$. This decomposition is sketched on Figure 4. Let $H_{i,j}(Y)$ denote the module $H_i(Q) \otimes \Lambda_j$. Then $(\dot{E}^j_Y)^r_{p,q} \Rightarrow H_{p+q-j,j}(Y)$.

Let us construct the corresponding ⌉-shaped spectral subsequences in $(\dot{E}_X)^*_{*,*}$. Consider the bigraded vector subspaces $(\dot{E}_X^j)^1_{*,*}$:

$$(\dot{E}_X^j)^1_{p,q} = \begin{cases} (\dot{E}_X)^1_{p,q}, & \text{if } q = j \text{ and } p < n, \\ 0, & \text{if } q \neq j \text{ and } p < n, \\ H_{q+n-j}(Q, \partial Q) \otimes \Lambda_j, & \text{if } p = n. \end{cases}$$

In the last case we used the isomorphism of Theorem 4.3. Theorem 4.3 implies that all differentials of $(\dot{E}_X)^*_{*,*}$ preserve the subspace $(\dot{E}_X^j)^1_{*,*}$, thus spectral subsequences $(\dot{E}_X^j)^r_{*,*}$ are well defined for $r \geq 2$, and $(\dot{E}_X)^r_{*,*} = \bigoplus_{j=0}^n (\dot{E}_X^j)^r_{*,*}$.

Over a field, $H_k(X)$ can be identified with the associated module $\bigoplus_{p+q=k} (E_X)^\infty_{p,q}$, and therefore inherits the double grading:

$$H_k(X) \cong \bigoplus_{i+j=k} H_{i,j}(X), \quad \text{where}$$

$$H_{i,j}(X) \stackrel{\text{def}}{=} \bigoplus_{p+q=i+j} (\dot{E}_X^j)^\infty_{p,q}.$$

Hence we have $(\dot{E}_X^j)^r_{p,q} \Rightarrow H_{p+q-j,j}(X)$. The map $\dot{f}^r_*\colon (\dot{E}_Y)^r_{*,*} \to (\dot{E}_X)^r_{*,*}$ sends $(\dot{E}_Y^j)^r_{*,*}$ to $(\dot{E}_X^j)^r_{*,*}$ for each $j \in \{0, \dots, n\}$. The map $f_*\colon H_*(Y) \to H_*(X)$ sends $H_{i,j}(Y)$ to $H_{i,j}(X)$.

Theorem 6.2.

(1) *If $i > j$, then $f_*\colon H_{i,j}(Y) \to H_{i,j}(X)$ is an isomorphism. As a consequence, we get $H_{i,j}(X) \cong H_i(Q) \otimes \Lambda_j$.*
(2) *If $i < j$, then there exists an isomorphism $H_{i,j}(X) \cong H_i(Q, \partial Q) \otimes \Lambda_j$.*
(3) *In case $i = j < n$, the module $H_{i,i}(X)$ fits in the exact sequence*

$$0 \to (\dot{E}_X)^\infty_{i,i} \to H_{i,i}(X) \to H_i(Q, \partial Q) \otimes \Lambda_i \to 0,$$

or, equivalently,

$$0 \to \operatorname{Im} \delta_{i+1} \otimes \Lambda_i \to (\dot{E}_X)^1_{i,i} \to H_{i,i}(X) \to H_i(Q, \partial Q) \otimes \Lambda_i \to 0.$$

(4) *If $i = j = n$, then*

$$H_{n,n}(X) = (\dot{E}_X)^\infty_{n,n} = (\dot{E}_X)^1_{n,n} \cong H_n(Q, \partial Q).$$

Proof. According to Theorem 4.3, the map $\dot{f}^1_*\colon (\dot{E}_Y^j)^1_{i,q} \to (\dot{E}_X^j)^1_{i,q}$ is an isomorphism if $i > j$ or $i = j = n$, and injective if $i = j < n$. For each j both spectral sequences $(\dot{E}_Y^j)$ and $(\dot{E}_X^j)$ are $\urcorner$-shaped, and therefore unfold in the long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & (\dot{E}^j_Y)^1_{i,j} \arrow[r] \arrow[d, "\dot{f}^1_*"] & H_{i,j}(Y) \arrow[r] \arrow[d, "f_*"] & (\dot{E}^j_Y)^1_{n,i-n+j} \arrow[r, "(\dot{d}_Y)^{n-i+1}"] \arrow[d, "\dot{f}^1_*", "\cong"'] & (\dot{E}^j_Y)^1_{i-1,j} \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & (\dot{E}^j_X)^1_{i,j} \arrow[r] & H_{i,j}(X) \arrow[r] & (\dot{E}^j_X)^1_{n,i-n+j} \arrow[r, "(\dot{d}_X)^{n-i+1}"] & (\dot{E}^j_X)^1_{i-1,j} \arrow[r] & \cdots
\end{tikzcd} \tag{6.1}$$

Application of five lemma in the case $i > j$ proves (1). For $i < j$, the groups $(\dot{E}^j_X)^1_{i,j}$, $(\dot{E}^j_X)^1_{i-1,j}$ vanish by dimensional reasons, thus $H_{i,j}(X) \cong (\dot{E}^j_X)^1_{n,i-n+j} \cong (\dot{E}^j_Y)^1_{n,i-n+j} \cong H_i(Q,\partial Q) \otimes \Lambda_j$. Case $i = j$ also follows from (6.1) by a simple diagram chase. □

In the case of manifolds Theorem 6.2 reveals a bigraded duality. If Q is a nice manifold with corners, $Y = Q \times T^n$, and λ is a characteristic function over $\mathbb{Z}$, then $X = Y/{\sim}$ is a compact orientable topological manifold with the locally standard torus action. In this case Poincare duality respects the double grading.

PROPOSITION 6.3. *Let Q be a Buchsbaum manifold with corners, and let X be a quotient construction over Q determined by a $\mathbb{Z}$-characteristic function. Then $H_{i,j}(X;\Bbbk) \cong H_{n-i,n-j}(X;\Bbbk)$ for any field $\Bbbk$.*

PROOF. When $i < j$, we have

$$H_{i,j}(X) \cong H_i(Q,\partial Q) \otimes \Lambda_j \cong H_{n-i}(Q) \otimes \Lambda_{n-j} \cong H_{n-i,n-j}(X),$$

by the Poincare–Lefschetz duality applied to Q and Poincare duality applied to torus. The remaining isomorphism $H_{i,i}(X) \cong H_{n-i,n-i}(X)$ now follows from the ordinary Poincare duality on X. □

REMARK 6.4. If X is determined by $\mathbb{Q}$-characteristic function, then it is a homology $\mathbb{Q}$-manifold. In this case Proposition 6.3 holds over $\mathbb{Q}$.

7. One example with non-acyclic proper faces.

CLAIM 7.1. *In general, Betti numbers of manifolds with locally standard torus actions may depend not only on the orbit space, but also on the values of characteristic function.*

Let Q be the product of a circle S^1 with the closed interval $\mathbb{I} = [-1,1] \subset \mathbb{R}^1$. Then Q is a nice manifold with corners having two proper faces: $F_1 = S^1 \times \{-1\}$ and $F_2 = S^1 \times \{1\}$. The faces are not acyclic, so the arguments of the paper are not applicable. Consider the 2-torus T^2 with a given coordinate splitting $T^2 = T^{(\{1\})} \times T^{(\{2\})}$.

First, define the characteristic function λ on Q by

$$\lambda(F_1) = T^{(\{1\})}, \quad \lambda(F_2) = T^{(\{2\})}.$$

The corresponding manifold with locally standard action is

$$X = (S^1 \times \mathbb{I} \times T^2)/{\sim} = S^1 \times (\mathbb{I} \times T^2/{\sim}) = S^1 \times \mathcal{Z}_{\mathbb{I}} \cong S^1 \times S^3.$$

Here $\mathcal{Z}_{\mathbb{I}}$ is the moment-angle manifold of the interval $\mathbb{I}$, see [**4**] or [**6**].

Next, consider the characteristic function λ' on Q determined by

$$\lambda'(F_1) = \lambda'(F_2) = T^{(\{1\})}.$$

The corresponding manifold is

$$X' = (S^1 \times \mathbb{I} \times T^2)/\sim\, = S^1 \times T^{(\{2\})} \times (\mathbb{I} \times T^{(\{1\})}/\sim) \cong S^1 \times S^1 \times S^2.$$

We see that the same manifold with corners $S^1 \times \mathbb{I}$ may be the orbit space of two manifolds with locally standard actions having different Betti numbers.

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