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Periodicity and the values of the real Buchstaber invariants

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Abstract. The Buchstaber invariant $s(K)$ is defined to be the maximum integer for which there is a subtorus of dimension $s(K)$ acting freely on the moment-angle complex associated with a finite simplicial complex K. Analogously, its real version $s_{\mathbb{R}}(K)$ can also be defined by using the real moment-angle complex instead of the moment-angle complex. The importance of these invariants comes from the fact that $s(K)$ and $s_{\mathbb{R}}(K)$ distinguish two simplicial complexes and are the source of nontrivial and interesting combinatorial tasks. The ultimate goal of this paper is to compute the real Buchstaber invariants of skeleta $K = \Delta^{m-1}_{m-p-1}$ of the simplex Δ^{m-1} by making a formula. In fact, it can be solved by integer linear programming. We also give a counterexample to the conjecture which is proposed in [**6**] and we provide an adjusted formula which can be thought of as a preperiodicity of some numbers related to the real Buchstaber invariants.

1. Introduction.

Let K be a simplicial complex, that is a collection of simplices $\sigma \subset [m] = \{1, 2, \ldots, m\}$ such that:

(a) Any face of a simplex σ in K is also in K; and
(b) The intersection of any two simplices $\sigma, \tau \in K$ is a face of both σ and τ, or is empty.

In toric topology, moment-angle space $\mathcal{Z}_K$ is a special topological space. Davis and Januszkiewicz introduced $\mathcal{Z}_K$ in [**4**] and Buchstaber and Panov studied the topology of this space [**2**]. For the pair of topological spaces (X, A), the K-power is defined as follows [**8**]. Define

$$(X, A)^\sigma := \{(x_1, \ldots, x_m) \in X^m \mid x_i \in A \text{ for } i \notin \sigma\}.$$

Then the K-power of (X, A) is defined by

$$(X, A)^K := \bigcup_{\sigma \in K} (X, A)^\sigma.$$

The moment-angle complex $\mathcal{Z}_K$ is a special example of K-power. For a pair of unit disk $D^2 \subset \mathbb{C}$ and unit circle $S^1 \subset \mathbb{C}$, we obtain $\mathcal{Z}_K = (D^2, S^1)^K$. Analogously, for a pair of interval $D^1 = [-1, 1] \subset \mathbb{R}$ and $S^0 = \{\pm 1\}$, we obtain *real moment-angle complex*

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$\mathbb{R}\mathcal{Z}_K = (D^1, S^0)^K$. In particular, if K is a $(m-p-1)$-skeleton Δ^{m-1}_{m-p-1} of the $(m-1)$-simplex Δ^{m-1}, then $\mathcal{Z}_K$ is given by

$$\mathcal{Z}_K = \bigcup (D^2)^{m-p} \times (S^1)^p \subset (D^2)^m,$$

where the union is taken over all $(m-p)$ products of D^2 in $(D^2)^m$.

Clearly the natural action of $(S^1)^m$ on $\mathbb{C}^m$ leaves the moment-angle complex $\mathcal{Z}_K$ invariant and its action on $\mathcal{Z}_K$ is, in general, non-free. However, there is a subtorus of $(S^1)^m$ whose restriction acts freely on $\mathcal{Z}_K$. The *Buchstaber invariant*, denoted $s(K)$, is defined to be the maximal dimension $s(K)$ of a toric subgroup $G \subset (S^1)^m$ which acts freely on $\mathcal{Z}_K$. It is known [**5**] that $s(P) := s(\partial P^*) \geq m - \gamma(P) + s(\Delta^{\gamma-1}_{n-1})$, where $\gamma(P)$ is the chromatic number of the polytope P and ∂P^* is the boundary of the dual polytope P^*.

In an analogous way, we may consider a real version of the moment-angle complex of $K = \Delta^{m-1}_{m-p-1}$. The real moment-angle complex $\mathbb{R}\mathcal{Z}_K$ is given by

$$\mathbb{R}\mathcal{Z}_K = \bigcup (D^1)^{m-p} \times (S^0)^p \subset (D^1)^m,$$

where the union is taken over all $(m-p)$ products of D^1 in $(D^1)^m$. The coordinatewise action of $\mathbb{Z}_2^m$ on $(D^1)^m$ leaves $\mathbb{R}\mathcal{Z}_K$ invariant. Here, the group $\mathbb{Z}_2$ acts on D^1 by changing sign. With these said, the *real Buchstaber invariant*, denoted $s_{\mathbb{R}}(K)$, is defined to be the maximal rank of a subgroup $G \subset (\mathbb{Z}/2)^m$ which acts freely on $\mathbb{R}\mathcal{Z}_K$. If the subgroup $G \subset (S^1)^m$ acts freely on $\mathcal{Z}_K$, then $G_2 = G \cap (S^0)^m \subset (S^0)^m$ acts freely on $\mathbb{R}\mathcal{Z}_K$. By using the isomorphism $(\mathbb{Z}/2)^m \simeq (S^0)^m : (z_1, \ldots, z_m) \to ((-1)^{z_1}, \ldots, (-1)^{z_m})$, we can find a subgroup $G_2' \subset (\mathbb{Z}/2)^m$ which acts freely on $\mathbb{R}\mathcal{Z}_K$. Therefore, $s(K) \leq s_{\mathbb{R}}(K)$. Actually there exists a simplicial complex K such that $s(K) \neq s_{\mathbb{R}}(K)$ [**1**]. There exist simplicial complexes with the same f-vector, chromatic number, bigraded Betti numbers but with different (real) Buchstaber invariants [**1**], [**5**]. The importance of the study of these invariants comes from the fact that Buchstaber invariants distinguish two simplicial complexes and are the source of nontrivial and interesting combinatorial tasks. Victor Buchstaber posed the problem [**3**].

Problem. Find a combinatorial description of $s(K)$.

For simplicity, let $s_{\mathbb{R}}(m,p) := s_{\mathbb{R}}(\Delta^{m-1}_{m-p-1})$. In [**6**], the authors reformulate the problem to find $s_{\mathbb{R}}(m,p)$ as a problem of the integer linear programming (see Lemma 2.1 below). For a nonnegative integer $b \geq 0$, let $m_k(b)$ be the maximum of $\sum a_v$ over all nonnegative integer a_v, $v \in (\mathbb{Z}/2)^k \setminus \{0\}$ satisfying

$$\sum_{(u,v)=0} a_v \leq b \quad \text{for each } u \in (\mathbb{Z}/2)^k \setminus \{0\}.$$

It is known that $s_{\mathbb{R}}(m,p) = k$ if and only if $m_{k+1}(p-1) < m \leq m_k(p-1)$ [**6**]. So the problem to find $s_{\mathbb{R}}(m,p)$ is equivalent to that to find $m_k(b)$ for all k.

This problem of integer linear programming can be written as $\boldsymbol{A_k x} \le \boldsymbol{b}$, where $\boldsymbol{A_k}$ is $(2^k-1)\times(2^k-1)$ binary matrix, $\boldsymbol{x} = (x_1,\dots,x_{2^k-1})^T \in \mathbb{Z}_{\ge 0}^{2^k-1}$ and $\boldsymbol{b} = (b,\dots,b)^T$. Then $m_k(b) = \sum x_i^*$, where $\boldsymbol{x}^*$ is an optimal solution.

Let us call the function f *preperiodic* up to S with period T if $f(x+T) = f(x)+S$. The first goal of this paper is to prove that $m_k(b)$, viewed as a function of b, is preperiodic up to 2^k-1 with period $2^{k-1}-1$ for large values of b. Moreover, there is a precise formula for $m_k(b)$ for large values of b. The moment at which the behavior of $m_k(b)$ becomes predictable and periodical depends on the remainder of b modulo the period. This moment is encoded by the constant $d_k(b)$ which will be defined later. Theorem 1.1 below is the first main result.

Theorem 1.1. *Let $k \ge 2$.*

$$m_k((2^{k-1}-1)\ell_2 + b) - m_k((2^{k-1}-1)\ell_1 + b) = (2^k-1)(\ell_2-\ell_1),$$

for any nonnegative integers $0 \le b \le 2^{k-1}-2$ and ℓ_1,ℓ_2 satisfying $\ell_2 \ge \ell_1 \ge d_k(2^{k-1}-1-b)$.

We give a formula for the value of $m_k(b)$.

Theorem 1.2. *Let $k \ge 2$. If $0 \le (2^{k-1}-1)-b < 2^{k-1}-1$ is expanded as*

$$(2^{k-1}-1)-b = c_2(2^{k-2}-1) + c_3(2^{k-3}-1) + \cdots + c_{k-1}(2^1-1),$$

where

$$\begin{aligned}
c_2 &= \left[\frac{(2^{k-1}-1)-b}{2^{k-2}-1}\right],\\
c_3 &= \left[\frac{(2^{k-1}-1)-b-c_2(2^{k-2}-1)}{2^{k-3}-1}\right],\\
&\vdots\\
c_{k-2} &= \left[\frac{(2^{k-1}-1)-b-c_2(2^{k-2}-1)-\cdots-c_{k-3}(2^3-1)}{2^2-1}\right],\\
c_{k-1} &= (2^{k-1}-1)-b-c_2(2^{k-2}-1)-\cdots-c_{k-2}(2^2-1),
\end{aligned}$$

$[x]$ is the largest integer less than or equal to x, then

$$m_k((2^{k-1}-1)\ell + b) = (2^k-1)(\ell+1) - c_2(2^{k-1}-1) - \cdots - c_{k-1}(2^2-1) \tag{1}$$

for $\ell \ge d_k(2^{k-1}-1-b)$.

We will find upper bound of $d_k(b)$ in Section 5. Except two cases, we can deduce that $d_k(b) \le k-5$. The above theorem guarantees that if $b = (2^{k-1}-1)\ell + a \ge (2^{k-1}-1)d_k((2^{k-1}-1)-a)+a$, then $m_k(b)$ is preperiodic up to (2^k-1).

Every problem of linear programming can be converted into a dual problem. i.e., $\boldsymbol{A_k^T y} \geq \boldsymbol{b}$. Let $m_k^*(b) = \sum y_i^*$ for an optimal integer solution $\boldsymbol{y}^*$ of the dual problem. The main technical tool used in the proofs is the auxiliary function $m_k^*(b)$ which is closely related to $m_k(b)$, but has better properties. In particular, $m_k^*(b)$ is preperiodic up to $2^k - 1$ with period $2^{k-1} - 1$ for all $b \geq 0$, and $m_k^*(b)$ can be studied by induction on k. The following (2) is the relation between $m_k(b)$ and $m_k^*(b)$. There exists $\ell_0 > 0$ such that for $\ell \geq \ell_0$,

$$m_k((2^{k-1} - 1)(\ell - 1) + b) + m_k^*((2^{k-1} - 1) - b) = (2^k - 1)\ell. \tag{2}$$

Fukukawa and Masuda in [**6**] stated the following conjecture.

$$m_k((2^{k-1} - 1)\ell + b) = (2^k - 1)\ell + m_k(b) \tag{3}$$

for any nonnegative integers $\ell \geq 0$ and $0 \leq b \leq 2^{k-1} - 2$. If we suppose their conjecture is true, then the formula in Theorem 1.2 for large values of b works for small b as well. But for small b some values of $m_k(b)$ were calculated in [**6**], and they do not coincide with the values predicted by the formula. Thus the conjecture does not hold.

We organize this paper as follows. In Section 2, we give some preliminaries on $m_k(b)$ and the problem of linear programming associated with $m_k(b)$. In Section 3, we prove preperiodicity of $m_k^*(b)$ and find the formula for the values of $m_k^*(b)$. In Section 4 we prove Theorem 1.1 and Theorem 1.2. In Section 5 we find upper bound of $d_k(b)$ for each $0 \leq b \leq 2^{k-1} - 1$. In Section 6 we describe the counterexamples to Conjecture 2.4 and give concluding remarks.

2. Preliminaries.

In [**6**], Y. Fukukawa and M. Masuda found a necessary and sufficient condition for $s_{\mathbb{R}}(m, p) \geq k$, as follows.

Lemma 2.1 ([**6**, Lemma 3.3]). *Suppose $k \geq 2$. Then $s_{\mathbb{R}}(m,p) \geq k$ if and only if there is a set of non-negative integers $\{a_v \mid v \in (\mathbb{Z}/2)^k \setminus \{0\}\}$ with $\sum a_v = m$ which satisfies the following $(2^k - 1)$ inequalities*

$$\sum_{(u,v)=0} a_v \leq p - 1 \quad \textit{for each } u \in (\mathbb{Z}/2)^k \setminus \{0\}, \tag{4}$$

where $(\ ,\ )$ denotes the standard bilinear form on $(\mathbb{Z}/2)^k$.

Motivated by Lemma 2.1, they gave the following definition.

Definition 2.2 ([**6**]). For a nonnegative integer $b \geq 0$, let $m_k(b)$ be the maximal value of $\sum a_v$ over all nonnegative integers satisfying

$$\sum_{(u,v)=0} a_v \leq b \quad \text{for each } u \in (\mathbb{Z}/2)^k \setminus \{0\}.$$

It turns out that Lemma 2.1 gives certain close connection between $m_k(b)$ and $s_{\mathbb{R}}(m,p)$. In fact, a necessary and sufficient condition for $s_{\mathbb{R}}(m,p) \geq k$ can be explicitly formulated by using the values of $m_k(b)$.

REMARK 2.3. Given a nonnegative integer $m \geq 0$, we can write $m = (2^k - 1)\ell + s$ for some $\ell \geq 0$ and $0 \leq s \leq 2^k - 2$. Since $m_k(2^{k-1} - 1) = 2^k - 1$ [**6**] and $m_k(b)$ is strictly increasing function on b, there exists $0 \leq a \leq 2^{k-1} - 1$ such that

$$m_k((2^{k-1} - 1)\ell + a - 1) < m \leq m_k((2^{k-1} - 1)\ell + a).$$

Then it is easy to see that $m \leq m_k(p-1)$ if and only if $(2^{k-1} - 1)\ell + a \leq p - 1$. As mentioned in the introduction that $m \leq m_k(p-1)$ is equivalent to $s_{\mathbb{R}}(m,p) \geq k$. So if we replace ℓ by $(m - s) \times 1/(2^k - 1)$, then

$$s_{\mathbb{R}}(m,p) \geq k \quad \text{if and only if}$$
$$(2^{k-1} - 1)m \leq (2^k - 1)(p - 1) + (2^{k-1} - 1)s - (2^k - 1)a.$$

So a necessary and sufficient condition for $s_{\mathbb{R}}(m,p) \geq k$ can be explicitly formulated in terms of m and p by using the values of $m_k(b)$.

There is an interesting conjecture which is closely related with real Buchstaber invariant. The authors in [**6**] proposed that

CONJECTURE 2.4 ([**6**]). *Given $k \geq 2$, for $b \geq 0$,*

$$m_k((2^{k-1} - 1) + b) = (2^k - 1) + m_k(b).$$

They proved that Conjecture 2.4 holds for $b \geq (2^{k-1} - 1)(2^{k-2} - 1)$.

As we know, to find the values of $m_k(b)$ is a problem of integer linear programming. So we will modify the problem of linear programming as follows.

Given $k \geq 2$, consider an ordered set of $2^k - 1$ vectors $\{v_1, \ldots, v_{2^k-1}\} = (\mathbb{Z}/2)^k \setminus \{0\}$ and a $(2^k - 1) \times (2^k - 1)$ binary matrix $\boldsymbol{A_k}$ which is defined by

$$\boldsymbol{A_k}(i,j) := \begin{cases} 0, & \text{if } (v_i, v_j) = 1, \\ 1, & \text{if } (v_i, v_j) = 0, \end{cases}$$

where (,) is the standard bilinear form on $(\mathbb{Z}/2)^k$. For convenience, we introduce the lexicographic order to $(\mathbb{Z}/2)^k \setminus \{0\}$.

EXAMPLE 2.5. In case of $k = 3$, the lexicographic order of $(\mathbb{Z}/2)^3 \setminus \{0\}$ is given by:

$$(1,0,0) < (0,1,0) < (1,1,0) < (0,0,1) < (1,0,1) < (0,1,1) < (1,1,1).$$

Then it is easy to see that the matrix $\boldsymbol{A_k}$ has the following properties.

PROPOSITION 2.6. *The following properties hold for $\boldsymbol{A_k}$.*

(1) $\boldsymbol{A_k}$ *is symmetric.*
(2) $\boldsymbol{A_k}(i, 2^{k-1}) = \boldsymbol{A_k}(2^{k-1}, i) = 1$ *for* $1 \leq i \leq 2^{k-1} - 1$.
(3) *Write*

$$\boldsymbol{A_k} = \left(\begin{array}{ccc|c|ccc} & & & 1 & & & \\ & \boldsymbol{P} & & \vdots & & \boldsymbol{Q} & \\ & & & 1 & & & \\ \hline 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ \hline & & & 0 & & & \\ & \boldsymbol{R} & & \vdots & & \boldsymbol{S} & \\ & & & 0 & & & \end{array}\right),$$

where $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ *and* $\boldsymbol{S}$ *are square matrices of size* $(2^{k-1} - 1)$. *Then,* $\boldsymbol{P} = \boldsymbol{Q} = \boldsymbol{R} = \boldsymbol{A_{k-1}}$ *and*

$$\boldsymbol{S} + \boldsymbol{A_{k-1}} = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}.$$

PROOF. (1) $\boldsymbol{A_k}$ is symmetric by definition.
(2) Note that $v_{2^{k-1}} = (0, \dots, 0, 1)$. So $(v_{2^{k-1}}, w) = (w, v_{2^{k-1}}) = 0$ for all $w < (0, \dots, 0, 1)$ in $(\mathbb{Z}/2)^k \setminus \{0\}$.
(3) Clearly $\boldsymbol{P} = \boldsymbol{A_{k-1}}$. Note that $v_{2^{k-1}+r} = v_r + v_{2^{k-1}}$ for $1 \leq r < 2^{k-1}$. So $(v_r, v_i) = (v_{2^{k-1}+r}, v_i)$ for $1 \leq i \leq 2^{k-1} - 1$. Thus we have $\boldsymbol{P} = \boldsymbol{R}$. By symmetricity, $\boldsymbol{Q} = \boldsymbol{R}$. Since $(v_{2^{k-1}+r}, v_{2^{k-1}+r'}) + (v_r, v_{r'}) = 2(v_r, v_{r'}) + (v_{2^{k-1}}, v_{2^{k-1}}) = 1$ for $1 \leq r, r' < 2^{k-1}$, (3) is proved.
This completes the proof of Proposition 2.6. $\square$

EXAMPLE 2.7. For $k = 3$, it is easy to see that the matrix $\boldsymbol{A_3}$ is

$$\boldsymbol{A_3} = \left(\begin{array}{ccc|c|ccc} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right) = \left(\begin{array}{ccc|c|ccc} & & & 1 & & & \\ & \boldsymbol{A_2} & & 1 & & \boldsymbol{A_2} & \\ & & & 1 & & & \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline & & & 0 & & & \\ & \boldsymbol{A_2} & & 0 & & \boldsymbol{1} - \boldsymbol{A_2} & \\ & & & 0 & & & \end{array}\right),$$

where $\boldsymbol{1}$ is the 3×3 matrix with all entries 1.

Fix a nonnegative integer $b \geq 0$ and consider three special linear programming problems.

(1) Given $k \geq 2$ and nonnegative integer $b \geq 0$, maximize $\sum x_i$ subject to $\boldsymbol{A_k x} \leq \boldsymbol{b} := (b, b, \dots, b)^T$ and $\boldsymbol{x} = (x_1, x_2, \dots, x_{2^k-1})^T \geq \boldsymbol{0}$.
(2) Given $k \geq 2$ and nonnegative integer $b \geq 0$, minimize $\sum by_i$ subject to $\boldsymbol{A_k^T y} \geq \boldsymbol{c} := (1, 1, \dots, 1)^T$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_{2^k-1})^T \geq \boldsymbol{0}$.
(3) Given $k \geq 2$ and nonnegative integer $b \geq 0$, minimize $\sum y_i$ subject to $\boldsymbol{A_k y} \geq \boldsymbol{b} = (b, b, \dots, b)^T$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_{2^k-1})^T \geq \boldsymbol{0}$.

Note that the linear programming (2) is the standard form of dual problem of (1) [**7**] and (3) is the same problem as (2). So we will regard (3) as a dual problem of (1). In general, the optimal solution of the primal and dual problem is $\boldsymbol{x}^* = \boldsymbol{y}^* = (b/(2^{k-1}-1), b/(2^{k-1}-1), \dots, b/(2^{k-1}-1))$ and $\sum x_i^* = \sum y_i^* = ((2^k-1)/(2^{k-1}-1))b$ (strong duality theorem). If we add a restriction that $\boldsymbol{x}$ is an integer vector to (1) above, then finding the solution $\sum x_i$ is equivalent to finding $m_k(b)$. i.e.,

$$(\boldsymbol{A_k x})_i = \sum_{(v_i, v_j)=0} x_j \leq b \quad \text{for each } v_i \in (\mathbb{Z}/2)^k \setminus \{0\},$$

$$\sum x_i = m_k(b).$$

For convenience, let's introduce the notation for the problems of integer linear programming.

Definition 2.8. Given $k \geq 2$ and nonnegative integer $b \geq 0$, let $\mathrm{IP}(k, b)$ be the problem of integer linear programming:

> Maximize $\sum x_i$ subject to $\boldsymbol{A_k x} \leq \boldsymbol{b} = (b, b, \dots, b)^T$ and integer vector $\boldsymbol{x} = (x_1, x_2, \dots, x_{2^k-1})^T \geq \boldsymbol{0}$.

And let $\mathrm{IP}^*(k, b)$ be the problem of integer linear programming:

> Minimize $\sum y_i$ subject to $\boldsymbol{A_k y} \geq \boldsymbol{b} = (b, b, \dots, b)^T$ and integer vector $\boldsymbol{y} = (y_1, y_2, \dots, y_{2^k-1})^T \geq \boldsymbol{0}$.

3. The values of $\boldsymbol{m_k^*(b)}$.

In this section, we will define $m_k^*(b)$ which is *dual* to $m_k(b)$. The value $m_k^*(b)$ has interesting properties. Some of them are different to that of $m_k(b)$. The most important fact is that $m_k^*(b)$ is preperiodic for $b \geq 0$. From preperiodicity, we can find the formula for the values of $m_k^*(b)$. We begin with the definition.

Definition 3.1. Define $m_k(b) := \sum x_i$ for an optimal solution $\boldsymbol{x}^*$ of $\mathrm{IP}(k, b)$ and define $m_k^*(b) := \sum y_i^*$ for an optimal solution $\boldsymbol{y}^*$ of $\mathrm{IP}^*(k, b)$.

The following lemmas and corollary show the relation between $m_k^*(b)$ and $m_{k-1}^*(b)$ associated with the values of b.

Lemma 3.2. *For any nonnegative integer* $1 \leq q \leq b$,

$$m_k^*(b) \geq m_k^*(b-q)+q$$

and $m_k^*(b-q)+q$ *non-strictly decreases as* q *increases.*

Proof. Let $\boldsymbol{y}^*$ be an optimal solution to $\mathrm{IP}^*(k, b+1)$. Then subtracting 1 from one of the coordinates of $\boldsymbol{y}^*$ gives a feasible solution to $\mathrm{IP}^*(k,b)$. Thus

$$m_k^*(b)+1 \leq m_k^*(b+1). \tag{5}$$

If we use (5) repeatedly, it follows that

$$b = m_k^*(0)+b \leq m_k^*(b-q)+q \leq m_k^*(b-1)+1 \leq m_k^*(b). \qquad \square$$

Lemma 3.3. *If* $\ell = [b/(2^{k-1}-1)]$, *the biggest integer less than or equal to* $b/(2^{k-1}-1)$, *then*

$$m_k^*(b) \geq m_{k-1}^*(b-q)+q$$

for any integer $\ell \leq q \leq b$.

Proof. Let $\boldsymbol{y}^*$ be an optimal solution to $\mathrm{IP}^*(k,b)$. Then $y_i^* \leq \ell$ for some $1 \leq i \leq 2^k-1$. Because if $y_i^* \geq \ell+1$ for all $1 \leq i \leq 2^k-1$, then $(\boldsymbol{A_k y}^*)_i \geq (2^{k-1}-1)(\ell+1) > b$ for all $1 \leq i \leq 2^k-1$. So, $\boldsymbol{y}^*$ is a feasible solution to $\mathrm{IP}^*(k,b+1)$ which contradicts to the relation (5). We may assume that $y_{2^{k-1}}^* \leq \ell$.

Let $\boldsymbol{y'}$ be a vector in $\mathbb{Z}^{(2^{k-1}-1)}$ such that $y_i' := y_i^* + y_{i+2^{k-1}}^*$ for $1 \leq i \leq 2^{k-1}-1$. If we let $\pi : \mathbb{Z}^{(2^k-1)} \to \mathbb{Z}^{(2^{k-1}-1)}$ be the natural projection, then by Proposition 2.6,

$$\begin{aligned}
\boldsymbol{A_{k-1}y'} &= \boldsymbol{A_{k-1}}(y_1^*,\dots,y_{2^{k-1}-1}^*)^T + \boldsymbol{A_{k-1}}(y_{2^{k-1}+1}^*,\dots,y_{2^k-1}^*)^T \\
&= \pi(\boldsymbol{A_k}(y_1^*,\dots,y_{2^{k-1}-1}^*,0,\dots,0)^T + \boldsymbol{A_k}(0,\dots,0,y_{2^{k-1}+1}^*,\dots,y_{2^k-1}^*)^T) \\
&= \pi(\boldsymbol{A_k y}^*) - \pi(\boldsymbol{A_k}(0,\dots,0,y_{2^{k-1}}^*,0,\dots,0)^T) \\
&\geq (b,\dots,b)^T - (y_{2^{k-1}}^*,\dots,y_{2^{k-1}}^*)^T \\
&= (b-y_{2^{k-1}}^*,\dots,b-y_{2^{k-1}}^*)^T.
\end{aligned}$$

So $\boldsymbol{y'}$ is a feasible solution of $\mathrm{IP}^*(k-1, b-y_{2^{k-1}}^*)$ and hence

$$m_{k-1}^*(b-y_{2^{k-1}}^*) \leq \sum y_i'.$$

Therefore,

$$\begin{aligned}
m_k^*(b) = \sum y_i^* = y_{2^{k-1}}^* + \sum_{i \neq 2^{k-1}} y_i^* &= y_{2^{k-1}}^* + \sum y_i' \\
&\geq y_{2^{k-1}}^* + m_{k-1}^*(b-y_{2^{k-1}}^*).
\end{aligned} \tag{6}$$

By Lemma 3.2, $q + m^*_{k-1}(b-q)$ decreases as q increases. Therefore, the inequality in Lemma 3.3 follows from (6). $\square$

LEMMA 3.4. *For $k \geq 3$, $m^*_k(b) \leq m^*_{k-1}(b)$.*

PROOF. Let $\boldsymbol{y}^*$ be an optimal solution of $\mathrm{IP}^*(k-1, b)$ and let

$$\boldsymbol{y'} := (y^*_1, \ldots, y^*_{2^{k-1}-1}, 0, \ldots, 0)^T \in \mathbb{Z}^{2^k-1}.$$

Then by Proposition 2.6, the i-th component of $\boldsymbol{A_k y'}$

$$(\boldsymbol{A_k y'})_i = \begin{cases} (\boldsymbol{A_{k-1} y^*})_i, & \text{if } i \neq 2^{k-1} \\ \sum y^*_j = m^*_{k-1}(b), & \text{if } i = 2^{k-1}. \end{cases}$$

Here, $m^*_{k-1}(b) \geq b$ from the definition. So we have

$$\boldsymbol{A_k y'_i} \geq b \quad \text{for all } 1 \leq i \leq 2^k - 1.$$

Therefore, $\boldsymbol{y'}$ is a feasible solution of $\mathrm{IP}^*(k, b)$ so that

$$m^*_k(b) \leq \sum y'_i = \sum y^*_i = m^*_{k-1}(b). \qquad \square$$

COROLLARY 3.5. *Let $k \geq 3$. Then $m^*_k(b) = m^*_{k-1}(b)$ for $0 \leq b \leq 2^{k-1} - 2$.*

PROOF. The idea of the current proof below was proposed by Mikiya Masuda.

Since $[b/(2^{k-1}-1)] = 0$, one can take $q = 0$ in Lemma 3.3. Therefore we have $m^*_k(b) \geq m^*_{k-1}(b)$. On the other hand, $m^*_k(b) \leq m^*_{k-1}(b)$ by Lemma 3.4. Therefore the corollary follows. $\square$

LEMMA 3.6. *Let $k \geq 2$. Then*

$$m^*_k(b + b') \leq m^*_k(b) + m^*_k(b').$$

In particular,

$$m^*_k((2^{k-1}-1)Q + R) \leq (2^k - 1)Q + m^*_k(R)$$

for any $Q \geq 0$ and $0 \leq R \leq 2^{k-1} - 2$.

PROOF. Let $\boldsymbol{x} = \boldsymbol{x_1} + \boldsymbol{x_2}$, where $\boldsymbol{x_1}$ (resp. $\boldsymbol{x_2}$) is an optimal solution of $\mathrm{IP}^*(k, b)$ (resp. $\mathrm{IP}^*(k, b')$). Then $\boldsymbol{A_k x} \geq (b + b', \ldots, b + b')^T$ so that

$$m^*_k(b + b') \leq \sum x_i = \sum x_{1i} + \sum x_{2i} = m^*_k(b) + m^*_k(b').$$

Let $\boldsymbol{y} = (Q, \ldots, Q)^T + \boldsymbol{y_1}$, where $\boldsymbol{y_1}$ is an optimal solution of $\mathrm{IP}^*(k, R)$. Then

$\boldsymbol{A_k y} \geq ((2^{k-1}-1)Q+R, \ldots, (2^{k-1}-1)Q+R)$ so that

$$m_k^*((2^{k-1}-1)Q+R) \leq \sum y_i = (2^k-1)Q + m_k^*(R). \qquad \square$$

PROPOSITION 3.7. *Let $k \geq 2$. Then for any $b > 0$*

$$m_k^*((2^{k-1}-1)+b) = (2^k-1) + m_k^*(b).$$

PROOF. The idea of the current proof below was proposed by Mikiya Masuda. We will prove by using induction on k. If $k=2$, then

$$\boldsymbol{A_2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So it is easy to see that $m_2^*(b) = 3b$ and $m_2^*(1+b) = 3(1+b) = 3 + m_2^*(b)$. Suppose that the proposition holds for $k-1 (k \geq 3)$. We note that the proposition is equivalent to the statement that

$$m_k^*((2^{k-1}-1)Q+R) = (2^k-1)Q + m_k^*(R)$$

for any $Q \geq 0$ and $0 \leq R \leq 2^{k-1}-2$. Since $[((2^{k-1}-1)Q+R)/(2^{k-1}-1)] = Q$, one can take $q = Q$ in Lemma 3.3. It follows that

$$\begin{aligned}
&m_k^*((2^{k-1}-1)Q+R) \\
&\quad \geq m_{k-1}^*((2^{k-1}-1)Q+R-Q)+Q \quad \text{(by Lemma 3.3)} \\
&\quad = m_{k-1}^*((2^{k-2}-1)2Q+R)+Q \\
&\quad = (2^{k-1}-1)2Q + m_{k-1}^*(R) + Q \quad \text{(by the induction assumption)} \\
&\quad = (2^k-1)Q + m_{k-1}^*(R) \\
&\quad = (2^k-1)Q + m_k^*(R) \quad \text{(by Corollary 3.5)}.
\end{aligned}$$

On the other hand, by Lemma 3.6

$$m_k^*((2^{k-1}-1)Q+R) \leq (2^k-1)Q + m_k^*(R)$$

proving the proposition. $\square$

The following proposition gives us a useful tool to calculate $m_k^*(b)$. If we know the values of $m_k^*(b)$ for half of small $0 \leq b < 2^{k-1}-1$, then we can calculate that of $m_k^*(b)$ for the other half. Precisely,

PROPOSITION 3.8. *Let $k \geq 3$. Then*

$$m_k^*((2^{k-2}-1)+b) = (2^{k-1}-1) + m_k^*(b)$$

for $0 \le b \le 2^{k-2}-1$.

PROOF. Since $(2^{k-2}-1)+b \le 2^{k-1}-2$, by Corollary 3.5 $m_k^*((2^{k-2}-1)+b) = m_{k-1}^*((2^{k-2}-1)+b)$ and $m_k^*(b) = m_{k-1}^*(b)$. Thus we have

$$\begin{aligned} & m_k^*((2^{k-2}-1)+b) \\ & \quad = m_{k-1}^*((2^{k-2}-1)+b) \\ & \quad = (2^{k-1}-1) + m_{k-1}^*(b) \quad \text{(by Proposition 3.7)} \\ & \quad = (2^{k-1}-1) + m_k^*(b). \end{aligned}$$

□

Proposition 3.7 and Corollary 3.5 enable us to find all the values of $m_k^*(b)$ inductively.

LEMMA 3.9. *For* $0 \le b < 2^{k-1}-1$, b *can be expanded uniquely as*

$$b = c_2(2^{k-2}-1) + c_3(2^{k-3}-1) + \cdots + c_{k-1}(2^1-1),$$

where

$$\begin{aligned} c_2 &= \left[\frac{b}{2^{k-2}-1}\right], \\ c_3 &= \left[\frac{b - c_2(2^{k-2}-1)}{2^{k-3}-1}\right], \\ &\vdots \\ c_{k-2} &= \left[\frac{b - c_2(2^{k-2}-1) - \cdots - c_{k-3}(2^3-1)}{2^2-1}\right], \\ c_{k-1} &= b - c_2(2^{k-2}-1) - \cdots - c_{k-2}(2^2-1). \end{aligned}$$

PROOF. The c_2 is quotient of b divided by $2^{k-2}-1$. The c_3 is quotient of the remainder $b - c_2(2^{k-2}-1)$ divided by $2^{k-3}-1$ and so on. This guarantees uniqueness. □

For convenience, we use the notation $b = \overline{c_2, \ldots, c_{k-1}}$ for the expansion $b = c_2(2^{k-2}-1) + c_3(2^{k-3}-1) + \cdots + c_{k-1}(2^1-1)$.

THEOREM 3.10. *Let* $k \ge 2$. *If* $b = \overline{c_2, \ldots, c_{k-1}}$, *then*

$$m_k^*((2^{k-1}-1)\ell + b) = (2^k-1)\ell + \overline{c_2, \ldots, c_{k-1}, 0}$$

for any $\ell \ge 0$.

PROOF. By Proposition 3.7, it is enough to show that

$$m_k^*(b) = \overline{c_2, \ldots, c_{k-1}, 0}. \tag{7}$$

We will prove relation (7) by induction on k. Obviously $m_2^*(0) = 0$. Suppose that (7) holds for $k-1$. Then

$$\begin{aligned} m_k^*(b) &= m_k^*(\overline{c_2, \ldots, c_{k-1}}) \\ &= m_{k-1}^*(\overline{c_2, \ldots, c_{k-1}}) \quad \text{(by Corollary 3.5)} \\ &= c_2(2^{k-1}-1) + m_{k-1}^*(\overline{c_3, \ldots, c_{k-1}}) \quad \text{(by Proposition 3.7)} \\ &= \overline{c_2, \ldots, c_{k-1}, 0} \quad \text{(by the induction assumption).} \end{aligned}$$

Therefore, (7) holds for k. □

COROLLARY 3.11. *Let $k \geq 2$ and $0 \leq b \leq 2^{k-1} - 2$. Then*

$$m_k^*(b+1) - m_k^*(b) = 1 \text{ or } 3.$$

Precisely, if $b = \overline{c_2, \ldots, c_{k-1}}$, then $m_k^*(b+1) - m_k^*(b) = 1$ if and only if $c_i = 2$ for some $2 \leq i \leq k-1$.

PROOF. Note that if $c_i = 2$, then $c_j = 0$ for $j > i$. If $c_i \neq 2$ for all $2 \leq i \leq (k-1)$, then

$$b + 1 = \overline{c_2, \ldots, c_{k-1}} + 1 = \overline{c_2, \ldots, c_{k-1} + 1}.$$

By Theorem 3.10 we have

$$m_k^*(b+1) - m_k^*(b) = \overline{c_2, \ldots, c_{k-1} + 1, 0} - \overline{c_2, \ldots, c_{k-1}, 0} = 2^2 - 1 = 3.$$

If $c_{i_0} = 2$ for some $2 \leq i_0 \leq k-1$. Then

$$b + 1 = \overline{c_2, \ldots, c_{i_0 - 1}, 2, 0, \ldots, 0} + 1 = \overline{c_2, \ldots, c_{i_0-1} + 1, 0, \ldots, 0}.$$

By Theorem 3.10 we have

$$m_k^*(b+1) - m_k^*(b) = (2^{k-i_0+2} - 1) - 2(2^{k-i_0+1} - 1) = 1. \qquad \square$$

Since $m_k^*(b)$ is preperiodic, Corollary 3.11 implies that $m_k^*(b+1) - m_k^*(b) = 1$ or 3 for any $b > 0$. The following lemma is a generalization of Corollary 3.11.

LEMMA 3.12. *For $0 \leq b \leq 2^{k-1} - 2$ and $p \geq 1$, suppose that b is expanded as $b = \overline{c_2, \ldots, c_{k-1}}$. Then $m_{k-1}^*(b+p) = m_{k-1}^*(b) + p$ if and only if $p \leq k-3$ and $c_{i_0} = 2$ for some $3 \leq i_0 \leq k-1$ and (only for $p \geq 2$) $c_{i_0-(p-1)} = c_{i_0-(p-2)} = \cdots = c_{i_0-2} = c_{i_0-1} = 1$.*

PROOF. It is easy to see that $m^*_{k-1}(b+p) = m^*_{k-1}(b) + p$ if and only if $m^*_{k-1}(b+q+1) = m^*_{k-1}(b+q) + 1$ for any $0 \le q \le p-1$. If $q = 0$, then by Corollary 3.11, $m^*_{k-1}(b+1) = m^*_{k-1}(b) + 1$ if and only if $c_{i_0} = 2$ for some $3 \le i_0 \le k-1$. Since $\overline{c_2, \ldots, c_{i_0-1}, 2, 0, \ldots, 0} + 1 = \overline{c_2, \ldots, c_{i_0-1} + 1, 0, \ldots, 0}$, whenever we increase q by 1, we move rightmost 2 to the left step by step so that $c_{i_0-(p-1)} = \cdots = c_{i_0-1} = 1$. Since $i_0 - (p-1) \ge 2$ and $i_0 \le k-1$, we have $p \le k-2$. If $p = k-2$, then $i_0 = k-1$ so that the integer $b \le 2^{k-1} - 2$ which satisfies $c_{i_0} = c_{k-1} = 2$ and $c_2 = \cdots = c_{k-2} = 1$ is $b = \overline{1, \ldots, 1, 2}$. So, $b + (k-2) = 2^{k-1} - 1$ and $m^*_{k-1}(2^{k-1}-1) - m^*_{k-1}(2(2^{k-2}-1)) = 3$ by Corollary 3.11. Thus we have $m^*_{k-1}(b+p) < m^*_{k-1}(b) + p$. Therefore, $p \le k-3$. □

We already saw that $m^*_k(b) = m^*_{k-1}(b)$ for $0 \le b \le 2^{k-1} - 2$. The following Lemma 3.13 is about relation between $m^*_k(b)$ and $m^*_{k-1}(b)$ for $b \ge 2^{k-1} - 1$. This lemma shows that $m^*_k((2^{k-1}-1)\ell + b) = m^*_{k-1}((2^{k-1}-1)\ell + b)$ is equivalent to $m^*_{k-1}(b+\ell) = m^*_{k-1}(b) + \ell$. From Lemma 3.12 we can decide whether $m^*_{k-1}(b+\ell) = m^*_{k-1}(b) + \ell$ by expanding $b = \overline{c_2, \ldots, c_{k-1}}$.

LEMMA 3.13. *Let $k \ge 3$. Then for any $0 \le b \le 2^{k-1} - 2$ and $\ell \ge 0$, the relation $m^*_k((2^{k-1}-1)\ell + b) = m^*_{k-1}((2^{k-1}-1)\ell + b)$ is equivalent to the relation $m^*_{k-1}(b+\ell) = m^*_{k-1}(b) + \ell$.*

PROOF. We have

$$\begin{aligned}
&m^*_{k-1}((2^{k-1}-1)\ell + b)\\
&\quad = m^*_{k-1}((2^{k-2}-1)2\ell + (b+\ell))\\
&\quad = (2^{k-1}-1)2\ell + m^*_{k-1}(b+\ell) \quad \text{(by Theorem 3.7)}\\
&\quad \ge (2^{k-1}-1)2\ell + m^*_{k-1}(b) + \ell \quad \text{(by Lemma 3.2)}\\
&\quad = (2^k-1)\ell + m^*_k(b) \quad \text{(by Corollary 3.5)}\\
&\quad = m^*_k((2^{k-1}-1)\ell + b) \quad \text{(by Theorem 3.7).}
\end{aligned}$$

Thus $m^*_{k-1}((2^{k-1}-1)\ell + b) = m^*_k((2^{k-1}-1)\ell + b)$ implies $m^*_{k-1}(b+\ell) = m^*_{k-1}(b) + \ell$ and vice versa. □

EXAMPLE 3.14. All the values of $m^*_k(b)$ can be computed by Theorem 3.10. Table 1 below is a table of values of $m^*_k((2^{k-1}-1)\ell + b)$ for $k = 2, 3, 4, 5, 6$.

4. Relation between $m_k(b)$ and $m^*_k(b)$.

In this section, we define the invariant $d_k(b)$ estimating the moment at which the function $m_k(b)$ becomes preperiodic, and prove Theorem 1.1 and Theorem 1.2.

DEFINITION 4.1. Let $k \ge 2$. For $0 \le b \le 2^{k-1} - 1$, define

$$d_k(b) := \min\{\max\{y^*_i\} \mid \boldsymbol{y}^* \text{ is optimal solution of } \mathrm{IP}^*(k,b)\} - 1.$$

$b\backslash k$	2	3	4	5	6
0	3ℓ	7ℓ	15ℓ	31ℓ	63ℓ
1		$7\ell+3$	$15\ell+3$	$31\ell+3$	$63\ell+3$
2		$7\ell+6$	$15\ell+6$	$31\ell+6$	$63\ell+6$
3			$15\ell+7$	$31\ell+7$	$63\ell+7$
4			$15\ell+10$	$31\ell+10$	$63\ell+10$
5			$15\ell+13$	$31\ell+13$	$63\ell+13$
6			$15\ell+14$	$31\ell+14$	$63\ell+14$
7				$31\ell+15$	$63\ell+15$
8				$31\ell+18$	$63\ell+18$
9				$31\ell+21$	$63\ell+21$
10				$31\ell+22$	$63\ell+22$
11				$31\ell+25$	$63\ell+25$
12				$31\ell+28$	$63\ell+28$
13				$31\ell+29$	$63\ell+29$
14				$31\ell+30$	$63\ell+30$
15					$63\ell+31$
16					$63\ell+34$
17					$63\ell+37$
18					$63\ell+38$
19					$63\ell+41$
20					$63\ell+44$
21					$63\ell+45$
22					$63\ell+46$
23					$63\ell+49$
24					$63\ell+52$
25					$63\ell+53$
26					$63\ell+56$
27					$63\ell+59$
28					$63\ell+60$
29					$63\ell+61$
30					$63\ell+62$

Table 1. $m_k^*((2^{k-1}-1)\ell+b)$ for $k=2,3,4,5,6$.

Lemma 4.2. *Let $k \geq 2$ and $0 \leq b < 2^{k-1}-1$. Let $\boldsymbol{y}$ be an optimal solution of* $\mathrm{IP}^*(k,b)$. *Then there exists an integer $\ell_0 > 0$ such that for $\ell \geq \ell_0$, $\boldsymbol{x} := (\ell,\dots,\ell)^T - \boldsymbol{y}$ is an optimal solution of* $\mathrm{IP}(k,(2^{k-1}-1)\ell-b)$.

Proof. Let's take $\ell_0 = \max\{y_i\}$ so that $\boldsymbol{x} \geq \mathbf{0}$. Since $\boldsymbol{A_k y} \geq (b,\dots,b)^T$, we have

$$\begin{aligned}\boldsymbol{A_k x} &\leq \boldsymbol{A_k}(\ell,\dots,\ell)^T - (b,\dots,b)^T \\ &= ((2^{k-1}-1)\ell-b,\dots,(2^{k-1}-1)\ell-b)^T.\end{aligned}$$

Then $\boldsymbol{x}$ is an optimal solution of $\mathrm{IP}(k,(2^{k-1}-1)\ell-b)$. Otherwise, let $\boldsymbol{x'}$ be an optimal solution of $\mathrm{IP}(k,(2^{k-1}-1)\ell-b)$. Consider $\boldsymbol{y'}=(Q,\dots,Q)^T-\boldsymbol{x'}\geq\mathbf{0}$. We may assume that $Q\geq\ell$. Let $Q'=Q-\ell$. Then

$$\begin{aligned}\boldsymbol{A_k y'}&\geq\boldsymbol{A_k}(Q,\dots,Q)^T-((2^{k-1}-1)\ell-b,\dots,(2^{k-1}-1)\ell-b)^T\\&=((2^{k-1}-1)Q'+b,\dots,(2^{k-1}-1)Q'+b)^T.\end{aligned}$$

Since $\sum x'_i+\sum y'_i=\sum x_i+\sum y_i+(2^k-1)Q'$ and $\sum x_i<\sum x'_i$,

$$\begin{aligned}\sum y'_i&<\sum y_i+(2^k-1)Q'=m^*_k(b)+(2^k-1)Q'\\&=m^*_k((2^{k-1}-1)Q'+b).\end{aligned}$$

The last equality holds by Proposition 3.7. This is contradiction. □

Proposition 4.3. *Let $k\geq 2$. For $\ell\geq d_k((2^{k-1}-1)-b)$,*

$$m_k((2^{k-1}-1)\ell+b)+m^*_k(2^{k-1}-1-b)=(2^k-1)(\ell+1).$$

Proof. Let $\boldsymbol{y}^*$ be an optimal solution of $\mathrm{IP}^*(k,2^{k-1}-1-b)$ with $\max\{y_i\}=d_k(2^{k-1}-1-b)+1$. By Lemma 4.2, $\boldsymbol{x}:=(\ell+1,\dots,\ell+1)^T-\boldsymbol{y}^*$ is an optimal solution of $\mathrm{IP}(k,(2^{k-1}-1)\ell+b)$. Then

$$m_k((2^{k-1}-1)\ell+b)+m^*_k(2^{k-1}-1-b)=\sum x_i+\sum y^*_i=(2^k-1)(\ell+1).$$ □

Now let us prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 4.3,

$$\begin{aligned}&m_k((2^{k-1}-1)\ell_2+b)-m_k((2^{k-1}-1)\ell_1+b)\\&\quad=(2^k-1)(\ell_2+1)-(2^k-1)(\ell_1+1)\\&\quad=(2^k-1)(\ell_2-\ell_1).\end{aligned}$$

for $\ell_2\geq\ell_1\geq d_k((2^{k-1}-1)-b)$. □

Remark 4.4. $m_k((2^{k-1}-1)\ell+b)$ is preperiodic up to (2^k-1) for $\ell\geq d_k((2^{k-1}-1)-b)$.

Corollary 4.5. *Let $0\leq b\leq 2^{k-1}-2$. Then*

$$m_k((2^{k-1}-1)\ell+b+1)-m_k((2^{k-1}-1)\ell+b)=1\ \textit{or}\ 3$$

for $\ell\geq\max\{d_k((2^{k-1}-1)-b),d_k((2^{k-1}-1)-(b+1))\}$. Precisely, if $0\leq(2^{k-1}-1)-(b+1)<2^{k-1}-1$ is expanded as

$$(2^{k-1}-1)-(b+1)=\overline{c_2,\dots,c_{k-1}},$$

then

$$m_k((2^{k-1}-1)\ell+b+1)-m_k((2^{k-1}-1)\ell+b)=1$$

if and only if $c_i=2$ *for some* $2\le i\le k-1$.

PROOF. By Proposition 4.3

$$\begin{aligned}&m_k((2^{k-1}-1)\ell+b+1)-m_k((2^{k-1}-1)\ell+b)\\&\quad=m_k^*((2^{k-1}-1)-b)-m_k^*((2^{k-1}-1)-(b+1))\\&\quad=1 \text{ or } 3.\end{aligned}$$

The last equality is given by Corollary 3.11, and we have 1 if and only if $c_i=2$ for some $2\le i\le k-1$. □

Let us prove Theorem 1.2.

PROOF OF THEOREM 1.2. By Proposition 4.3 and Theorem 3.10, we get equation (1) and Theorem 1.2 is proved. □

REMARK 4.6. The values of $m_k((2^{k-1}-1)\ell+b)$ for $\ell<d_k(2^{k-1}-1-b)$ are unknown in many cases. However, since $(2^k-1)\ell+m_k(b)\le m_k((2^{k-1}-1)\ell+b)$ [**6**, Lemma 4.1], we have an upper bound of unknown values of $m_k(b)$. For $\ell<d_k((2^{k-1}-1)-b)$,

$$m_k((2^{k-1}-1)\ell+b)\le m_k((2^{k-1}-1)d_k(b)+b)-(2^k-1)(d_k(b)-\ell).$$

THEOREM 4.7. *Let* $k\ge 3$. *For* $a,b>0$ *with* $a+b=2^{k-1}-1$ *and* $d_k(a)\ge 1$, *suppose that*

$$m_{k-1}^*(a+1)\ne m_{k-1}^*(a)+1. \tag{8}$$

Then

$$m_k((2^{k-1}-1)\ell+b)<(2^k-1)(\ell+1)-m_k^*(a) \tag{9}$$

for $0\le\ell\le d_k(a)-1$.

PROOF. Let $\boldsymbol{x}$ be an optimal solution of $\mathrm{IP}(k,(2^{k-1}-1)(d_k(a)-1)+b)$ and let $s+1:=\max\{x_i\}$. Suppose that

$$m_k((2^{k-1}-1)(d_k(a)-1)+b)\ge(2^k-1)(d_k(a)-1)+((2^k-1)-m_k^*(a)).$$

Let $\boldsymbol{y}:=(s+1,\dots,s+1)-\boldsymbol{x}$. Then

$$\begin{aligned}(\boldsymbol{A_k y})_i &\geq (2^{k-1}-1)(s+1) - (2^{k-1}-1)(d_k(a)-1) + b \\ &= (2^{k-1}-1)(s - d_k(a) + 1) + a\end{aligned}$$

for $1 \leq i \leq 2^k - 1$ and

$$\begin{aligned}\sum y_i &= (2^k - 1)(s+1) - \sum x_i \\ &= (2^k-1)(s+1) - m_k((2^{k-1}-1)(d_k(a)-1) + b) \\ &\leq (2^k - 1)(s - d_k(a) + 1) + m_k^*(a) \\ &= m_k^*((2^{k-1}-1)(s - d_k(a) + 1) + a),\end{aligned}$$

where the inequality is by the assumption at the beginning of the proof and the last equality is by Proposition 3.7. Thus we have

$$\sum y_i = m_k^*((2^{k-1}-1)(s - d_k(a) + 1) + a)$$

so that $\boldsymbol{y}$ is an optimal solution of $\mathrm{IP}^*(k, (2^{k-1}-1)(s - d_k(a) + 1) + a)$.

Since $b > 0$, we have $\sum x_i = m_k((2^{k-1}-1)(d_k(a)-1)+b) > (2^k-1)(d_k(a)-1)$ so that $\max\{x_i\} = s+1 \geq d_k(a)$. If $s+1 = d_k(a)$, then $\boldsymbol{y}$ is an optimal solution of $\mathrm{IP}^*(k,a)$ such that $\max\{y_i\} \leq s+1 = d_k(a)$. This contradicts to the definition of $d_k(a)$. Suppose that $s+1 \geq d_k(a)+1$. Note that $y_i = 0$ for some $1 \leq i \leq 2^k - 1$ and we may assume that $y_{2^{k-1}} = 0$. Let $\boldsymbol{y'} \in \mathbb{Z}_{\geq 0}^{(2^{k-1}-1)}$ be defined by $y_i' := y_i + y_{i+2^{k-1}}$ for $1 \leq i \leq 2^{k-1}-1$. Then $\sum y_i = \sum y_i'$ and by Proposition 2.6,

$$(\boldsymbol{A_{k-1} y'})_i = (\boldsymbol{A_k y})_i \geq (2^{k-1}-1)(s - d_k(a) + 1) + a$$

for $1 \leq i \leq 2^{k-1}-1$. Thus $\sum y_i' \geq m_{k-1}^*((2^{k-1}-1)(s - d_k(a)+1) + a)$ and

$$m_k^*((2^{k-1}-1)(s-d_k(a)+1)+a) = \sum y_i = \sum y_i' \geq m_{k-1}^*((2^{k-1}-1)(s-d_k(a)+1)+a).$$

Combining this with Lemma 3.4, we get

$$m_k^*((2^{k-1}-1)(s - d_k(a) + 1) + a) = m_{k-1}^*((2^{k-1}-1)(s - d_k(a) + 1) + a). \tag{10}$$

Since $s - d_k(a) + 1 \geq 1$, by Lemma 3.13, relation (10) is equivalent to

$$m_{k-1}^*(a + s - d_k(a) + 1) = m_{k-1}^*(a) + s - d_k(a) + 1.$$

However, the hypothesis (8) implies that there is no $s \geq 1$ such that $m_{k-1}^*(a+s-d_k(a)+1) = m_{k-1}^*(a) + s - d_k(a) + 1$. Therefore,

$$m_k((2^{k-1}-1)(d_k(a)-1)+b) < (2^k-1)(d_k(a)-1) + ((2^k-1) - m_k^*(a)).$$

Moreover, for $0 \le t \le d_k(a)-1$, by [**6**, Lemma 4.1]

$$\begin{aligned}(2^k-1)(d_k(a)-1-t)&+m_k((2^{k-1}-1)t+b)\\ &\le m_k((2^{k-1}-1)(d_k(a)-1)+b)\\ &< (2^k-1)(d_k(a)-1)+(2^k-1)-m_k^*(a).\end{aligned}$$

Thus we have

$$m_k((2^{k-1}-1)t+b) < (2^k-1)(t+1)-m_k^*(a)$$

for $0 \le t \le d_k(a)-1$. □

Remark 4.8. For nonnegative integers $a, b > 0$ with $a+b = 2^{k-1}-1$ and $d_k(a) \ge 1$, from Theorem 1.1 we know that $m_k((2^{k-1}-1)\ell+b)$ is preperiodic up to (2^k-1) for $\ell \ge d_k(a)$. When $m_{k-1}^*(a+1) \ne m_{k-1}^*(a)+1$, Theorem 4.7 guarantees that $m_k((2^{k-1}-1)\ell+b)$ is preperiodic *definitely* for $\ell \ge d_k(a)$. Because by Proposition 4.3 for $0 \le \ell \le d_k(a)-1$,

$$\begin{aligned}m_k((2^{k-1}-1)&d_k(a)+b)-m_k((2^{k-1}-1)\ell+b)\\ &= ((2^k-1)(d_k(a)+1)-m_k^*(a))-m_k((2^{k-1}-1)\ell+b)\\ &> ((2^k-1)(d_k(a)+1)-m_k^*(a))-((2^k-1)(\ell+1)-m_k^*(a))\\ &= (2^k-1)(d_k(a)-\ell),\end{aligned}$$

where the inequality is by equation (9).

5. Upper bound of $d_k(b)$.

In this section, we will find upper bound of $d_k(b)$ for each $0 \le b \le 2^{k-1}-1$.

Lemma 5.1. *Suppose $k \ge 2$. The following holds*

(1) $d_k(0) = -1$.
(2) $d_k(b) \le d_{k-1}(b)$ *for* $0 \le b \le 2^{k-2}-1$.
(3) $d_k((2^{k-2}-1)+b) \le d_k(b)+1$ *for* $0 \le b \le 2^{k-2}-1$.
(4) $d_k(b) = 0$ *for* $2^{k-1}-k \le b \le 2^{k-1}-1$ *and* $b > 0$.

Proof. Obviously, $\mathbf{0}$ is the unique optimal solution of $\mathrm{IP}^*(k,0)$. Hence $d_k(0) = -1$.

Let $\boldsymbol{y}$ be an optimal solution of $\mathrm{IP}^*(k-1,b)$ such that $\max\{y_i\} = d_{k-1}(b)+1$ and let $\eta : \mathbb{Z}^{2^{(k-1)}-1} \hookrightarrow \mathbb{Z}^{2^k-1}$ be the natural inclusion. Then by Proposition 2.6, $\boldsymbol{A_k}\eta(\boldsymbol{y}) \ge \boldsymbol{b}$ so that $\eta(\boldsymbol{y})$ is a feasible solution of $\mathrm{IP}^*(k,b)$. Thus

$$d_k(b) \le \max\{\eta(\boldsymbol{y})_i\}-1 = \max\{y_i\}-1 = d_{k-1}(b).$$

This proves (2).

Let $\boldsymbol{z}$ be an optimal solution of $\mathrm{IP}^*(k,b)$ such that $\max\{z_i\} = d_k(b)+1$ for $0 \le b \le 2^{k-2}-1$ and let

$$\boldsymbol{z}' := \boldsymbol{z} + (\underbrace{1,\dots,1}_{2^{k-1}-1},0,\dots,0)^T.$$

Then we have

$$\begin{aligned}\boldsymbol{A_k z'} &= \boldsymbol{A_k z} + \boldsymbol{A_k}(1,\dots,1,0,\dots,0)^T\\ &\ge \boldsymbol{b} + (2^{k-2}-1,\dots,2^{k-2}-1)^T\end{aligned}$$

and by Proposition 3.8

$$\sum z'_i = m_k^*(b) + (2^{k-1}-1) = m_k^*((2^{k-2}-1)+b). \tag{11}$$

Thus $\boldsymbol{z}'$ is an optimal solution of $\mathrm{IP}^*(k,(2^{k-2}-1)+b)$ and

$$d_k((2^{k-2}-1)+b) \le \max\{z'_i\}+1 = \max\{z_i\}+2 = d_k(b)+1.$$

This proves (3).

Note that for $1 \le a \le k-1$ we can expand $(2^{k-1}-1)-a$ as

$$\begin{aligned}(2^{k-1}-1)-1 &= \overline{2,0,\dots,0},\\ (2^{k-1}-1)-2 &= \overline{1,2,0,\dots,0},\\ (2^{k-1}-1)-3 &= \overline{1,1,2,0,\dots,0},\\ &\vdots\\ (2^{k-1}-1)-(k-2) &= \overline{1,\dots,1,2},\\ (2^{k-1}-1)-(k-1) &= \overline{1,\dots,1}.\end{aligned}$$

Thus we have $c_{i+a}=2$ for $1 \le a \le k-2$ so that by Corollary 3.11

$$m_k^*((2^{k-1}-1)-(a-1)) - m_k^*((2^{k-1}-1)-a) = 1$$

and

$$m_k^*((2^{k-1}-1)-(k-2)) - m_k^*((2^{k-1}-1)-(k-1)) = 3.$$

This implies that for $0 \le a \le k-2$,

$$m_k^*((2^{k-1}-1)-a) = m_k^*(2^{k-1}-1)-a = (2^k-1)-a$$

and

$$m_k^*((2^{k-1}-1)-(k-1)) = m_k^*((2^{k-1}-1)-(k-2)) - 3 = (2^k-1)-(k+1).$$

Let $\{\boldsymbol{e_j}\}_{\boldsymbol{j}=1,\dots,k}$ be a standard basis of $(\mathbb{Z}/2)^k$ and let $\{\boldsymbol{f_i}\}_{\boldsymbol{i}=1,\dots,2^k-1}$ be a standard basis of $\mathbb{Z}^{2^k-1}$ defined by

$$(\boldsymbol{f_i})_j := \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Consider vectors in $\mathbb{Z}^{2^k-1}$ of the form

$$\boldsymbol{x}_a := \begin{cases} (1,\dots,1)^T - \boldsymbol{f_{2^0}} - \cdots - \boldsymbol{f_{2^{a-1}}} & \text{if } 0 \le a \le k-2, \\ (1,\dots,1)^T - \boldsymbol{f_{2^0}} - \cdots - \boldsymbol{f_{2^{k-1}}} - \boldsymbol{f_{2^k-1}} & \text{if } a = k-1. \end{cases}$$

Note that $(\boldsymbol{A_k f_i})_j = 1$ if and only if $(v_j, v_i) = 0$. Then since any k vectors in $\{v_{2^0}, v_{2^1}, \dots, v_{2^{k-1}}, v_{2^k-1}\} = \{\boldsymbol{e_1}, \dots, \boldsymbol{e_k}, \sum_{j=1}^k \boldsymbol{e_j}\}$ span $(\mathbb{Z}/2)^k$, at most $k-1$ elements of $\{ v_{2^0}, v_{2^1}, \dots, v_{2^{k-1}}, v_{2^k-1}\}$ can be in hyperplane of $(\mathbb{Z}/2)^k$. If $\boldsymbol{A_k}(\boldsymbol{f_{2^0}} + \cdots + \boldsymbol{f_{2^{k-1}}} + \boldsymbol{f_{2^k-1}}) \ge k$, then there exists $v_{j_0} \in (\mathbb{Z}/2)^k \setminus \{0\}$ such that v_{j_0} is orthogonal to at least k elements of $\{ v_{2^0}, v_{2^1}, \dots, v_{2^{k-1}}, v_{2^k-1}\}$. This is impossible. Thus for $0 \le a \le k-1$,

$$\begin{aligned} \boldsymbol{A_k x}_a &\ge (2^{k-1}-1, \dots, 2^{k-1}-1)^T - (a, \dots, a)^T \\ &= ((2^{k-1}-1)-a, \dots, (2^{k-1}-1)-a)^T \end{aligned}$$

and

$$\sum (x_a)_i = \begin{cases} (2^k-1)-a = m_k^*((2^{k-1}-1)-a) & \text{if } a \le k-2, \\ (2^k-1)-(k+1) = m_k^*((2^{k-1}-1)-(k-1)) & \text{if } a = k-1. \end{cases}$$

Thus $\boldsymbol{x}_a$ is an optimal solution of $\mathrm{IP}^*(k, (2^{k-1}-1)-a)$ so that

$$d_k((2^{k-1}-1)-a) \le \max\{(x_a)_i\} - 1 = 0.$$

Therefore, $d_k((2^{k-1}-1)-a) = 0$ for $0 \le a \le k-1$. This proves (4). $\square$

As mentioned in the introduction, $s_{\mathbb{R}}(m,p) = k$ if and only if $m_{k+1}(p-1) < m \le m_k(p-1)$. Thus given p, we can decide the range of m which satisfies $s_{\mathbb{R}}(m,p) = k$ by calculating $m_{k+1}(p-1)$ and $m_k(p-1)$. However, the formula (1) holds for $\ell \ge d_k((2^{k-1}-1)-b)$. More precisely, if we write $p-1$ as $p-1 = (2^{k-1}-1)\ell_1 + b_1$ or $p-1 = (2^k-1)\ell_2 + b_2$, then we can apply the formula (1) for $\ell_1 \ge d_k((2^{k-1}-1)-b_1)$ and $\ell_2 \ge d_{k+1}((2^k-1)-b_2)$. So we need to find the value or good estimation of $d_k(b)$ for each $0 \le b \le 2^{k-1}-2$. In the remaining part of this section, we will care about that.

Lemma 5.2. *$d_5(9) = 0$ and $d_5(10) = 1$. Moreover,*

$$d_k((2^{k-1}-1)-(k+1)) \neq 0 \quad \textit{for } k \geq 12,$$
$$d_k((2^{k-1}-1)-k) \neq 0 \quad \textit{for } k \geq 5.$$

Proof. Let $\boldsymbol{x} \in \mathbb{Z}^{2^4-1}$ be a vector such that

$$x_i := \begin{cases} 1 & \text{if } i = 1, 2, 4, 8, 15 \\ 0 & \text{otherwise.} \end{cases}$$

If we represent $\boldsymbol{x}$ with the basis (12), then $\boldsymbol{x} = \boldsymbol{f_{2^0}} + \cdots + \boldsymbol{f_{2^3}} + \boldsymbol{f_{2^4-1}}$. Thus we have $\boldsymbol{A_4 x} \leq (3, \ldots, 3)^T$ and $\sum x_i = 5 = m_4(3)$ so that $\boldsymbol{x}$ is an optimal solution of IP$(4,3)$. Let $\boldsymbol{y} := (\boldsymbol{x}, 0, \boldsymbol{x})^T \in \mathbb{Z}^{2^5-1}$. Then by Proposition 2.6

$$\boldsymbol{A_5 y} = \left(\begin{array}{c|c|c} & 1 & \\ \boldsymbol{A_4} & 1 & \boldsymbol{A_4} \\ & 1 & \\ \hline 1 \quad 1 \quad 1 & 0 & 0 \quad 0 \quad 0 \\ \hline & 0 & \\ \boldsymbol{A_4} & 0 & \mathbf{1} - \boldsymbol{A_4} \\ & 0 & \end{array}\right) \begin{pmatrix} \boldsymbol{x} \\ \\ 0 \\ \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 2\boldsymbol{A_4 x} \\ \\ \sum x_i \\ \vdots \\ \sum x_i \end{pmatrix} \leq \begin{pmatrix} 6 \\ \vdots \\ 6 \\ 5 \\ \vdots \\ 5 \end{pmatrix},$$

where $\mathbf{1}$ is $(2^4-1) \times (2^4-1)$ matrix with all entries 1. So, if we let $\boldsymbol{y'} = (1, \ldots, 1)^T - \boldsymbol{y}$, then $\sum y'_i = 21 = m^*_5(9)$ and $\boldsymbol{A_5 y'} \geq (9, \ldots, 9, 10, \ldots, 10)^T$. Thus $\boldsymbol{y'}$ is an optimal solution of IP$^*(5,9)$ and $d_5(9) = 0$.

Before we prove the remaining part, we compute $m^*_k(2^{k-1}-1-k)$ and $m^*_k(2^{k-1}-1-(k+1))$. Note that

$$(2^{k-1}-1)-k = (2^{k-2}-1) + \cdots + (2^2-1),$$
$$(2^{k-1}-1)-(k+1) = (2^{k-2}-1) + \cdots + (2^3-1) + 2(2^1-1)$$

so that

$$m^*_k((2^{k-1}-1)-k) = (2^{k-1}-1) + \cdots + (2^3-1) = (2^k-1)-(k+4),$$
$$m^*_k((2^{k-1}-1)-(k+1)) = m^*_k((2^{k-1}-1)-k) - 1 = (2^k-1)-(k+5).$$

We already know that $m_k(k) = k+2$ for $k \geq 5$ by [**6**, Theorem 5.2]. Suppose that $d_k((2^{k-1}-1)-k) = 0$ for $k \geq 5$. Then there exists an optimal solution $\boldsymbol{x}$ of IP$^*(k, (2^{k-1}-1)-k)$ such that $\max\{x_i\} = d_k((2^{k-1}-1)-k)+1 = 1$. By Lemma 4.2, $\boldsymbol{x'} := (1, \ldots, 1)^T - \boldsymbol{x}$ is an optimal solution of IP(k,k) so that

$$m_k(k) = \sum x'_i = (2^k-1) - \sum x_i = (2^k-1) - m^*_k((2^{k-1}-1)-k) = k+4$$

which is contradiction. Thus we have $d_k((2^{k-1}-1)-k) \geq 1$ for $k \geq 5$. Similarly, we already know that $m_k(k+1) = k+3$ for $k \geq 12$ by [**6**, Theorem 5.3]. In the same way, we can prove that $d_k((2^{k-1}-1)-(k+1)) \geq 1$ for $k \geq 12$.

On the other hand, by Lemma 5.1

$$d_5(10) = d_5(7+3) \leq d_5(3) + 1 \leq d_4(3) + 1 = 1.$$

Therefore, $d_5(10) = 1$. □

REMARK 5.3. By Lemma 5.1 we have $d_5(8) \leq d_5(1) + 1 = 1$ so that $d_5(8) = 0$ or 1. If $d_5(8) = 0$, then $m_5(15\ell + 7)$ is preperiodic for $\ell \geq 0$. However, we obtain the values $m_5(7) = 11$ and $m_5(15+7) = 31+13$ by computer programming.[1] Therefore, $d_5(8) = 1$.

We summarize the upper bound of $d_k(b)$ by using Lemma 5.1, Lemma 5.2 and Remark 5.3 repeatedly in the following propositions.

PROPOSITION 5.4. *Let $b > 0$. If $k \leq 4$, then $d_k(b) = 0$. If $k = 5$, then $d_5(b) = 0$ for $b \neq 8, 10$. In exceptional cases, $d_5(8) = 1$ and $d_5(10) = 1$. If $k \geq 6$, then $d_k(b) = 0$ if $b = 9$ or*

$$(2^n - 1) - n \leq b \leq 2^n - 1 \quad \textit{for } 2 \leq n \leq k-1.$$

And $d_k(b) \leq 1$ if $b = 8$ or $b = 10$ or for any $4 \leq p \leq k-2$

$$(2^p - 1) + (2^n - 1) - n \leq b \leq (2^p - 1) + (2^n - 1) \quad \textit{for } 2 \leq n \leq p-1.$$

PROOF. It is straightforward for $k \leq 4$ by Lemma 5.1.

If $k = 5$, then by Lemma 5.1

$$d_k(b) \leq \begin{cases} 1 & \text{if } 8 \leq b \leq 10, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 5.2 we have $d_5(9) = 0$ and $d_5(10) = 2$.

If $k \geq 6$, then by Lemma 5.1 (2) $d_k(9) \leq d_5(9) = 1$. Moreover, if $(2^n - 1) - n \leq b \leq 2^n - 1$ for some $2 \leq n \leq k-1$, then by Lemma 5.1 (4)

$$d_k(b) \leq d_{k-1}(b) \leq \cdots \leq d_{n+1}(b) = 0.$$

If $b = 0$ or $b = 10$, then by Lemma 5.1 (2) $d_k(b) \leq d_5(b) = 1$. Moreover, if $4 \leq p \leq k-2$, then by Lemma 5.1 (2) and (3)

$$d_k((2^p - 1) + a) \leq \cdots \leq d_{p+2}((2^p - 1) + a) \leq d_{p+2}(a) + 1 = 1,$$

where $(2^n - 1) - n \leq a \leq 2^n - 1$ for some $2 \leq n \leq p-1$. □

[1] This value is calculated by Python LP Solver. The author got this program from Cho, Jin-Hwan.

PROPOSITION 5.5. *Let $k \geq 6$. Given $r \geq 2$, $d_k(b) \leq r$ if there exist $4 \leq q_{r-1} < \cdots < q_1 \leq k-2$ and $4 \leq p_r < \cdots < p_1 \leq k-2$ such that b satisfies one of the following*

(1) $b = (2^{q_1} - 1) + \cdots + (2^{q_{r-1}} - 1) + (8 \text{ or } 10)$.
(2) $b = (2^{p_1} - 1) + \cdots + (2^{p_r} - 1) + 9$.
(3) *For* $2 \leq n \leq (p_r - 1)$,

$$(2^{p_1} - 1) + \cdots + (2^{p_r} - 1) + (2^n - 1) - n \leq b \leq (2^{p_1} - 1) + \cdots + (2^{p_r} - 1) + (2^n - 1)$$

for $2 \leq n \leq p_r - 1$.

When $r = 2$, we use the convention $4 \leq q_1 \leq k-2$.

PROOF. We will prove case by case.
Case (1). By Lemma 5.1 (2) and (3)

$$\begin{aligned}
&d_k((2^{q_1} - 1) + \cdots + (2^{q_{r-1}} - 1) + (8 \text{ or } 10)) \\
&\quad \leq d_{q_1+2}((2^{q_1} - 1) + \cdots + (2^{q_{r-1}} - 1) + (8 \text{ or } 10)) \\
&\quad \leq d_{q_1+2}((2^{q_2} - 1) + \cdots + (2^{q_{r-1}} - 1) + (8 \text{ or } 10)) + 1 \\
&\quad \leq d_{q_2+2}((2^{q_2} - 1) + \cdots + (2^{q_{r-1}} - 1) + (8 \text{ or } 10)) + 1 \\
&\quad \vdots \\
&\quad \leq d_{q_{r-1}+2}((2^{q_{r-1}} - 1) + (8 \text{ or } 10)) + (r-2) \\
&\quad \leq d_{q_{r-1}+2}((8 \text{ or } 10)) + (r-1) \\
&\quad \leq d_5(8 \text{ or } 10) + (r-1) \\
&\quad = r.
\end{aligned}$$

Case (2). In a similar way as in Case (1),

$$\begin{aligned}
&d_k((2^{p_1} - 1) + \cdots + (2^{p_r} - 1) + 9) \\
&\quad \leq d_5(9) + r \\
&\quad = r.
\end{aligned}$$

Case (3). By Lemma 5.1 (2) and (3)

$$\begin{aligned}
&d_k((2^{p_1} - 1) + \cdots + (2^{p_r} - 1) + a) \\
&\quad \leq d_{p_1+2}((2^{p_1} - 1) + \cdots + (2^{p_r} - 1) + a) \\
&\quad \leq d_{p_1+2}((2^{p_2} - 1) + \cdots + (2^{p_r} - 1) + a) + 1 \\
&\quad \leq d_{p_2+2}((2^{p_2} - 1) + \cdots + (2^{p_r} - 1) + a) + 1 \\
&\quad \vdots \\
&\quad \leq d_{p_r+2}((2^{p_r} - 1) + a) + (r-1) \\
&\quad = r,
\end{aligned}$$

where $(2^n-1)-n \le a \le 2^n-1$ for some $2 \le n \le p_r-1$. □

Note that Proposition 5.4 covers all the cases of $1 \le b \le 2^{k-1}-1$ for $k \le 5$.

THEOREM 5.6. *If $k \le 4$, then $m_k(b)$ is preperiodic up to (2^k-1). If $k \ge 5$, then*

$$d_k(b) \le k-5$$

except $b=(2^{k-1}-1)-k$ and $b=(2^{k-1}-1)-(k+2)$ (in each exceptional case, $d_k(b) \le k-4$). In particular, $m_k(b)$ is preperiodic up to (2^k-1). i.e.,

$$m_k((2^{k-1}-1)\ell_2+b)-m_k((2^{k-1}-1)\ell_1+b)=(2^k-1)(\ell_2-\ell_1) \tag{13}$$

for $\ell_2 \ge \ell_1 \ge k-5$ except $b=k$ and $b=k+2$ (in each exceptional case, $\ell_2 \ge \ell_1 \ge k-4$).

PROOF. If $k \le 4$, then $d_k(b)=0$ for $0 \le b \le 2^{k-1}-2$. So, $m_k((2^{k-1}-1)+b)=(2^k-1)+m_k(b)$ by Theorem 1.1. The case of $b=0$ is also treated in [**6**, Corollary 3.6]. If $k=5$, by Proposition 5.4 $d_5(b)=0$ except $b=8,10$ and in each exceptional case, $d_5(8) \le 1$ and $d_5(10)=1$.

Fix $k \ge 6$. Suppose that $d_k(b) \le r$ for some $r \ge 2$. Then by Proposition 5.5 $r \le k-\alpha$ for some α. In case (1), at least there exists q_i's such that $4 \le q_{r-1} < \cdots < q_1 \le k-2$ so that $r \le k-4$. Similarly in case (2) and (3), $r \le k-5$. Moreover, if $r=k-4$, then there exists only two b's such that $d_k(b) \le k-4$ (case (1)). These cases happen only when $b=(2^{k-2}-1)+\cdots+(2^4-1)+8=(2^{k-1}-1)-(k+2)$ or $b=(2^{k-2}-1)+\cdots+(2^4-1)+10=(2^{k-1}-1)-k$. Therefore, except these two cases, $d_k(b) \le k-5$. The conclusion follows from Theorem 1.1. □

6. Conclusion.

In this section, we will show a counterexample to Conjecture 2.4 and give a table of the values of $m_k(b)$ which is calculated by Theorem 1.2. We give examples, which were calculated using Remark 2.3 and Theorem 4.7. We conclude with some questions at the end of this section.

EXAMPLE 6.1. Here is a counterexample to Conjecture 2.4. As we saw in the proof of Lemma 5.2,

$$m_k^*((2^{k-1}-1)-k)=(2^k-1)-(k+4),$$
$$m_k^*((2^{k-1}-1)-(k+1))=(2^k-1)-(k+5).$$

So, by Theorem 1.2 for $\ell_1 \ge d_k((2^{k-1}-1)-k)$ and $\ell_2 \ge d_k((2^{k-1}-1)-(k+1))$,

$$m_k((2^{k-1}-1)\ell_1+k)=(2^k-1)\ell_1+k+4,$$
$$m_k((2^{k-1}-1)\ell_2+k+1)=(2^k-1)\ell_2+k+5.$$

However, we already know that $m_k(k)=k+2$ for $k \ge 5$ and $m_k(k+1)=k+3$ for

$k \geq 12$ [**6**]. Therefore,

$$m_k((2^{k-1}-1)\ell_1 + k) \neq (2^k - 1)\ell_1 + m_k(k) \quad \text{for } k \geq 5,$$
$$m_k((2^{k-1}-1)\ell_2 + k + 1) \neq (2^k - 1)\ell_2 + m_k(k+1) \quad \text{for } k \geq 12.$$

Conjecture 2.4 does not hold in these cases.

EXAMPLE 6.2. Table 2 below is a table of the values of $m_k((2^{k-1}-1)\ell + b)$ for $k = 2,3,4,5,6$ and $\ell \geq u_k((2^{k-1}-1) - b)$, where $u_k((2^{k-1}-1) - b)$ is an upper bound of $d_k((2^{k-1}-1) - b)$ which we calculate in Proposition 5.4 and Proposition 5.5. The values

$b \backslash k$	2	$\ell \geq$	3	$\ell \geq$	4	$\ell \geq$	5	$\ell \geq$	6	$\ell \geq$
0	3ℓ	0	7ℓ	0	15ℓ	0	31ℓ	0	63ℓ	0
1			$7\ell+1$	0	$15\ell+1$	0	$31\ell+1$	0	$63\ell+1$	0
2			$7\ell+4$	0	$15\ell+2$	0	$31\ell+2$	0	$63\ell+2$	0
3					$15\ell+5$	0	$31\ell+3$	0	$63\ell+3$	0
4					$15\ell+8$	0	$31\ell+6$	0	$63\ell+4$	0
5					$15\ell+9$	0	$31\ell+9$	1^*	$63\ell+7$	0
6					$15\ell+12$	0	$31\ell+10$	0	$63\ell+10$	2
7							$31\ell+13$	1	$63\ell+11$	$1^\sharp$
8							$31\ell+16$	0	$63\ell+14$	2
9							$31\ell+17$	0	$63\ell+17$	1
10							$31\ell+18$	0	$63\ell+18$	1
11							$31\ell+21$	0	$63\ell+19$	1
12							$31\ell+24$	0	$63\ell+22$	1
13							$31\ell+25$	0	$63\ell+25$	1
14							$31\ell+28$	0	$63\ell+26$	1
15									$63\ell+29$	1
16									$63\ell+32$	0
17									$63\ell+33$	0
18									$63\ell+34$	0
19									$63\ell+35$	0
20									$63\ell+38$	0
21									$63\ell+41$	1
22									$63\ell+42$	0
23									$63\ell+45$	1
24									$63\ell+48$	0
25									$63\ell+49$	0
26									$63\ell+50$	0
27									$63\ell+53$	0
28									$63\ell+56$	0
29									$63\ell+57$	0
30									$63\ell+60$	0

Table 2. $m_k((2^{k-1}-1)\ell + b)$ for $\ell \geq u_k((2^{k-1}-1) - b)$ and $k = 2,3,4,5,6$.

of $m_k((2^{k-1}-1)\ell+b)$ for $\ell \geq u_k((2^{k-1}-1)-b)$ can be obtained by Theorem 1.2.

Note the asterisk mark on $m_5(15\ell+5) = 31\ell+9$, for $\ell \geq 1^*$. We already know $m_5(5) \neq 9$ so that $u_5(10) = 1 = d_5(10)$ (cf. Example 6.1). Also note the sharp mark on $m_6(31\ell+7) = 63\ell+11$ for $\ell \geq 1^\sharp$. In fact, it is known that $m_6(31\ell+7) = 63\ell+11$ for all $\ell \geq 0$ [**6**] (cf. Lemma 5.2).

The following example describes the value of m and p for which $s_{\mathbb{R}}(m,p) \geq 5$.

EXAMPLE 6.3. Let $k = 5$. According to Table 2 and by Remark 2.3 we have $s_{\mathbb{R}}(m,p) \geq 5$ if and only if for $\ell \geq 0$

$$15m \leq \begin{cases} 31(p-1) & \text{if } m = 31\ell, \\ 31(p-1)-16 & \text{if } m = 31\ell+1, \\ 31(p-1)-32 & \text{if } m = 31\ell+2, \\ 31(p-1)-48 & \text{if } m = 31\ell+3, \\ 31(p-1)-64 & \text{if } m = 31\ell+4, \\ 31(p-1)-49 & \text{if } m = 31\ell+5, \\ 31(p-1)-34 & \text{if } m = 31\ell+6, \\ 31(p-1)-50 & \text{if } m = 31\ell+7, \\ 31(p-1)-66 & \text{if } m = 8, \\ 31(p-1)-35 & \text{if } m = 31(\ell+1)+8, \\ 31(p-1)-51 & \text{if } m = 9, \\ 31(p-1)-20 & \text{if } m = 31(\ell+1)+9, \\ 31(p-1)-36 & \text{if } m = 31\ell+10, \\ 31(p-1)-52 & \text{if } m = 31\ell+11, \\ 31(p-1)-68 & \text{if } m = 12, \\ 31(p-1)-37 & \text{if } m = 31(\ell+1)+12, \\ 31(p-1)-53 & \text{if } m = 13, \\ 31(p-1)-22 & \text{if } m = 31(\ell+1)+13, \\ 31(p-1)-38 & \text{if } m = 31\ell+14, \\ 31(p-1)-23 & \text{if } m = 31\ell+15, \\ 31(p-1)-8 & \text{if } m = 31\ell+16, \\ 31(p-1)-24 & \text{if } m = 31\ell+17, \\ 31(p-1)-40 & \text{if } m = 31\ell+18, \\ 31(p-1)-56 & \text{if } m = 31\ell+19, \\ 31(p-1)-41 & \text{if } m = 31\ell+20, \\ 31(p-1)-26 & \text{if } m = 31\ell+21, \\ 31(p-1)-42 & \text{if } m = 31\ell+22, \\ 31(p-1)-27 & \text{if } m = 31\ell+23, \\ 31(p-1)-12 & \text{if } m = 31\ell+24, \\ 31(p-1)-28 & \text{if } m = 31\ell+25, \\ 31(p-1)-44 & \text{if } m = 31\ell+26, \\ 31(p-1)-29 & \text{if } m = 31\ell+27, \\ 31(p-1)-14 & \text{if } m = 31\ell+28, \\ 31(p-1)-30 & \text{if } m = 31\ell+29, \\ 31(p-1)-15 & \text{if } m = 31\ell+30. \end{cases}$$

In two cases for $m = 12$ and $m = 13$, we use the fact $m_5(7) = 11$ which is obtained by computer program (see Remark 5.3).

EXAMPLE 6.4. In Section 5 we computed upper bound of $d_k(b)$. There are many b's such that upper bound of $d_k(b)$ is non-zero. In such cases, $d_k(b)$ may be non-zero and the values of $m_k((2^{k-1}-1)\ell + b)$ for $\ell < d_k(b)$ are unknown. If $d_k(b) > 0$, then we can find some of these unknown values by Theorem 4.7. Moreover such values of $m_k(b)$ give us counterexamples to Conjecture 2.4 (cf. Table 2).

For example, when $k = 6$ and $b = 31 + 8$ (resp. $b = 8$), it is known [**6**] that $m_6(31+8) = 63+12$ or $63+14$ (resp. $m_6(8) = 12$ or 14). By Proposition 5.5, $d_6(31-8) \leq 2$. Suppose that $d_6(23) \geq 2$ (resp. $d_6(23) \geq 1$). Then since $(2^{6-1}-1) - 8 = 23 = (2^4-1)+(2^3-1)+(2^1-1)$, we have $m^*_{6-1}(23+1) \neq m^*_{6-1}(23)+1$. By Theorem 4.7, $m_6(31+8) < 63\times 2 - m^*_6(23) = 63+14$ (resp. $m_6(8) < 63 - m^*_6(23) = 14$) and thus $m_6(31+8) = 63+12$ (resp. $m_6(8) = 12$). With assumption $d_k(b) \geq 1$ or 2, we can calculate other values in this way. Here is the list of examples.

$$m_5(5) = 7\,[\mathbf{6}],\quad m_5(7) = 11,\quad m_6(6) = 8\,[\mathbf{6}],\quad m_6(31+6) = 63+8,$$

$$m_6(8) = 12,\quad m_6(31+8) = 63+12,\quad m_6(9) = 13 \text{ or } 15,\quad m_6(12) = 20,$$

$$m_6(13) = 21 \text{ or } 23,\quad m_6(15) = 27,\quad m_6(21) = 39,\quad m_6(23) = 43.$$

Now we conclude with some questions. The invariant $d_k(\cdot)$ is important. For convenience, let $a > 0$ be a positive integer such that $a + b = 2^{k-1}-1$. If $d_k(a) = 0$, then $m_k(b)$ is preperiodic up to $2^k - 1$ for $b \geq 0$ and Conjecture 2.4 holds. Moreover in this case, we can calculate all the values of $m_k(b)$ for all $b \geq 0$ by Theorem 1.2. We consider the converse.

QUESTION 1 ([**9**]). Let k, b be the numbers for which the statement of Conjecture 2.4 holds, and let $a = (2^{k-1}-1) - b$. Is it true that $d_k(a) = 0$?

If $d_k(a) \geq 1$, then $m_k(b)$ is preperiodic and equation (3) holds for $\ell \geq d_k(a)$. When $m^*_{k-1}(a+1) \neq m^*_{k-1}(a)+1$, by Remark 4.8 equation (3) holds *definitely* for $\ell \geq d_k(a)$. So more restrictive but equivalent to the original question is as follows

QUESTION 2. Suppose $m^*_{k-1}(a+1) = m^*_{k-1}(a)+1$ and Conjecture 2.4 holds. Is it true that $d_k(a) = 0$?

The author does not have an answer to Question 2 neither negative or positive. Note that

$$(2^{k-1}-1) - (k+1) = (2^{k-2}-1) + \cdots + (2^3-1) + 2(2^1-1)$$

so that

$$m^*_{k-1}((2^{k-1}-1)-k) = m^*_{k-1}((2^{k-1}-1)-(k+1)) + 1$$

for $k \geq 4$. However we already know that $d_k((2^{k-1}-1)-(k+1)) \neq 0$ for $k \geq 12$ by

Lemma 5.2.

Another important issue is finding accurate form of optimal solutions of $\mathrm{IP}(k,b)$. An optimal solution $\boldsymbol{x}=(x_i)$ of $\mathrm{IP}(k,b)$ corresponds to a subgroup of $(S^0)^m$ which acts freely on $\mathbb{R}\mathcal{Z}_{\Delta^{m-1}_{m-b}}$ in the following way. Let $\boldsymbol{M}=(\boldsymbol{a_1},\dots,\boldsymbol{a_m})$ be a $k\times m$ matrix, such that each vector $v_i\in(\mathbb{Z}/2)^k\setminus\{0\}$ appears among its columns x_i times and let $\rho_{\boldsymbol{M}}:(S^0)^k\to(S^0)^m$ be a homomorphism defined by $\rho_{\boldsymbol{M}}(\mathrm{g})=(\mathrm{g}^{\boldsymbol{a_1}},\dots,\mathrm{g}^{\boldsymbol{a_m}})$, where $\mathrm{g}^{\boldsymbol{a}}=\prod_{i=1}^k {\mathrm{g_i}}^{a^i}$ for $\mathrm{g}=(\mathrm{g_1},\dots,\mathrm{g_k})\in(S^0)^k$ and $\boldsymbol{a}=(a^1,\dots,a^k)^T$. Then $\rho_{\boldsymbol{M}}((S^0)^k)$ is a rank k subgroup of $(S^0)^m$ which acts freely on $\mathbb{R}\mathcal{Z}_{\Delta^{m-1}_{m-b}}$ (refer to [**6**, Lemma 2.1] for more details). If $\boldsymbol{y}^*$ is an optimal solution of $\mathrm{IP}^*(k,b)$, then by Lemma 4.2, $\boldsymbol{y'}:=(\ell,\dots,\ell)^T-\boldsymbol{y}^*$ is an optimal solution of $\mathrm{IP}(k,(2^{k-1}-1)\ell-b)$ for $\ell\ge d_k(b)+1$. Moreover, all the optimal solutions $\boldsymbol{y'}$ of $\mathrm{IP}(k,(2^{k-1}-1)\ell-b)$ can be obtained in the form $\boldsymbol{y'}=(\ell,\dots,\ell)^T-\boldsymbol{y}^*$. Thus it is significant to find all the optimal solutions of $\mathrm{IP}^*(k,b)$.

Let $b=c_2(2^{k-2}-1)+\cdots+c_{k-1}(2^1-1)$ be the expansion of $0\le b\le 2^{k-1}-2$. Then there is a natural optimal solution of $\mathrm{IP}^*(k,b)$. Note that $\boldsymbol{z_i}=(1,\dots,1)^T\in\mathbb{Z}^{2^{k-i+1}-1}$ is an optimal solution of $\mathrm{IP}^*(k,2^{k-i}-1)$. Let

$$\boldsymbol{z}:=c_2\eta_2(\boldsymbol{z_2})+\cdots+c_{k-1}\eta_{k-1}(\boldsymbol{z_{k-1}})\in\mathbb{Z}^{2^k-1},$$

where $\eta_i:\mathbb{Z}^{2^{k-i+1}-1}\hookrightarrow\mathbb{Z}^{2^k-1}$ are natural inclusions. Then $\boldsymbol{z}$ is an optimal solution of $\mathrm{IP}^*(k,b)$ by Proposition 2.6. Now, we have a question.

Question 3 ([**10**]). Let $\boldsymbol{y}^*$ be an optimal solution of $\mathrm{IP}^*(k,2^r-1)$ for $r\le k-1$. Is it true that $\boldsymbol{y}^*=(y_i^*)$ has the form

$$y_i^*=\begin{cases}1 & \text{if } v_i\in L\setminus\{0\}\\ 0 & \text{otherwise,}\end{cases}$$

where L is some $(r+1)$ dimensional subspace of $(\mathbb{Z}/2)^k$?

Question 3 has affirmative answer when $k=3$ and $r=1$. If $r=k-1$, then the answer of the Question 3 is positive. Indeed, $\boldsymbol{A_k}\boldsymbol{y}^*\ge(2^{k-1}-1,\dots,2^{k-1}-1)^T$ implies

$$(2^{k-1}-1)\times\sum y_i^*\ge(2^{k-1}-1)\times(2^k-1). \tag{14}$$

On the other hand, $\sum y_i^*=m_k^*(2^{k-1}-1)=2^k-1$. So the inequality in (14) becomes equality. Moreover, $\boldsymbol{A_k}\boldsymbol{y}^*=(2^{k-1}-1,\dots,2^{k-1}-1)^T$. Since $\boldsymbol{A_k}$ is invertible, there is only one solution which is $\boldsymbol{y}^*=(1,\dots,1)^T$.

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