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Hadamard variational formula for the Green function of the Stokes equations under the general second order perturbation

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Abstract. We consider the Hadamard variational formula for the Green function of the Stokes equations with the Dirichlet boundary condition under the smooth perturbation without assuming the volume preserving property. We establish the formula for the first and the second variation for the second order perturbation.

Introduction.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with the smooth boundary $\partial\Omega$. We consider the stationary Stokes equations on Ω with the Dirichlet boundary condition;

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{v} = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where $\boldsymbol{v} = (v^1, v^2, v^3)$ and p are unknown functions for the velocity and for the pressure respecitvely, while $\boldsymbol{f} = (f^1, f^2, f^3)$ is the given external force. Our aim of the present paper is to establish the Hadamard variational formula for the Green function of the Stokes equations with the Dirichlet boundary condition under the smooth perturbation. For any real parameter ε and for any function $\rho_1, \rho_2 \in C^\infty(\partial\Omega)$, we denote the perturbed domain by Ω_ε whose boundary is expressed as

$$\partial\Omega_\varepsilon := \left\{ x + \rho_1(x)\nu_x\varepsilon + \frac{1}{2}\rho_2(x)\nu_x\varepsilon^2 \ ; \ x \in \partial\Omega \right\}, \tag{0.2}$$

where $\nu_x = (\nu_x^1, \nu_x^2, \nu_x^3)$ is the unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$. The Green function $\{\boldsymbol{G}_{\varepsilon,m}, P_{\varepsilon,m}\}_{m=1,2,3}$ of the Stokes equations on Ω_ε is subject to

$$\begin{cases} -\Delta \boldsymbol{G}_{\varepsilon,m}(x,z) + \nabla P_{\varepsilon,m}(x,z) = \delta(x-z)e_m, & (x,z) \in \Omega_\varepsilon \times \Omega_\varepsilon, \\ \operatorname{div} \boldsymbol{G}_{\varepsilon,m}(x,z) = 0, & (x,z) \in \Omega_\varepsilon \times \Omega_\varepsilon, \\ \boldsymbol{G}_{\varepsilon,m}(x,z) = 0, & x \in \partial\Omega, z \in \Omega_\varepsilon \end{cases} \tag{0.3}$$

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for $m = 1, 2, 3$, where $\{e_m\}_{m=1,2,3}$ denotes the canonical basis in $\mathbb{R}^3$. For simplicity, we abbreviate the Green function $\{\boldsymbol{G}_{0,m}, P_{0,m}\}_{m=1,2,3}$ as $\{\boldsymbol{G}_m, P_m\}_{m=1,2,3}$.

The purpose of this paper is to establish the representation formula for the first and the second order terms of the expansion for the parameter ε as

$$\boldsymbol{G}_{\varepsilon,m}(x,z) = \boldsymbol{G}_m(x,z) + \delta\boldsymbol{G}_m(x,z)\varepsilon + \frac{1}{2}\delta^2\boldsymbol{G}_m(x,z)\varepsilon^2 + \cdots$$
$$P_{\varepsilon,m}(x,z) = P_m(x,z) + \delta P_m(x,z)\varepsilon + \frac{1}{2}\delta^2 P_m(x,z)\varepsilon^2 + \cdots$$

for $m = 1, 2, 3$. More precisely, we construct the Hadamard variational formula for both of $\{\delta\boldsymbol{G}_m, \delta P_m\}_{m=1,2,3}$ and $\{\delta^2\boldsymbol{G}_m, \delta^2 P_m\}_{m=1,2,3}$ under the general smooth second order perturbation as (0.2) with respect to ε.

The Hadamard variational formula has been studied under the first order perturbation Ω_ε of Ω with the boundary $\partial\Omega_\varepsilon$ defined by

$$\partial\Omega_\varepsilon := \{x + \varepsilon\rho(x)\nu_x; x \in \partial\Omega\}, \tag{0.4}$$

where $\rho \in C^\infty(\partial\Omega)$. Under such a perturbation, Hadamard [**7**] first introduced the representation formula for the Green function of the usual Laplace equation, and later on Garabedian–Schiffer [**4**] and Garabedian [**3**] gave a rigorous proof of that formula not only for the first variation δG but also for the second variation $\delta^2 G$ as

$$\delta G(y,z) = \int_{\partial\Omega} \frac{\partial G}{\partial\nu_x}(x,y)\frac{\partial G}{\partial\nu_x}(x,z)\rho(x)\,d\sigma_x, \tag{0.5}$$

$$\begin{aligned}\delta^2 G(y,z) = 2\int_\Omega \nabla_x\delta G(x,y)\cdot\nabla_x\delta G(x,z)\,dx \\ + \int_{\partial\Omega} \frac{\partial G}{\partial\nu_x}(x,y)\frac{\partial G}{\partial\nu_x}(x,z)H(x)\rho(x)\,d\sigma_x,\end{aligned} \tag{0.6}$$

where H is the mean curvature at $x \in \partial\Omega$. The Hadamard variational formula is indispensable for the perturbation problems. Indeed, the formula for the Laplace equation has been applied to various problems as Aomoto [**1**], Grinfeld [**5**], [**6**], Ozawa [**15**] and [**16**], for instance. Furthermore, Fujiwara–Ozawa [**2**] and Peetre [**17**] generalized (0.5) for some normal elliptic boundary problem with the higher order differentiation. Recently, the author and Kozono [**10**] treated the Stokes equations (0.1), and established the representation formula for the Green function of that as

$$\delta G_m^n(y,z) = \int_{\partial\Omega}\sum_{i=1}^d \frac{\partial G_m^i}{\partial\nu_x}(x,z)\frac{\partial G_n^i}{\partial\nu_x}(x,y)\rho(x)\,d\sigma_x, \quad m,n = 1,\dots,d, \tag{0.7}$$

where d is the dimension of the original domain Ω. Furthermore, the author herself [**20**] succeed to construct its formula for the second order variation $\{\delta^2\boldsymbol{G}_m, \delta^2 P_m\}_{m=1,\dots,d}$. In [**10**], [**19**] and [**20**], there is a restriction on the domain perturbation of Ω. Namely, in order to preserve the divergence free property when the domain has been changed, we need to handle the volume preserving diffeomorphism $\Phi_\varepsilon : \Omega \to \Omega_\varepsilon$ satisfying $\det J_\varepsilon(x) = 1$ for all $x \in \overline{\Omega}$ and all $\varepsilon \geq 0$, where $\det J_\varepsilon$ is the Jacobian of Φ_ε. Such a method was first introduced by Inoue–Wakimoto [**8**]. Jimbo–Ushikoshi [**9**] succeeded to remove such a restriction as volume preserving property, and derived the variational formula for the eigenvalues even with the multiplicity of the Stokes equations under the Dirichlet boundary condition. They made use of the *piola transform* which enables us to make a divergence free property invariant under the domain perturbation (see, e.g., Marsden–Hughes [**12**]). In the present paper, by means of the *piola transform*, we may consider the general smooth perturbation without assuming the volume preserving property. Furthermore, we discuss the second order perturbations, which may be regarded as a certain generalization of (0.4).

For the proof of the variational formula, the author [**20**] clarified that there are two essential procedures; the first one is to show the existence of $\{\delta \boldsymbol{G}_m, \delta P_m\}_{m=1,2,3}$ and $\{\delta^2 \boldsymbol{G}_m, \delta^2 P_m\}_{m=1,2,3}$ by investigating an ε-dependence of $\{\boldsymbol{G}_{\varepsilon,m}, P_{\varepsilon,m}\}_{m=1,2,3}$ as $\varepsilon \to 0$. To this end, the Schauder estimate of solutions to (0.1) due to Solonnikov [**18**] plays an important role. The second one is the heart matter to prove that the variation $\{\delta^k \boldsymbol{G}_m, \delta^k P_m\}_{m=1,2,3}$ $(k = 1, 2)$ satisfies, in fact, the Stokes equations

$$-\Delta_x \delta^k \boldsymbol{G}_m(x, z) + \nabla_x \delta^k P_m(x, z) = 0, \quad \operatorname{div}_x \delta^k \boldsymbol{G}_m(x, z) = 0, \quad x \in \Omega \text{ with } x \neq z \tag{0.8}$$

for $m = 1, 2, 3$, with an inhomogeneous boundary condition on $\partial\Omega$. Then by the standard method of integral representation formula to solution (0.1), we may derive the explicit integral representation to $\{\delta^k \boldsymbol{G}_m, \delta^k P_m\}_{m=1,2,3}$ by means of the boundary integral on $\partial\Omega$. For the second procedure, we need to deal with $\delta \boldsymbol{G}'_m(x, z) := (d/d\varepsilon)(\Phi_\varepsilon^{-1})_* \boldsymbol{G}_{\varepsilon,m}(x, z)|_{\varepsilon=0}$ and $\delta P'_m(x, z) := (d/d\varepsilon)(\Phi_\varepsilon^{-1})_* P_{\varepsilon,m}(x, z)|_{\varepsilon=0}$, and then obtain the error estimate for $\delta \boldsymbol{G}_m - \delta \boldsymbol{G}'_m$ and $\delta P_m - \delta P'_m$. It should be noted that the derivation of such an error estimate requires a complicated calculation which seems to be difficult to handle the variation $\{\delta^k \boldsymbol{G}_m, \delta^k P_m\}_{m=1,2,3}$ for higher order $k \geq 2$. In the present paper, we shall establish a simple argument to derive (0.8). Indeed, if we assume formally commutatively of differentiation between spacial variables and the parameter ε for (0.2) with respect to the family Ω_ε of domains, it is easy to verify (0.8). However, to make such a formal argument rigorous, we need to show a uniform bound $\{(d/d\varepsilon)\boldsymbol{G}_{\varepsilon,m}, (d/d\varepsilon)P_{\varepsilon,m}\}$ with respect to ε in some neighborhood of $\varepsilon = 0$. Indeed, we shall establish a certain uniform bound for ε in function spaces with Hölder continuity, which ensures validity of (0.8).

The paper is organized as follows. In Section 1, we introduce the assumption for the domain Ω and state our main results. Section 2 introduces the Green function of the Stokes equations and some useful identities related to the Stokes equations. By means of a diffeomorphism constructed in Section 1, the Stokes equations is reduced to the problem on Ω in Section 3. The analysis of the ε-dependence is an essential part of this paper. By the Schauder estimate for the Stokes operator, we discuss the differentiability of the Green function $\{\boldsymbol{G}_{\varepsilon,m}, P_{\varepsilon,m}\}_{m=1,2,3}$ for the parameter ε in Section 4. Finally, we construct the representation formula for the first and the second variation in Section 5.

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1. Results.

We first introduce the assumption for the perturbation $\{\Omega_\varepsilon\}_{\varepsilon\geq 0}$ of domains. Let Ω be a bounded domain in $\mathbb{R}^3$ with the smooth boundary $\partial\Omega$ of class C^∞, and let ρ_1, ρ_2 be smooth functions on $\partial\Omega$. For any $\varepsilon \geq 0$, we define Ω_ε whose boundary $\partial\Omega_\varepsilon$ is defined by

$$\partial\Omega_\varepsilon := \left\{x + \rho_1(x)\nu_x\varepsilon + \frac{1}{2}\rho_2(x)\nu_x\varepsilon^2\ ;\ x \in \partial\Omega\right\},$$

where ν_x is the unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$. Then, we may construct some diffeomorphism from $\Omega \to \Omega_\varepsilon$ as follows.

Lemma 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary $\partial\Omega$. For any $z \in \Omega$ there exists $\delta = \delta(\Omega, \rho_1, \rho_2, z) > 0$ such that for any $\varepsilon \in [0, \delta)$, there exsits a diffeomorphism $\Phi_\varepsilon(\cdot; z) : \overline{\Omega} \to \overline{\Omega}_\varepsilon$ satisfying in the following properties*

$$\Phi_0(x; z) = x, \quad x \in \overline{\Omega}; \tag{1.1}$$

$$\left.\frac{d^k\Phi_\varepsilon(x; z)}{d\varepsilon^k}\right|_{\varepsilon=0} = \rho_k(x)\nu_x, \quad x \in \partial\Omega, \quad (k = 1, 2); \tag{1.2}$$

$$\begin{aligned}&\Phi_\varepsilon(x; z) = x + \boldsymbol{S}_1(x, z)\varepsilon + \frac{1}{2}\boldsymbol{S}_2(x, z)\varepsilon^2 \text{ for all } x \in \overline{\Omega}, \text{ with some functions}\\ &\boldsymbol{S}_1(\cdot, z), \boldsymbol{S}_2(\cdot, z) \in C^\infty(\overline{\Omega}) \text{ satisfying } \boldsymbol{S}_1(x, z) = \boldsymbol{S}_2(x, z) = 0 \text{ for all } x \in B_{d_z/3}(z),\\ &\text{where } d_z := dist(z, \partial\Omega).\end{aligned} \tag{1.3}$$

For the proof, see Appendix.

For the diffeomorphism $\Phi_\varepsilon(\cdot; z) = \tilde{x} = (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \in \Omega_\varepsilon$ for $x = (x^1, x^2, x^3) \in \Omega$, we denote the Jacobian matrix of Φ_ε by J_ε, i.e.,

$$J_\varepsilon(x, z) := \left(\frac{\partial\tilde{x}^i}{\partial x^j}\right)_{1\leq i,j\leq 3}. \tag{1.4}$$

Then, the inverse J_ε^{-1} of J_ε may be written as

$$J_\varepsilon^{-1}(x, z) := \left(\frac{\partial x^i}{\partial\tilde{x}^j}\right)_{1\leq i,j\leq 3}. \tag{1.5}$$

Hence, it holds that

$$\sum_{i=1}^{3} \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial x^i}{\partial \tilde{x}^k} = \delta^{jk}, \quad j, k = 1, 2, 3.$$

Furthermore, we introduce the fundamental solution $\{p_m\}_{m=1,2,3}$ for the pressure by

$$r_m(x, z) := \frac{1}{4\pi} \frac{(x^m - z^m)}{|x - z|^3}, \quad m = 1, 2, 3. \tag{1.6}$$

Now, we state our results.

THEOREM 1.1. *Let $\{\boldsymbol{G}_{\varepsilon,m}, P_{\varepsilon,m}\}_{m=1,2,3}$ be the Green function of the Dirichlet boundary value problem for* (0.3) *satisfying*

$$\int_{\Omega_\varepsilon} (P_{\varepsilon,m}(x, z) - r_m(x, z)) \det(J_\varepsilon{}^{-1}(x, z))\, dx = 0, \quad m = 1, 2, 3 \tag{1.7}$$

for all $z \in \Omega_\varepsilon$ and for all $\varepsilon \geq 0$, where J_ε^{-1} is defined by (1.5). *Then there exist*

$$\delta G_m^n(y, z) := \lim_{\varepsilon \to 0} \frac{G_{\varepsilon,m}^n(y, z) - G_m^n(y, z)}{\varepsilon}, \tag{1.8}$$

$$\delta P_m(y, z) := \lim_{\varepsilon \to 0} \frac{P_{\varepsilon,m}(y, z) - P_m(y, z)}{\varepsilon} \tag{1.9}$$

for all $y, z \in \Omega$ with $y \neq z$, with an explicit representation as

$$\delta G_m^n(y, z) = \int_{\partial\Omega} \sum_{i=1}^{3} \frac{\partial G_n^i}{\partial \nu_x}(x, y) \frac{\partial G_m^i}{\partial \nu_x}(x, z) \rho_1(x)\, d\sigma_x, \tag{1.10}$$

$$\delta P_m(y, z) = \int_{\partial\Omega} \sum_{i=1}^{3} \frac{\partial P_i(y, x)}{\partial \nu_x} \frac{\partial G_m^i}{\partial \nu_x}(x, z) \rho_1(x)\, d\sigma_x \tag{1.11}$$

for $m, n = 1, 2, 3$, where σ_x denotes the surface element of $\partial\Omega$.

Concerning the representation formula for the second variation, we have

THEOREM 1.2. *Let $\{\boldsymbol{G}_{\varepsilon,m}, P_{\varepsilon,m}\}_{m=1,2,3}$ be the Green function of the Dirichlet boundary value problem for* (0.3) *satisfying* (1.7) *for all $z \in \Omega_\varepsilon$ and for all $\varepsilon \geq 0$. Then there exist*

$$\delta^2 G_m^n(y, z) := 2 \lim_{\varepsilon \to 0} \frac{G_{\varepsilon,m}^n(y, z) - G_m^n(y, z) - \varepsilon \delta G_m^n(y, z)}{\varepsilon^2}, \tag{1.12}$$

$$\delta^2 P_m(y, z) := 2 \lim_{\varepsilon \to 0} \frac{P_{\varepsilon,m}(y, z) - P_m(y, z) - \varepsilon \delta P_m(y, z)}{\varepsilon^2} \tag{1.13}$$

for all $y, z \in \Omega$ with $y \neq z$, with an explicit representation as

$$
\begin{aligned}
\delta^2 G^n_m(y,z) = {} & 2\int_{\partial\Omega}\int_{\partial\Omega}\sum_{i,j=1}^{3}\frac{\partial G^i_n}{\partial\nu_x}(x,y)\frac{\partial^2 G^j_i}{\partial\nu_x\partial\nu_w}(w,x)\frac{\partial G^j_m}{\partial\nu_x}(w,z)\rho_1(x)\rho_1(w)\,d\sigma_x d\sigma_w \\
& + \int_{\partial\Omega}\sum_{i=1}^{3}\frac{\partial G^i_n}{\partial\nu_x}(x,y)\frac{\partial^2 G^i_m}{\partial\nu_x^2}(x,z)\rho_1^2(x)\,d\sigma_x \\
& + \int_{\partial\Omega}\sum_{i=1}^{3}\frac{\partial G^i_n}{\partial\nu_x}(x,y)\frac{\partial G^i_m}{\partial\nu_x}(x,z)\rho_2(x)\,d\sigma_x, \qquad (1.14)
\end{aligned}
$$

$$
\begin{aligned}
\delta^2 P_m(y,z) = {} & 2\int_{\partial\Omega}\int_{\partial\Omega}\sum_{i,j=1}^{3}\frac{\partial P_i(y,x)}{\partial\nu_x}\frac{\partial^2 G^j_i}{\partial\nu_x\partial\nu_w}(w,x)\frac{\partial G^j_m}{\partial\nu_x}(w,z)\rho_1(x)\rho_1(w)\,d\sigma_x d\sigma_w \\
& + \int_{\partial\Omega}\sum_{i=1}^{3}\frac{\partial P_i(y,x)}{\partial\nu_x}\frac{\partial^2 G^i_m}{\partial\nu_x^2}(x,z)\rho_1^2(x)\,d\sigma_x \\
& + \int_{\partial\Omega}\sum_{i=1}^{3}\frac{\partial P_i(y,x)}{\partial\nu_x}\frac{\partial G^i_m}{\partial\nu_x}(x,z)\rho_2(x)\,d\sigma_x, \qquad (1.15)
\end{aligned}
$$

for $m, n = 1, 2, 3$.

REMARK 1.1. (i) It is well known that for each $z \in \Omega$, the Green function $P_{\varepsilon,m}(x,z)$ of the pressure for (0.3) is uniquely determined as the functions of x up to an additive constant $C_m(z)$ depending on $z \in \Omega$. Since we may have some freedom to choose $C_m(z)$, by the assumption (1.7) we see that the compensation term $q_{\varepsilon,m}(\tilde{x},z) := P_{\varepsilon,m}(\tilde{x},z) - r_m(\tilde{x},z)$ satisfies

$$\int_{\Omega}(\Phi_\varepsilon^* q_{\varepsilon,m})(x,z)\,dx = 0,$$

which yields

$$\|\Phi_\varepsilon^* q_{\varepsilon,m}(\cdot,z)\|_{L^2(\Omega)} \leq C\|\nabla\Phi_\varepsilon^* q_{\varepsilon,m}(\cdot,z)\|_{L^2(\Omega)}$$

with the constant C independent of ε and $z \in \Omega$, where Φ_ε is a diffeomorphism constructed in Lemma 1.1. Such an estimate as the Poincaré's inequality enables us to obtain the unique existence of $\delta P'_m(x,z) := (d/d\varepsilon)(\Phi_\varepsilon^{-1})_* P_{\varepsilon,m}(x,z)|_{\varepsilon=0}$.

(ii) Our theorem states that for each $z \in \Omega$, if we take a suitable diffeomorphism $\Phi_\varepsilon(\cdot,z)$; $\overline{\Omega} \to \overline{\Omega}_\varepsilon$ and a canonical pressure $P_{\varepsilon,m}(x,z)$, with the condition (1.7), both the first variation $\{\delta\boldsymbol{G}_m, \delta P_m\}$ and the second one $\{\delta^2\boldsymbol{G}_m, \delta^2 P_m\}$ can be represented like (1.10), (1.11), (1.14) and (1.15) in terms of the boundary integral on $\partial\Omega$. It should be noted that the expression $\delta\boldsymbol{G}_m(x,z)$ and $\delta^2\boldsymbol{G}_m(x,z)$ are independent of choice of $\Phi_\varepsilon(\cdot;z)$ and $P_\varepsilon(x,z)$. However, the suitable choice of $P_{\varepsilon,m}(x,z)$ as in (1.7) makes it sure that the pair $\{\delta^k\boldsymbol{G}_m, \delta^k P_m\}_{m=1,2,3}$ fulfills the homogeneous Stokes equations in

Ω with a certain inhomogeneous condition on $\partial\Omega$. Hence, because of the assumption (1.7), the proof of the unique existence of the first and second variation becomes more refined and we may establish a systematic treatment to handle higher variation with an explicit representation in terms of the boundary integral on $\partial\Omega$. Another approach without assuming such an condition as (1.7) was employed by Kozono–Ushikoshi [**10**], which requires more complicated technique in comparison with our method.

(iii) We consider the second order perturbation with respect to ε. If $\rho_2(x) = 0$ for all $x \in \partial\Omega$, the representation formula for $\{\delta^2 \boldsymbol{G}_m, \delta^2 P_m\}_{m=1,2,3}$ agrees to that of [**20**]. Our method is also applicable for the higher order perturbation.

2. Green function for the Stokes equations.

In this section, we introduce the Green function of the Stokes equations derived by Odqvist [**13**]. The Green function $\{\boldsymbol{G}_m, P_m\}_{m=1,2,3}$ for the velocity and pressure are composed by the fundamental tensor $\{\boldsymbol{u}_m, r_m\}_{m=1,2,3}$ of the Stokes equations (0.3) with the compensation term $\{\boldsymbol{q}_m, q_m\}_{m=1,2,3}$;

$$\begin{cases} \boldsymbol{G}_m(x,z) = \boldsymbol{u}_m(x,z) - \boldsymbol{q}_m(x,z), \\ P_m(x,z) = r_m(x,z) - q_m(x,z), \qquad m = 1,2,3. \end{cases} \tag{2.1}$$

Here, the fundamental tensor $\{\boldsymbol{u}_m\}_{m=1,2,3}$ for the velocity is represented by

$$u_m^i(x,z) := \frac{1}{8\pi}\left(\frac{\delta^{im}}{|x-z|} + \frac{(x^i - z^i)(x^m - z^m)}{|x-z|^3}\right), \quad i,m = 1,2,3, \tag{2.2}$$

and the pressure $\{r_m(x,z)\}_{m=1,2,3}$ is as in (1.6). For any fixed $z \in \Omega$, the compensation term $\{\boldsymbol{q}_m, q_m\}_{m=1,2,3}$ is analytic function in Ω and continuous in $\overline{\Omega}$, which is chosen so that (0.3) is satisfied, i.e.,

$$\begin{cases} -\Delta_x \boldsymbol{q}_m(x,z) + \nabla_x q_m(x,z) = 0, & x \in \Omega, \\ \operatorname{div} \boldsymbol{q}_m(x,z) = 0, & x \in \Omega, \\ \boldsymbol{q}_m(x,z) = \boldsymbol{u}_m(x,z), & x \in \partial\Omega, \quad m = 1,2,3. \end{cases} \tag{2.3}$$

We next introduce the Green integral formula for the Stokes operator $\mathcal{L}$ as follows (see [**13**] and [**11**]),

$$\begin{aligned} &\int_\Omega \sum_{i=1}^3 \left\{\mathcal{L}^i(\boldsymbol{v},\pi)(x) w^i(x) - \mathcal{L}^i(\boldsymbol{w},-\tilde{\pi})(x) v^i(x)\right\} dx \\ &\quad = \int_{\partial\Omega} \sum_{i,j=1}^3 \left\{T^{ij}(\boldsymbol{v},\pi)(x) w^i(x) - T^{ij}(\boldsymbol{w},-\tilde{\pi})(x) v^i(x)\right\} \nu_x^j \, d\sigma_x, \end{aligned} \tag{2.4}$$

where $\{T^{ij}\}_{i,j=1,2,3}$ is the stress tensor defined by

$$T^{ij}(\boldsymbol{v}, \pi)(x) := -\left(\frac{\partial v^i}{\partial x^j}(x) + \frac{\partial v^j}{\partial x^i}(x)\right) + \delta^{ij}\pi(x), \quad i, j = 1, 2, 3 \tag{2.5}$$

for the vector functions $\boldsymbol{v}, \boldsymbol{w} \in C^2(\overline{\Omega})^3$ with $\operatorname{div} \boldsymbol{v} = \operatorname{div} \boldsymbol{w} = 0$ in Ω and scalar functions $\pi, \tilde{\pi} \in C^1(\overline{\Omega})$, $\nu_x = (\nu_x^1, \nu_x^2, \nu_x^3)$ is the unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$ and σ_x denotes the surface element of $\partial\Omega$.

Lemma 2.1. *Let $\{\boldsymbol{G}_m, P_m\}_{m=1,2,3}$ be the Green function for the Stokes equations as in* (2.1). *Then it holds that*

$$\lim_{r\to 0} \int_{\partial B_r(z)} \sum_{i,j=1}^{3} T^{ij}(\boldsymbol{G}_m, P_m)(x, z) v^i(x) \nu_x^j \, d\sigma_x = -v^m(z), \quad m = 1, 2, 3 \tag{2.6}$$

for all $z \in \Omega$ and all smooth vector functions $\boldsymbol{v} = (v^1, v^2, v^3)$ near z, where $\{T^{ij}\}_{i,j=1,2,3}$ is the stress tensor defined by (2.5), *$\partial B_r(z)$ denotes the surface centered at z with the radius r, ν_x is the unit inner normal vector to $\partial B_r(z)$ at x and σ_x denotes the surface element of $\partial B_r(z)$.*

For the proof, see Ushikoshi [**19**, Lemma 2.1].

3. Reduction of the problem by the piola transform.

In this section, we reduce the problem (0.3) in Ω_ε to the fixed domain Ω by *piola transform*. For that purpose, we prepare the several symbols and useful identities for the diffeomorphism $\Phi_\varepsilon; \overline{\Omega} \to \overline{\Omega}_\varepsilon$ which is constructed in Lemma 1.1;

Proposition 3.1. *Let Φ_ε be the diffeomorphism as in Lemma* 1.1. *Suppose that $\{a_{\varepsilon,ij}\}_{i,j=1,2,3}$ and $\{a_\varepsilon^{ij}\}_{i,j=1,2,3}$ are respectively defined by*

$$a_\varepsilon^{ij} := \sum_{l=1}^{3} \frac{\partial x^i}{\partial \tilde{x}^l} \frac{\partial x^j}{\partial \tilde{x}^l}, \quad a_{\varepsilon,ij} := \sum_{l=1}^{3} \frac{\partial x^l}{\partial x^i} \frac{\partial x^l}{\partial \tilde{x}^j}, \quad i, j = 1, 2, 3. \tag{3.1}$$

Then it holds that

$$\begin{aligned}
\frac{\partial \tilde{x}^i}{\partial x^j} &= \delta^{ij} + \frac{\partial S_1^i}{\partial x^j}\varepsilon + \frac{1}{2}\frac{\partial S_2^i}{\partial x^j}\varepsilon^2, \\
a_{\varepsilon,ij} &= \delta^{ij} + \delta a_{ij}\varepsilon + \frac{1}{2}\delta^2 a_{ij}\varepsilon^2, \\
\frac{\partial x^i}{\partial \tilde{x}^j} &= \delta^{ij} - \frac{\partial S_1^i}{\partial x^j}\varepsilon + \frac{1}{4}\left(\sum_{l=1}^{3} \frac{\partial S_1^l}{\partial x^j}\frac{\partial S_1^i}{\partial x^l} - \frac{\partial S_2^i}{\partial x^j}\right)\varepsilon^2 + O(\varepsilon^3), \\
a_\varepsilon^{ij} &= \delta^{ij} + \delta a^{ij}\varepsilon + \frac{1}{2}\delta^2 a^{ij}\varepsilon^2 + O(\varepsilon^3), \quad \text{as } \varepsilon \to 0
\end{aligned}$$

for $i, j = 1, 2, 3$, where $\tilde{x} = \Phi_\varepsilon(x; z)$ for $x \in \overline{\Omega}$, $\{\delta a_{ij}\}_{i,j=1,2,3}$ and $\{\delta a^{ij}\}_{i,j=1,2,3}$ are *represented by*

$$\delta a_{ij} := \left(\frac{\partial S_1^i}{\partial x^j} + \frac{\partial S_1^j}{\partial x^i}\right), \qquad \delta^2 a_{ij} := 2\sum_{l=1}^{3} \frac{\partial S_1^l}{\partial x^i}\frac{\partial S_1^l}{\partial x^j} + \left(\frac{\partial S_2^i}{\partial x^j} + \frac{\partial S_2^j}{\partial x^i}\right), \tag{3.2}$$

$$\delta a^{ij} := -\left(\frac{\partial S_1^i}{\partial x^j} + \frac{\partial S_1^j}{\partial x^i}\right), \quad \delta^2 a^{ij} := 2\sum_{l=1}^{3}\left(\frac{\partial S_1^i}{\partial x^l}\frac{\partial S_1^j}{\partial x^l} + \frac{\partial S_1^l}{\partial x^j}\frac{\partial S_1^i}{\partial x^l} + \frac{\partial S_1^l}{\partial x^i}\frac{\partial S_1^j}{\partial x^l}\right) + \left(\frac{\partial S_2^i}{\partial x^j} + \frac{\partial S_2^j}{\partial x^i}\right),$$

$i, j = 1, 2, 3$. *Here, the vector function* $\boldsymbol{S}_1$ *and* $\boldsymbol{S}_2$ *are defined by* (A.2). *Furthermore, the Jacobian of* J_ε *and* J_ε^{-1} *as in* (1.4) *and* (1.5) *is expanded with respect to* ε *as*

$$\det J_\varepsilon(x,z) = 1 + J_1(x,z)\varepsilon + \frac{1}{2}J_2(x,z)\varepsilon^2 + O(\varepsilon^3), \tag{3.3}$$

$$\det(J_\varepsilon^{-1}(x,z)) = 1 + K_1(x,z)\varepsilon + \frac{1}{2}K_2(x,z)\varepsilon^2 + O(\varepsilon^3), \quad \text{as } \ \varepsilon \to 0$$

for $x \in \overline{\Omega}$, *where* J_i *for each* $i = 1, 2$ *are respectively expressed by*

$$J_1 = \mathrm{div}_x\, \boldsymbol{S}_1, \tag{3.4}$$

$$J_2 = \frac{1}{2}\left[2\left\{\frac{\partial S_1^1}{\partial x^1}\frac{\partial S_1^2}{\partial x^2} + \frac{\partial S_1^2}{\partial x^2}\frac{\partial S_1^3}{\partial x^3} + \frac{\partial S_1^3}{\partial x^3}\frac{\partial S_1^1}{\partial x^1} - \left(\frac{\partial S_1^1}{\partial x^3}\frac{\partial S_1^3}{\partial x^1} + \frac{\partial S_1^1}{\partial x^2}\frac{\partial S_1^2}{\partial x^1} + \frac{\partial S_1^2}{\partial x^3}\frac{\partial S_1^3}{\partial x^2}\right)\right\} + \mathrm{div}_x\, \boldsymbol{S}_2\right],$$

and K_i *for each* $i = 1, 2$ *are as*

$$K_1 = -\mathrm{div}_x\, \boldsymbol{S}_1, \tag{3.5}$$

$$K_2 = \frac{1}{2}\left[2\left\{ -\left(\frac{\partial S_1^1}{\partial x^1}\frac{\partial S_1^2}{\partial x^2} + \frac{\partial S_1^2}{\partial x^2}\frac{\partial S_1^3}{\partial x^3} + \frac{\partial S_1^3}{\partial x^3}\frac{\partial S_1^1}{\partial x^1}\right) + \frac{\partial S_1^1}{\partial x^3}\frac{\partial S_1^3}{\partial x^1} + \frac{\partial S_1^1}{\partial x^2}\frac{\partial S_1^2}{\partial x^1} + \frac{\partial S_1^2}{\partial x^3}\frac{\partial S_1^3}{\partial x^2} + (\mathrm{div}_x\, \boldsymbol{S}_1)^2\right\} - \mathrm{div}_x\, \boldsymbol{S}_2\right].$$

The proof is an immediate consequence of (1.3). So, we may omit it.

We next introduce the *piola transform*, which makes a divergence free condition invariant under the perturbation of a domain. Indeed, it holds that;

Lemma 3.1. *For a parameter* $\varepsilon \geq 0$ *and any vector function* $\boldsymbol{W}_\varepsilon \in C^\infty(\Omega_\varepsilon)$ *satisfying* $\mathrm{div}_x\, \boldsymbol{W}_\varepsilon(\tilde{x}) = 0$ *for* $\tilde{x} \in \Omega_\varepsilon$, *we define the vector function* $\boldsymbol{w}_\varepsilon \in C^\infty(\Omega)$ *by*

$$\boldsymbol{w}_\varepsilon(x) := \det(J_\varepsilon(x,z)) \sum_{j=1}^{3} \frac{\partial x}{\partial \tilde{x}^j} W_\varepsilon^j(\tilde{x}) \quad \text{for } \ \tilde{x} \in \Omega_\varepsilon,$$

where $\tilde{x} = \Phi(x, \varepsilon)$ *and* J_ε *is as in* (1.4)*. Then, it holds that*

$$\operatorname{div}_x \boldsymbol{w}_\varepsilon(x) = 0 \quad \textit{for all} \quad x \in \Omega.$$

For the proof, see [**9**] and [**12**], for instance.
By Lemma 3.1, we see that the transformed Green function $\{\boldsymbol{g}_{\varepsilon,m}\}_{m=1,2,3}$ defined by

$$\boldsymbol{g}_{\varepsilon,m}(x,z) := \det(J_\varepsilon(x,z)) \sum_{j=1}^{3} \frac{\partial x}{\partial \tilde{x}^j} G^j_{\varepsilon,m}(\tilde{x}, \tilde{z}), \quad m = 1,2,3 \tag{3.6}$$

satisfies the divergence free condition in Ω. Hence, we have by a standard procedure the transformed Stokes equations on Ω as

$$\begin{cases} \mathcal{L}_\varepsilon(\boldsymbol{g}_{\varepsilon,m}, p_{\varepsilon,m})(x,z) = 0, & x \in \Omega \setminus \{z\}, \\ \operatorname{div}_x \boldsymbol{g}_{\varepsilon,m}(x,z) = 0, & x \in \Omega \setminus \{z\}, \\ \boldsymbol{g}_{\varepsilon,m}(x,z) = 0, & x \in \partial\Omega, \quad m = 1,2,3, \end{cases} \tag{3.7}$$

where $\mathcal{L}_\varepsilon(\boldsymbol{v}, \pi) = (\mathcal{L}^1_\varepsilon(\boldsymbol{v}, \pi), \mathcal{L}^2_\varepsilon(\boldsymbol{v}, \pi), \mathcal{L}^3_\varepsilon(\boldsymbol{v}, \pi))$ has an expression as

$$\begin{aligned} \mathcal{L}^q_\varepsilon(\boldsymbol{v}, \pi)(x) := -\sum_{i,k,l,p,s=1}^{3} \frac{\partial}{\partial x^k} \biggl\{ (\det J_\varepsilon) a^{ks}_\varepsilon \frac{\partial}{\partial x^s} \biggl((\det J_\varepsilon^{-1}) \frac{\partial \tilde{x}^i}{\partial x^l} v^l(x) \biggr) \biggr\} (\det J_\varepsilon^{-1}) \frac{\partial \tilde{x}^i}{\partial x^p} a^{pq}_\varepsilon \\ + \sum_{i=1}^{3} a^{qi}_\varepsilon \frac{\partial \pi}{\partial x^i}(x) \end{aligned} \tag{3.8}$$

for $q = 1,2,3$ with a variable coefficient J_ε, J_ε^{-1} and $\{a^{ks}_\varepsilon\}_{k,s=1,2,3}$ as in (1.4), (1.5) and (3.1). Furthermore, the function $\{p_{\varepsilon,m}\}_{m=1,2,3}$ is defined by

$$p_{\varepsilon,m}(x,z) := P_{\varepsilon,m}(\tilde{x}, \tilde{z}), \quad m = 1,2,3, \tag{3.9}$$

which is the transformed Green function for the pressure on Ω. Here, it should be noted that $\mathcal{L}_\varepsilon$ is the Stokes operator on the Riemannian manifold $(\overline{\Omega}, a_\varepsilon)$, and $\mathcal{L}_0$ is the operator on the $(\overline{\Omega}, \delta)$, where $a_\varepsilon = \{a_{\varepsilon,ij}\}_{i,j=1,2,3}$ is a metric defined by (3.1).

4. Analysis for the ε-dependence of the Green function.

In this section, we consider the ε-dependence of the Green function $\{\boldsymbol{G}_{\varepsilon,m}, P_{\varepsilon,m}\}_{m=1,2,3}$ under the domain perturbation.

4.1. The n-th order expansion of the transformed Green function.

We first study the ε-dependence of the transformed Green function $\{\boldsymbol{g}_{\varepsilon,m}, p_{\varepsilon,m}\}_{m=1,2,3}$ as in (3.6) and (3.9). By means of a priori estimate, Ushikoshi [**20**] has analyzed the ε-dependence of the transformed Green function $\{\boldsymbol{\mathcal{G}}_{\varepsilon,m}\}_{m=1,2,3}$ defined by

$$\mathcal{G}^i_{\varepsilon,m}(x,z) := \sum_{l=1}^{3} \frac{\partial x^i}{\partial \tilde{x}^l} G^l_{\varepsilon,m}(\tilde{x},\tilde{z}), \quad i = 1,2,3, \tag{4.1}$$

whose method was first introduced by Fujiwara–Ozawa [**2**] and Ozawa [**14**] for the case of the Laplace equation. Similarly to [**19**] and [**20**], we shall discuss the differentiability of the Green function by the *piola transform* for the parameter ε so that we introduce some notations.

For $k \in \mathbb{N}$, the k-th order term of the expanded Green function $\{\boldsymbol{g}_{\varepsilon,m}, p_{\varepsilon,m}\}_{m=1,2,3}$ around $\varepsilon = \varepsilon'$, is defined by $\{\delta^k_{\varepsilon'}\boldsymbol{G}'_m, \delta^k_{\varepsilon'}P'_m\}_{m=1,2,3}$ as

$$\delta^k_{\varepsilon'}\boldsymbol{G}'_m(x,z) := k! \lim_{\varepsilon\to\varepsilon'} (\varepsilon-\varepsilon')^{-k}\left(\boldsymbol{g}_{\varepsilon,m}(x,z) - \sum_{l=0}^{k-1}\frac{1}{l!}\delta^l_{\varepsilon'}\boldsymbol{G}'_m(x,z)(\varepsilon-\varepsilon')^l\right), \tag{4.2}$$

$$\delta^k_{\varepsilon'}P'_m(x,z) := k! \lim_{\varepsilon\to\varepsilon'} (\varepsilon-\varepsilon')^{-k}\left(p_{\varepsilon,m}(x,z) - \sum_{l=0}^{k-1}\frac{1}{l!}\delta^l_{\varepsilon'}P'_m(x,z)(\varepsilon-\varepsilon')^l\right) \tag{4.3}$$

for $m = 1,2,3$. Here, $\{\delta^0_{\varepsilon'}\boldsymbol{G}'_m, \delta^0_{\varepsilon'}P'_m\}_{m=1,2,3} = \{\boldsymbol{G}_{\varepsilon',m}, P_{\varepsilon',m}\}_{m=1,2,3}$, and $\{\delta^k_0\boldsymbol{G}'_m, \delta^k_0P'_m\}_{m=1,2,3}$ is abbreviated as $\{\delta^k\boldsymbol{G}'_m, \delta^kP'_m\}_{m=1,2,3}$.

The purpose of this subsection is to prove the following theorem according to the method of Fujiwara–Ozawa [**2**], Ozawa [**14**] and Ushikoshi [**20**].

THEOREM 4.1. *For any $z \in \Omega$, there exists $\varepsilon_1 = \varepsilon_1(z,\Omega)$ such that for any $\varepsilon' \in [0,\varepsilon_1)$ and for each $k = 1,2$, $\{\delta^k_{\varepsilon'}\boldsymbol{G}'_m(\cdot,z)\}_{m=1,2,3} \in C^{2+\theta}(\overline{\Omega}\setminus\{z\})$ and $\{\delta^k_{\varepsilon'}P'_m(\cdot,z)\}_{m=1,2,3} \in C^{1+\theta}(\overline{\Omega}\setminus\{z\})$ with $0<\theta<1$, which is defined by* (4.2) *and* (4.3).

For the proof of Theorem 4.1, it is important to study the ε-dependence of the compensation term $\{\boldsymbol{q}'_{\varepsilon,m}(\cdot,z), q'_{\varepsilon,m}(\cdot,z)\}_{m=1,2,3}$, which is defined by

$$\begin{aligned}\boldsymbol{q}'_{\varepsilon,m}(x,z) &:= \boldsymbol{u}_{\varepsilon,m}(x,z) - \boldsymbol{g}_{\varepsilon,m}(x,z), \\ q'_{\varepsilon,m}(x,z) &:= r_{\varepsilon,m}(x,z) - p_{\varepsilon,m}(x,z), \quad m=1,2,3,\end{aligned} \tag{4.4}$$

where $\{\boldsymbol{u}_{\varepsilon,m}(\cdot,z), r_{\varepsilon,m}(\cdot,z)\}_{m=1,2,3}$ is as in

$$u^i_{\varepsilon,m}(x,z) := \det(J_\varepsilon(x,z)) \sum_{l=1}^{3} \frac{\partial x^i}{\partial \tilde{x}^l} u^l_m(\tilde{x},\tilde{z}), \tag{4.5}$$

$$r_{\varepsilon,m}(x,z) := r_m(\tilde{x},\tilde{z}), \quad i,m = 1,2,3 \tag{4.6}$$

with the fundamental solution of the Stokes equations $\{\boldsymbol{u}_m, r_m\}_{m=1,2,3}$ as in (1.6) and (2.2), respectively and $\tilde{x} = \Phi_\varepsilon(x;z)$, $\tilde{z} = \Phi_\varepsilon(z;z)$ for $x,z\in\Omega$. For simplicity, the compensation term $\{\boldsymbol{q}'_{0,m}, q'_{0,m}\}_{m=1,2,3}$ is abbreviated as $\{\boldsymbol{q}_m, q_m\}_{m=1,2,3}$, and $\{\boldsymbol{w}_{0,m}, r_{0,m}\}_{m=1,2,3} = \{\boldsymbol{u}_m, r_m\}_{m=1,2,3}$.

The following lemma analyzes the differentiability of the compensation term (4.4) with respect to ε;

Lemma 4.1. *For any $z \in \Omega$, take $\varepsilon_1 = \varepsilon_1(z,\Omega)$ in Theorem 4.1, then for any $\varepsilon' \in [0,\varepsilon_1)$ and for each $k \in \mathbb{N}$, $\{\delta^k_{\varepsilon'}\boldsymbol{q}'_m(\cdot,z)\}_{m=1,2,3} \in C^{2+\theta}(\overline{\Omega})$ and $\{\delta^k_{\varepsilon'}q'_m(\cdot,z)\}_{m=1,2,3} \in C^{1+\theta}(\overline{\Omega})$ with $0<\theta<1$, where*

$$\delta^k_{\varepsilon'}\boldsymbol{q}'_m(x,z) := k!\lim_{\varepsilon\to\varepsilon'}(\varepsilon-\varepsilon')^{-k}\biggl(\boldsymbol{q}'_{\varepsilon,m}(x,z) - \sum_{l=0}^{k-1}\frac{1}{l!}\delta^l_{\varepsilon'}\boldsymbol{q}'_m(x,z)(\varepsilon-\varepsilon')^l\biggr), \tag{4.7}$$

$$\delta^k_{\varepsilon'}q'_m(x,z) := k!\lim_{\varepsilon\to\varepsilon'}(\varepsilon-\varepsilon')^{-k}\biggl(q'_{\varepsilon,m}(x,z) - \sum_{l=0}^{k-1}\frac{1}{l!}\delta^l_{\varepsilon'}q'_m(x,z)(\varepsilon-\varepsilon')^l\biggr) \tag{4.8}$$

for $m=1,2,3$. Here, $\{\boldsymbol{q}'_{\varepsilon,m}, q'_{\varepsilon,m}\}_{m=1,2,3}$ is as in (4.4) respectively.

Remark 4.1. Similarly to (4.2) and (4.3), $\{\delta^0_{\varepsilon'}\boldsymbol{q}'_m, \delta^0_{\varepsilon'}q'_m\}_{m=1,2,3} = \{\boldsymbol{q}_{\varepsilon',m}, q_{\varepsilon',m}\}_{m=1,2,3}$, and $\{\delta^k_0\boldsymbol{q}'_m, \delta^k_0 q'_m\}_{m=1,2,3}$ is abbreviated as $\{\delta^k\boldsymbol{q}'_m, \delta^k q'_m\}_{m=1,2,3}$ in (4.7) and (4.8).

Proof of Theorem 4.1. Since by Proposition 3.1, the transformed fundamental solution $\{\boldsymbol{u}_{\varepsilon,m}, r_{\varepsilon,m}\}_{m=1,2,3}$ is differentiable at $\varepsilon=\varepsilon'$ for $x \in \overline{\Omega}$ with $x \neq z$, it is easy to see that Lemma 4.1 yields Theorem 4.1. □

For the proof of Lemma 4.1, we need to establish the uniform estimate for the parameter ε according to [**2**], [**14**] and [**20**].

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with the $C^{2+\theta}$-boundary $\partial\Omega$, $0<\theta<1$. There exist $\varepsilon_1 = \varepsilon_1(\Omega) > 0$ and $C = C(\Omega) > 0$ such that if $\varepsilon \le \varepsilon_1$, then it holds that*

$$\|\boldsymbol{v}\|_{C^{2+\theta}(\overline{\Omega})} + \|\pi\|_{C^{1+\theta}(\overline{\Omega})} \le C\bigl(\|\mathcal{L}_\varepsilon(\boldsymbol{v},\pi)\|_{C^\theta(\overline{\Omega})} + \|\boldsymbol{v}\|_{C^{2+\theta}(\partial\Omega)}\bigr) \tag{4.9}$$

for all $\boldsymbol{v} \in C^{2+\theta}(\overline{\Omega})$ and $\pi \in C^{1+\theta}(\overline{\Omega})$ satisfying $\int_\Omega \pi(x)\,dx = 0$, where $\mathcal{L}_\varepsilon$ is defined by (3.8).

Proof. We give a proof by a contradiction argument. Suppose that (4.9) is not true. Then for any $m=1,2,\ldots$, there exist $\boldsymbol{v}_m \in C^{2+\theta}(\overline{\Omega})$ and $\pi_m \in C^{1+\theta}(\overline{\Omega})$ with $\|\boldsymbol{v}_m\|_{C^{2+\theta}(\overline{\Omega})} + \|\pi_m\|_{C^\theta(\overline{\Omega})} \equiv 1$ such that

$$\frac{1}{m} > \bigl(\|\mathcal{L}_{1/m}(\boldsymbol{v}_m,\pi_m)\|_{C^\theta(\overline{\Omega})} + \|\boldsymbol{v}_m\|_{C^{2+\theta}(\partial\Omega)}\bigr), \quad m=1,2,\ldots, \tag{4.10}$$

while by (1.3), (4.10) and an a priori estimate for the Stokes equations (see e.g., (Solonnikov [**18**, Theorem 3.1]), it holds that

$$\begin{aligned}&\|\boldsymbol{v}_m\|_{C^{2+\theta}(\overline{\Omega})} + \|\pi_m\|_{C^\theta(\overline{\Omega})}\\ &\quad\le M\bigl(\|\mathcal{L}(\boldsymbol{v}_m,\pi_m)\|_{C^\theta(\overline{\Omega})} + \|\boldsymbol{v}_m\|_{C^{2+\theta}(\partial\Omega)}\bigr)\end{aligned}$$

$$
\begin{aligned}
&\le M\big(\|(\mathcal{L}_0-\mathcal{L}_{1/m})(\boldsymbol{v}_m,\pi_m)\|_{C^\theta(\overline{\Omega})}+\|\mathcal{L}_{1/m}(\boldsymbol{v}_m,\pi_m)\|_{C^\theta(\overline{\Omega})}+\|\boldsymbol{v}_m\|_{C^{2+\theta}(\partial\Omega)}\big)\\
&\le M\bigg\{\frac{1}{m}(\|\boldsymbol{v}_m\|_{C^{2+\theta}(\overline{\Omega})}+\|\pi_m\|_{C^{1+\theta}(\overline{\Omega})})+(\|\mathcal{L}_{1/m}(\boldsymbol{v}_m,\pi_m)\|_{C^\theta(\overline{\Omega})}+\|\boldsymbol{v}_m\|_{C^{2+\theta}(\partial\Omega)})\bigg\}\\
&\le 2M\frac{1}{m}\to 0,\quad \text{as}\ \ m\to\infty, \qquad (4.11)
\end{aligned}
$$

where M is a constant independent of m. In the above calculation, we have used the fact that $\|(\mathcal{L}_0-\mathcal{L}_\varepsilon)(\boldsymbol{v},\pi)\|_{C^\theta(\overline{\Omega})}=O(\varepsilon)$, as $\varepsilon\to 0$ for all $\boldsymbol{v}\in C^{2+\theta}(\Omega)$ and $\pi\in C^{1+\theta}(\Omega)$, which is derived by (1.3). Therefore, since $\|\boldsymbol{v}_m\|_{C^{2+\theta}(\overline{\Omega})}+\|\pi_m\|_{C^{1+\theta}(\overline{\Omega})}\equiv 1$ for $m=1,2,\dots,$ this causes a contradiction. □

We have the boundedness and continuity of the compensation term $\{\boldsymbol{q}'_{\varepsilon,m},q'_{\varepsilon,m}\}_{m=1,2,3}$ for the parameter ε by Proposition 4.2. Indeed, it holds;

Corollary 4.1. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with the $C^{2+\theta}$-boundary $\partial\Omega$, $0<\theta<1$. For any fixed $z\in\Omega$, there exist $\varepsilon_2=\varepsilon_2(z)>0$ and $C=C(z)>0$ such that if $\varepsilon\le\varepsilon_2$, then it holds that*

$$\|\boldsymbol{q}'_{\varepsilon,m}(\cdot,z)\|_{C^{2+\theta}(\overline{\Omega})}+\|q'_{\varepsilon,m}(\cdot,z)\|_{C^{1+\theta}(\overline{\Omega})}\le C,\quad m=1,2,3, \qquad (4.12)$$

where $\{\boldsymbol{q}'_{\varepsilon,m},q'_{\varepsilon,m}\}_{m=1,2,3}$ is the compensation term defined by (4.4).

Proof. Since $\{q'_{\varepsilon,m}\}_{m=1,2,3}$ satisfies $\int_\Omega q'_{\varepsilon,m}(x,z)\,dx=0$ for all $z\in\Omega$, applying $\{\boldsymbol{q}'_{\varepsilon,m},q'_{\varepsilon,m}\}_{m=1,2,3}$ to Proposition 4.2, we have that

$$
\begin{aligned}
&\|\boldsymbol{q}'_{\varepsilon,m}(\cdot,z)\|_{C^{2+\theta}(\overline{\Omega})}+\|q'_{\varepsilon,m}(\cdot,z)\|_{C^{1+\theta}(\overline{\Omega})}\\
&\qquad\le M\big(\|\mathcal{L}_\varepsilon(\boldsymbol{q}'_{\varepsilon,m},q'_{\varepsilon,m})(\cdot,z)\|_{C^\theta(\overline{\Omega})}+\|\boldsymbol{q}'_{\varepsilon,m}(\cdot,z)\|_{C^{2+\theta}(\partial\Omega)}\big) \qquad (4.13)
\end{aligned}
$$

for all $\varepsilon\le\varepsilon_1$, where $\varepsilon_1=\varepsilon_1(\Omega)$ as in Proposition 4.2 and $M=M(\Omega)>0$ are constants independent of ε. On the other hand, for each fixed $z\in\Omega$, there exists an $\varepsilon'_0(z)>0$ such that

$$\|\boldsymbol{q}'_{\varepsilon,m}(\cdot,z)\|_{C^{2+\theta}(\partial\Omega)}=\|\boldsymbol{u}_{\varepsilon,m}(\cdot,z)\|_{C^{2+\theta}(\partial\Omega)}<2\|\boldsymbol{u}_m(\cdot,z)\|_{C^{2+\theta}(\partial\Omega)} \qquad (4.14)$$

for all $0<\varepsilon\le\varepsilon'_0(z)$. Since $\mathcal{L}_\varepsilon(\boldsymbol{q}'_{\varepsilon,m},q'_{\varepsilon,m})(\cdot,z)=0$, implied by (3.7) and (4.4), it follows from (4.13) and (4.14) that

$$\|\boldsymbol{q}'_{\varepsilon,m}(\cdot,z)\|_{C^{2+\theta}(\overline{\Omega})}+\|q'_{\varepsilon,m}(\cdot,z)\|_{C^{1+\theta}(\overline{\Omega})}\le 2M\|\boldsymbol{u}_m(\cdot,z)\|_{C^{2+\theta}(\partial\Omega)}:=M'$$

for all $0<\varepsilon\le\varepsilon_2$, where $\varepsilon_2:=\min\{\varepsilon_1,\varepsilon'_0(z)\}$, and M' is a constant depending only on $z\in\Omega$. This proves Corollary 4.1. □

By Proposition 4.2 and Corollary 4.1, we obtain the continuity with respect to ε for $\{\tilde{\boldsymbol{q}}_{\varepsilon,m},\tilde{q}_{\varepsilon,m}\}_{m=1,2,3}$. Indeed, it holds;

Proposition 4.3. *For every $z \in \Omega$, take ε_1 as in Proposition 4.2, then for any $\varepsilon' \in [0, \varepsilon_1)$, we have that*

$$\|\boldsymbol{q}'_{\varepsilon,m}(\cdot, z) - \boldsymbol{q}'_{\varepsilon',m}(\cdot, z)\|_{C^{2+\theta}(\overline{\Omega})} \to 0, \tag{4.15}$$

$$\|q'_{\varepsilon,m}(\cdot, z) - q'_{\varepsilon',m}(\cdot, z)\|_{C^{1+\theta}(\overline{\Omega})} \to 0, \quad m = 1, 2, 3, \quad \text{as } \varepsilon \to \varepsilon', \tag{4.16}$$

where $\{\boldsymbol{q}'_{\varepsilon,m}, q'_{\varepsilon,m}\}_{m=1,2,3}$ is the compensation term defined by (4.4).

Proof. By a direct calculation, we see that the pair $\{\boldsymbol{q}'_{\varepsilon,m} - \boldsymbol{q}'_{\varepsilon',m}, q'_{\varepsilon,m} - q'_{\varepsilon',m}\}$ satisfies the following equations.

$$\begin{cases} \mathcal{L}_{\varepsilon'}(\boldsymbol{q}'_{\varepsilon,m} - \boldsymbol{q}'_{\varepsilon',m}, q'_{\varepsilon,m} - q'_{\varepsilon',m})(x, z) = (\mathcal{L}_{\varepsilon'} - \mathcal{L}_{\varepsilon})(\boldsymbol{q}'_{\varepsilon,m}, q'_{\varepsilon,m})(x, z), & x \in \Omega, \\ \operatorname{div}_x (\boldsymbol{q}'_{\varepsilon,m} - \boldsymbol{q}'_{\varepsilon',m})(x, z) = 0, & x \in \Omega, \\ (\boldsymbol{q}'_{\varepsilon,m} - \boldsymbol{q}'_{\varepsilon',m})(x, z) = (-\boldsymbol{u}_{\varepsilon,m} + \boldsymbol{u}_{\varepsilon',m})(x, z), & x \in \partial\Omega \end{cases}$$

for $m = 1, 2, 3$. Hence by (1.3), (4.12) and Proposition 4.2, we have that for any $\varepsilon' < \varepsilon_1$,

$$\begin{aligned} &\|(\boldsymbol{q}'_{\varepsilon,m} - \boldsymbol{q}'_{\varepsilon',m})(\cdot, z)\|_{C^{2+\theta}(\overline{\Omega})} + \|(q'_{\varepsilon,m} - q'_{\varepsilon',m})(\cdot, z)\|_{C^{1+\theta}(\overline{\Omega})} \\ &\quad \le M\big(\|(\mathcal{L}_{\varepsilon'} - \mathcal{L}_{\varepsilon})(\boldsymbol{q}'_{\varepsilon,m}, q'_{\varepsilon,m})(\cdot, z)\|_{C^{\theta}(\overline{\Omega})} + \|(-\boldsymbol{u}_{\varepsilon,m} + \boldsymbol{u}_{\varepsilon',m})(\cdot, z)\|_{C^{2+\theta}(\partial\Omega)}\big) \\ &\quad \le M\big\{(\varepsilon' - \varepsilon)(\|\boldsymbol{q}'_{\varepsilon,m}(\cdot, z)\|_{C^{2+\theta}(\overline{\Omega})} + \|q'_{\varepsilon,m}(\cdot, z)\|_{C^{1+\theta}(\overline{\Omega})}) \\ &\qquad\quad + \|(-\boldsymbol{u}_{\varepsilon,m} + \boldsymbol{u}_{\varepsilon',m})(\cdot, z)\|_{C^{2+\theta}(\partial\Omega)}\big\} \\ &\quad \le M((\varepsilon' - \varepsilon)C_z + \|(-\boldsymbol{u}_{\varepsilon,m} + \boldsymbol{u}_{\varepsilon',m})(\cdot, z)\|_{C^{2+\theta}(\partial\Omega)}), \quad m = 1, 2, 3 \end{aligned}$$

for all $\varepsilon < \varepsilon_2$, where ε_2 is as in Corollary 4.1. Since M is the constant dependent only on z, we thus have by (1.3) and this estimate, the desired result (4.15) and (4.16). □

Now, we are in position to prove Lemma 4.1.

Proof of Lemma 4.1. Concerning the case for $k = 1$, for any fixed $\varepsilon > 0$ with $\varepsilon \neq \varepsilon'$, we see that the pair $\{\boldsymbol{V}_{\varepsilon,\varepsilon',m}, \Pi_{\varepsilon,\varepsilon',m}\}_{m=1,2,3}$ defined by

$$\begin{aligned} \boldsymbol{V}_{\varepsilon,\varepsilon',m}(x, z) &:= (\varepsilon - \varepsilon')^{-1}(\boldsymbol{q}'_{\varepsilon,m}(x, z) - \boldsymbol{q}'_{\varepsilon',m}(x, z)), \\ \Pi_{\varepsilon,\varepsilon',m}(x, z) &:= (\varepsilon - \varepsilon')^{-1}(q'_{\varepsilon,m}(x, z) - q'_{\varepsilon',m}(x, z)), \quad m = 1, 2, 3, \end{aligned}$$

is subject to the following identities.

$$\begin{cases} \mathcal{L}_{\varepsilon'}(\boldsymbol{V}_{\varepsilon,\varepsilon',m}, \Pi_{\varepsilon,\varepsilon',m})(x, z) \\ \quad = (\mathcal{L}_{\varepsilon'} - \mathcal{L}_{\varepsilon})((\varepsilon - \varepsilon')^{-1}\boldsymbol{q}'_{\varepsilon,m}, (\varepsilon - \varepsilon')^{-1}q'_{\varepsilon,m})(x, z), & x \in \Omega, \\ \operatorname{div}_x \boldsymbol{V}_{\varepsilon,\varepsilon',m}(x, z) = 0, & x \in \Omega, \\ \boldsymbol{V}_{\varepsilon,\varepsilon',m}(x, z) = (\varepsilon - \varepsilon')^{-1}(-\boldsymbol{u}_{\varepsilon,m} + \boldsymbol{u}_{\varepsilon',m})(x, z), & x \in \partial\Omega. \end{cases} \tag{4.17}$$

On the other hand, there exist $\{\boldsymbol{h}_{\varepsilon',m}(\cdot,z)\}_{m=1,2,3}\in C^{2+\theta}(\overline{\Omega})$ and $\{h_{\varepsilon',m}(\cdot,z)\}_{m=1,2,3}\in C^{1+\theta}(\overline{\Omega})$ satisfying $\int_\Omega h_{\varepsilon',m}(x,z)\,dx=0$ such that

$$\begin{cases}\mathcal{L}_{\varepsilon'}(\boldsymbol{h}_{\varepsilon',m},h_{\varepsilon',m})(x,z)=\delta_{\varepsilon'}\mathcal{L}(\boldsymbol{q}'_{\varepsilon',m},q'_{\varepsilon',m})(x,z), & x\in\Omega,\\ \operatorname{div}_x\boldsymbol{h}_{\varepsilon',m}(x,z)=0, & x\in\Omega,\\ \boldsymbol{h}_{\varepsilon',m}(x,z)=\delta_{\varepsilon'}\boldsymbol{u}_m(x,z), & x\in\partial\Omega,\end{cases}\tag{4.18}$$

where

$$\delta_{\varepsilon'}\mathcal{L}^i(\boldsymbol{v},\pi):=\frac{d}{d\varepsilon}\mathcal{L}^i_\varepsilon(\boldsymbol{v},\pi)\bigg|_{\varepsilon=\varepsilon'},\quad i=1,2,3,\tag{4.19}$$

$$\delta_{\varepsilon'}\boldsymbol{u}_m(x,z):=\lim_{\varepsilon\to\varepsilon'}\frac{\boldsymbol{u}_{\varepsilon,m}(x,z)-\boldsymbol{u}_{\varepsilon',m}(x,z)}{\varepsilon-\varepsilon'},\quad m=1,2,3\tag{4.20}$$

with $\{\boldsymbol{u}_{\varepsilon,m}\}_{m=1,2,3}$ in (4.5) (see e.g., Ladyzhenskaya [**11**]). By (4.17), (4.18) and Proposition 4.2, there exists $C(\Omega)>0$ such that for any $\varepsilon'<\varepsilon_1$,

$$\begin{aligned}&\|(\boldsymbol{V}_{\varepsilon,\varepsilon',m}-\boldsymbol{h}_{\varepsilon',m})(\cdot,z)\|_{C^{2+\theta}(\overline{\Omega})}+\|(\Pi_{\varepsilon,\varepsilon',m}-h_{\varepsilon',m})(\cdot,z)\|_{C^{1+\theta}(\overline{\Omega})}\\ &\quad\le C\big(\|\mathcal{L}_{\varepsilon'}(\boldsymbol{V}_{\varepsilon,\varepsilon',m}-\boldsymbol{h}_{\varepsilon',m},\Pi_{\varepsilon,\varepsilon',m}-h_{\varepsilon',m})(\cdot,z)\|_{C^\theta\overline{\Omega})}\\ &\qquad\qquad+\|(\boldsymbol{V}_{\varepsilon,\varepsilon',m}-\boldsymbol{h}_{\varepsilon',m})(\cdot,z)\|_{C^{2+\theta}(\partial\Omega)}\big)\\ &\quad=C\big(\|(\mathcal{L}_{\varepsilon'}-\mathcal{L}_\varepsilon)((\varepsilon-\varepsilon')^{-1}\boldsymbol{q}'_{\varepsilon,m},(\varepsilon-\varepsilon')^{-1}q'_{\varepsilon,m})(\cdot,z)-\delta_{\varepsilon'}\mathcal{L}(\boldsymbol{q}'_{\varepsilon',m},q'_{\varepsilon',m})(\cdot,z)\|_{C^\theta(\overline{\Omega})}\\ &\qquad\qquad+\|(\varepsilon-\varepsilon')^{-1}(-\boldsymbol{u}_{\varepsilon,m}+\boldsymbol{u}_{\varepsilon',m})(\cdot,z)-\boldsymbol{h}_{\varepsilon',m}(\cdot,z)\|_{C^{2+\theta}(\partial\Omega)}\big)\\ &\quad\to 0,\quad\text{as }\ \varepsilon\to\varepsilon',\qquad m=1,2,3,\end{aligned}\tag{4.21}$$

which implies Lemma 4.1 for the case $k=1$.

In the same manner as the case for $k=1$, we may handle the case for $k\ge 2$. Taking $\{\boldsymbol{V}_{\varepsilon,\varepsilon',m},\Pi_{\varepsilon,\varepsilon',m}\}_{m=1,2,3}$ defined by

$$\begin{aligned}\boldsymbol{V}_{\varepsilon,\varepsilon',m}(x,z)&:=(\varepsilon-\varepsilon')^{-k}\bigg(\boldsymbol{q}'_{\varepsilon,m}(x,z)-\sum_{l=0}^{k-1}\frac{1}{l!}\delta^l_{\varepsilon'}\boldsymbol{q}'_m(x,z)(\varepsilon-\varepsilon')^l\bigg),\\ \Pi_{\varepsilon,\varepsilon',m}(x,z)&:=(\varepsilon-\varepsilon')^{-k}\bigg(q'_{\varepsilon,m}(x,z)-\sum_{l=0}^{k-1}\frac{1}{l!}\delta^l_{\varepsilon'}q'_m(x,z)(\varepsilon-\varepsilon')^l\bigg)\end{aligned}\tag{4.22}$$

for $m=1,2,3$, we iterate the inequality (4.21) for (4.22), which complete Lemma 4.1. □

Theorem 4.1 enables us to expand the transformed Green function $\{\boldsymbol{g}_{\varepsilon,m},r_{\varepsilon,m}\}_{m=1,2,3}$ up to the k-th order with respect to ε. We next discuss the criterion of their differentiability for the variable x and parameter ε.

Theorem 4.4. *Let $\{\boldsymbol{g}_{\varepsilon,m},p_{\varepsilon,m}\}_{m=1,2,3}$ be the transformed Green function defined by* (3.6) *and* (3.9)*. For any $z\in\Omega$, it holds that*

$$\nabla_{x^i}^{k_1}\left(\frac{d^{k_2}}{d\varepsilon^{k_2}}g_{\varepsilon,m}^l(x,z)\bigg|_{\varepsilon=0}\right)=\frac{d^{k_2}}{d\varepsilon^{k_2}}\left(\nabla_{x^i}^{k_1}g_{\varepsilon,m}^l(x,z)\right)\bigg|_{\varepsilon=0}, \tag{4.23}$$

$$\nabla_{x^i}\left(\frac{d^{k_2}}{d\varepsilon^{k_2}}p_{\varepsilon,m}(x,z)\bigg|_{\varepsilon=0}\right)=\frac{d^{k_2}}{d\varepsilon^{k_2}}\left(\nabla_{x^i}p_{\varepsilon,m}(x,z)\right)\bigg|_{\varepsilon=0}, \quad i,l=1,2,3 \tag{4.24}$$

for all $x\in\overline{\Omega}$ with $x\neq z$ and for every $k_1,k_2=1,2$. Here, $\nabla_{x^i}:=\partial/\partial x^i, i=1,2,3$.

In order to prove Theorem 4.4, we need to investigate the ε-dependence of the compensation term for $\{\boldsymbol{q}'_{\varepsilon,m}(\cdot,z),q'_{\varepsilon,m}(\cdot,z)\}_{m=1,2,3}$ in more detail. For that purpose, we shall introduce some functions for the velocity and pressure as follows.

Let $z\in\Omega$ and $m=1,2,3$ be fixed. Then, the transformed compensation term $\{\boldsymbol{q}'_{\varepsilon,m}(\cdot,z),q'_{\varepsilon,m}(\cdot,z)\}_{m=1,2,3}$ as in (4.4) is considered as the function on $\Omega\times[0,\varepsilon_1)$. Here, ε_1 is as in Theorem 4.1. Therefore, we define the function $\{\boldsymbol{\mathfrak{q}},\mathfrak{q}\}$ respectively by $\boldsymbol{\mathfrak{q}}_{z,m}(x,\varepsilon):=\boldsymbol{q}'_{\varepsilon,m}(x,z)$ and $\mathfrak{q}_{z,m}(x,\varepsilon):=q'_{\varepsilon,m}(x,z)$ for $x\in\Omega$ and $\varepsilon\in[0,\varepsilon_1)$. For simplicity, we shall abbreviate $\boldsymbol{\mathfrak{q}}_{z,m}(x,\varepsilon)$ and $\mathfrak{q}_{z,m}(x,\varepsilon)$ as

$$\boldsymbol{\mathfrak{q}}(x,\varepsilon):=\boldsymbol{\mathfrak{q}}_{z,m}(x,\varepsilon),\quad \mathfrak{q}(x,\varepsilon):=\mathfrak{q}_{z,m}(x,\varepsilon). \tag{4.25}$$

For these functions $\boldsymbol{\mathfrak{q}}$ and $\mathfrak{q}$, we have the following lemma, which plays an important role to prove Theorem 4.4.

Lemma 4.2. *$\nabla_{x^i}^{k_1}(d\mathfrak{q}^l/d\varepsilon)$ and $\nabla_{x^i}(d\mathfrak{q}/d\varepsilon)$ is continuous at any $(x',\varepsilon')\in\overline{\Omega}\times[0,\varepsilon_3)$ for $i,l=1,2,3$ and $k_1=1,2$ with some positive constant ε_3, where $\nabla_{x^i}=\partial/\partial x^i$. Namely, it holds that*

$$\lim_{(h,k)\to(0,0)}\left(\nabla_{x^i}^{k_1}\left(\frac{d}{d\varepsilon}\mathfrak{q}^l\right)(x'+h,\varepsilon'+k)\right)=\nabla_{x^i}^{k_1}\left(\frac{d}{d\varepsilon}\mathfrak{q}^l\right)(x',\varepsilon'), \tag{4.26}$$

$$\lim_{(h,k)\to(0,0)}\left(\nabla_{x^i}\left(\frac{d}{d\varepsilon}\mathfrak{q}\right)(x'+h,\varepsilon'+k)\right)=\nabla_{x^i}\left(\frac{d}{d\varepsilon}\mathfrak{q}\right)(x',\varepsilon') \tag{4.27}$$

for $i,l=1,2,3$ and for $k_1=1,2$.

For the proof of Lemma 4.2, we need to obtain the uniform estimate of the first and the second variation with respect to ε in the following proposition.

Proposition 4.5. *For any $z\in\Omega$, there exist $\varepsilon_3=\varepsilon_3(z)$ and a constant $C=C(z,\Omega)>0$ such that if $\varepsilon'\leq\varepsilon_3$, it holds that*

$$\left\|\delta_{\varepsilon'}^k\boldsymbol{q}'_m(\cdot,z)\right\|_{C^{2+\theta}(\overline{\Omega})}+\left\|\delta_{\varepsilon'}^k q'_m(\cdot,z)\right\|_{C^{1+\theta}(\overline{\Omega})}<C,\quad m=1,2,3,\ k=1,2,3, \tag{4.28}$$

where $\{\delta_{\varepsilon'}^k\boldsymbol{q}'_m(\cdot,z),\delta_{\varepsilon'}^k q'_m(\cdot,z)\}_{m=1,2,3}$ is defined by (4.7) and (4.8), respectively.

Proof. We first prove the case for $k=1$. Taking ε_1 as in Proposition 4.2, it holds that for any $\varepsilon'\leq\varepsilon_1$,

$$\begin{aligned}&\|\delta_{\varepsilon'}\boldsymbol{q}'_m(\cdot,z)\|_{C^{2+\theta}(\overline{\Omega})}+\|\delta_{\varepsilon'}q'_m(\cdot,z)\|_{C^{1+\theta}(\overline{\Omega})}\\&\quad\le C_1\big(\|\mathcal{L}_{\varepsilon'}(\delta_{\varepsilon'}\boldsymbol{q}'_m,\delta_{\varepsilon'}q'_m)(\cdot,z)\|_{C^{2+\theta}(\overline{\Omega})}+\|\delta_{\varepsilon'}\boldsymbol{q}'_m(\cdot,z)\|_{C^{2+\theta}(\partial\Omega)}\big),\end{aligned}\tag{4.29}$$

where a constant C_1 is independent of ε'. Furthermore, it follows from (4.18) that

$$\mathcal{L}_{\varepsilon'}(\delta_{\varepsilon'}\boldsymbol{q}'_m,\delta_{\varepsilon'}q'_m)(x,z)=\delta_{\varepsilon'}\mathcal{L}(\boldsymbol{q}'_{\varepsilon',m},q'_{\varepsilon',m})(x,z),\quad m=1,2,3\tag{4.30}$$

for all $x\in\Omega$, where $\delta_{\varepsilon'}\mathcal{L}$ is defined by (4.19). Moreover, there exists $\varepsilon'_0=\varepsilon'_0(z)>0$ such that if $\varepsilon'<\varepsilon'_0$, then it holds that

$$\|\delta_{\varepsilon'}\boldsymbol{q}'_m(\cdot,z)\|_{C^{2+\theta}(\partial\Omega)}=\|\delta_{\varepsilon'}\boldsymbol{u}_m(\cdot,z)\|_{C^{2+\theta}(\partial\Omega)}<C_2(z)=:C_2,\quad m=1,2,3,\tag{4.31}$$

where a constant C_2 is independent of ε'. Substituting (4.30) and (4.31) to the right hand side of (4.29), we obtain from Proposition 3.1 and Corollary 4.1 that

$$\begin{aligned}&\|\delta_{\varepsilon'}\boldsymbol{q}'_m(\cdot,z)\|_{C^{2+\theta}(\overline{\Omega})}+\|\delta_{\varepsilon'}q'_m(\cdot,z)\|_{C^{1+\theta}(\overline{\Omega})}\\&\quad\le C_1\big(\|\delta_{\varepsilon'}\mathcal{L}(\boldsymbol{q}'_{\varepsilon',m},q'_{\varepsilon',m})(\cdot,z)\|_{C^{2+\theta}(\overline{\Omega})}+C_2\big)\\&\quad\le C_1\big((1+\varepsilon_3+C_3\varepsilon_3^2)(\|\boldsymbol{q}'_{\varepsilon',m}(\cdot,z)\|_{C^{2+\theta}(\overline{\Omega})}+\|q'_{\varepsilon',m}(\cdot,z)\|_{C^{1+\theta}(\overline{\Omega})})+C_2\big)\\&\quad\le C\end{aligned}\tag{4.32}$$

for $m=1,2,3$ and for all $\varepsilon'\le\varepsilon_3:=\min\{\varepsilon_1,\varepsilon_2,\varepsilon'_0\}$, where ε_2 is as in Corollary 4.1 and a constant C is independent of ε'. This completes (4.28) for $k=1$. We can prove the case for $k\ge 2$ in the same manner as (4.32). □

By means of Proposition 4.5, we shall prove Lemma 4.2.

Proof of Lemma 4.2. We first consider (4.26) for $k_1=1$. For any $(x',\varepsilon')\in\overline{\Omega}\times[0,\varepsilon_3)$ and for any sufficiently small $|h|,k>0$, it holds that

$$\begin{aligned}&\left|\frac{\partial}{\partial x^i}\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x'+h,\varepsilon'+k)-\frac{\partial}{\partial x^i}\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x',\varepsilon')\right|\\&\quad=\left|\frac{\partial}{\partial x^i}\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x'+h,\varepsilon'+k)-\frac{\partial}{\partial x^i}\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x',\varepsilon'+k)\right.\\&\qquad\left.+\frac{\partial}{\partial x^i}\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x',\varepsilon'+k)-\frac{\partial}{\partial x^i}\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x',\varepsilon')\right|\\&\quad\le\left|\int_0^1\frac{d}{d\theta_1}\left\{\frac{\partial}{\partial x^i}\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)((1-\theta_1)x'+\theta_1 h,\varepsilon'+k)\right\}d\theta\right|\\&\qquad+\left\|\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(\cdot,\varepsilon'+k)-\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(\cdot,\varepsilon')\right\|_{C^{2+\theta}(\overline{\Omega})}\end{aligned}$$

$$\leq h\left\|\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(\cdot,\varepsilon'+k)\right\|_{C^{2+\theta}(\overline{\Omega})}+k\left\|\left(\frac{d^2\mathfrak{q}^l}{d\varepsilon^2}\right)(\cdot,\varepsilon')\right\|_{C^{1+\theta}(\overline{\Omega})}, \quad i,l=1,2,3 \tag{4.33}$$

for some $0<\theta_1<1$. In the above arguments, we have used the fact that by Lemma 4.1, it holds that

$$\left|\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x,\varepsilon'+k)-\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x,\varepsilon')\right|\leq kC\left\|\left(\frac{d^2\mathfrak{q}^l}{d\varepsilon^2}\right)(\cdot,\varepsilon')\right\|_{C^{2+\theta}(\overline{\Omega})}, \quad l=1,2,3$$

for all $x\in\overline{\Omega}$, where a constant C is independent of k. Then, it follows from Lemma 4.1 and Proposition 4.5 that

$$\left\|\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(\cdot,\varepsilon'+k)\right\|_{C^{\theta}(\overline{\Omega})}\leq C_1, \quad \left\|\left(\frac{d^2\mathfrak{q}^l}{d\varepsilon^2}\right)(\cdot,\varepsilon')\right\|_{C^{1+\theta}(\overline{\Omega})}\leq C_2 \tag{4.34}$$

for some constants $C_1=C_1(\Omega,z)>0$ which is independent of k, and $C_2=C_2(\Omega,z)>0$. Hence, applying (4.34) to the right hand side of (4.33), we have that

$$\left|\frac{\partial}{\partial x^i}\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x'+h,\varepsilon'+k)-\frac{\partial}{\partial x^i}\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x',\varepsilon')\right|\leq C(h+k)\to 0,$$

as $(h,k)\to(0,0)$, where $C:=\max(C_1,C_2)$. This implies (4.26) for $k_1=1$. Furthermore, the identity (4.26) for $k_1=2$ and (4.27) may be handled in the same way as (4.26) for $k_1=1$. We thus have Lemma 4.2. $\square$

Analogously with Lemma 4.2, we can discuss the continuity for the *second* variation with respect to ε.

Corollary 4.2. *$\nabla_{x^i}^{k_1}(d^2\mathfrak{q}^l/d\varepsilon^2)$ and $\nabla_{x^i}(d^2\mathfrak{q}/d\varepsilon^2)$ is continuous at any $(x',\varepsilon')\in\overline{\Omega}\times[0,\varepsilon_3)$ for $i,l=1,2,3$ and $k_1=1,2$ with some positive constant ε_3, where $\nabla_{x^i}=\partial/\partial x^i$. Namely, it holds that*

$$\lim_{(h,k)\to(0,0)}\left(\nabla_{x^i}^{k_1}\left(\frac{d^2}{d\varepsilon^2}\mathfrak{q}^l\right)(x'+h,\varepsilon'+k)\right)=\nabla_{x^i}^{k_1}\left(\frac{d^2}{d\varepsilon^2}\mathfrak{q}^l\right)(x',\varepsilon'), \tag{4.35}$$

$$\lim_{(h,k)\to(0,0)}\left(\nabla_{x^i}\left(\frac{d^2}{d\varepsilon^2}\mathfrak{q}\right)(x'+h,\varepsilon'+k)\right)=\nabla_{x^i}\left(\frac{d^2}{d\varepsilon^2}\mathfrak{q}\right)(x',\varepsilon') \tag{4.36}$$

for $i,l=1,2,3$ and for $k_1=1,2$.

Proof. Since for any $\varepsilon'\in[0,\varepsilon_3)$ and for sufficiently $k>0$, it follows from Lemma 4.1 and Proposition 4.5 that

$$\left\|\left(\frac{d^2\mathfrak{q}^l}{d\varepsilon^2}\right)(\cdot,\varepsilon'+k)\right\|_{C^{\theta}(\overline{\Omega})}\leq C_1, \quad \left\|\left(\frac{d^3\mathfrak{q}^l}{d\varepsilon^3}\right)(\cdot,\varepsilon')\right\|_{C^{1+\theta}(\overline{\Omega})}\leq C_2, \tag{4.37}$$

for some constants $C_1 = C_1(z, \Omega), C_2 = C_2(z, \Omega) > 0$ independent of k, we can handle (4.35) and (4.36) in the same manner respectively as (4.26) and (4.27). $\square$

Now, we are in position to prove Theorem 4.4.

PROOF OF THEOREM 4.4. For any $k_1 = 1, 2$, and $k_2 = 1, 2$, it follows from (4.4) that

$$\begin{aligned}
&\frac{\partial^{k_1}}{\partial x^{i^{k_1}}}\left(\frac{d^{k_2}}{d\varepsilon^{k_2}}\boldsymbol{g}_{\varepsilon,m}(x,z)\bigg|_{\varepsilon=0}\right) \\
&= \frac{\partial^{k_1}}{\partial x^{i^{k_1}}}\left\{\frac{d^{k_2}}{d\varepsilon^{k_2}}(\boldsymbol{u}_{\varepsilon,m}(x,z) - \boldsymbol{q}'_{\varepsilon,m}(x,z))\bigg|_{\varepsilon=0}\right\} \\
&= \frac{\partial^{k_1}}{\partial x^{i^{k_1}}}\left(\frac{d^{k_2}}{d\varepsilon^{k_2}}\boldsymbol{u}_{\varepsilon,m}(x,z)\bigg|_{\varepsilon=0}\right) - \frac{\partial^{k_1}}{\partial x^{i^{k_1}}}\left(\frac{d^{k_2}}{d\varepsilon^{k_2}}\mathfrak{q}(x,\varepsilon)\bigg|_{\varepsilon=0}\right)
\end{aligned} \tag{4.38}$$

for $i, l, m = 1, 2, 3$. For the fundamental solution $\{\boldsymbol{u}_{\varepsilon,m}\}_{m=1,2,3}$ as in (4.5) and $\mathfrak{q}$ is defined by (4.25), we immediately have by a direct calculation and (1.3) that

$$\frac{\partial^{k_1}}{\partial x^{i^{k_1}}}\left(\frac{d^{k_2}}{d\varepsilon^{k_2}}\boldsymbol{u}_{\varepsilon,m}(x,z)\bigg|_{\varepsilon=0}\right) = \frac{d^{k_2}}{d\varepsilon^{k_2}}\left(\frac{\partial^{k_1}}{\partial x^{i^{k_1}}}\boldsymbol{u}_{\varepsilon,m}(x,z)\right)\bigg|_{\varepsilon=0}, \quad i, l, m = 1, 2, 3 \tag{4.39}$$

for $x \in \overline{\Omega}$ and for $k_1 = 1, 2$ and $k_2 = 1, 2$. Therefore, it suffices to prove the following identity as

$$\frac{\partial^{k_1}}{\partial x^{i^{k_1}}}\left(\frac{d^{k_2}}{d\varepsilon^{k_2}}\mathfrak{q}(x,\varepsilon)\bigg|_{\varepsilon=0}\right) = \frac{d^{k_2}}{d\varepsilon^{k_2}}\left(\frac{\partial^{k_1}}{\partial x^{i^{k_1}}}\mathfrak{q}(x,\varepsilon)\right)\bigg|_{\varepsilon=0}, \quad m = 1, 2, 3 \tag{4.40}$$

for $x \in \overline{\Omega}$ with $x \neq z$ and for $k_1 = 1, 2$ and $k_2 = 1, 2$. Concerning the identity of (4.40), we first consider $k_2 = 1$.

(i) $(k_1, k_2) = (1, 1)$.
For every $\varepsilon' \in (0, \varepsilon_3)$ and for sufficiently small k, we define the function D by

$$D := \boldsymbol{\psi}(\varepsilon' + k) - \boldsymbol{\psi}(\varepsilon'), \tag{4.41}$$

where ε_3 is a constant of Lemma 4.2. Here, for sufficiently small h, $\boldsymbol{\psi}$ is denoted by

$$\boldsymbol{\psi}(\varepsilon') := \mathfrak{q}(x + he_i, \varepsilon') - \mathfrak{q}(x, \varepsilon'),$$

where $\{e_i\}_{i=1,2,3}$ is a canonical basis of $\mathbb{R}^3$. By means of Lemma 4.1, we apply the mean value theorem to the function ψ. Then, it holds that

$$\psi^l(\varepsilon' + k) - \psi^l(\varepsilon') = \left(\frac{d\psi^l}{d\varepsilon}\right)(\varepsilon' + \theta_1 k)k, \quad l = 1, 2, 3 \tag{4.42}$$

for some $0 < \theta_1 < 1$. Here, it should be noted that

$$\left(\frac{d\psi^l}{d\varepsilon}\right)(\varepsilon' + \theta_1 k) = \left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x + he_i, \varepsilon' + \theta_1 k) - \left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x, \varepsilon' + \theta_1 k).$$

Furthermore, applying the mean value theorem to the right hand side of the above argument, we have also by Lemma 4.1 that

$$\begin{aligned}\left(\frac{d\psi^l}{d\varepsilon}\right)(\varepsilon' + \theta_1 k) &= \left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x + he_i, \varepsilon' + \theta_1 k) - \left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x, \varepsilon' + \theta_1 k) \\ &= \frac{\partial}{\partial x^i}\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x + \theta_2 he_i, \varepsilon' + \theta_1 k)h, \quad i, l = 1, 2, 3 \end{aligned} \tag{4.43}$$

for some $0 < \theta_2 < 1$. Substituting (4.43) to (4.42), it holds that

$$\psi^l(\varepsilon' + k) - \psi^l(\varepsilon') = \frac{\partial}{\partial x^i}\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x + \theta_2 he_i, \varepsilon' + \theta_1 k)hk, \quad i, l = 1, 2, 3.$$

Therefore, we have by (4.26) that

$$\lim_{(h,k)\to(0,0)} \frac{\psi^l(\varepsilon' + k) - \psi^l(\varepsilon')}{hk} = \frac{\partial}{\partial x^i}\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x, \varepsilon'), \quad i, l = 1, 2, 3. \tag{4.44}$$

Furthermore, since the vector function $\mathfrak{q}(\cdot, \varepsilon)$ is smooth in Ω, we see that

$$\begin{aligned}\frac{D}{hk} &= \frac{1}{k}\left\{\frac{\mathfrak{q}^l(x + he_i, \varepsilon' + k) - \mathfrak{q}^l(x + he_i, \varepsilon') - \mathfrak{q}^l(x, \varepsilon' + k) + \mathfrak{q}^l(x, \varepsilon')}{h}\right\} \\ &= \frac{1}{k}\left\{\frac{\mathfrak{q}^l(x + he_i, \varepsilon' + k) - \mathfrak{q}^l(x, \varepsilon' + k) - (\mathfrak{q}^l(x + he_i, \varepsilon') - \mathfrak{q}^l(x, \varepsilon'))}{h}\right\} \\ &\to \frac{1}{k}\left(\frac{\partial \mathfrak{q}^l}{\partial x^i}(x, \varepsilon' + k) - \frac{\partial \mathfrak{q}^l}{\partial x^i}(x, \varepsilon')\right), \quad i, l = 1, 2, 3, \quad \text{as} \ \ h \to 0. \end{aligned} \tag{4.45}$$

Taking a limit of (4.45) as $k \to 0$, we have by (4.44) that

$$\frac{\partial}{\partial x^i}\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)(x, \varepsilon') = \frac{d}{d\varepsilon}\left(\frac{\partial \mathfrak{q}^l}{\partial x^i}\right)(x, \varepsilon'), \quad i, l = 1, 2, 3. \tag{4.46}$$

Taking $\varepsilon' = 0$, we thus have (4.40) for $(k_1, k_2) = (1, 1)$. Then, it is easy to see that (4.39) for $(k_1, k_2) = (1, 1)$ and (4.40) yield the desired identity (4.23) for $(k_1, k_2) = (1, 1)$. In the same manner as (4.46), we may prove (4.24) for $k_2 = 1$.

(ii) $(k_1, k_2) = (2, 1)$.
By (4.46), we see that there exists

$$\frac{d}{d\varepsilon}\left(\frac{\partial \mathfrak{q}^l}{\partial x^i}\right)(x,\varepsilon), \quad i,l=1,2,3 \tag{4.47}$$

for all $(x,\varepsilon) \in \Omega \times (0,\varepsilon_3)$, where ε_3 is a constant as in Lemma 4.2. Furthermore taking $\varepsilon' = 0$ in (4.46), we have by Lemma 4.2 and (4.46) that the function

$$\frac{\partial}{\partial x^i}\left\{\frac{d}{d\varepsilon}\left(\frac{\partial q^l}{\partial x^i}\right)\right\}, \quad i,l=1,2,3$$

is continuous at $(x',0)$. Therefore, similarly to the case for (i), it holds that

$$\frac{\partial}{\partial x^i}\left\{\frac{d}{d\varepsilon}\left(\frac{\partial \mathfrak{q}^l}{\partial x^i}\right)\right\}(x',0) = \frac{d}{d\varepsilon}\left(\frac{\partial^2}{\partial {x^i}^2}\mathfrak{q}^l\right)(x',0), \quad i,l=1,2,3,$$

which implies (4.40). We thus have by (4.39), the desired identity (4.23) for $(k_1,k_2) = (2,1)$.

In the next step, we consider the case for $k_2 = 2$.
(iii) $(k_1,k_2) = (1,2)$.
By Lemma 4.1, we see that for any $\varepsilon \in (0,\varepsilon_3)$, there exists a function

$$\frac{\partial}{\partial x^i}\left\{\frac{d}{d\varepsilon}\left(\frac{d\mathfrak{q}^l}{d\varepsilon}\right)\right\}(x,\varepsilon), \quad i,l=1,2,3 \tag{4.48}$$

for all $x \in \Omega$, where ε_3 is a constant as in Lemma 4.2. Furthermore, it follows from Corollary 4.2 that a function (4.48) is continuous at any point $(x',\varepsilon') \in \Omega \times (0,\varepsilon_3)$. Hence, we have by (4.46) that

$$\begin{aligned}\frac{\partial}{\partial x^i}\left(\frac{d^2\mathfrak{q}^l}{d\varepsilon^2}\right)(x',\varepsilon') &= \frac{d}{d\varepsilon}\left\{\frac{\partial}{\partial x^i}\left(\frac{d}{d\varepsilon}\mathfrak{q}^l\right)\right\}(x',\varepsilon') \\ &= \frac{d^2}{d\varepsilon^2}\left(\frac{\partial}{\partial x^i}\mathfrak{q}^l\right)(x',\varepsilon'), \quad i,l=1,2,3.\end{aligned} \tag{4.49}$$

Taking $\varepsilon' = 0$ in the above argument and by (4.39) for $(k_1,k_2) = (1,2)$, this implies (4.23) for $(k_1,k_2) = (1,2)$. Similarly to (4.49), we also have (4.24) for $k_2 = 2$.

(iv) $(k_1,k_2) = (2,2)$.
By Lemma 4.1 and (4.49), we may assure the existence of

$$\frac{\partial}{\partial x^i}\left\{\frac{d^2}{d\varepsilon^2}\left(\frac{\partial}{\partial x^i}\mathfrak{q}^l\right)\right\}(x,\varepsilon') \quad i,l=1,2,3 \tag{4.50}$$

for all $x \in \Omega$. Moreover, it follows from Corollary 4.2 and (4.49) that a function (4.50) is continuous at $(x',0)$. Therefore, by (4.49), it holds, similarly to the case for (iii), that

$$\begin{aligned}\frac{\partial^2}{\partial {x^i}^2}\left(\frac{d^2}{d\varepsilon^2}\mathfrak{q}^l\right)(x',0) &= \frac{\partial}{\partial x^i}\left\{\frac{\partial}{\partial x^i}\left(\frac{d^2}{d\varepsilon^2}\mathfrak{q}^l\right)\right\}(x',0)\\ &= \frac{\partial}{\partial x^i}\left\{\frac{d^2}{d\varepsilon^2}\left(\frac{\partial}{\partial x^i}\mathfrak{q}^l\right)\right\}(x',0)\\ &= \frac{d^2}{d\varepsilon^2}\left(\frac{\partial^2}{\partial {x^i}^2}\mathfrak{q}^l\right)(x',0), \quad i,l=1,2,3,\end{aligned}$$

which yields (4.40). Hence, by (4.39), we thus have (4.23) for $(k_1,k_2)=(2,2)$.

According to from (i) to (iv), we complete (4.23) for $k_1,k_2=1,2$, and (4.24) for $k_2=1,2$. We thus prove Theorem 4.4. □

4.2. The first order term of the expansion with respect to ε.

In this subsection, we discuss the ε-dependence of the Green function $\{\boldsymbol{G}_{\varepsilon,m},P_{\varepsilon,m}\}_{m=1,2,3}$. We first investigate the existence of the *first* order term of their expansion with respect to ε.

Lemma 4.3. *For any $z\in\Omega$, there exist $\{\delta\boldsymbol{G}_m(\cdot,z)\}_{m=1,2,3}\in C^{2+\theta}(\Omega\setminus\{z\})$ and $\{\delta P_m(\cdot,z)\}_{m=1,2,3}\in C^{2+\theta}(\Omega\setminus\{z\})$ with $0<\theta<1$. Moreover, we have the explicit expressions as*

$$\begin{aligned}\delta G_m^n(x,z) &= \delta {G''_m}^n(x,z) - (\mathrm{div}_x\,\boldsymbol{S}_1(x,z))G_m^n(x,z)\\ &\quad + \sum_{i=1}^3 \frac{\partial S_1^n}{\partial x^i}(x,z)G_m^i(x,z) - \nabla_x G_m^n(x,z)\cdot\boldsymbol{S}_1(x,z), \end{aligned}\tag{4.51}$$

$$\delta P_m(x,z) = \delta P'_m(x,z) - \nabla_x P_m(x,z)\cdot\boldsymbol{S}_1(x,z)\tag{4.52}$$

for $m,n=1,2,3$ and for all $x\in\Omega\setminus\{z\}$, where $\{\delta\boldsymbol{G}'_m,\delta P'_m\}_{m=1,2,3}$ are as in (4.2) *and* (4.3) *taking $\varepsilon'=0$ and $k=1$, and $\boldsymbol{S}_1$ is the vector function introduced by* (1.3).

Proof. By (3.6), Proposition 3.1 and the Taylor expansion of $\{g_{\varepsilon,m}^n(x,z)\}_{n,m=1,\dots,d}$ around $(x,z)\in\Omega\times\Omega$, we have that

$$\begin{aligned} & g_{\varepsilon,m}^n(x,z)\\ &= (\det J_\varepsilon)\sum_{i=1}^3\frac{\partial x^n}{\partial\tilde{x}^i}\Big\{G_{\varepsilon,m}^i(x,z)+\nabla_{\tilde{x}}G_{\varepsilon,m}^i(x,z)\cdot(\tilde{x}-x)+\nabla_{\tilde{z}}G_{\varepsilon,m}^i(x,z)\cdot(\tilde{z}-z)\\ &\qquad + \nabla_{\tilde{z}}\nabla_{\tilde{x}}G_{\varepsilon,m}^i(x,z+\theta_1(\tilde{z}-z))(\tilde{x}-x)\cdot(\tilde{z}-z)\\ &\qquad + \frac{1}{2}\nabla_{\tilde{x}}^2G_{\varepsilon,m}^i(x+\theta_2(\tilde{x}-x),\tilde{z})(\tilde{x}-x)\cdot(\tilde{x}-x)\\ &\qquad + \frac{1}{2}\nabla_{\tilde{z}}^2G_{\varepsilon,m}^i(x,z+\theta_3(\tilde{z}-z))(\tilde{z}-z)\cdot(\tilde{z}-z)\Big\}\end{aligned}$$

$$= (\det J_\varepsilon) \sum_{i=1}^{3} \frac{\partial x^n}{\partial \tilde{x}^i} \Big\{ G^i_{\varepsilon,m}(x,z) + \nabla_{\tilde{x}} G^i_{\varepsilon,m}(x,z) \cdot (\tilde{x} - x) + \frac{1}{2} \nabla^2_{\tilde{x}} G^i_{\varepsilon,m}(x + \theta_2(\tilde{x} - x), z)(\tilde{x} - x) \cdot (\tilde{x} - x) \Big\} \tag{4.53}$$

for $m, n = 1, 2, 3$ and some $0 < \theta_1, \theta_2, \theta_3 < 1$. Here, by (1.3) it should be noted that $\boldsymbol{S}_1(z,z) = \boldsymbol{S}_2(z,z) = 0$.

It follows from Corollary 4.1 that

$$\sup_{0<\varepsilon<\delta} \left| \nabla^k_{\tilde{x}} G^n_{\varepsilon,m}(x,z) \right| \le C \tag{4.54}$$

for $k = 1, 2$ with some constant C which may depend on $x, z \in \Omega$.

We substitute (4.53) to the right hand side of (4.2) taking $\varepsilon' = 0$ and $k = 1$, and it follows from Proposition 3.1 and Proposition 4.3 and (4.54) that

$$\begin{aligned}
&\delta {G'}^{\,n}_{m}(x,z) \\
&= \lim_{\varepsilon \to 0} \varepsilon^{-1} \Big\{ \det(J_\varepsilon(x,z)) \sum_{i=1}^{d} \frac{\partial x^n}{\partial \tilde{x}^i}(x) \big(G^i_{\varepsilon,m}(x,z) + \nabla_{\tilde{x}} G^i_{\varepsilon,m}(x,z) \cdot (\tilde{x} - x) \big) - G^n_m(x,z) \Big\} \\
&= \lim_{\varepsilon \to 0} \varepsilon^{-1} \Big\{ (1 + (\operatorname{div}_x \boldsymbol{S}_1(x,z))\varepsilon + O(\varepsilon^2)) \Big(\delta^{in} - \frac{\partial S^n_1}{\partial x^i}(x,z)\varepsilon + O(\varepsilon^2) \Big) \\
&\qquad \times \big(G^i_{\varepsilon,m}(x,z) + \varepsilon \nabla_{\tilde{x}} G^i_{\varepsilon,m}(x,z) \cdot \boldsymbol{S}_1(x,z) \big) - G^n_m(x,z) \Big\} \\
&= \lim_{\varepsilon \to 0} \varepsilon^{-1} \Big\{ G^n_{\varepsilon,m}(x,z) + \varepsilon \Big((\operatorname{div}_x \boldsymbol{S}_1(x,z)) G^n_{\varepsilon,m}(x,z) - \frac{\partial S^n_1}{\partial x^i}(x,z) G^i_{\varepsilon,m}(x,z) \\
&\qquad + \nabla_{\tilde{x}} G^n_{\varepsilon,m}(x,z) \cdot \boldsymbol{S}_1(x,z) \Big) - G^n_m(x,z) \Big\} \\
&= \lim_{\varepsilon \to 0} \frac{G^n_{\varepsilon,m}(x,z) - G^n_m(x,z)}{\varepsilon} \\
&\quad + \lim_{\varepsilon \to 0} \Big((\operatorname{div}_x \boldsymbol{S}_1(x,z)) G^n_{\varepsilon,m}(x,z) \\
&\qquad - \sum_{i=1}^{3} \frac{\partial S^n_1}{\partial x^i}(x,z) G^i_{\varepsilon,m}(x,z) + \nabla_{\tilde{x}} G^n_{\varepsilon,m}(x,z) \cdot \boldsymbol{S}_1(x,z) \Big) \\
&= \delta G^n_m(x,z) + (\operatorname{div}_x \boldsymbol{S}_1(x,z)) G^n_m(x,z) \\
&\quad - \sum_{i=1}^{3} \frac{\partial S^n_1}{\partial x^i}(x,z) G^i_m(x,z) + \nabla_x G^n_m(x,z) \cdot \boldsymbol{S}_1(x,z)
\end{aligned} \tag{4.55}$$

for $m, n = 1, 2, 3$ and for all $x, z \in \Omega$ with $x \ne z$, where $\tilde{x} = \Phi_\varepsilon(x; z)$, $\tilde{z} = \Phi_\varepsilon(z, \varepsilon)$. Here, we have used the fact that $\boldsymbol{S}_1(z,z)=0$, and it should be noted that $G^n_{\varepsilon,m}=G^n_{\varepsilon,m}(\tilde{x},\tilde{z})$ is regarded as a function on $\Omega_\varepsilon\times\Omega_\varepsilon$ with variables $(\tilde{x},\tilde{z})$. Then we have that

$$\begin{aligned}\delta G^n_m(x,z)&=\delta {G'}^n_m(x,z)-(\operatorname{div}_x \boldsymbol{S}_1(x,z))G^n_m(x,z)\\&\quad+\sum_{i=1}^3\frac{\partial S^n_1}{\partial x^i}(x,z)G^i_m(x,z)-\nabla_x G^n_m(x,z)\cdot\boldsymbol{S}_1(x,z)\end{aligned}\tag{4.56}$$

for $m,n=1,2,3$, which implies (4.51) for the velocity.

Similarly, by Taylor expansion of $\{P_{\varepsilon,m}(\tilde{x},\tilde{z})\}_{m=1,2,3}$, (4.54), Proposition 3.1 and Proposition 4.3, we have that

$$\begin{aligned}\delta P'_m(x,z)&=\lim_{\varepsilon\to0}\frac{P_{\varepsilon,m}(\tilde{x},\tilde{z})-P_m(x,z)}{\varepsilon}\\&=\lim_{\varepsilon\to0}\varepsilon^{-1}\big(P_{\varepsilon,m}(x,z)+\nabla_{\tilde{x}}P_{\varepsilon,m}(x,z)\cdot(\tilde{x}-x)\\&\qquad\qquad+\nabla_{\tilde{z}}P_{\varepsilon,m}(x,z)\cdot(\tilde{z}-z)-P_m(x,z)\big)\\&=\lim_{\varepsilon\to0}\varepsilon^{-1}\big(P_{\varepsilon,m}(x,z)+\varepsilon\nabla_{\tilde{x}}P_{\varepsilon,m}(x,z)\cdot\boldsymbol{S}_1(x,z)-P_m(x,z)\big)\\&=\lim_{\varepsilon\to0}\frac{P_{\varepsilon,m}(x,z)-P_m(x,z)}{\varepsilon}+\lim_{\varepsilon\to0}\nabla_{\tilde{x}}P_{\varepsilon,m}(x,z)\cdot\boldsymbol{S}_1(x,z)\\&=\delta P_m(x,z)+\nabla_x P_m(x,z)\cdot\boldsymbol{S}_1(x,z),\quad m=1,2,3\end{aligned}\tag{4.57}$$

for all $x,z\in\Omega$ with $x\neq z$, where $\tilde{z}=\Phi_\varepsilon(z,z)$. Here, we have also used the fact that $\boldsymbol{S}_1(z,z)=0$. Then it holds that

$$\delta P_m(x,z)=\delta P'_m(x,z)-\nabla_x P_m(x,z)\cdot\boldsymbol{S}_1(x,z),\quad m=1,2,3,\tag{4.58}$$

which implies Lemma 4.3 for the pressure. □

Similarly to Theorem 4.4, we shall discuss the criterion of the differentiability of the Green function $\{\boldsymbol{G}_{\varepsilon,m},P_{\varepsilon,m}\}_{m=1,2,3}$ for the variable x and the parameter ε.

Theorem 4.6. *Let $\{\boldsymbol{G}_{\varepsilon,m},P_{\varepsilon,m}\}_{m=1,2,3}$ be the Green function of the Stokes equations* (0.3)*. For any $z\in\Omega$, it holds that*

$$\nabla^{k_1}_{x^i}\left(\frac{d}{d\varepsilon}G^l_{\varepsilon,m}(x,z)\Big|_{\varepsilon=0}\right)=\frac{d}{d\varepsilon}\big(\nabla^{k_1}_{x^i}G^l_{\varepsilon,m}(x,z)\big)\Big|_{\varepsilon=0},\tag{4.59}$$

$$\nabla_{x^i}\left(\frac{d}{d\varepsilon}P_{\varepsilon,m}(x,z)\Big|_{\varepsilon=0}\right)=\frac{d}{d\varepsilon}(\nabla_{x^i}P_{\varepsilon,m}(x,z))\Big|_{\varepsilon=0},\qquad i,l,m=1,2,3\tag{4.60}$$

for all $x\in\Omega$ with $x\neq z$ and for $k_1=1,2$. Here, $\nabla_{x^i}:=\partial/\partial x^i, i=1,2,3$.

Proof. We first prove (4.59) for $k=1$. It follows form (4.55) that

$$
\begin{aligned}
&\nabla_{x^i}\{\varepsilon^{-1}(g^l_{\varepsilon,m}(x,z)-G^l_m(x,z))\}\\
&\quad=\nabla_{x^i}\Biggl\{\varepsilon^{-1}\Biggl(\det(J_\varepsilon(x,z))\sum_{i=1}^{3}\frac{\partial x^l}{\partial\tilde{x}^i}(x)\bigl(G^i_{\varepsilon,m}(x,z)+\nabla_{\tilde{x}}G^i_{\varepsilon,m}(x,z)\cdot(\tilde{x}-x)+O(\varepsilon^2)\bigr)\\
&\qquad\qquad\qquad-G^l_m(x,z)\Biggr)\Biggr\}\\
&\quad=\nabla_{x^i}\Biggl[\varepsilon^{-1}\Biggl\{(1+(\operatorname{div}_s\boldsymbol{S}_1(x,z))\varepsilon+O(\varepsilon^2))\sum_{i=1}^{3}\Biggl(\delta^{il}-\frac{\partial S^l_1}{\partial x^i}(x,z)\varepsilon+O(\varepsilon^2)\Biggr)\\
&\qquad\qquad\qquad\times\bigl(G^i_{\varepsilon,m}(x,z)+\varepsilon\nabla_{\tilde{x}}G^i_{\varepsilon,m}(x,z)\cdot\boldsymbol{S}_1(x,z)+O(\varepsilon^2)\bigr)-G^l_m(x,z)\Biggr\}\Biggr]\\
&\quad=\nabla_{x^i}\{\varepsilon^{-1}(G^l_{\varepsilon,m}(x,z)-G^l_m(x,z))\}\\
&\qquad+\nabla_{x^i}\Biggl((\operatorname{div}_x\boldsymbol{S}_1(x,z))G^l_{\varepsilon,m}(x,z)-\sum_{i=1}^{3}\frac{\partial S^l_1}{\partial x^i}(x,z)G^i_{\varepsilon,m}(x,z)\\
&\qquad\qquad+\nabla_{\tilde{x}}G^l_{\varepsilon,m}(x,z)\cdot\boldsymbol{S}_1(x,z)\Biggr)+O(\varepsilon),\quad i,l,m=1,2,3,\quad\text{as }\varepsilon\to0.
\end{aligned}
\tag{4.61}
$$

Letting $\varepsilon\to0$ in (4.61), we have by Proposition 4.3 that

$$
\begin{aligned}
&\lim_{\varepsilon\to0}\nabla_{x^i}\{\varepsilon^{-1}(G^l_{\varepsilon,m}(x,z)-G^l_m(x,z))\}\\
&\quad=\lim_{\varepsilon\to0}\nabla_{x^i}\{\varepsilon^{-1}(g^l_{\varepsilon,m}(x,z)-G^l_m(x,z))\}\\
&\qquad-\nabla_{x^i}\Biggl((\operatorname{div}_x\boldsymbol{S}_1(x,z))G^l_m(x,z)-\sum_{i=1}^{3}\frac{\partial S^l_1}{\partial x^i}(x,z)G^i_m(x,z)+\nabla_xG^l_m(x,z)\cdot\boldsymbol{S}_1(x,z)\Biggr)
\end{aligned}
\tag{4.62}
$$

for $i,l,m=1,2,3$ and for $x\in\Omega$ with $x\neq z$. On the other hand, in the same manner as (4.62), we have that

$$
\begin{aligned}
&\varepsilon^{-1}\{\nabla_{x^i}(g^l_{\varepsilon,m}(x,z)-G^l_m(x,z))\}\\
&\quad=\varepsilon^{-1}\{\nabla_{x^i}(G^l_{\varepsilon,m}(x,z)-G^l_m(x,z))\}\\
&\qquad+\nabla_{x^i}\Biggl((\operatorname{div}_x\boldsymbol{S}_1(x,z))G^l_{\varepsilon,m}(x,z)-\sum_{i=1}^{3}\frac{\partial S^l_1}{\partial x^i}(x,z)G^i_{\varepsilon,m}(x,z)\\
&\qquad\qquad+\nabla_{\tilde{x}}G^l_{\varepsilon,m}(x,z)\cdot\boldsymbol{S}_1(x,z)\Biggr)+O(\varepsilon)
\end{aligned}
$$

for $i,l,m=1,2,3$, as $\varepsilon\to0$. Letting $\varepsilon\to0$, we also have by Proposition 4.3 that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon^{-1}\{\nabla_{x^i}(G^l_{\varepsilon,m}(x,z)-G^l_m(x,z))\}\\&\quad=\lim_{\varepsilon\to 0}\varepsilon^{-1}\{\nabla_{x^i}(g^l_{\varepsilon,m}(x,z)-G^l_m(x,z))\}\\&\qquad-\nabla_{x^i}\biggl((\operatorname{div}_x \boldsymbol{S}_1(x,z))G^l_m(x,z)-\sum_{i=1}^{3}\frac{\partial S^l_1}{\partial x^i}(x,z)G^i_m(x,z)+\nabla_x G^l_m(x,z)\cdot\boldsymbol{S}_1(x,z)\biggr)\end{aligned}\tag{4.63}$$

for $i,l,m=1,2,3$ and for $x\in\Omega$ with $x\neq z$. Here, it follows from (4.62) and (4.63) that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\nabla_{x^i}\{\varepsilon^{-1}(G^l_{\varepsilon,m}(x,z)-G^l_m(x,z))\}-\lim_{\varepsilon\to 0}\varepsilon^{-1}\{\nabla_{x^i}(G^l_{\varepsilon,m}(x,z)-G^l_m(x,z))\}\\&\quad=\lim_{\varepsilon\to 0}\nabla_{x^i}\{\varepsilon^{-1}(g^l_{\varepsilon,m}(x,z)-G^l_m(x,z))\}-\lim_{\varepsilon\to 0}\varepsilon^{-1}\{\nabla_{x^i}(g^l_{\varepsilon,m}(x,z)-G^l_m(x,z))\}\end{aligned}$$

for $i,l,m=1,2,3$ and for $x\in\Omega$ with $x\neq z$. Hence, we have by (4.23) for $(k_1,k_2)=(1,1)$ that

$$\lim_{\varepsilon\to 0}\nabla_{x^i}\{\varepsilon^{-1}(G^l_{\varepsilon,m}(x,z)-G^l_m(x,z))\}=\lim_{\varepsilon\to 0}\varepsilon^{-1}\{\nabla_{x^i}(G^l_{\varepsilon,m}(x,z)-G^l_m(x,z))\},\tag{4.64}$$

which yields (4.59) for $k_1=1$. Similarly to (4.64), we may prove (4.59) for $k=2$ and (4.60) for $k=1$. This completes the proof of Theorem 4.6. □

4.3. The second order term of the expansion with respect to ε.

In the previous subsection, we assure the existence of the *first* order term of the Green function $\{\delta\boldsymbol{G}_m,\delta P_m\}_{m=1,2,3}$. In this subsection, we discuss the existence of the *second* order term, and analyze the regularity theory for the variable $x\in\Omega$ and parameter ε up to the *second* order term with analogue to Theorem 4.6.

We first investigate the existence of $\{\delta^2\boldsymbol{G}_m,\delta^2 P_m\}_{m=1,2,3}$ as in (1.12) and (1.13) in the following lemma.

Lemma 4.4. *For any fixed z in Ω, there exist $\{\delta^2\boldsymbol{G}_m(\cdot,z)\}_{m=1,2,3}\in C^{2+\theta}(\Omega\setminus\{z\})$ and $\{\delta^2 P_m(\cdot,z)\}_{m=1,2,3}\in C^{2+\theta}(\Omega\setminus\{z\})$ with $0<\theta<1$. Moreover, we have the explicit expressions as*

$$\begin{aligned}\delta^2 G^n_m(x,z)&=\delta^2 {G'}^n_m(x,z)+2\biggl\{-(\operatorname{div}_x\boldsymbol{S}_1(x,z))\delta G^n_m(x,z)+\sum_{i=1}^{3}\frac{\partial S^n_1}{\partial x^i}(x,z)\delta G^i_m(x,z)\\&\qquad-\nabla_x\delta G^n_m(x,z)\cdot\boldsymbol{S}_1(x,z)\biggr\}\\&\quad-J_2(x,z)G^n_m(x,z)+2(\operatorname{div}_x\boldsymbol{S}_1(x,z))\sum_{j=1}^{3}\frac{\partial S^n_1}{\partial x^j}(x,z)G^j_m(x,z)\\&\quad-2\sum_{j=1}^{3}\frac{\partial S^n_1}{\partial x^j}(x,z)(\nabla_x G^j_m)(x,z)\cdot\boldsymbol{S}_1(x,z)\end{aligned}$$

$$
\begin{aligned}
&+ 2(\operatorname{div}_x \boldsymbol{S}_1(x,z))(\nabla_x G_m^n)(x,z) \cdot \boldsymbol{S}_1(x,z) \\
&- (\nabla_x^2 G_m^n)(x,z) \cdot \boldsymbol{S}_1^2(x,z) - \sum_{j=1}^3 \frac{\partial S_1^n}{\partial x^j}(x,z) \nabla_x S_1^j(x,z) \cdot \boldsymbol{G}_m(x,z) \\
&- \nabla_x G_m^n(x,z) \cdot \boldsymbol{S}_2(x,z) + \nabla_x S_2^n(x,z) \cdot \boldsymbol{G}_m(x,z), \qquad (4.65)
\end{aligned}
$$

$$
\begin{aligned}
\delta^2 P_m(x,z) &= \delta^2 P_m'(x,z) - 2\nabla_x \delta P_m(x,z) - \nabla_x^2 P_m(x,z) \cdot \boldsymbol{S}_1^2(x,z) \\
&\quad - \nabla_x P_m(x,z) \cdot \boldsymbol{S}_2(x,z) \qquad (4.66)
\end{aligned}
$$

for $m,n = 1,2,3$ *and for all* $x \in \Omega$ *with* $x \neq z$, *where* $\{\delta^2 \boldsymbol{G}_m', \delta^2 P_m'\}_{m=1,2,3}$ *are as in* (4.2) *and* (4.3) *taking* $\varepsilon' = 0$ *and* $k = 2$, $\boldsymbol{S}_1$ *and* $\boldsymbol{S}_2$ *are the vector function introduced by* (1.3). *Here,* J_2 *is defined by* (3.4).

Proof. Similarly to (4.55), we have by Proposition 3.1, Proposition 4.3, (4.51) and (4.54) that

$$
\begin{aligned}
&\delta^2 {G'_m}^n(x,z) \\
&= 2 \lim_{\varepsilon \to 0} \varepsilon^{-2} \Bigg\{ \det(J_\varepsilon(x,z)) \sum_{i=1}^3 \frac{\partial x^n}{\partial \tilde{x}^i}(x) \Bigg(G_{\varepsilon,m}^i(x,z) + \nabla_{\tilde{x}} G_{\varepsilon,m}^i(x,z) \cdot (\tilde{x} - x) \\
&\qquad + \frac{1}{2} \nabla_{\tilde{x}}^2 G_{\varepsilon,m}^i(x,z)(\tilde{x}-x) \cdot (\tilde{x}-x) \Bigg) - G_m^n(x,z) - \varepsilon \delta {G'_m}^n(x,z) \Bigg\} \\
&= \delta^2 G_m^n(x,z) + 2 \Bigg\{ (\operatorname{div}_x \boldsymbol{S}_1(x,z)) \delta G_m^n(x,z) - \sum_{i=1}^3 \frac{\partial S_1^n}{\partial x^i}(x,z) \delta G_m^i(x,z) \\
&\qquad + \nabla_x \delta G_m^n(x,z) \cdot \boldsymbol{S}_1(x,z) \Bigg\} \\
&\quad + J_2(x,z) G_m^n(x,z) - 2(\operatorname{div}_x \boldsymbol{S}_1(x,z)) \sum_{j=1}^3 \frac{\partial S_1^n}{\partial x^j}(x,z) G_m^j(x,z) \\
&\quad + 2(\operatorname{div}_x \boldsymbol{S}_1(x,z))(\nabla_x G_m^n)(x,z) \cdot \boldsymbol{S}_1(x,z) - 2\sum_{j=1}^3 \frac{\partial S_1^n}{\partial x^j}(x,z)(\nabla_x G_m^j)(x,z) \cdot \boldsymbol{S}_1(x,z) \\
&\quad + (\nabla_x^2 G_m^n)(x,z) \cdot \boldsymbol{S}_1^2(x,z) + \sum_{j=1}^3 \frac{\partial S_1^n}{\partial x^j}(x,z) \nabla_x S_1^j(x,z) \cdot \boldsymbol{G}_m(x,z) \\
&\quad + \nabla_x G_m^n(x,z) \cdot \boldsymbol{S}_2(x,z) - \nabla_x S_2^n(x,z) \cdot \boldsymbol{G}_m(x,z), \quad m,n = 1,2,3
\end{aligned}
$$

for $x \in \Omega$ with $x \neq z$, where J_2 is defined by (3.4). In the above calculation, it should be noted that $\boldsymbol{S}_1(z,z) = \boldsymbol{S}_2(z,z) = 0$, and we have used the fact that

$$
\lim_{\varepsilon \to 0} \frac{\nabla_{\tilde{x}} G_{\varepsilon,m}^n(x,z) - \nabla_x G_m^n(x,z)}{\varepsilon} = \nabla_x \delta G_m^n(x,z), \quad m,n = 1,2,3
$$

for $x \in \Omega$ with $x \neq z$, which is derived by (4.59). Furthermore by (4.60), the identity (4.66) may be handled in the same manner as (4.65). We thus have Lemma 4.4. □

By Lemma 4.4, we can assure the existence of the second variation $\{\delta^2 \boldsymbol{G}_m, \delta^2 P_m\}_{m=1,2,3}$. Finally, we investigate their regularity criterion for a variable $x \in \Omega$ with analogue to Theorem 4.6.

THEOREM 4.7. *Let $\{\boldsymbol{G}_{\varepsilon,m}, P_{\varepsilon,m}\}_{m=1,2,3}$ be the Green function of the Stokes equations* (0.3). *For any $z \in \Omega$, it holds that*

$$\nabla_{x^i}^{k_1}\left(\frac{d^2}{d\varepsilon^2} G_{\varepsilon,m}^l(x,z)\bigg|_{\varepsilon=0}\right) = \frac{d^2}{d\varepsilon^2}\left(\nabla_{x^i}^{k_1} G_{\varepsilon,m}^l(x,z)\right)\bigg|_{\varepsilon=0}, \tag{4.67}$$

$$\nabla_{x^i}\left(\frac{d^2}{d\varepsilon^2} P_{\varepsilon,m}(x,z)\bigg|_{\varepsilon=0}\right) = \frac{d^2}{d\varepsilon^2}(\nabla_{x^i} P_{\varepsilon,m}(x,z))\bigg|_{\varepsilon=0}, \quad i,l,m = 1,2,3 \tag{4.68}$$

for all $x \in \Omega$ with $x \neq z$ and $k_1 = 1,2$. Here, $\nabla_{x^i} := \partial/\partial x^i, i = 1,2,3$.

PROOF. It is easy to see that Theorem 4.4 for $k_1 = 1,2$ and $k_2 = 2$ yields Theorem 4.7 in the same manner as Theorem 4.6. □

5. Construction of the representation formula.

In this section, we shall establish the represent formula for the first and the second variation $\{\delta \boldsymbol{G}_m, \delta P_m\}_{m=1,2,3}$ and $\{\delta^2 \boldsymbol{G}_m, \delta^2 P_m\}_{m=1,2,3}$ as in (1.8), (1.9), (1.12) and (1.13), respectively.

5.1. Properties of the first and the second variation.

We first prepare the key lemma for constructing the representation formula for $\{\delta^k \boldsymbol{G}_m, \delta^k P_m\}_{m=1,2,3}$ for $k = 1,2$. For that purpose, we introduce some functions.

Since by Theorem 4.1, we see that $\{\delta \boldsymbol{G}'_m\}_{m=1,2,3}$ as in (4.2) is a smooth function in $\overline{\Omega} \setminus \{z\}$, it follows from the definition of $\{\delta \boldsymbol{G}_m\}_{m=1,2,3}$ as in (1.8) and (1.9), and Lemma 4.3 that there exists $\boldsymbol{H}_m(\cdot, z) \in C^{2+\theta}(\overline{\Omega})$ such that

$$H_m^l(x,z) := \begin{cases} \delta G_m^l(x,z), & x \in \Omega \setminus \{z\}, \\ \delta {G'}_m^l(x,z) - (\mathrm{div}_x\, \boldsymbol{S}_1(x,z)) G_m^l(x,z) & \\ \quad + \displaystyle\sum_{i=1}^3 \frac{\partial S_1^l}{\partial x^i}(x,z) G_m^i(x,z) - \nabla_x G_m^l(x,z) \cdot \boldsymbol{S}_1(x,z), & x \in \partial\Omega \end{cases}$$

for $l, m = 1,2,3$, where $\boldsymbol{S}_1$ is defined by (A.2). If we define an extension $\{\delta \tilde{\boldsymbol{G}}_m\}_{m=1,2,3}$ of $\{\delta \boldsymbol{G}_m\}_{m=1,2,3}$ to the singularity $z \in \Omega$ and to the boundary $x \in \partial\Omega$ by

$$\delta \tilde{\boldsymbol{G}}_m(x,z) := \boldsymbol{H}_m(x,z), \qquad x \in \overline{\Omega}, \tag{5.1}$$

we may regard $\delta \boldsymbol{G}_m(\cdot, z) \in C^{2+\theta}(\overline{\Omega})$ with $0 < \theta < 1$. In the same manner as (5.1), we define the extensions of $\{\delta P_m(\cdot,z)\}_{m=1,2,3}$ and $\{\delta^2 \boldsymbol{G}_m(\cdot,z), \delta^2 P_m(\cdot,z)\}_{m=1,2,3}$ by $\{\delta \tilde{P}_m(\cdot,z)\}_{m=1,2,3}$ and $\{\delta^2 \tilde{\boldsymbol{G}}_m(\cdot,z), \delta^2 \tilde{P}_m(\cdot,z)\}_{m=1,2,3}$, respectively. For that extensions, we have the following theorem.

THEOREM 5.1. *Let $\{\delta^k \tilde{\boldsymbol{G}}_m, \delta^k \tilde{P}_m\}_{m=1,2,3}$ be the extensions of $\{\delta^k \boldsymbol{G}_m, \delta^k P_m\}_{m=1,2,3}$ for $k=1,2$, respectively. For any $z \in \Omega$, it holds that*

$$\begin{cases} -\Delta_x\, \delta^k \tilde{\boldsymbol{G}}_m(x,z) + \nabla_x \delta^k \tilde{P}_m(x,z) = 0, & x \in \Omega, \\ \operatorname{div}_x \delta^k \tilde{\boldsymbol{G}}_m(x,z) = 0, & x \in \Omega, \\ \delta^k \tilde{\boldsymbol{G}}_m(x,z) = \boldsymbol{\beta}_{m,k}(x,z), & x \in \partial\Omega \end{cases}$$

for $k=1,2$, where the boundary data $\boldsymbol{\beta}_{m,k}$ is defined by

$$\boldsymbol{\beta}^l_{m,k}(x,z) = \begin{cases} -\nabla_x G^l_m(x,z) \cdot \boldsymbol{S}_1(x,z), & k=1, \\ -2\nabla_x \delta G^l_m(x,z) \cdot \boldsymbol{S}_1(x,z) - \nabla_x^2 G^l_m(x,z) \cdot \boldsymbol{S}_1(x,z) & \\ \quad -\nabla_x G^l_m(x,z) \cdot \boldsymbol{S}_2(x,z), & k=2, \end{cases}$$

and the vector functions $\boldsymbol{S}_1, \boldsymbol{S}_2$ are defined by (A.2).

PROOF. Concerning the case for $k=1$, it follows from Theorem 4.6 that

$$\begin{aligned} &-\Delta_x \delta \boldsymbol{G}_m(x,z) + \nabla_x \delta P_m(x,z) \\ &= -\Delta_x \left(\frac{d}{d\varepsilon} \boldsymbol{G}_{\varepsilon,m}(x,z) \Big|_{\varepsilon=0} \right) + \nabla_x \left(\frac{d}{d\varepsilon} P_{\varepsilon,m}(x,z) \Big|_{\varepsilon=0} \right) \\ &= -\frac{d}{d\varepsilon} \big(\Delta_x \boldsymbol{G}_{\varepsilon,m}(x,z) + \nabla_x P_{\varepsilon,m}(x,z) \big) \Big|_{\varepsilon=0} \\ &= 0, \quad m=1,2,3 \end{aligned} \tag{5.2}$$

for all $x \in \Omega$ with $x \neq z$. Similarly to (5.2), we have by Theorem 4.6 that

$$\operatorname{div}_x \delta \boldsymbol{G}_m(x,z) = 0, \quad m=1,2,3 \tag{5.3}$$

for all $x \in \Omega$ with $x \neq z$. Furthermore, since $\delta \boldsymbol{G}'_m(x,z) = \boldsymbol{G}_m(x,z) = 0$ for $x \in \partial\Omega$, it follows from Lemma 4.3 that

$$\delta \tilde{G}^l_m(x,z) = -\nabla_x G^l_m(x,z) \cdot \boldsymbol{S}_1(x,z), \quad l,m = 1,2,3 \tag{5.4}$$

for $x \in \partial\Omega$, which implies Theorem 5.1 for $k=1$. We may handle the case for $k=2$ in the same way as (5.2) and (5.3). Indeed, it holds by Theorem 4.6 and 4.7 that

$$-\Delta_x \delta^2 \boldsymbol{G}_m(x,z) + \nabla_x \delta^2 P_m(x,z) = 0, \quad \operatorname{div}_x \delta^2 \boldsymbol{G}_m(x,z) = 0 \tag{5.5}$$

for $m=1,2,3$ and for all $x\in\Omega$. Furthermore, substituting (4.51) to the right had side of (4.65), we similarly to (5.4) have that

$$\begin{aligned}\delta\tilde{G}^l_m(x,z) &= -2\nabla_x\delta G^l_m(x,z)\cdot\boldsymbol{S}_1(x,z)-\nabla_x^2 G^l_m(x,z)\cdot\boldsymbol{S}_1(x,z)\\ &\quad-\nabla_x G^l_m(x,z)\cdot\boldsymbol{S}_2(x,z)\end{aligned}$$

for $l,m=1,2,3$ and for $x\in\partial\Omega$. Here, we have used the fact that $\delta^2\boldsymbol{G}'_m(x,z)=\delta\boldsymbol{G}'_m(x,z)=\boldsymbol{G}_m(x,z)=0$ for $x\in\partial\Omega$. □

5.2. Proof of Theorem 1.1.

In this subsection, we shall prove Theorem 1.1 by means of Theorem 5.1. Applying the Green integral formula for the Stokes operator $\mathcal{L}$ as in (2.4) on $\Omega\setminus B_\rho(y)$ to the functions

$$\begin{aligned}\boldsymbol{v}&=\delta\tilde{\boldsymbol{G}}_m(x,z), &\pi&=\delta\tilde{P}_m(x,z),\\ \boldsymbol{w}&=\boldsymbol{G}_n(x,y), &\tilde{\pi}&=-P_n(x,y)\end{aligned}$$

for $m,n=1,2,3$, we have by Theorem 5.1

$$\int_{\partial\{\Omega\setminus B_\rho(y)\}}\sum_{i,j=1}^{3}\big\{T^{ij}(\delta\tilde{\boldsymbol{G}}_m,\delta\tilde{P}_m)(x,z)G^i_n(x,y)-T^{ij}(\boldsymbol{G}_n,P_n)(x,y)\delta\tilde{G}^i_m(x,z)\big\}d\sigma_x=0 \tag{5.6}$$

for $m,n=1,2,3$. Here, we have used the fact that $\mathcal{L}(\boldsymbol{G}_n,P_n)(x,y)=0$ in $\Omega\setminus B_\rho(y)$ for $n=1,2,3$. Concerning the limit $\rho\to0$ at the point of y, it follows from Lemma 2.1 and (5.1) that

$$\begin{aligned}&\lim_{\rho\to0}\int_{\partial B_\rho(y)}\sum_{i,j=1}^{3}\big(T^{ij}(\delta\tilde{\boldsymbol{G}}_m,\delta\tilde{P}_m)(x,z)G^i_n(x,y)-T^{ij}(\boldsymbol{G}_n,P_n)(x,y)\delta\tilde{G}^i_m(x,z)\big)\nu^j_x\,d\sigma_x\\ &=\delta G^n_m(y,z),\quad m,n=1,2,3.\end{aligned}\tag{5.7}$$

Therefore, it follows from Theorem 5.1, (5.6) and (5.7) that

$$\delta G^n_m(y,z)=\int_{\partial\Omega}\sum_{i,j,k=1}^{3}T^{ij}(\boldsymbol{G}_n,P_n)(x,y)\frac{\partial G^i_m}{\partial x^k}(x,z)S^k_1(x,z)\nu^j_x\,d\sigma_x \tag{5.8}$$

for $m,n=1,2,3$ and for $y,z\in\Omega$. Furthermore, since $\boldsymbol{G}_m(x,z)=0$ for $x\in\partial\Omega$, it holds that

$$\frac{\partial G^i_m}{\partial x^k}(x,z)=\frac{\partial G^i_m}{\partial\nu_x}(x,z)\nu^k_x,\qquad i,k,m=1,2,3 \tag{5.9}$$

for $x \in \partial\Omega$. Applying (5.9) to the right hand side of (5.8), we have by (A.2) that

$$\begin{aligned}
&\delta G_m^n(y,z) \\
&\quad = \int_{\partial\Omega} \sum_{i,j=1}^{3} \bigg(\frac{\partial G_n^i}{\partial \nu_x}(x,y) \frac{\partial G_m^i}{\partial \nu_x}(x,z) \\
&\qquad\qquad + \operatorname{div}_x \boldsymbol{G}_n(x,y) \operatorname{div}_x \boldsymbol{G}_m(x,z) - P_n(x,y) \operatorname{div}_x \boldsymbol{G}_m(x,z) \bigg) \rho_1(x)\, d\sigma_x \\
&\quad = \int_{\partial\Omega} \sum_{i=1}^{3} \frac{\partial G_n^i}{\partial \nu_x}(x,y) \frac{\partial G_m^i}{\partial \nu_x}(x,z) \rho_1(x)\, d\sigma_x, \quad m,n = 1,2,3 \qquad (5.10)
\end{aligned}$$

for $y, z \in \Omega$. Furthermore, it follows from Theorem 5.1 at the point $x = y$ and (5.10) that

$$\nabla_{y^n} \delta P_m(y,z) = \Delta_y \int_{\partial\Omega} \sum_{i=1}^{3} \frac{\partial G_n^i}{\partial \nu_x}(x,y) \frac{\partial G_m^i}{\partial \nu_x}(x,z) \rho_1(x)\, d\sigma_x \qquad (5.11)$$

for $m, n = 1, 2, 3$. Since $\partial\Omega$ is sufficiently smooth, we have that for each $x \in \partial\Omega$,

$$\Delta_y G_m^n(y,x) = \nabla_{y^n} P_m(y.x), \quad m,n = 1,2,3 \qquad (5.12)$$

for $y \in \Omega$. Since $G_n^i(x,y) = G_i^n(y,x), i, n = 1, 2, 3$ for $(x,y) \in \overline{\Omega} \times \Omega$, we apply (5.12) to the right hand side of (5.11), which completes Theorem 1.1.

5.3. Proof of Theorem 1.2.

Similarly to Theorem 1.1, we have by Theorem 5.1 that

$$\delta^2 G_m^n(y,z) = \int_{\partial\Omega} \sum_{i,j=1}^{3} T^{ij}(\boldsymbol{G}_n, R_n)(x,y) \delta^2 \tilde{G}_m^i(x,z) \nu_x^j \, d\sigma_x, \quad m,n = 1,2,3. \qquad (5.13)$$

Applying (A.2) and (5.9) to (5.13), it holds that

$$\begin{aligned}
\delta^2 G_m^n(y,z) = 2 &\int_{\partial\Omega} \sum_{i,j=1}^{3} \frac{\partial G_n^i}{\partial \nu_x}(x,y) \frac{\partial^2 G_i^j}{\partial \nu_x \partial \nu_w}(w,x) \frac{\partial G_m^j}{\partial \nu_x}(w,z) \rho_1(x)\rho_1(w)\, d\sigma_x d\sigma_w \\
&+ \int_{\partial\Omega} \sum_{i=1}^{3} \frac{\partial G_n^i}{\partial \nu_x}(x,y) \frac{\partial^2 G_m^i}{\partial \nu_x^2}(x,z) \rho_1^2(x)\, d\sigma_x \\
&+ \int_{\partial\Omega} \sum_{i=1}^{3} \frac{\partial G_n^i}{\partial \nu_x}(x,y) \frac{\partial G_m^i}{\partial \nu_x}(x,z) \rho_2(x)\, d\sigma_x, \quad m,n = 1,2,3 \qquad (5.14)
\end{aligned}$$

for $y, z \in \Omega$. Here, we have used the fact that the representation formula for $\delta \boldsymbol{G}_m$ as

$$\delta G_m^j(x,z) = \int_{\partial\Omega} \sum_{i=1}^{3} \frac{\partial G_m^i}{\partial \nu_w}(w,z) \frac{\partial G_j^i}{\partial \nu_w}(w,x) \rho_1(w)\, d\sigma_w, \quad j,m=1,2,3 \tag{5.15}$$

holds for all $x \in \overline{\Omega}$. Furthermore, similarly to the proof of Theorem 1.1, by Theorem 5.1, (5.12) and (5.14), we thus complete Theorem 1.2. $\square$

A. Proof of Lemma 1.1.

We first define the bounded domains Ω_1 and Ω_2 by

$$\Omega_1 := \left\{ x + s\nu_x ; x \in \partial\Omega, -\frac{d_z}{3} < s < 0 \right\},$$
$$\Omega_2 := \left\{ x \in \Omega ; d_x \geq \frac{d_z}{3} \right\},$$

where $d_x := dist(x, \partial\Omega)$, so that $\Omega = \Omega_1 \cup \Omega_2$. For any $\chi(\cdot) \in C^\infty([-d_z/3, 0])$ with $0 \leq \chi(s) \leq 1$ for $s \in [-d_z/3, 0]$ satisfying

$$\chi(s) = \begin{cases} 1, & -\dfrac{d_z}{27} \leq s \leq 0, \\ 0, & -\dfrac{d_z}{3} \leq s \leq -\dfrac{d_z}{9}, \end{cases}$$

we may construct a mapping from $\overline{\Omega}$ to $\overline{\Omega}_\varepsilon$ in such a way that

$$\Phi_\varepsilon(x;z) := \begin{cases} x, & x \in \Omega_2, \\ x + \left(s + \left(\rho_1(x)\varepsilon + \dfrac{1}{2}\rho_2(x)\varepsilon^2 \right) \chi(s) \right) \nu_x, & \\ & x + s\nu_x \in \Omega_1,\ x \in \partial\Omega,\ s \in \left(-\dfrac{d_z}{3}, 0 \right). \end{cases} \tag{A.1}$$

Taking the vector functions $\boldsymbol{S}_1(\cdot, z)$ and $\boldsymbol{S}_2(\cdot, z)$ as

$$\boldsymbol{S}_1(x,z) := \begin{cases} 0, & x \in \Omega_2, \\ \rho_1(x)\chi(s)\nu_x, & x + s\nu_x \in \Omega_1,\ x \in \partial\Omega,\ s \in \left(-\dfrac{d_z}{3}, 0 \right), \end{cases} \tag{A.2}$$
$$\boldsymbol{S}_2(x,z) := \begin{cases} 0, & x \in \Omega_2, \\ \rho_2(x)\chi(s)\nu_x, & x + s\nu_x \in \Omega_1,\ x \in \partial\Omega,\ s \in \left(-\dfrac{d_z}{3}, 0 \right), \end{cases}$$

we immediately know that $\Phi_\varepsilon(\cdot; z) \in C^\infty(\overline{\Omega})$ satisfies the conditions from (1.1) to (1.3).

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