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On divergence of expectations of the Feynman–Kac type with singular potentials

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Abstract. Motivated by the work of Baras–Goldstein (1984), we discuss when expectations of the Feynman–Kac type with singular potentials are divergent. Underlying processes are Brownian motion and α-stable process. In connection with the work of Ishige–Ishiwata (2012) concerned with the heat equation in the half-space with a singular potential on the boundary, we also discuss the same problem in the half-space for the case of Brownian motion.

1. Introduction.

For $N \geq 3$, let V be a nonnegative measurable function on $\mathbb{R}^N$ and consider the following heat equation:

$$\begin{cases} \dfrac{\partial}{\partial t} u = \dfrac{1}{2}\Delta u + Vu & \text{in} \quad (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) \geq (\not\equiv)\, 0 & \text{in} \quad \mathbb{R}^N. \end{cases} \tag{1.1}$$

We assume $u_0 \in C_0(\mathbb{R}^N)$ for simplicity. In [**2**], Baras and Goldstein derived a sufficient condition on the potential function V for the nonexistence of solutions to the initial value problem (1.1) by using the Feynman–Kac formula. In the sequel we let ν be a nonnegative measurable function on $(0,\infty)$ that is nonincreasing near the origin.

Theorem 1.1 ([**2**, Theorem 6.1]). *Suppose that ν satisfies*

$$\liminf_{r \to 0+} r^2 \nu(r) > \frac{\pi^2}{8} N^2 \tag{1.2}$$

and that V satisfies $V(x) \geq \nu(|x|)$ for a.e. $x \in \mathbb{R}^N$. Then for any initial datum u_0, the equation (1.1) *does not have a solution.*

The precise meaning of the equation (1.1) not having a solution will be recalled in Section 2; in view of the Feynman–Kac formula, it may be regarded as the divergence of the expectation

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$$E_x\left[u_0(B_t)\exp\left(\int_0^t V(B_s)\,ds\right)\right] \tag{1.3}$$

for any $x \in \mathbb{R}^N$ and $t > 0$, where $(\{B_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^N})$ is an N-dimensional Brownian motion and E_x denotes the expectation with respect to the probability measure P_x.

One of the objectives of the paper is to show that the condition (1.2) can be relaxed as

$$\liminf_{r\to 0+} r^2\nu(r) > \frac{1}{2}j^2_{(N-2)/2,1}. \tag{1.4}$$

See Theorem 2.1 below. Here and in the sequel, we denote by $j_{\mu,1}$ the first positive zero of the Bessel function J_μ of the first kind with index μ for $\mu > -1$. Baras and Goldstein proved Theorem 1.1 probabilistically, while in [**2**, Theorem 2.2] they showed, employing an analytic approach not dependent on the Feynman–Kac formula, that in the case $V(x) = c/|x|^2$ with c a positive constant, the number $C_N = (1/2)((N-2)/2)^2$ is the threshold for the existence and nonexistence of solutions to the problem; that is, for any initial datum $u_0 \in C_0(\mathbb{R}^N)$, the equation (1.1) has a solution if $c \leq C_N$ and has no solution otherwise. Since $j_{\mu,1}/\mu \to 1$ as $\mu \to \infty$, our condition (1.4) is asymptotically optimal with respect to the dimension N, in the sense that as $N \to \infty$,

$$\frac{1}{2}j^2_{(N-2)/2,1} \times \frac{1}{C_N} \to 1.$$

The critical value C_N also appears as the best constant of Hardy's inequality in $\mathbb{R}^N$ as will be remarked in Section 2. We derive the condition (1.4) by adopting the same reasoning as in the proof of Theorem 1.1 by Baras–Goldstein, with improvement and simplification of estimates given there. The following lemma is a key ingredient in the derivation:

Lemma 1.1. *It holds that for all $T > 0$,*

$$\int_{\{\xi\in\mathbb{R}^N;|\xi|<1\}} P_\xi\Big(\max_{0\leq s\leq T}|B_s| < 1\Big)d\xi \geq \frac{2\varpi_N}{j^2_{(N-2)/2,1}}\exp\left(-\frac{1}{2}j^2_{(N-2)/2,1}T\right),$$

where $\varpi_N = 2\pi^{N/2}/\Gamma(N/2)$ is the surface area of the $(N-1)$-dimensional unit sphere. This estimate is also valid when $N = 1, 2$.

This lemma is proved by using eigenvalue expansions given in [**12**] for hitting distributions of Bessel processes. Note that the constant $(1/2)j^2_{(N-2)/2,1}$ is equal to the smallest eigenvalue of minus one half the Dirichlet Laplacian in the unit ball in $\mathbb{R}^N$.

Another objective of the paper is, with replacing $(1/2)\Delta$ in the equation (1.1) by the fractional Laplacian $-(-\Delta)^{\alpha/2}$ for $0 < \alpha < 2$, to give a sufficient condition on V for the nonexistence of solutions to the equation. To be more precise, we replace in the expectation (1.3) the Brownian motion $(\{B_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^N})$ by an N-dimensional rotationally invariant α-stable process, where we allow the dimension N to be less than

3, and of concern is the transient case $N > \alpha$; we prove that the expectation diverges for any $x \in \mathbb{R}^N$ and $t > 0$ if

$$\liminf_{r \to 0+} r^{\alpha} \nu(r) > j^{\alpha}_{(N-2)/2,1} \tag{1.5}$$

and $V(x) \geq \nu(|x|)$ for a.e. $x \in \mathbb{R}^N$. See Theorem 3.1. The proof is based on the representation of α-stable process as a subordinated Brownian motion and Lemma 1.1 stated above. Similarly to the case of Brownian motion (i.e., the case $\alpha = 2$), the constant $j^{\alpha}_{(N-2)/2,1}$ in (1.5) asymptotically coincides with the best constant of the Hardy-type inequality for the fractional Laplacian as will be seen in Section 3.

Let $N \geq 3$ as in the case of Brownian motion. In [**10**], Ishige and Ishiwata studied the existence and nonexistence of solutions to the heat equation in the half-space $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times (0, \infty)$ with a singular potential on the boundary. In connection with their work, we are also concerned with expectations of the type

$$\boldsymbol{E}_x \left[u_0(B'_t, |B^N_t|) \exp\left\{ \int_0^t V(B'_s, 0)\, dL^N_s \right\} \right] \tag{1.6}$$

for $x = (x', x_N) \in \mathbb{R}^N_+$ and $t > 0$, where under the probability measure $\boldsymbol{P}_x$, $\{B'_t\}_{t \geq 0}$ is an $(N-1)$-dimensional Brownian motion starting from x', $\{B^N_t\}_{t \geq 0}$ is a one-dimensional Brownian motion starting from x_N and independent of B', and $\{L^N_t\}_{t \geq 0}$ is the local time process of B^N at the origin; V is a measurable function on the boundary of $\mathbb{R}^N_+$ and we assume that u_0 is in $C_0(\mathbb{R}^N_+)$, nonnegative and not identically equal to 0. We show in Theorem 4.1 that if

$$\liminf_{r \to 0+} r\nu(r) > j_{(N-3)/2,1} \tag{1.7}$$

and $V(x', 0) \geq \nu(|x'|)$ for a.e. $x' \in \mathbb{R}^{N-1}$, then the expectation (1.6) diverges for any $x \in \mathbb{R}^N_+$ and $t > 0$. We also discuss a connection of the condition (1.7) with the best constant of Kato's inequality in $\mathbb{R}^N_+$.

This paper is organized as follows: In Section 2, we prove Theorem 2.1 which asserts that Theorem 1.1 holds true with the condition (1.2) replaced by (1.4). In Section 3, we deal with the case of fractional Laplacians and see how the condition (1.5) is derived in the proof of Theorem 3.1. Section 4 concerns expectations of the form (1.6), which are seen in Theorem 4.1 to be divergent if the condition (1.7) is fulfilled. Those three Theorems 2.1, 3.1 and 4.1 are proved in a unified manner by using Lemma 1.1. The proof of Lemma 1.1 is given in the appendix, where we also discuss a connection of the expression (1.6) with relativistic 1-stable process in terms of the Laplace transform.

Throughout the paper, for every positive integer $d \in \mathbb{N}$ and every $t > 0$, we denote by $g_d(t, \cdot)$ the Gaussian kernel on $\mathbb{R}^d$:

$$g_d(t, x) := \frac{1}{\sqrt{(2\pi t)^d}} \exp\left(-\frac{|x|^2}{2t} \right), \quad x \in \mathbb{R}^d.$$

For given two sequences $\{a_n\}$, $\{b_n\}$ of real numbers with $a_n \neq 0$ for all n, we write

$$a_n \sim b_n \quad \text{as } n \to \infty$$

to mean that $\lim_{n\to\infty} b_n/a_n = 1$. The symbol ν denotes a nonnegative measurable function on $(0,\infty)$ that is nonincreasing near the origin as mentioned above. Other notation will be introduced as needed.

2. Improvement of the condition (1.2).

In this section we let $N \geq 3$ and V a measurable function on $\mathbb{R}^N$. The purpose of this section is to give a proof of

Theorem 2.1. *Suppose that ν satisfies* (1.4) *and that $V(x) \geq \nu(|x|)$ for a.e. $x \in \mathbb{R}^N$. Then the equation* (1.1) *does not have a solution for any initial datum $u_0 \in C_0(\mathbb{R}^N)$.*

For each $m \in \mathbb{N}$, we set $V_m(x) = \min\{m, V(x)\}$, $x \in \mathbb{R}^N$. Then the equation (1.1) with V replaced by V_m has a unique solution u_m, and by the Feynman–Kac formula, it admits the representation

$$u_m(t,x) = E_x\left[u_0(B_t)\exp\left(\int_0^t V_m(B_s)\,ds\right)\right], \quad t>0,\ x\in\mathbb{R}^N. \tag{2.1}$$

Here $\{B_t\}_{t\geq 0}$ is an N-dimensional Brownian motion starting from x under the probability measure P_x. Following Baras–Goldstein [**2**], we say that the equation (1.1) does not have a solution if

$$\lim_{m\to\infty} u_m(t,x) = \infty \tag{2.2}$$

for all $t > 0$ and $x \in \mathbb{R}^N$. Note that by the representation (2.1) and the monotone convergence theorem, (2.2) is restated as the divergence of the expectation (1.3), to which we are going to give a proof from now on. Fix $t > 0$ and $x \in \mathbb{R}^N$ arbitrarily. Since we assume that u_0 is continuous and $u_0 \geq (\not\equiv)\, 0$, there exist $\epsilon_0 > 0$ and a nonempty open disc $D \subset \mathbb{R}^N$ such that

$$u_0(y) \geq \epsilon_0 \quad \text{for all } y \in D. \tag{2.3}$$

We fix $a \in (0,1/2)$. Following the proof of Theorem 1.1 by [**2**], we set an event A_n for each $n \in \mathbb{N}$ by

$$A_n = \left\{\max_{at\leq s\leq(1-a)t} |B_s| < \frac{1}{n},\ B_t \in D\right\}.$$

We take $n_0 \in \mathbb{N}$ so that ν is nonincreasing on $(0, 1/n_0]$. Then for $n \geq n_0$, by restricting the P_x-expectation in (1.3) to A_n and using (2.3), we see that (1.3) is bounded from below by

$$\epsilon_0 E_x\left[\exp\left\{\int_{at}^{(1-a)t} V(B_s)\,ds\right\};\,A_n\right] \geq \epsilon_0 \exp\left\{\nu\left(\frac{1}{n}\right)\gamma t\right\} P_x(A_n), \tag{2.4}$$

where we set $\gamma = 1 - 2a$. For $P_x(A_n)$, we have the following estimate: set $\mu = (N-2)/2$.

Proposition 2.1. *There exists a positive constant $C \equiv C(x,t,a,D,N)$ independent of n such that*

$$P_x(A_n) \geq C\left(\frac{1}{n}\right)^N \exp\left(-\frac{1}{2} j_{\mu,1}^2 n^2 \gamma t\right) \quad \textit{for all } n \in \mathbb{N}.$$

This estimate also holds true in the case $N = 1, 2$.

Once this proposition is shown, the proof of Theorem 2.1 is immediate:

Proof of Theorem 2.1. By (2.4) and Proposition 2.1, the expectation (1.3) is bounded from below by

$$\epsilon_0 C\left(\frac{1}{n}\right)^N \exp\left\{\left(\nu\left(\frac{1}{n}\right) - \frac{1}{2} j_{\mu,1}^2 n^2\right)\gamma t\right\},$$

which tends to infinity as $n \to \infty$ under the condition (1.4). Therefore the assertion is proved. □

It remains to prove Proposition 2.1.

Proof of Proposition 2.1. By the Markov property of Brownian motion, we have

$$P_x(A_n) = E_x\left[\varphi(B_{at});\, |B_{at}| < \frac{1}{n}\right],$$

where we set

$$\varphi(y) = P_y\left(\max_{0\leq s\leq \gamma t} |B_s| < \frac{1}{n},\ B_{(1-a)t} \in D\right), \quad y \in \mathbb{R}^N.$$

Using the Markov property again, we further have for all $y \in \mathbb{R}^N$,

$$\begin{aligned}\varphi(y) &= E_y\left[P_{B_{\gamma t}}(B_{at} \in D);\, \max_{0\leq s\leq \gamma t} |B_s| < \frac{1}{n}\right]\\ &\geq \inf_{|z|\leq 1/n} P_z(B_{at} \in D) \times P_y\left(\max_{0\leq s\leq \gamma t} |B_s| < \frac{1}{n}\right)\\ &\geq c_1 P_y\left(\max_{0\leq s\leq \gamma t} |B_s| < \frac{1}{n}\right),\end{aligned}$$

where $c_1 := \inf_{|z|\le 1} P_z(B_{at} \in D)$, which is positive since $\mathbb{R}^N \ni z \mapsto P_z(B_{at} \in D)$ is continuous. Therefore we have the estimate

$$\begin{aligned} P_x(A_n) &\ge c_1 E_x\left[P_{B_{at}}\left(\max_{0\le s\le \gamma t}|B_s| < \frac{1}{n}\right); \, |B_{at}| < \frac{1}{n}\right] \\ &= c_1 \int_{|y|<1/n} dy\, g_N(at, y-x) P_y\left(\max_{0\le s\le \gamma t}|B_s| < \frac{1}{n}\right) \\ &= c_1 \left(\frac{1}{n}\right)^N \int_{|\xi|<1} d\xi\, g_N\left(at, \frac{\xi}{n} - x\right) P_{\xi/n}\left(\max_{0\le s\le \gamma t}|B_s| < \frac{1}{n}\right) \\ &\ge c_1 c_2 \left(\frac{1}{n}\right)^N \int_{|\xi|<1} d\xi\, P_\xi\left(\max_{0\le s\le n^2\gamma t}|B_s| < 1\right) \end{aligned}$$

with $c_2 := \inf_{|\xi|\le 1} g_N(at, \xi - x) > 0$ in the last line, where we also used the scaling property of Brownian motion. The proposition follows by taking $T = n^2\gamma t$ in Lemma 1.1. □

We end this section with a remark on Theorem 2.1.

Remark 2.1. (1) For every real $\delta \ge 2$ and $r > 0$, we denote by $(\{R_t\}_{t\ge 0}, P_r^{(\delta)})$ a δ-dimensional Bessel process starting from r. It is known [**20**] that Bessel processes enjoy the following absolute continuity relationship: for every $t > 0$ and every nonnegative measurable functional F on the space $C([0,t];\mathbb{R})$ of real-valued continuous paths over $[0,t]$,

$$E_r^{(\delta)}[F(R_s, s\le t)] = E_r^{(2)}\left[F(R_s, s\le t)\left(\frac{R_t}{r}\right)^{\mu} \exp\left(-\frac{1}{2}\mu^2\int_0^t \frac{ds}{R_s^2}\right)\right],$$

where $\mu = \delta/2 - 1$. Take $\delta = N$ with $N \ge 3$. In the expression (1.3), suppose that u_0 is rotationally invariant, namely $u_0(x) = f(|x|)$ for all $x \in \mathbb{R}^N$ for some nonnegative function f on $(0,\infty)$, and that V is of the form $V(x) = c/|x|^2$ with c a positive constant. Then by the above relationship, (1.3) is written as

$$\begin{aligned} &E_x\left[f(|B_t|)\exp\left(c\int_0^t \frac{ds}{|B_s|^2}\right)\right] \\ &\quad = E_{|x|}^{(2)}\left[f(R_t)\left(\frac{R_t}{|x|}\right)^{N/2-1} \exp\left\{(c - C_N)\int_0^t \frac{ds}{R_s^2}\right\}\right] \end{aligned} \tag{2.5}$$

when $x \ne 0$. Here $C_N = (1/2)((N-2)/2)^2$ as introduced in Section 1. It is clear that if $c \le C_N$ and f is compactly supported, then (2.5) is finite; moreover, by the fact that

$$E_{|x|}^{(2)}\left[\frac{1}{R_s^2}\,\middle|\, R_t = y\right] = \infty \quad \text{for a.e. } y > 0 \tag{2.6}$$

for any $0 < s < t$, the expectation (2.5) is divergent as long as $|\{f > 0\}| > 0$ in the case $c > C_N$. This observation agrees with [**2**, Theorem 2.2]. The fact (2.6) is easily deduced from the explicit representation for the transition density functions of Bessel process (see, e.g., [**18**, Chapter XI]). See also Remark 3.1 (2) in the next section.

(2) Also explicitly known is the following joint distribution [**3**, p. 386, Formula 1.20.8]:

$$P_r^{(\delta)}\left(\int_0^t \frac{ds}{R_s^2} \in dz,\, R_t \in d\xi\right) = \frac{1}{t}\left(\frac{\xi}{r}\right)^{\mu} \xi \exp\left(-\frac{1}{2}\mu^2 z - \frac{r^2+\xi^2}{2t}\right)\theta_{r\xi/t}(z)\, dz d\xi, \quad z, \xi > 0, \tag{2.7}$$

for any $r > 0$ and $t > 0$, where for every $\rho > 0$, θ_ρ is a constant multiple of the density function of the Hartman–Watson distribution on $(0, \infty)$, whose integral representation is given in [**20**]:

$$\theta_\rho(z) = \frac{\rho}{\sqrt{2\pi^3 z}} \int_0^\infty dy \, \exp\left(\frac{\pi^2 - y^2}{2z}\right) \exp(-\rho\cosh y)\sinh y \sin\left(\frac{\pi y}{z}\right), \quad z > 0.$$

By this expression, we have in particular

$$\begin{aligned}\lim_{z\to\infty} \sqrt{2\pi z^3}\theta_\rho(z) &= \rho\int_0^\infty dy\, y\exp(-\rho\cosh y)\sinh y \\ &= \int_0^\infty dy \, \exp(-\rho\cosh y) \\ &= K_0(\rho),\end{aligned}$$

where K_0 is the modified Bessel function of the third kind (Macdonald function) with index 0. From this asymptotics and (2.7), we see that for every $x \in \mathbb{R}^N$ $(x \neq 0)$ and $t > 0$,

$$E_x\left[\exp\left(c\int_0^t \frac{ds}{|B_s|^2}\right)\right] \begin{cases} < \infty & \text{if } c \le C_N, \\ = \infty & \text{if } c > C_N, \end{cases} \tag{2.8}$$

which is consistent with the observation in (1). We remark that since by the scaling property,

$$\begin{aligned}E_0\left[\int_0^t \frac{ds}{|B_s|^2}\right] &= \int_0^t \frac{ds}{s} \times E_0\left[\frac{1}{|B_1|^2}\right] \\ &= \infty\end{aligned}$$

for any $t > 0$, we cannot draw a sufficient condition on c for the finiteness of expectations in (2.8) from Khas'minskii's well-known lemma (see, e.g., [**6**, Lemma 3.7]).

(3) The constant C_N coincides with the best constant of Hardy's inequality:

$$C_N \int_{\mathbb{R}^N} \frac{|\phi(x)|^2}{|x|^2}\, dx \le \int_{\mathbb{R}^N} \phi(x)\Big(-\frac{1}{2}\Delta\phi(x)\Big) dx, \quad \phi \in C_0^\infty(\mathbb{R}^N).$$

The factor $1/2$ in the right-hand side is put in accordance with (1.1). Theorem 2.1 indicates that $(1/2)j^2_{(N-2)/2,1} \ge C_N$; in fact, the following upper and lower estimates are known [**5**], [**15**] as to $j_{\mu,1}$ for $\mu > -1$:

$$\sqrt{(\mu+1)(\mu+5)} \le j_{\mu,1} \le \sqrt{\mu+1}\big(\sqrt{\mu+2}+1\big). \tag{2.9}$$

For more precise bounds, see, e.g., [**17**] (see also [**14**, Chapter 5] for detailed descriptions of Bessel functions). These estimates reveal that the constant $(1/2)j^2_{(N-2)/2,1}$ is asymptotically optimal in the sense that

$$\frac{1}{2} j^2_{(N-2)/2,1} \sim C_N \quad \text{as } N \to \infty.$$

3. The case of fractional Laplacians.

In this section the dimension N is allowed to be less than 3. Fix $0 < \alpha < 2$. For each $x \in \mathbb{R}^N$, we denote by $(\{X_t\}_{t\ge 0}, P_x)$ an N-dimensional rotationally invariant α-stable process starting from x, that is, under the probability measure P_x, the process $X_t - x, t \ge 0$, is a Lévy process whose characteristic function is given by

$$E_x[\exp\{i\xi\cdot(X_t - x)\}] = e^{-t|\xi|^\alpha}, \quad t \ge 0,\ \xi \in \mathbb{R}^N;$$

recall that the process $(\{X_t\}_{t\ge 0}, \{P_x\}_{x\in\mathbb{R}^N})$ is a right-continuous Markov process with infinitesimal generator $-(-\Delta)^{\alpha/2}$. Throughout the section, unless otherwise stated, we assume $N > \alpha$, i.e., we deal with the transient case (see Remark 3.1 (2) as to this condition on N). The same as in the previous section, we let V be a measurable function on $\mathbb{R}^N$ and assume that $u_0 \in C_0(\mathbb{R}^N)$ is nonnegative and not identically equal to 0. The purpose of this section is to prove

Theorem 3.1. *Suppose that ν satisfies the condition* (1.5) *and that $V(x) \ge \nu(|x|)$ for a.e. $x \in \mathbb{R}^N$. Then*

$$E_x\left[u_0(X_t)\exp\left(\int_0^t V(X_s)\, ds\right)\right] = \infty \tag{3.1}$$

for any $x \in \mathbb{R}^N$ and $t > 0$.

To prove the theorem, we first recall that the α-stable process X is identical in law with a subordinated Brownian motion. Let $\{T_t^\alpha\}_{t\ge 0}$ be an $\alpha/2$-stable subordinator under a probability measure P, that is, T^α is a nondecreasing Lévy process characterized by

$$E\big[e^{-\lambda T_t^\alpha}\big] = e^{-t\lambda^{\alpha/2}} \quad \text{for all } \lambda, t \ge 0. \tag{3.2}$$

Let $\{W(t)\}_{t\ge 0}$ be an N-dimensional standard Brownian motion under P, independent of T^α. Then it is known that the following identity in law holds:

$$(\{X_t\}_{t\ge 0}, P_x) \overset{(d)}{=} \big(\{x + W(2T_t^\alpha)\}_{t\ge 0}, P\big); \tag{3.3}$$

for subordinators and stable processes, see [**1**, Chapter 1]. Using this identity and Lemma 1.1, we prove Theorem 3.1. As in the previous section, we fix $a \in (0, 1/2)$ and set $\gamma = 1 - 2a$; we also let a positive ϵ_0 and a nonempty open disc $D \subset \mathbb{R}^N$ be such that u_0 fulfills (2.3).

Proof of Theorem 3.1. For each $n \in \mathbb{N}$, set

$$A_n = \left\{ \max_{at\le s\le (1-a)t} |X_s| < \frac{1}{n},\, X_t \in D \right\}.$$

Then by arguing in the same way as in the proof of Theorem 2.1, the left-hand side of (3.1) is bounded from below by

$$\epsilon_0 \exp\left\{ \nu\left(\frac{1}{n}\right)\gamma t \right\} P_x(A_n) \tag{3.4}$$

for every sufficiently large n. By the Markov property of α-stable process,

$$\begin{aligned} P_x(A_n) &= E_x\left[P_{X_{(1-a)t}}(X_{at} \in D);\, \max_{at\le s\le (1-a)t} |X_s| < \frac{1}{n} \right] \\ &\ge c_1 P_x\left(\max_{at\le s\le (1-a)t} |X_s| < \frac{1}{n} \right), \end{aligned} \tag{3.5}$$

where $c_1 := \inf_{|z|\le 1} P_z(X_{at} \in D)$, which is positive since by (3.3),

$$\begin{aligned} c_1 &= \inf_{|z|\le 1} \int_0^\infty P(T_{at}^\alpha \in ds) P(z + W(2s) \in D) \\ &\ge c_1' \times P(1 \le T_{at}^\alpha \le 2) \times |D| \end{aligned}$$

with

$$c_1' := \inf\{ g_N(2s, y - z);\, 1 \le s \le 2,\, y \in \overline{D},\, |z| \le 1 \} > 0.$$

By the Markov property and (3.3), the probability in the right-hand side of (3.5) is written as

$$E_x\left[P_{X_{at}}\left(\max_{0\le s\le \gamma t} |X_s| < \frac{1}{n} \right);\, |X_{at}| < \frac{1}{n} \right]$$

$$= \int_0^\infty P(T^\alpha_{at} \in ds) \int_{|y|<1/n} dy\, g_N(2s, y-x) P_y\Big(\max_{0\le s\le \gamma t} |X_s| < \frac{1}{n} \Big).$$

Therefore setting a positive constant c_2 by

$$c_2 = P(1 \le T^\alpha_{at} \le 2) \times \inf\{g_N(2s, y-x);\ 1 \le s \le 2,\ |y| \le 1\},$$

we see from (3.5) that

$$P_x(A_n) \ge c_1 c_2 \int_{|y|<1/n} dy\, P_y\Big(\max_{0\le s\le \gamma t} |X_s| < \frac{1}{n} \Big). \tag{3.6}$$

By (3.3), the integrand in the right-hand side of (3.6) is rewritten and estimated as

$$\begin{aligned} &P\Big(\max_{0\le s\le \gamma t} |y + W(2T^\alpha_s)| < \frac{1}{n} \Big) \\ &\quad \ge P\Big(\max_{0\le s\le 2T^\alpha_{\gamma t}} |y + W(s)| < \frac{1}{n} \Big) \\ &\quad = \int_0^\infty P(T^\alpha_{\gamma t} \in d\tau) P\Big(\max_{0\le s\le 2\tau} |y + W(s)| < \frac{1}{n} \Big), \end{aligned}$$

where the inequality is due to the fact that T^α may have a jump. Plugging this estimate into (3.6), we have by Fubini's theorem and the scaling property of Brownian motion,

$$\begin{aligned} P_x(A_n) &\ge c_1 c_2 \Big(\frac{1}{n}\Big)^N \int_0^\infty P(T^\alpha_{\gamma t} \in d\tau) \int_{|\xi|<1} d\xi\, P\Big(\max_{0\le s\le 2n^2\tau} |\xi + W(s)| < 1 \Big) \\ &\ge c_1 c_2 \Big(\frac{1}{n}\Big)^N \times \frac{2\varpi_N}{j^2_{(N-2)/2,1}} \int_0^\infty P(T^\alpha_{\gamma t} \in d\tau) \exp\big(-j^2_{(N-2)/2,1} n^2 \tau \big) \\ &= \frac{2\varpi_N}{j^2_{(N-2)/2,1}} c_1 c_2 \Big(\frac{1}{n}\Big)^N \exp\big(-j^\alpha_{(N-2)/2,1} n^\alpha \gamma t \big), \end{aligned} \tag{3.7}$$

where we used Lemma 1.1 with $T = 2n^2\tau$ for the second line and (3.2) for the third. By (3.7), we see that (3.4) diverges as $n \to \infty$ under the condition (1.5), which ends the proof. □

We conclude this section with a remark on Theorem 3.1.

Remark 3.1. (1) We recall the Hardy-type inequality for the fractional Laplacian $-(-\Delta)^{\alpha/2}$ in $\mathbb{R}^N$ with $N > \alpha$:

$$C_{N,\alpha} \int_{\mathbb{R}^N} \frac{|\phi(x)|^2}{|x|^2}\, dx \le \int_{\mathbb{R}^N} \phi(x)\big((-\Delta)^{\alpha/2}\phi(x)\big) dx, \quad \phi \in C_0^\infty(\mathbb{R}^N),$$

where

$$C_{N,\alpha} := 2^{\alpha} \frac{\Gamma^2((N+\alpha)/4)}{\Gamma^2((N-\alpha)/4)} \tag{3.8}$$

with Γ denoting the gamma function, is the best constant; see, e.g., [**8**], [**9**]. The constant $j^{\alpha}_{(N-2)/2,1}$ in the condition (1.5) asymptotically recovers this optimal $C_{N,\alpha}$:

$$j^{\alpha}_{(N-2)/2,1} \sim C_{N,\alpha} \quad \text{as } N \to \infty.$$

Indeed, the estimates (2.9) on $j_{\mu,1}$ shows the asymptotics

$$j^{\alpha}_{(N-2)/2,1} \sim \left(\frac{N}{2}\right)^{\alpha},$$

which $C_{N,\alpha}$ admits as well by Stirling's formula. In view of (2.8), it is plausible that for every $x \in \mathbb{R}^N$ ($x \neq 0$) and $t > 0$,

$$E_x\left[\exp\left(c\int_0^t \frac{ds}{|X_s|^{\alpha}}\right)\right] \begin{cases} < \infty & \text{if } c \leq C_{N,\alpha}, \\ = \infty & \text{if } c > C_{N,\alpha}. \end{cases}$$

(2) In the case $N \leq \alpha$ it holds that for any $\epsilon > 0$,

$$E_x\left[\frac{1}{|X_s|^{\alpha}}\mathbf{1}_{\{|X_s|<\epsilon\}} \,\middle|\, X_t = y\right] = \infty \quad \text{for a.e. } y \in \mathbb{R}^N \tag{3.9}$$

for every $0 < s < t$. Indeed, by denoting the transition density function of X by $p^{\alpha}_t(x,y)$, $t > 0$, $x, y \in \mathbb{R}^N$, the left-hand side of (3.9) is written, for a.e. y, as

$$\int_{|z|<\epsilon} \frac{dz}{|z|^{\alpha}} \frac{p^{\alpha}_s(x,z)p^{\alpha}_{t-s}(z,y)}{p^{\alpha}_t(x,y)},$$

which is rewritten, by changing to polar coordinates, as

$$\int_{(0,\epsilon)} dr\, r^{N-\alpha-1} \int_{\mathbb{S}^{N-1}} \sigma(dw)\, \frac{p^{\alpha}_s(x,rw)p^{\alpha}_{t-s}(rw,y)}{p^{\alpha}_t(x,y)}$$

with $\mathbb{S}^{N-1}$ and σ being the $(N-1)$-dimensional unit sphere and the surface element on $\mathbb{S}^{N-1}$, respectively. By this expression, we have (3.9) if $N - \alpha - 1 \leq -1$, i.e., $N \leq \alpha$.

4. Heat equation with a singular potential on the boundary.

In this section we let $N \geq 3$. We denote by $(\{\boldsymbol{B}_t\}_{t\geq 0}, \{\boldsymbol{P}_x\}_{x\in\mathbb{R}^N})$ an N-dimensional Brownian motion and by $\boldsymbol{E}_x$ the expectation relative to the probability measure $\boldsymbol{P}_x$. Set $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times (0,\infty)$. For $x = (x', x_N) \in \mathbb{R}^N_+$, we write $\boldsymbol{B}_t = (B'_t, B^N_t)$, $t \geq 0$, where under $\boldsymbol{P}_x$, B' is the $(N-1)$-dimensional Brownian motion starting from $x' \in \mathbb{R}^{N-1}$ that consists of the first $(N-1)$ coordinates of $\boldsymbol{B}$, and B^N is the one-dimensional Brownian motion starting from $x_N > 0$, given as the Nth coordinate of $\boldsymbol{B}$. Note that two processes B' and B^N are independent. We denote by $\{L_t^N\}_{t\ge 0}$ the local time process of B^N at the origin, which is given through Tanaka's formula:

$$|B_t^N| = x_N + \int_0^t \operatorname{sgn} B_s^N \, dB_s^N + L_t^N, \quad t \ge 0 \quad \boldsymbol{P}_x\text{-a.s.}, \tag{4.1}$$

where $\operatorname{sgn} a$ denotes the signature of $a \in \mathbb{R}$. Let V be a measurable function on $\partial\mathbb{R}_+^N = \mathbb{R}^{N-1} \times \{0\}$. The purpose of this section is to prove the following theorem:

Theorem 4.1. *Let $u_0 \in C_0(\mathbb{R}_+^N)$ be nonnegative and not identically equal to* 0*. Suppose that ν satisfies the condition* (1.7) *and that $V(x', 0) \ge \nu(|x'|)$ for a.e. $x' \in \mathbb{R}^{N-1}$. Then the expectation* (1.6) *diverges for any $x \in \mathbb{R}_+^N$ and $t > 0$.*

4.1. Feynman–Kac formula for a boundary value problem.

Before giving a proof of Theorem 4.1, we explain where expectations of the form (1.6) arise from. We consider the following initial-boundary value problem for the heat equation in $\mathbb{R}_+^N$:

$$\begin{cases} \dfrac{\partial}{\partial t} u - \dfrac{1}{2}\Delta u = 0 & \text{in} \quad (0,\infty) \times \mathbb{R}_+^N, \\ \dfrac{\partial}{\partial x_N} u + V u = 0 & \text{on} \quad (0,\infty) \times \partial\mathbb{R}_+^N, \\ u(0,x) = u_0(x) & \text{in} \quad \mathbb{R}_+^N. \end{cases} \tag{4.2}$$

In what follows we often write $u(t,x) = u(t,x',x_N)$ for $x = (x', x_N) \in \mathbb{R}_+^N$.

Proposition 4.1. *Assume that V is bounded and that the continuous function $u : [0,\infty) \times \mathbb{R}_+^N \to [0,\infty)$ is of class $C^{1,2}$ on $(0,\infty) \times \mathbb{R}_+^N$ and satisfies* (4.2)*. Moreover, assume that for each finite $T > 0$, there exist constants $K > 0$ and $0 < \lambda < 1/(2NT)$ such that*

$$\max_{0 \le t \le T} u(t,x) \le K e^{\lambda |x|^2} \quad \textit{for all } x \in \mathbb{R}_+^N. \tag{4.3}$$

Then for every $t \ge 0$ and $x \in \mathbb{R}_+^N$, $u(t,x)$ admits the representation (1.6)*.*

Proof. Let $T > 0$ be fixed and set

$$M_t := e^{A_t} u(T-t, B_t', |B_t^N|), \quad 0 \le t \le T,$$

where

$$A_t := \int_0^t V(B_s', 0)\, dL_s^N.$$

By Itô's formula, it holds that $\boldsymbol{P}_x$-a.s.,

$$\begin{aligned} M_t = u(T,x) &- \int_0^t e^{A_s} \frac{\partial u}{\partial t}(T-s, B_s', |B_s^N|)\, ds + \int_0^t e^{A_s} u(T-s, B_s', |B_s^N|)\, dA_s \\ &+ \int_0^t e^{A_s} \nabla_{x'} u(T-s, B_s', |B_s^N|) \cdot dB_s' + \int_0^t e^{A_s} \frac{\partial u}{\partial x_N}(T-s, B_s', |B_s^N|)\, d|B_s^N| \\ &+ \frac{1}{2}\int_0^t e^{A_s} \Delta u(T-s, B_s', |B_s^N|)\, ds \end{aligned}$$

for all $0 \le t \le T$. As u solves (4.2), the second and sixth terms on the right-hand side are cancelled. Moreover, by Tanaka's formula (4.1) and by the boundary condition in (4.2), the sum of the third and fifth terms is equal to

$$\begin{aligned} &\int_0^t e^{A_s} u(T-s, B_s', 0) V(B_s', 0)\, dL_s^N \\ &\quad + \int_0^t e^{A_s} \frac{\partial u}{\partial x_N}(T-s, B_s', |B_s^N|) \operatorname{sgn} B_s^N \, dB_s^N \\ &\quad + \int_0^t e^{A_s} \frac{\partial u}{\partial x_N}(T-s, B_s', 0)\, dL_s^N \\ &= \int_0^t e^{A_s} \frac{\partial u}{\partial x_N}(T-s, B_s', |B_s^N|) \operatorname{sgn} B_s^N \, dB_s^N. \end{aligned}$$

Here we used the fact that dL_s^N is carried by the set $\{s \ge 0; B_s^N = 0\}$. Therefore we have $\boldsymbol{P}_x$-a.s.,

$$\begin{aligned} M_t = u(T,x) &+ \int_0^t e^{A_s} \nabla_{x'} u(T-s, B_s', |B_s^N|) \cdot dB_s' \\ &+ \int_0^t e^{A_s} \frac{\partial u}{\partial x_N}(T-s, B_s', |B_s^N|) \operatorname{sgn} B_s^N \, dB_s^N \end{aligned}$$

for all $0 \le t \le T$. We follow the notation in the proof of [**11**, Theorem 4.4.2] to define $S_n := \inf\{t > 0; |\boldsymbol{B}_t| \ge n\sqrt{N}\}$, $n \in \mathbb{N}$. By the continuity of $\nabla_{x'} u$ and $\partial u/\partial x_N$, and by the boundedness of V, we deduce that

$$\boldsymbol{E}_x[M_{T\wedge S_n}] = u(T,x)$$

for every $n \in \mathbb{N}$. In fact, as $\{L_t^N\}_{t\ge 0}$ satisfies

$$\boldsymbol{E}_x\left[e^{\kappa L_t^N}\right] < \infty \tag{4.4}$$

for all $\kappa > 0$ and $t \ge 0$ (see (4.11) below), the process $\{M_{t\wedge S_n}\}_{0\le t\le T}$ is a square-integrable martingale, from which we have $\boldsymbol{E}_x[M_{T\wedge S_n}] = \boldsymbol{E}_x[M_0] = u(T,x)$. Since

$$M_T = e^{A_T} u_0(B'_T, |B^N_T|)$$

by definition, it remains to prove

$$\lim_{n\to\infty} \boldsymbol{E}_x[M_{T\wedge S_n}] = \boldsymbol{E}_x[M_T]. \tag{4.5}$$

To this end, we divide $\boldsymbol{E}_x[M_{T\wedge S_n}]$ into the sum

$$\boldsymbol{E}_x\bigl[M_T \mathbf{1}_{\{S_n>T\}}\bigr] + \boldsymbol{E}_x\bigl[M_{S_n}\mathbf{1}_{\{S_n\le T\}}\bigr].$$

Due to the nonnegativity of u_0, the first term converges to $\boldsymbol{E}_x[M_T]$ as $n\to\infty$ by the monotone convergence theorem. To see that the second term converges to 0, we fix an exponent $p>1$ so that $\lambda p < 1/(2NT)$ for λ given in the condition (4.3), and use the Hölder inequality to obtain

$$\begin{aligned}\boldsymbol{E}_x\bigl[M_{S_n}\mathbf{1}_{\{S_n\le T\}}\bigr] &= \boldsymbol{E}_x\bigl[e^{A_{S_n}} u\bigl(T-S_n, B'_{S_n}, |B^N_{S_n}|\bigr)\mathbf{1}_{\{S_n\le T\}}\bigr]\\ &\le \bigl\{\boldsymbol{E}_x\bigl[e^{qA_{S_n}}\mathbf{1}_{\{S_n\le T\}}\bigr]\bigr\}^{1/q} \times \bigl\{K e^{\lambda p N n^2}\boldsymbol{P}_x(S_n\le T)\bigr\}^{1/p},\end{aligned}$$

where q is the conjugate of p. Note that the first factor of the last member is bounded because of (4.4) and the boundedness of V. The second factor converges to 0 as $n\to\infty$ by the same argument as in the proof of [**11**, Theorem 4.4.2] since $\lambda p < 1/(2NT)$. Therefore (4.5) is proved, which ends the proof of the proposition. □

Remark 4.1. For the solvability of (4.2) and a priori estimates on the unique solution, see [**13**, Chapter IV].

In [**10**], Ishige and Ishiwata studied the problem (4.2) in the case of a singular potential given by $V(x) = c/|x|$, $c>0$; employing a PDE approach, they showed the existence of the threshold number C^*_N such that for any nonnegative initial datum u_0 ($\not\equiv 0$) in $C_0(\mathbb{R}^N_+)$, the equation (4.2) has a solution if $c\le C^*_N$ and has no solution otherwise. The constant C^*_N is characterized as the best constant of Kato's inequality in $\mathbb{R}^N_+$:

$$C^*_N \int_{\partial\mathbb{R}^N_+} \frac{|\phi(x)|^2}{|x|}\sigma(dx) \le \int_{\mathbb{R}^N_+} |\nabla\phi(x)|^2\,dx, \quad \phi\in C_0^\infty(\mathbb{R}^N_+),$$

where $\sigma(dx)$ denotes the $(N-1)$-dimensional Lebesgue measure on $\partial\mathbb{R}^N_+$. It is known [**9**], [**7**] that

$$C^*_N = 2\frac{\Gamma^2(N/4)}{\Gamma^2((N-2)/4)}.$$

The constant $j_{(N-3)/2,1}$ in the condition (1.7) of Theorem 4.1 asymptotically coincides with C^*_N; indeed, Stirling's formula and (2.9) entail that

$$\lim_{N\to\infty}\frac{1}{N}C_N^* = \lim_{N\to\infty}\frac{1}{N}j_{(N-3)/2,1} = \frac{1}{2}.$$

In view of the fact (2.8), we conjecture that

$$\boldsymbol{E}_x\left[\exp\left(c\int_0^t \frac{dL_s^N}{|B_s'|}\right)\right]\begin{cases} <\infty & \text{if } c\le C_N^*,\\ =\infty & \text{if } c> C_N^*,\end{cases}$$

for any $x\in\mathbb{R}_+^N$ $(x\neq 0)$ and $t>0$. We also note that C_N^* is equal to $C_{N,\alpha}$ given in (3.8), with $\alpha=1$ and with N replaced by $N-1$. We show a connection of the representation (1.6) with $(N-1)$-dimensional (relativistic) 1-stable process in Subsection A.2.

4.2. Proof of Theorem 4.1.

We proceed to the proof of Theorem 4.1. From now on, we fix $x=(x',x_N)\in\mathbb{R}_+^N$ and $t>0$. As u_0 is continuous and $u_0\ge(\not\equiv)\,0$, we may assume that there exist $\epsilon_0>0$, a nonempty open disc $D\subset\mathbb{R}^{N-1}$ and an interval $J=(l,r)\subset(0,\infty)$ $(l<r)$ such that

$$u_0(y)\ge\epsilon_0\quad\text{for all } y\in D\times J. \tag{4.6}$$

We fix an $a\in(0,1/2)$ and set $\gamma=1-2a$ as in preceding sections. For each $n\in\mathbb{N}$ we set an event A_n by

$$A_n=\left\{\max_{at\le s\le(1-a)t}|B_s'|<\frac{1}{n},\ B_t'\in D\right\}.$$

Let $n_0\in\mathbb{N}$ be such that ν is nonincreasing on $(0,1/n_0]$. Then, for $n\ge n_0$, by restricting the $\boldsymbol{P}_x$-expectation to the event $A_n\cap\{|B_t^N|\in J\}$ and using (4.6), the expectation (1.6) is bounded from below by

$$\begin{aligned}&\epsilon_0\boldsymbol{E}_x\left[\exp\left\{\int_{at}^{(1-a)t}V(B_s',0)\,dL_s^N\right\};\,A_n\cap\{|B_t^N|\in J\}\right]\\ &\qquad\ge\epsilon_0\boldsymbol{P}_x(A_n)\times I_n,\end{aligned}\tag{4.7}$$

where

$$I_n:=\boldsymbol{E}_x\left[\exp\left\{\nu\left(\frac{1}{n}\right)\left(L_{(1-a)t}^N-L_{at}^N\right)\right\};\,|B_t^N|\in J\right].$$

Here we used the independence of B' and B^N. Applying Proposition 2.1 with $N-1$ replacing N, we have the following estimate for $\boldsymbol{P}_x(A_n)$:

$$\boldsymbol{P}_x(A_n)\ge C\left(\frac{1}{n}\right)^{N-1}\exp\left(-\frac{1}{2}j_{(N-3)/2,1}^2n^2\gamma t\right)\quad\text{for all } n\in\mathbb{N}, \tag{4.8}$$

with some positive constant C independent of n. As to I_n, we have

Proposition 4.2. *There exists a positive constant $C' \equiv C'(x_N, t, a, J)$ independent of n such that*

$$I_n \geq C'\nu\left(\frac{1}{n}\right)\exp\left\{\frac{1}{2}\nu^2\left(\frac{1}{n}\right)\gamma t - 2\nu\left(\frac{1}{n}\right)\right\} \quad \text{for all } n \in \mathbb{N}.$$

Combining these two estimates leads to Theorem 4.1:

Proof of Theorem 4.1. By (4.8), Proposition 4.2 and the condition (1.7), the right-hand side of (4.7) diverges as $n \to \infty$, which concludes the theorem. $\square$

It remains to prove Proposition 4.2. For the rest of the section, we denote by the pair $(\{B_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}})$ a one-dimensional Brownian motion and by $\{L_t\}_{t\geq 0}$ the local time process of $\{B_t\}_{t\geq 0}$ at the origin, so that we may write

$$I_n = E_{x_N}\left[\exp\left\{\nu\left(\frac{1}{n}\right)\left(L_{(1-a)t} - L_{at}\right)\right\}; |B_t| \in J\right].$$

Here E_{x_N} denotes the expectation relative to P_{x_N} as above.

Proof of Proposition 4.2. Restricting the P_{x_N}-expectation to the event $\{|B_{at}| < 1\}$ and using the Markov property, we have

$$\begin{aligned} I_n &\geq E_{x_N}[\psi(B_{at}); \, |B_{at}| < 1] \\ &= \int_{-1}^{1} dx \, g_1(at, x - x_N)\psi(x), \end{aligned} \tag{4.9}$$

where we set

$$\psi(x) := E_x\left[\exp\left\{\nu\left(\frac{1}{n}\right)L_{\gamma t}\right\}; \, |B_{(1-a)t}| \in J\right], \quad x \in \mathbb{R}.$$

Restricting the expectation to the event $\{|B_{\gamma t}| < 1\}$ in the definition of ψ, and using the Markov property again, we see that for every $x \in \mathbb{R}$,

$$\begin{aligned} \psi(x) &\geq E_x\left[\exp\left\{\nu\left(\frac{1}{n}\right)L_{\gamma t}\right\}P_{B_{\gamma t}}(|B_{at}| \in J); \, |B_{\gamma t}| < 1\right] \\ &\geq c_1 E_x\left[\exp\left\{\nu\left(\frac{1}{n}\right)L_{\gamma t}\right\}; \, |B_{\gamma t}| < 1\right], \end{aligned} \tag{4.10}$$

where $c_1 := \inf_{|z|\leq 1} P_z(|B_{at}| \in J) > 0$. We recall that for every $x \in \mathbb{R}$ and $s > 0$, the joint distribution of L_s and B_s under P_x is given by

$$P_x(L_s = 0, B_s \in dz) = \frac{1}{\sqrt{2\pi s}}\exp\left\{-\frac{(z-x)^2}{2s}\right\}\left\{1 - \exp\left(-\frac{2xz}{s}\right)\right\}dz$$

for $z \in \{xz \geq 0\}$, and

$$P_x(L_s \in dy, B_s \in dz) = \frac{1}{\sqrt{2\pi s^3}}(y + |z| + |x|) \exp\left\{ -\frac{(y + |z| + |x|)^2}{2s} \right\} dydz \tag{4.11}$$

for $y > 0, z \in \mathbb{R}$; see [**3**, p. 155, Formula 1.3.8] and also Exercise (3.8) in [**18**, Chapter XII]. Using this expression of the joint distribution, we see that the expectation in (4.10) is estimated as, for all $|x| < 1$,

$$\begin{aligned}
E_x&\left[\exp\left\{\nu\left(\frac{1}{n}\right)L_{\gamma t}\right\}; |B_{\gamma t}| < 1\right] \\
&= \int_{-1}^{1} dz\, g_1(\gamma t, z - x) \\
&\quad + \frac{1}{2}\nu\left(\frac{1}{n}\right)\int_{-1}^{1} dz \exp\left\{\frac{1}{2}\nu^2\left(\frac{1}{n}\right)\gamma t - \nu\left(\frac{1}{n}\right)(|z| + |x|)\right\} \\
&\quad \times \mathrm{Erfc}\left(\frac{|z| + |x|}{\sqrt{2\gamma t}} - \nu\left(\frac{1}{n}\right)\sqrt{\frac{\gamma t}{2}}\right) \\
&\geq \nu\left(\frac{1}{n}\right) \exp\left\{\frac{1}{2}\nu^2\left(\frac{1}{n}\right)\gamma t - 2\nu\left(\frac{1}{n}\right)\right\}\mathrm{Erfc}\left(\sqrt{\frac{2}{\gamma t}}\right)
\end{aligned}$$

with

$$\mathrm{Erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-y^2}\, dy, \quad z \in \mathbb{R}.$$

For the first equality in the above estimate, refer also to [**3**, p. 155, Formula 1.3.7]. Combining this estimate with (4.10), we see from (4.9) that

$$I_n \geq c_1 c_2 \nu\left(\frac{1}{n}\right) \exp\left\{\frac{1}{2}\nu^2\left(\frac{1}{n}\right)\gamma t - 2\nu\left(\frac{1}{n}\right)\right\},$$

where

$$c_2 := \mathrm{Erfc}\left(\sqrt{\frac{2}{\gamma t}}\right)\int_{-1}^{1} dx\, g_1(at, x - x_N).$$

The proof is complete. □

Appendix.

A.1. Proof of Lemma 1.1.

In this subsection we give a proof of Lemma 1.1. For every $\mu > -1$, we denote by

$$0 < j_{\mu,1} < \cdots < j_{\mu,k} < \cdots$$

the positive zeros of J_μ. It is known that

$$j_{\mu,k} = \left(k + \frac{1}{2}\mu - \frac{1}{4}\right)\pi + O\left(\frac{1}{k}\right) \quad \text{as } k \to \infty$$

when $\mu \neq \pm 1/2$; see, e.g., [**19**, p. 506]. Recall also $J_{1/2}(z) = \sqrt{2/(\pi z)}\sin z$, $J_{-1/2}(z) = \sqrt{2/(\pi z)}\cos z$. To prove the lemma, we need the following:

Lemma A.1. *For $\mu > -1/2$, it holds that*

$$\lim_{k\to\infty} \sqrt{\frac{\pi j_{\mu,k}}{2}}|J_{\mu+1}(j_{\mu,k})| = 1.$$

Proof. By the asymptotic expansion [**14**, Equation (5.11.6)] of J_μ with $\mu > -1/2$, for any $\epsilon \in (0,1)$, there exists an $L > 0$ such that for all $z > L$, both

$$\left|\sqrt{\frac{\pi z}{2}}J_\mu(z) - \cos\left(z - \frac{1}{2}\mu\pi - \frac{1}{4}\pi\right)\right| < \epsilon$$

and

$$\left|\sqrt{\frac{\pi z}{2}}J_{\mu+1}(z) - \cos\left\{z - \frac{1}{2}(\mu+1)\pi - \frac{1}{4}\pi\right\}\right| < \epsilon$$

hold. Then, for all k such that $j_{\mu,k} > L$, we have

$$\left|\cos\left(j_{\mu,k} - \frac{1}{2}\mu\pi - \frac{1}{4}\pi\right)\right| < \epsilon \quad \text{and} \quad \left|\sqrt{\frac{\pi j_{\mu,k}}{2}}J_{\mu+1}(j_{\mu,k}) - \sin\left(j_{\mu,k} - \frac{1}{2}\mu\pi - \frac{1}{4}\pi\right)\right| < \epsilon.$$

Therefore, for sufficiently large k,

$$\sqrt{1-\epsilon^2} - \epsilon < \sqrt{\frac{\pi j_{\mu,k}}{2}}|J_{\mu+1}(j_{\mu,k})| < 1 + \epsilon,$$

from which the assertion of the lemma follows. □

We are in a position to prove Lemma 1.1. For every positive integer N, set $\mu = (N-2)/2$.

Proof of Lemma 1.1. As it is known [**12**, Section 8], [**3**, p. 373, Formula 1.1.4] that

$$P_\xi\Big(\max_{0\le s\le T}|B_s| < 1\Big) = \frac{2}{|\xi|^\mu}\sum_{k=1}^{\infty}\frac{J_\mu\left(j_{\mu,k}|\xi|\right)}{j_{\mu,k}J_{\mu+1}(j_{\mu,k})}\exp\left(-\frac{1}{2}j_{\mu,k}^2 T\right) \tag{A.1}$$

for all $|\xi| < 1$, we have

$$\int_{|\xi|<1} d\xi\, P_\xi\Big(\max_{0\le s\le T}|B_s| < 1\Big)$$
$$= 2\varpi_N \int_0^1 dr\, r^{\mu+1} \sum_{k=1}^\infty \frac{J_\mu(j_{\mu,k}r)}{j_{\mu,k}J_{\mu+1}(j_{\mu,k})} \exp\Big(-\frac{1}{2}j_{\mu,k}^2 T\Big). \tag{A.2}$$

First we consider the case $\mu \ge 0$ (i.e., $N \ge 2$). By Lemma A.1 and by the fact that J_μ is a bounded function for $\mu \ge 0$, we see that the series in the integrand relative to r converges uniformly on the interval $[0,1]$, hence the termwise integration is possible. By the relation $\{z^{\mu+1}J_{\mu+1}(z)\}' = z^{\mu+1}J_\mu(z)$, we have

$$\int_0^1 r^{\mu+1} J_\mu(j_{\mu,k}r)\, dr = \frac{J_{\mu+1}(j_{\mu,k})}{j_{\mu,k}}.$$

Therefore the right-hand side of (A.2) is equal to

$$2\varpi_N \sum_{k=1}^\infty \frac{1}{j_{\mu,k}^2} \exp\Big(-\frac{1}{2}j_{\mu,k}^2 T\Big),$$

which yields the lemma for $N \ge 2$. By writing down the right-hand side of (A.1) into

$$\frac{4}{\pi}\sum_{k=1}^\infty (-1)^{k-1}\frac{\cos((2k-1)\pi\xi/2)}{2k-1}\exp\Big\{-\frac{\pi^2}{8}(2k-1)^2 T\Big\}$$

for $\mu = -1/2$, the case $N = 1$ is similarly proved. □

A.2. A connection of (1.6) with 1-stable processes.

In this subsection we explore a connection of the Feynman–Kac representation (1.6) with 1-stable processes. For ease of exposition, we start the one-dimensional Brownian motion B^N from the origin, that is, we consider the expression (1.6) on the boundary $\partial\mathbb{R}^N_+$, with which we define the function $u : [0,\infty) \times \mathbb{R}^{N-1} \to [0,\infty)$ by

$$u(t,x) = \boldsymbol{E}_{(x,0)}\Big[u_0(B'_t, |B^N_t|)\exp\Big\{\int_0^t V(B'_s)\, dL^N_s\Big\}\Big]. \tag{A.3}$$

Here and below we regard $V : \partial\mathbb{R}^N_+ \to \mathbb{R}$ as a function on $\mathbb{R}^{N-1}$ and simply write $V(x,0) = V(x)$ for $(x,0) \in \partial\mathbb{R}^N_+$.

For every real-valued continuous function w on $[0,\infty)$ vanishing at the origin, we write

$$\overline{w}_t = \max_{0\le s\le t} w_s, \quad t \ge 0,$$

and denote by $\tau_\cdot(w)$ the right-continuous inverse of $\overline{w}$:

$$\tau_a(w) = \inf\{t > 0; \overline{w}_t > a\}, \quad a \geq 0.$$

Let $\{\beta_t\}_{t\geq 0}$ together with a probability measure P, be a one-dimensional standard Brownian motion and $(\{W(t)\}_{t\geq 0}, \{Q_x\}_{x\in\mathbb{R}^{N-1}})$ an $(N-1)$-dimensional Brownian motion. We assume that these two processes are defined on distinct measurable spaces. By the equivalence in law between L^N and $\overline{\beta}$ due to Lévy, we have the following identity as to the additive functional in (A.3):

$$\int_0^\cdot V(B'_s)\, dL^N_s \overset{(d)}{=} \int_0^\cdot V(W(s))\, d\overline{\beta}_s,$$

where in the right-hand side, the law is with respect to the product probability measure $Q_x \otimes P$. We make the change of variables with $s = \tau_a(\beta)$ to see that for all $t \geq 0$,

$$\int_0^t V(W(s))\, d\overline{\beta}_s = \int_0^{\overline{\beta}_t} V(W(\tau_a(\beta)))da. \tag{A.4}$$

It is well known that the process $\{W(\tau_a(\beta))\}_{a\geq 0}$ has the same law as a rotationally invariant 1-stable process (or Cauchy process) starting from x; indeed, for every $a \geq 0$ and $\xi \in \mathbb{R}^{N-1}$,

$$\begin{aligned} &Q_x \otimes P[\exp\{i\xi \cdot (W(\tau_a(\beta)) - x)\}] \\ &= P\Big[\exp\Big\{-\frac{1}{2}|\xi|^2\tau_a(\beta)\Big\}\Big] \\ &= \exp(-a|\xi|), \end{aligned} \tag{A.5}$$

where the last equality follows from the fact

$$P(\tau_a(\beta) \in ds) = \frac{a}{\sqrt{2\pi s^3}} \exp\Big(-\frac{a^2}{2s}\Big)ds, \quad s > 0,$$

when $a > 0$. In (A.5) and in the remainder of this section, for any probability measure μ, the notation $\mu[\,\cdot\,]$ stands for the expectation with respect to μ.

The connection will be clearer if we take the Laplace transform of (A.3) in variable t. Given a positive real m, let $(\{X_t^{(m)}\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^{N-1}})$ be an $(N-1)$-dimensional relativistic 1-stable process with mass m, that is, under P_x, the process $X^{(m)} - x$ is a Lévy process with characteristic function

$$E_x\big[\exp\{i\xi \cdot (X_t^{(m)} - x)\}\big] = \exp\big\{-t\big(\sqrt{|\xi|^2 + m^2} - m\big)\big\}, \quad t \geq 0, \xi \in \mathbb{R}^{N-1}. \tag{A.6}$$

The infinitesimal generator of $X^{(m)}$ is the relativistic Schrödinger operator $m - \sqrt{-\Delta + m^2}$ (cf. [**4**]). For each $x \in \mathbb{R}^{N-1}$, set

$$u_m(x) := \int_0^\infty dt\, e^{-(1/2)m^2 t} u(t,x).$$

Then the function u_m is related with the process $X^{(m)}$ in the following fashion:

PROPOSITION A.1. *It holds that for all $x \in \mathbb{R}^{N-1}$,*

$$u_m(x) = \int_0^\infty dt\, e^{-mt} E_x\left[f_m(X_t^{(m)}) \exp\left\{ \int_0^t V(X_s^{(m)})\, ds \right\}\right], \tag{A.7}$$

where $f_m : \mathbb{R}^{N-1} \to [0,\infty)$ is given by

$$f_m(x) = \int_0^\infty dt\, e^{-(1/2)m^2 t} f_0(t,x)$$

with

$$f_0(t,x) := \int_{\mathbb{R}^{N-1}} dz\, g_{N-1}(t, z-x) \int_{\mathbb{R}} \frac{dy}{t} |y| g_1(t,y) u_0(z,|y|), \quad t>0,\ x \in \mathbb{R}^{N-1}.$$

For the Brownian motion β introduced above, we denote its local time at level 0 by $\{L_t\}_{t\geq 0}$, to which we associate the measure μ_L on $(0,\infty)$ via

$$\begin{aligned} \mu_L((a,b]) &:= P[L_b] - P[L_a] \\ &= \int_a^b \frac{ds}{\sqrt{2\pi s}} \end{aligned}$$

for all $0 < a < b$. For each $v > 0$ and $y \in \mathbb{R}$, we denote by $P_{v,y}$ the regular version of conditional probability $P(\,\cdot\,|\,\beta_v = y)$, namely under $P_{v,y}$, the process $\{\beta_s\}_{0\leq s\leq v}$ is a Brownian bridge over $[0,v]$ starting from 0 and ending at y. From now on, we fix $x \in \mathbb{R}^{N-1}$. We start the proof of Proposition A.1 with the following lemma:

LEMMA A.2. *It holds that for every $t > 0$,*

$$u(t,x) = \int_0^t \mu_L(dv)\, Q_x \otimes P_{v,0}\left[f_0(t-v, W(v)) \exp\left(\int_0^v V(W(s))\, dL_s \right)\right].$$

In order to prove this lemma, we recall some facts on the path decomposition of Brownian motion at the last zero before a fixed time. For every given $t > 0$, we set

$$\gamma_t = \sup\{s \leq t;\ \beta_s = 0\}.$$

Then it holds that conditionally on $\gamma_t = v$ $(0 < v < t)$:

(i) $\{\beta_s\}_{0\leq s\leq v}$ is identical in law with a Brownian bridge $\{b_s\}_{0\leq s\leq v}$ such that $b_0 = b_v = 0$;

(ii) $\{\beta_{s+v}\}_{0\le s\le t-v}$ is identical in law with

$$\{\boldsymbol{n}M_s\}_{0\le s\le t-v},$$

where $\boldsymbol{n}$ is a Bernoulli distributed random variable with parameter $1/2$ and M is a Brownian meander of duration $t-v$,

with these three elements $b, \boldsymbol{n}, M$ being independent. It is also known that γ_t follows the arcsine law:

$$P(\gamma_t \in dv) = \frac{dv}{\pi\sqrt{v(t-v)}}, \quad v \in (0,t).$$

For descriptions of the decomposition, see [**16**, Section 3.1] and references therein.

Proof of Lemma A.2. By the equivalence in law and by the fact that the local time L does not increase when β is away from 0, we may write

$$\begin{aligned} u(t,x) &= Q_x \otimes P\left[u_0(W(t),|\beta_t|)\exp\left(\int_0^t V(W(s))\,dL_s\right)\right] \\ &= Q_x \otimes P\left[u_0(W(t),|\beta_t|)\exp\left(\int_0^{\gamma_t} V(W(s))\,dL_s\right)\right], \end{aligned}$$

which is rewritten, by using the above facts and the Markov property of W, as

$$\begin{aligned} \int_0^t \frac{dv}{\pi\sqrt{v(t-v)}}\, Q_x\bigg[P_{v,0}\bigg[\exp\left(\int_0^v V(W(s))\,dL_s\right)\bigg] \\ \times Q_{W(v)} \otimes P[u_0(W(t-v),|\boldsymbol{n}M_{t-v}|)]\bigg]. \end{aligned} \tag{A.8}$$

Since

$$P(M_{t-v}\in dy) = \sqrt{\frac{2\pi}{t-v}}\, y g_1(t-v,y)\,dy, \quad y>0,$$

we have in (A.8)

$$\begin{aligned} &Q_{W(v)} \otimes P[u_0(W(t-v),|\boldsymbol{n}M_{t-v}|)] \\ &= \sqrt{\frac{\pi}{2(t-v)}}\int_{\mathbb{R}^{N-1}} dz\, g_{N-1}(t-v, z-W(v)) \int_{-\infty}^{\infty} dy\, |y| g_1(t-v,y) u_0(z,|y|) \\ &= \sqrt{\frac{\pi(t-v)}{2}} f_0(t-v, W(v)) \end{aligned}$$

by the definition of f_0. Plugging this into (A.8), we obtain the claimed representation for $u(t,x)$. □

Using Lemma A.2, we prove Proposition A.1. To this end, we set $\beta_t^{(m)} = \beta_t + mt$, $t \geq 0$, and recall the identity in law:

$$\big(\{X_t^{(m)}\}_{t\geq 0}, P_x\big) \overset{(d)}{=} \big(\{W(\tau_t(\beta^{(m)}))\}_{t\geq 0}, Q_x \otimes P\big), \tag{A.9}$$

which can easily be checked by similar calculation to (A.5), upon using the Cameron–Martin relation; indeed, for every $t \geq 0$ and $\xi \in \mathbb{R}^{N-1}$,

$$\begin{aligned}
&Q_x \otimes P\big[\exp\big\{i\xi \cdot (W(\tau_t(\beta^{(m)})) - x)\big\}\big] \\
&\quad = Q_x \otimes P\left[\exp\left(mt - \frac{1}{2}m^2\tau_t(\beta)\right)\exp\{i\xi \cdot (W(\tau_t(\beta)) - x)\}\right] \\
&\quad = P\left[\exp\left\{mt - \frac{1}{2}(|\xi|^2 + m^2)\tau_t(\beta)\right\}\right] \\
&\quad = \exp\big\{t\big(m - \sqrt{|\xi|^2 + m^2}\big)\big\},
\end{aligned}$$

in agreement with (A.6). We are in a position to prove Proposition A.1.

Proof of Proposition A.1. By (A.9), we rewrite the P_x-expectation in the right-hand side of (A.7) as

$$\begin{aligned}
&Q_x \otimes P\left[f_m(W(\tau_t(\beta^{(m)})))\exp\left\{\int_0^t V(W(\tau_s(\beta^{(m)})))ds\right\}\right] \\
&\quad = Q_x \otimes P\left[\exp\left(mt - \frac{1}{2}m^2\tau_t(\beta)\right)f_m(W(\tau_t(\beta)))\exp\left\{\int_0^t V(W(\tau_s(\beta)))ds\right\}\right],
\end{aligned}$$

where for the second line, we used the Cameron–Martin relation under P. Hence by Fubini's theorem, the right-hand side of (A.7) is equal to

$$Q_x \otimes P\left[\int_0^\infty dt\, \exp\left(-\frac{1}{2}m^2\tau_t(\beta)\right)f_m(W(\tau_t(\beta)))\exp\left\{\int_0^t V(W(\tau_s(\beta)))ds\right\}\right].$$

By changing variables with $t = \overline{\beta}_v$ and noting (A.4), the above expression is further rewritten as

$$\begin{aligned}
&Q_x \otimes P\left[\int_0^\infty d\overline{\beta}_v\, e^{-(1/2)m^2 v}f_m(W(v))\exp\left(\int_0^v V(W(s))\,d\overline{\beta}_s\right)\right] \\
&\quad = Q_x \otimes P\left[\int_0^\infty dL_v\, e^{-(1/2)m^2 v}f_m(W(v))\exp\left(\int_0^v V(W(s))\,dL_s\right)\right] \\
&\quad = \int_0^\infty \mu_L(dv)\, e^{-(1/2)m^2 v}Q_x \otimes P_{v,0}\left[f_m(W(v))\exp\left(\int_0^v V(W(s))\,dL_s\right)\right],
\end{aligned} \tag{A.10}$$

where the first equality is due to Lévy's equivalence, and the second follows from the definition of μ_L and the fact that dL_v is carried by the set $\{v \geq 0;\, \beta_v = 0\}$; for the validity of the latter computation, refer to Exercise (2.29) in [**18**, Chapter VI] (closely related is the theory of Brownian excursions, see Chapter XII of the same reference). By the definition of f_m, we may write

$$f_m(W(v)) = \int_v^\infty dt\, e^{-(1/2)m^2(t-v)} f_0(t-v, W(v)).$$

Inserting this expression into (A.10) and using Fubini's theorem, we see that (A.10) is equal to

$$\int_0^\infty dt\, e^{-(1/2)m^2 t} \int_0^t \mu_L(dv)\, Q_x \otimes P_{v,0}\Big[f_0(t-v, W(v)) \exp\Big(\int_0^v V(W(s))\, dL_s\Big)\Big],$$

which agrees with $u_m(x)$ by Lemma A.2. This ends the proof of the proposition. □

Remark A.1. (1) A point of the above computation is the nonnegativity of u_0, which allows us to use Fubini's theorem without taking the integrability into account, and hence we may take $u_0 \equiv 1$ to obtain for all $x \in \mathbb{R}^{N-1}$,

$$\begin{aligned}\frac{m^2}{2} \int_0^\infty dt\, e^{-(1/2)m^2 t} \boldsymbol{E}_{(x,0)}\Big[\exp\Big\{\int_0^t V(B'_s)\, dL^N_s\Big\}\Big] \\ = m \int_0^\infty dt\, e^{-mt} E_x\Big[\exp\Big\{\int_0^t V(X^{(m)}_s)\, ds\Big\}\Big].\end{aligned}$$

(2) If we take $x = (x', x_N)$ with $x_N > 0$ in (1.6), then its Laplace transform admits the following representation:

$$\begin{aligned}&\int_0^\infty dt\, e^{-(1/2)m^2 t} \boldsymbol{E}_x\Big[u_0(B'_t, |B^N_t|) \exp\Big\{\int_0^t V(B'_s)\, dL^N_s\Big\}\Big] \\ &= m^N \int_{\mathbb{R}^{N-1}} dz \Big\{x_N \Phi_N\Big(m\sqrt{|z-x'|^2 + x_N^2}\Big) u_m(z) \\ &\qquad + \int_0^\infty dr\, u_0(z,r) \int_{|r-x_N|}^{r+x_N} d\eta\, \eta \Phi_N\Big(m\sqrt{|z-x'|^2+\eta^2}\Big)\Big\}, \qquad \text{(A.11)}\end{aligned}$$

where we set

$$\Phi_N(y) = \frac{2}{\sqrt{(2\pi y)^N}} K_{N/2}(y), \quad y > 0,$$

with $K_{N/2}$ the modified Bessel function of the third kind of index $N/2$, and u_m is defined as above and expressed as (A.7). The representation (A.11) is seen by decomposing (1.6) into the sum

$$\boldsymbol{E}_x\left[u_0(B'_t,|B^N_t|)\exp\left\{\int_{\sigma_0^N}^t V(B'_s)\,dL^N_s\right\};\,\sigma_0^N\le t\right]+\boldsymbol{E}_x\left[u_0(B'_t,|B^N_t|);\,\sigma_0^N>t\right],\quad \text{(A.12)}$$

where σ_0^N is the first hitting time of B^N to the origin. By conditioning on σ_0^N and using the (strong) Markov property of Brownian motion, we may see that the first term of (A.12) is rewritten as

$$\int_0^t dv\,\frac{x_N}{\sqrt{2\pi v^3}}\exp\left(-\frac{x_N^2}{2v}\right)\int_{\mathbb{R}^{N-1}}\frac{dz}{\sqrt{(2\pi v)^{N-1}}}\exp\left\{-\frac{|z-x'|^2}{2v}\right\}u(t-v,z)$$

with u the function defined by (A.3). We use the explicit representation of the transition density of one-dimensional Brownian motion absorbed at the origin (see, e.g., [**11**, Problem 2.8.6]) to rewrite the second term of (A.12) as

$$\int_{\mathbb{R}^{N-1}}dz\int_0^\infty dr\,u_0(z,r)\int_{|r-x_N|}^{r+x_N}\frac{d\eta}{\sqrt{(2\pi t)^N}}\,\frac{\eta}{t}\exp\left(-\frac{|z-x'|^2+\eta^2}{2t}\right).$$

Combining these expressions and noting the relation that

$$\int_0^\infty dt\,t^{-N/2-1}\exp\left(-\frac{1}{2}m^2t-\frac{a^2}{2t}\right)=2\left(\frac{m}{a}\right)^{N/2}K_{N/2}(am)$$

for any $a>0$ (cf. [**14**, Equation (5.10.25)]), we obtain (A.11).

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