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Generalized commutators of multilinear Calderón–Zygmund type operators

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Abstract. Let T be an m-linear Calderón–Zygmund operator with kernel K and T^* be the maximal operator of T. Let S be a finite subset of $Z^+ \times \{1, \dots, m\}$ and denote $d\vec{y} = dy_1 \cdots dy_m$. Define the commutator $T_{\vec{b},S}$ of T, and $T^*_{\vec{b},S}$ of T^* by $T_{\vec{b},S}(\vec{f})(x) = \int_{\mathbb{R}^{nm}} \prod_{(i,j)\in S}(b_i(x) - b_i(y_j)) \cdot K(x, y_1, \dots, y_m) \prod_{j=1}^m f_j(y_j) d\vec{y}$ and $T^*_{\vec{b},S}(\vec{f})(x) = \sup_{\delta>0} \big| \int_{\sum_{j=1}^m |x-y_j|^2>\delta^2} \cdot \prod_{(i,j)\in S}(b_i(x) - b_i(y_j))K(x, y_1, \dots, y_m) \prod_{j=1}^m f_j(y_j) d\vec{y}\big|$. These commutators are reflexible enough to generalize several kinds of commutators which already existed. We obtain the weighted strong and endpoint estimates for $T_{\vec{b},S}$ and $T^*_{\vec{b},S}$ with multiple weights. These results are based on an estimate of the Fefferman–Stein sharp maximal function of the commutators, which is proved in a pretty much more organized way than some known proofs. Similar results for the commutators of vector-valued multilinear Calderón–Zygmund operators are also given.

1. Introduction.

Let T be a Calderón–Zygmund operator and b be a locally integrable function on $\mathbb{R}^n$. The commutator of T was defined by Coifman, Rochberg and Weiss [**7**] in 1976, for smooth functions, in the following way

$$T_b f = [b, T]f = bT(f) - T(bf). \tag{1.1}$$

This operator turns out to be bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ [**7**], and not to be of weak type $(1, 1)$ [**21**] when b is a BMO function. Moreover, Pérez [**21**] obtained the following alternative endpoint estimate,

$$|\{y \in \mathbb{R}^n : |[b, T]f(y)| > \lambda\}| \le C_{\|b\|_{BMO}} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\left(1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right) dx$$

for each smooth function f with compact support and all $\lambda > 0$.

A simple proof of the L^p boundedness of T_b given by Coifman, Rochberg and Weiss [**7**] combines the theory of weights and properties of Cauchy integral. This idea was further developed by Alvarez, Bagby, Kurtz and Pérez [**1**]. Although this method is general

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enough to prove the weighted boundedness of commutators of any linear operators, it seems that one at least cannot get the Coifman–Fefferman type estimates as well as the endpoint estimates with this method.

An important idea in [**21**] is to relate the commutator to the sharp maximal operator of Fefferman–Stein (This idea may go back to Strömberg [**13**]). One advantage of such method is that we can deduce both the weighted strong and endpoint estimate (see e.g. [**2**], [**21**]), and by duality, we can have estimates for the commutator with two weights. We refer to [**22**] and [**24**] for more information about this. This idea or technique is also available when we consider generalizations of the commutator. In [**27**], Pérez and Trujillo-González introduced the following more general type commutator

$$T_{\vec{b}}(f)(x) = \int_{\mathbb{R}^n} \left[\prod_{j=1}^{m} (b_j(x) - b_j(y)) \right] K(x,y) f(y) dy \tag{1.2}$$

and obtained both the weighted strong and the weighted endpoint estimates. Obviously, $T_{\vec{b}}$ is a generalization of both the one defined in (1.1) and the higher order commutator studied in [**21**].

It is natural to extend the definition of the commutators to the case of multilinear Calderón–Zygmund operators, for which the theory has been undergoing a rapid progress; see [**11**] for a detailed account and Section 2 for the definition of multilinear C-Z operators. The multiple weights theory was established in [**15**], where the authors considered the following type of commutators

$$T_{\vec{b}}(\vec{f})(x) = \sum_{i=1}^{m} \big(b_i T(\vec{f})(x) - T(f_1, \dots, b_i f_i, \dots, f_m)(x) \big), \tag{1.3}$$

and proved its weighted strong and weak type end-point estimates. One key theme in [**15**] was to obtain an improved estimate of sharp maximal function of $T(\vec{f})$ and $T_{\vec{b}}(\vec{f})$, which was controlled by a new maximal operator (see Section 2.1). And by this new maximal operator one can deduce better estimates.

More recently, in [**25**], iterated commutator $T_{\prod b}$ was introduced as below

$$\begin{aligned} T_{\prod b}(\vec{f})(x) &= [b_1, [b_2, \dots, [b_{m-1}, [b_m, T]_m]_{m-1} \cdots]_2]_1 (\vec{f})(x) \\ &= \int_{\mathbb{R}^{nm}} \prod_{j=1}^{m} (b_j(x) - b_j(y_j)) K(x, y_1, \dots, y_m) \prod_{j=1}^{m} f_j(y_j) d\vec{y}, \end{aligned} \tag{1.4}$$

where $d\vec{y} = dy_1 \cdots dy_m$. This operator enjoys similar weighted strong and weak type endpoint estimats to those of the operators mentioned above.

All the commutators mentioned above have maximal partners. Segovia and Torrea [**32**] first studied the weighted strong boundedness of T_b^* defined by

$$T_b^*(f)(x) = \sup_{\delta > 0} \left| \int_{|x-y|>\delta} (b(x) - b(y)) K(x,y) f(y) dy \right|, \tag{1.5}$$

which was the maximal operator of T_b in (1.1). The weak type endpoint estimate of T_b^* was given by Alphonse [**17**]. Zhang [**37**] then considered the maximal operator of $T_{\vec{b}}$ in (1.2). Recently, in [**36**], Xue considered the following iterated type commutator of multilinear operator T^* which was defined by:

$$T^*_{\prod b}(\vec{f})(x) = \sup_{\delta>0} \left| \int_{(\sum_{j=1}^m |x-y_j|^2)^{1/2}>\delta} \prod_{j=1}^m (b_j(x)-b_j(y_j))K(x,y_1,\dots,y_m) \prod_{j=1}^m f_j(y_j)d\vec{y} \right|,$$

and proved both the strong and weak endpoint estimates.

In this paper, we want to find a general frame that contains the results of all the above operators. Note that the operators, for instance, defined in (1.2), (1.3) and (1.4) are all independent of each other. We are lead to the following definition.

Definition 1.1. Let T be an m-linear Calderón–Zygmund operator with kernel K. Let S be a finite subset of $Z^+ \times \{1,\dots,m\}$. The commutator $T_{\vec{b},S}$ of T and its maximal operator $T^*_{\vec{b},S}$ are defined by

$$T_{\vec{b},S}(\vec{f})(x) = \int_{\mathbb{R}^{nm}} \prod_{(i,j)\in S} (b_i(x)-b_i(y_j))K(x,y_1,\dots,y_m) \prod_{j=1}^m f_j(y_j)d\vec{y}, \tag{1.6}$$

and

$$T^*_{\vec{b},S}(\vec{f})(x) = \sup_{\delta>0} \left| \int_{(\sum_{j=1}^m |x-y_j|^2)^{1/2}>\delta} \prod_{(i,j)\in S} (b_i(x)-b_i(y_j))K(x,y_1,\dots,y_m) \prod_{j=1}^m f_j(y_j)d\vec{y} \right| \tag{1.7}$$

for all $f_j \in \mathcal{S}(\mathbb{R}^n)$, $j=1,\dots,m$, and all $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$. If $S=\emptyset$, we simply denote $T_{\vec{b},\emptyset} = T$ and $T^*_{\vec{b},\emptyset} = T^*$.

Remark 1.2. The commutators defined in (1.6) and (1.7) indeed contain the operators defined as in (1.2), (1.3) and (1.4). Moreover, (1.6) and (1.7) in fact contain more than that. Take, for example, $S = \{(i,j) : i \in \{1,\dots,l\}, j \in \{1,\dots,m\}\}$, then $T_{\vec{b},S}(\vec{f})(x) = \int_{\mathbb{R}^{nm}} \prod_{i=1}^l \prod_{j=1}^m (b_i(x)-b_i(y_j))K(x,y_1,\dots,y_m)d\vec{y}$. In this case, if take $m=1$, then this operator coincides with the commutator defined in (1.2). If we take $S = \{(j,j) : j \in \{1,\dots,m\}\}$, then (1.6) coincides with (1.4).

It turns out, not very surprisingly, that both $T_{\vec{b},S}$ and $T^*_{\vec{b},S}$ satisfy similar boundedness as all their earlier brothers such as (1.1) and (1.2). We will only state and prove the results for $T^*_{\vec{b},S}$. The same reasoning applies to the case of $T_{\vec{b},S}$, with small and straightforward modifications.

For a finite subset S of $Z^+ \times \{1,\dots,m\}$, $|S|$ will denote the cardinal number of S. For a map R from S to the set of positive numbers that are bigger that one. Denote $r_{ij} = R(i,j)$, $1/r_j = \sum_{i:(i,j)\in S}(1/r_{ij})$ and $\vec{R}_S = (1/r_1,\dots,1/r_m)$.

The first result of this paper is:

THEOREM 1.1. *Let $0 < p < \infty$, $\omega \in A_\infty$. Let $T^*_{\vec{b},S}$ be the commutator defined in (1.7), and R be such a map defined above, $b_i \in Osc_{expL^{r_{ij}}}$. Then there exists a constant C such that*

$$\begin{aligned}&\big\|T^*_{\vec{b},S}(\vec{f})\big\|_{L^p(\omega)}\\&\quad\le C\big(\|T^*\|[\omega]^{|S|}_{A_\infty}+\|K\|_{|S|+1}\big)[\omega]_{A_\infty}\prod_{(i,j)\in S}\|b_i\|_{Osc_{expL^{r_{ij}}}}\big\|\mathcal{M}_{L(logL)^{\vec{R}_S}}(\vec{f})\big\|_{L^p(\omega)}\end{aligned}$$

for any bounded and compact supported functions f_j $(j = 1, \dots, m)$.

For the definition of $\|T^*\|$, $\|K\|_k$ with $k \in N^+$, $[\omega]_{A_\infty}$, $\|b_i\|_{Osc_{expL^{r_{ij}}}}$, $\mathcal{M}_{L(logL)^{\vec{R}_S}}(\vec{f})$, $A_{\vec{p}}$ and $\|T^*\|_k$ in the following corollary, see Section 2.

Theorem 1.1 implies the boundedness of $T^*_{\vec{b},S}$ on $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ with $\vec{\omega} = (w_1, \dots, w_m) \in A_{\vec{p}}$.

COROLLARY 1.2. *Let $\vec{\omega} \in A_{\vec{p}}$ with $1/p = \sum_{j=1}^m (1/p_j)$, $1 < p_j < \infty$, $b_i \in Osc_{expL^{r_{ij}}}$, $r_{ij} \ge 1$ $(j = 1, \dots, m)$. Then for any $f_j \in L^p(\omega_j)$, there exists a constant C depending on $\vec{\omega}$ such that*

$$\big\|T^*_{\vec{b},S}(\vec{f})\big\|_{L^p(v_{\vec{\omega}})} \le C\|T^*\|_{|S|+1}\prod_{(i,j)\in S}\|b_i\|_{Osc_{expL^{r_{ij}}}}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\omega_j)}.$$

We obtain the following weak type estimate of $T^*_{\vec{b},S}$.

THEOREM 1.3. *Suppose $p > 0$ and $\omega \in A_\infty$. Let $\varphi : (0, \infty) \to (0, \infty)$ be doubling and there exists some constant C_1 such that for any $t \in (0, \infty)$, $\varphi(t) < C_1 t$. Suppose that $b_i \in Osc_{expL^{r_{ij}}}$, $r_{ij} \ge 1$ $(j = 1, \dots, m)$. Then, for any bounded and compact supported functions f_j, there exists a constant $C > 0$ depending on the A_∞ constant of ω, such that*

$$\begin{aligned}&\sup_{\lambda>0}\varphi(\lambda)\omega\big\{x\in\mathbb{R}^n : |T^*_{\vec{b},S}(\vec{f})(x)| > \lambda^m\big\}\\&\quad\le C\sup_{\lambda>0}\varphi(\lambda)\omega\bigg\{x\in\mathbb{R}^n : \mathcal{M}_{L(logL)^{\vec{R}_S}}(\vec{f})(x) > \frac{\lambda^m}{\|T^*\|_{\|S\|+1}\prod_{(i,j)\in S}\|b_i\|_{Osc_{expL^{r_{ij}}}}}\bigg\}.\end{aligned}\tag{1.8}$$

Based on Theorem 1.3, we can get the following weighted weak type estimate.

THEOREM 1.4. *Let $(\omega_1, \dots, \omega_m) \in A_{(1,\dots,1)}$, $b_i \in Osc_{expL^{r_{ij}}}$, $r_{ij} \ge 1$ and $1/r_j = \sum_{(i,j)\in S}(1/r_{ij})$ $(j = 1, \dots, m)$. Denote $\Phi_j(t) = t(1 + \log^+ t)^{1/r_j}$ and $\Phi(t) = t(1 + \log^+ t)^{\sum_{j=1}^m (1/r_j)}$. Then, for any bounded and compact supported functions f_j, there exists a constant C depending on $\vec{\omega}$ and T such that for any $t > 0$*

$$v_{\vec{\omega}}\{x \in \mathbb{R}^n : |T^*_{\vec{b},S}(\vec{f})(x)| > t^m\}$$
$$\leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} \Phi\left(\frac{|f_j(x)| \prod_{i:(i,j)\in S} \|b_i\|_{Osc_{expL^{r_{ij}}}}}{t} \right) \omega_j(x)dx \right)^{1/m}. \quad (1.9)$$

Moreover, when $r_{ij} \equiv 1$ for any $(i,j) \in S$, this result is sharp in the sense that it does not hold for $\Phi(t) = t(1+\log^+ t)^\alpha$ with $\alpha < \sum_{j=1}^m (1/r_j)$.

The proof of Theorem 1.4 will also be based on the weighted weak type estimate of $\mathcal{M}_{L(logL)^{\vec{\alpha}}}(\vec{f})$ as follows.

THEOREM 1.5. *Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m)$ and $(\omega_1, \ldots, \omega_m) \in A_{(1,\ldots,1)}$. Denote $\Phi(t) = t(1+\log^+ t)^{\sum_{j=1}^m \alpha_j}$. Then, for any $t > 0$, there exists a constant $C > 0$ such that*

$$v_\omega\{x \in \mathbb{R}^n : \mathcal{M}_{L(logL)^{\vec{\alpha}}}(\vec{f})(x) > t^m\} \leq C[\vec{\omega}]_{A_{\vec{1}}}^{1/m} \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} \Phi\left(\frac{|f_j(x)|}{t} \right) \omega_j(x)dx \right)^{1/m}.$$

We can extend almost all the above results to the commutators of vector-valued (l^r valued) operators; see Section 5 for the statements.

The article is organized as follows. In Section 2, some preliminaries will be given. A key lemma is given and proved in Section 3. Section 4 will be devoted to the proofs of the theorems stated above. We will state and prove the weak type endpoint estimates for the commutators of vector-valued operators in Section 5.

Throughout this paper except in Section 2, the constant C may depend on the n, m, p, $|S|$ but will not depend on T, b_i, ω, or $\vec{f}$ unless it is indicated explicitly, and may vary from line to line.

2. Some preliminaries.

2.1. Multilinear C-Z operators and multiple weights.

Multilinear Calderón–Zygmund operators was originated in the work of Coifman and Mayer [**4**], [**5**], [**6**]. See also [**14**]. We follow in this paper, with minor modification, the definition introduced in [**11**] by Grafakos and Torres.

DEFINITION 2.1. Let T be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values in the space of tempered distributions,

$$T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n).$$

We say that T is an m-linear Calderón–Zygmund operator if it can be extended to a bounded multilinear operator from $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$, and if there exists a function K, defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) \prod_{j=1}^{m} f_j(y_j) d\vec{y}$$

for all $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$; and there exists, for some $\varepsilon > 0$, a constant A_ε such that

$$|K(x, y_1, \ldots, y_m) - K(x', y_1, \ldots, y_m)| \le \frac{A_\varepsilon |x - x'|^\varepsilon}{\left(\sum_{j=1}^m |x - y_j|\right)^{mn+\varepsilon}} \tag{2.1}$$

whenever $|x - x'| \le (1/2) \max_{1 \le j \le m} |x - y_j|$, and

$$|K(y_0, y_1, \ldots, y_m)| \le \frac{A_\varepsilon}{\left(\sum_{k,l=0}^m |y_k - y_l|\right)^{mn}}. \tag{2.2}$$

The maximal operator of T is defined by

$$T^*(\vec{f})(x) = \sup_{\delta>0} \left| \int_{(\sum_{j=1}^m |x-y_j|^2)^{1/2} > \delta} K(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j(y_j) d\vec{y} \right|.$$

We define, for $k \in N^+$, $\|K\|_k = \inf\{A_\varepsilon/\varepsilon^k : \text{(2.1) and (2.2) hold}\}$ and $\|T^*\|_k = \|T^*\|_{L^1 \times \cdots \times L^1 \to L^{1/m,\infty}} + \inf\{A_\varepsilon/\varepsilon^k : \text{(2.1) and (2.2) hold}\}$. Denote $\|T^*\|_1$ by $\|T^*\|$.

The operator T satisfies the weighted boundedness given by Lerner Ombrosi, Pérez, Torres and Trujillo-González [**15**], where multiple weights was first introduced in the following way.

DEFINITION 2.2 ([**15**] (Multilinear $A_{\vec{p}}$ condition)). Let $1 \le p_1, \ldots, p_m < \infty$. Given $\vec{\omega} = (\omega_1, \ldots, \omega_m)$, set $\nu_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{p/p_j}$. We say that $\vec{\omega}$ satisfies the $A_{\vec{p}}$ condition if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \prod_{j=1}^m {\omega_j}^{p/p_i} \right)^{1/p} \prod_{j=1}^m \left(\frac{1}{|Q|} \int_Q {\omega_j}^{1-p'_j} \right)^{1/p'_j} < \infty.$$

When $p_j = 1$, $\left((1/|Q|) \int_Q {\omega_j}^{1-p'_j}\right)^{1/p'_i}$ is understood as $(\inf_Q \omega_j)^{-1}$.

When $m = 1$, this coincides with the classical A_p weight [**18**]. The $A_{\vec{p}}$ condition turns out to be able to characterize the strong-type inequalities for a more refined multilinear maximal function $\mathcal{M}$ with multiple weights defined by

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{j=1}^m \frac{1}{|Q|} \int_Q |f_j(y_j)| \, dy_j.$$

Let T be an m-linear Calderón–Zygmund operator, $1/p = 1/p_1 + \cdots + 1/p_m$, and $\vec{\omega}$ satisfy the $A_{\vec{p}}$ condition. It was shown in [**15**], as an important application of the boundedness of $\mathcal{M}$, that T is bounded from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^p(\nu_{\vec{\omega}})$ when $1 < p_j < \infty$ and to $L^{p,\infty}(\nu_{\vec{\omega}})$ when $1 \le p_j < \infty$.

2.2. Sharp maximal functions.

M will always denote the Hardy-Littlewood maximal operator throughout the paper. For $\delta > 0$, M_δ is defined by

$$M_\delta(f)(x) = (M_\delta(|f|^\delta)(x))^{1/\delta} = \left(\sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|^\delta dy\right)^{1/\delta}.$$

Sharp maximal function $M^\#$ of Fefferman and Stein is defined by

$$M^\#(f)(x) = \sup_{x\in Q} \inf_{c\in\mathbb{R}^1} \frac{1}{|Q|}\int_Q |f(y)-c|dy.$$

This gives directly that

$$M^\#(f)(x) \sim \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)-f_Q|dy,$$

where f_Q denotes the average of f on Q, and we write $M_\delta^\#(f)(x) = (M^\#(|f|^\delta)(x))^{1/\delta}$.

We will need the following lemma.

Lemma 2.1 ([**20**]). (a). *Let $0<p<\infty$, $0<\delta<1$, and let $\omega\in A_\infty$. Then there exists a constant C_n only depending on n*

$$\|M_\delta f\|_{L^p(\omega)} \le C_n \max\{1,p\}[\omega]_{A_\infty} \left\|M_\delta^\#(f)\right\|_{L^p(\omega)}$$

for any function f such that the left hand side is finite.
(b). *Let $\varphi:(0,\infty)\to(0,\infty)$ doubling. Then, there exists a constant C depending upon the A_∞ constant of ω and the doubling condition of φ such that*

$$\sup_{\lambda>0}\varphi(\lambda)\omega(y\in\mathbb{R}^n : M_\delta f(y)>\lambda) \le C\sup_{\lambda>0}\varphi(\lambda)\omega\big(y\in\mathbb{R}^n : M_\delta^\# f(y)>\lambda\big)$$

for any function such that the left hand side is finite.

Remark 2.3. For the strong boundedness, it is the dyadic version that was proved in [**20**], which was based on the estimate by Ortiz-Caraballo [**19**]

$$M_\delta^{\#,d}(M_\varepsilon^d(f))(x) \le CM_\varepsilon^{\#,d}(f)(x) \quad \text{if} \quad 0<\delta<\varepsilon<1.$$

However, the non-dyadic version of the above inequality follows from the same line. $[\omega]_{A_\infty}$ in the above lemma was defined by Wilson [**35**] as $[\omega]_\infty = \sup_Q (1/\omega(Q))\int_Q M(\omega\chi_Q)$. It is shown in [**12**] that $[\omega]_{A_\infty}$ is smaller, and sometimes much smaller than $\|\omega\|_{A_\infty}$ defined by $\|\omega\|_{A_\infty} = \sup_Q \big((1/|Q|)\int_Q \omega\big)\exp\big((1/|Q|)\int_Q \log\omega^{-1}\big)$.

2.3. Orlicz spaces and a lemma.

We present here only what we will need in this paper, see [**29**] for a detailed account. Define a function $\Phi:[0,\infty)\to[0,\infty)$ to be a Young function if it is convex, increasing, and $\Phi(0)=0$. The Φ-norm of a function f over a cube Q is defined by

$$\|f\|_{\Phi,Q} = \inf\left\{\lambda : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\}.$$

The maximal operator M_Φ is defined by $M_\Phi(f)(x) = \sup_{Q\ni x}\|f\|_{\Phi,Q}$, where the supremum is take over all cubes containing x. When $\Phi_t = t\log^s(e+t)$ $(s>0)$, we denote $\|f\|_{\Phi,Q} = \|f\|_{L(\log L)^s,Q}$ and $M_\Phi = M_{L(\log L)^s}$. If $\Phi_t = e^{t^r}-1$, we denote $\|f\|_{\Phi,Q} = \|f\|_{\exp L^r,Q}$ and $M_\Phi = M_{\exp L^r}$. For a Young function Φ, the oscillation $Osc_\Phi(f,Q)$ of a function f is defined by $Osc_\Phi(f,Q) = \|f - f_Q\|_{\Phi,Q}$. Also, we define

$$\|f\|_{Osc_\Phi} = \sup_Q\{Osc_\Phi(f,Q)\},$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$. For $r\ge 1$, we define the space $Osc_{\exp L^r}$ by

$$Osc_{\exp L^r} = \{f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_{Osc_{\exp L^r}} < \infty\},$$

where

$$\|f\|_{Osc_{\exp L^r}} = \sup_Q \|f - f_Q\|_{\exp L^r,Q} = \sup_Q \|f - f_Q\|_{e^{t^r}-1,Q},$$

and the supremum is taken over all cubes in $\mathbb{R}^n$. It is easy to see that $Osc_{\exp L^1} = BMO(\mathbb{R}^n)$ and we know that for $r>1$, the space $Osc_{\exp L^r}$ is properly contained in $BMO(\mathbb{R}^n)$ with the norm $\|b\|_* \le C\|b\|_{Osc_{\exp L^r}}$.

It was shown in [**27**] that the following generalized Hölder's inequality holds,

$$\frac{1}{|Q|}\int_Q |f_1\cdots f_m g|dx \le C_m\prod_{j=1}^m \|f_j\|_{expL^{r_j},Q}\|g\|_{L(\log L)^{1/r},Q}, \tag{2.3}$$

where $r_1,\dots,r_m \ge 1$ and $1/r = \sum_{j=1}^m(1/r_j)$.

Note that $\Phi(t) = t(1+\log^+ t)^{1/\alpha}$ is submultiplicative, which meams that there exists a constant $C>0$ such that for any $s,t>0$, $\Phi(st) \le C\Phi(s)\Phi(t)$.

Given a vector $\vec{\alpha} = (\alpha_1,\dots,\alpha_m)$, define

$$\mathcal{M}_{L(\log L)^{\vec{\alpha}}}(\vec{f})(x) = \sup_{Q\ni x}\prod_{j=1}^m \|f_j\|_{L(\log L)^{\alpha_j},Q}.$$

We will need the following analogue in the case of vector-valued operators. For $\vec{q} = (q_1,\dots,q_m)$, $\mathcal{M}_{L(logL)^{\vec{\alpha}}}(\vec{f_{\vec{q}}})$ is defined by

$$\mathcal{M}_{L(logL)^{\vec{\alpha}}}(\vec{f_{\vec{q}}})(x) = \sup_{Q\ni x}\prod_{j=1}^m \|\{f_j\}_{l^{q_j}}\|_{L(\log L)^{\alpha_j},Q}.$$

When $\vec{\alpha} = (0,\dots,0)$, we write $\mathcal{M}_{L(\log L)^{\vec{0}}}(\vec{f_{\vec{q}}})(x) = \mathcal{M}(\vec{f_{\vec{q}}})(x)$.

The following Lemma 2.2 plays an important role in the proof of Theorem 1.5. For a general finite set R with $|R| = r$, $r \in R^+$, we denote the set of all bijective mappings from $\{1, 2, \dots, r\}$ onto R by B_R.

LEMMA 2.2. *Let Φ_j be a submultiplicative Young function for any $j \in \{1, \dots, m\}$ and E be any measurable set. For any $\beta \in B_{1,\dots,m}$, define*

$$\Phi_{\beta,j}(x) = \Phi_{\beta(1)} \circ \cdots \circ \Phi_{\beta(j)}(x).$$

Then there is a constant C such that whenever

$$1 < \prod_{j=1}^{m} \|f_j\|_{\Phi_j,E} \tag{2.4}$$

holds, we have

$$\prod_{j=1}^{m} \|f_j\|_{\Phi_j,E} \le C \max\left\{ \prod_{j=1}^{m} \frac{1}{|E|} \int_E \Phi_{\beta,j}(|f_{\beta(j)}(x)|)dx : \beta \in B_{\{1,\dots,m\}} \right\}. \tag{2.5}$$

PROOF. Consider first the case $m = 1$. We need to show that if Φ is a Young function, then for any measurable set E,

$$\|f\|_{\Phi,E} > 1 \implies \|f\|_{\Phi,E} \le \frac{1}{E} \int_E \Phi(|f(x)|)dx. \tag{2.6}$$

This is exactly Lemma 6.1 in [**9**].

For $m \ge 2$, we prove it by induction. Assume that for some $j \in \{1, \dots, m\}$ the result holds for $m - 1$. Given functions that (2.4) holds, assume

$$\|f_{j_0}\|_{\Phi_{j_0},E} = \min\{\|f_j\|_{\Phi_j,E} : j = 1, \dots, m\}.$$

Then

$$\prod_{j=1,\, j\ne j_0}^{m} \|f_j\|_{\Phi_j,E} > 1.$$

Therefore, by (2.6) and submultiplicativity, we have

$$\begin{aligned} 1 < \prod_{j=1}^{m} \|f_j\|_{\Phi_j,E} = \left\| f_{j_0} \prod_{j=1,\, j\ne j_0}^{m} \|f_j\|_{\Phi_j,E} \right\|_{\Phi_{j0},E} \le \frac{1}{E} \int_E \Phi_{j_0}\left(f_{j_0}(x) \prod_{j=1,\, j\ne j_0}^{m} \|f_j\|_{\Phi_j,E} \right) dx \\ \le \frac{C}{|E|} \int_E \Phi_{j_0}(|f_{j_0}(x)|)dx \Phi_{j_0}\left(\prod_{j=1,\, j\ne j_0}^{m} \|f_j\|_{\Phi_j,E} \right). \end{aligned}$$

By assumption,

$$\prod_{j=1,\, j\neq j_0}^{m} \|f_j\|_{\Phi_j,E} \leq C \max\left\{ \prod_{j=1}^{m-1} \frac{1}{|Q_k|} \int_E \Phi_{\beta,j}(|f_{\beta(j)}(x)|)dx : \beta \in B_{\{1,\dots,m\}/j_0} \right\}.$$

Thus,

$$\begin{aligned}
1 &< \prod_{j=1}^{m} \|f_j\|_{\Phi_j,E} \\
&\leq \frac{C}{|E|} \int_E \Phi_{j_0}(|f_{j_0}(x)|)dx \Phi_{j_0}\left(C \max\left\{ \prod_{j=1}^{m-1} \frac{1}{|Q_k|} \int_E \Phi_{\beta,j}(|f_{\beta(j)}(x)|)dx : \beta \in B_{\{1,\dots,m\}/j_0} \right\}\right) \\
&\leq \frac{C}{|E|} \int_E \Phi_{j_0}(|f_{j_0}(x)|)dx \max\left\{ \prod_{j=2}^{m} \frac{1}{|Q_k|} \int_E \Phi_{\beta,j}(|f_{\beta(j)}(x)|)dx : \beta \in B_{\{1,\dots,m\}}, \beta(1) = j_0 \right\} \\
&\leq C \max\left\{ \prod_{j=1}^{m} \frac{1}{|E|} \int_E \Phi_{\beta,j}(|f_{\beta(j)}(x)|)dx : \beta \in B_{\{1,\dots,m\}} \right\},
\end{aligned}$$

where the third inequality is due to Jensen's inequality. This is (2.5). □

2.4. Extrapolation.

We need the following extrapolation result:

LEMMA 2.3 ([**8**]). *Given a family of functions $\mathfrak{F}$, suppose that for some p_0, $0 < p_0 < \infty$, and for every $\omega \in A_\infty$, there exists a constant C_ω such that for all $(f, g) \in \mathfrak{F}$,*

$$\int_{\mathbb{R}^n} |f(x)|^{p_0} \omega(x)dx \leq C_\omega \int_{\mathbb{R}^n} |g(x)|^{p_0} \omega(x)dx.$$

Then for all $0 < p, q < \infty$, $0 < s \leq \infty$, and $\omega \in A_\infty$, there exists a constant C_ω such that

$$\left\| \left(\sum_j |f_j|^q \right)^{1/q} \right\|_{L^{p,s}(\omega)} \leq C_\omega \left\| \left(\sum_j |g_j|^q \right)^{1/q} \right\|_{L^{p,s}(\omega)},$$

for all $(f_j, g_j) \in \mathfrak{F}$, $j \in \mathbb{N}^+$.

3. A key Lemma.

As mentioned in the introduction, Theorem 1.1 will be a consequence of a Fefferman–Stein function estimate. To state it, we first define a maximal commutator. Let K_η satisfy (2.1) and (2.2) uniformly for any $\eta > 0$. Define

$$W^*_{\vec{b},S}(\vec{f})(x) = \sup_{\eta>0}\left|\int_{\mathbb{R}^n}\prod_{(i,j)\in S}(b_i(x)-b_i(y_j))K_\eta(x,y_1,\dots,y_m)\prod_{j=1}^m f_j(y_j)d\vec{y}\right|$$
$$= \sup_{\eta>0}\left|W_{\vec{b},S,\eta}(\vec{f})(x)\right| \tag{3.1}$$

and again write $W^*_{\vec{b},\emptyset}(\vec{f}) = W^*(\vec{f})$. The following inequality provides a foundation for our analysis.

LEMMA 3.1. *Let $0<\delta<1/m$, then for any number δ_0, $\delta<\delta_0<\infty$, there exists a constant C such that for any bounded and compact supported functions f_j $(j=1,\dots,m)$, one can obtain*

$$M^{\#}_\delta\big(W^*_{\vec{b},S}(\vec{f})\big)(x) \le C\|W^*\|_{|S|+1}\mathcal{M}_{L(logL)^{\vec{R}_S}}(\vec{f})(x)\prod_{(i,j)\in S}\|b_i\|_{Osc_{expL^{r_{ij}}}}$$
$$+ C\sum_{D\subset S} M_{\delta_0}\big(T_{\vec{b},D}(\vec{f})\big)(x)\prod_{(i,j)\in S\setminus D}\|b_i\|_{BMO}.$$

To prove Lemma 3.1, we will use the following elementary lemma.

LEMMA 3.2. *Let S be a set of $Z^+\times Z^+$, x_{ij} be a sequence of real numbers for $(i,j)\in S$. Then the identity*

$$\prod_{(i,j)\in S}(x_{i0}-x_{ij})$$
$$= \prod_{(i,j)\in S}(\lambda_i-x_{ij}) + \sum_{D\subset S}(-1)^{|S\setminus D|+1}\Big(\prod_{(i,j)\in D}(x_{i0}-x_{ij})\Big)\Big(\prod_{(i,j)\in S\setminus D}(x_{i0}-\lambda_i)\Big)$$

holds for any constants λ_i.

PROOF. Note first that, in general, for an arbitrary finite set E,

$$\prod_{i\in E}(a_i+b_i) = \sum_{A\subseteq E}\prod_{i\in A}a_i\prod_{i\in E/A}b_i. \tag{3.2}$$

Using (3.2) twice, we can write for any λ_i,

$$\prod_{(i,j)\in S}(x_{i0}-x_{ij}) = \prod_{(i,j)\in S}(x_{i0}-\lambda_i+\lambda_i-x_{ij})$$
$$= \prod_{(i,j)\in S}(\lambda_i-x_{ij}) + \sum_{A\subset S}\prod_{(i,j)\in A}(\lambda_i-x_{i0}+x_{i0}-x_{ij})\prod_{(i,j)\in S/A}(x_{i0}-\lambda_i)$$
$$= \prod_{(i,j)\in S}(\lambda_i-x_{ij}) + \sum_{A\subset S}\sum_{D\subseteq A}\prod_{(i,j)\in D}(x_{i0}-x_{ij})\prod_{(i,j)\in A/D}(\lambda_i-x_{i0})\prod_{(i,j)\in S/A}(x_{i0}-\lambda_i)$$

$$= \prod_{(i,j)\in S}(\lambda_i - x_{ij}) + \sum_{D\subset S}\sum_{A:D\subseteq A\subset S}(-1)^{|A/D|}\prod_{(i,j)\in D}(x_{i0}-x_{ij})\prod_{(i,j)\in S\setminus D}(x_{i0}-\lambda_i).$$

As for any subset $D\subset S$,

$$\sum_{A:D\subseteq A\subset S}(-1)^{|A/D|} = \sum_{D:D\subset S\setminus D}(-1)^{|D|} = \sum_{k=0}^{|S\setminus D|-1} C^k_{|S\setminus D|}(-1)^k = (-1)^{|S\setminus D|+1},$$

Lemma 3.2 is thus proved. □

PROOF OF LEMMA 3.1. Fix a point $x\in\mathbb{R}^n$ and a cube Q containing x, and let $0<\delta<1/m$. We need to show that there exists a constant c_Q^* such that

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q \big||W^*_{\vec{b},S}(\vec{f})(z)|^\delta - |c_Q^*|^\delta\big|dx\right)^{1/\delta}\\ &\quad\le C\|W^*\|_{|S|+1}\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})(x)\prod_{(i,j)\in S}\|b_i\|_{Osc_{expL^{r_{ij}}}}\\ &\quad + C\sum_{D\subset S} M_{\delta_0}\big(W^*_{\vec{b},D}(\vec{f})\big)(x)\prod_{(i,j)\in S\setminus D}\|b_i\|_{BMO},\end{aligned}$$

where C is independent of x and Q.

Let $c_Q^* = \sup_{\eta>0}|c_{Q,\eta}|$ ($c_{Q,\eta}$ will be choosen later) and define

$$I^*(z) = \sup_{\eta>0}\big|\mathcal{W}_{\vec{b},S,\eta}(\vec{f})(z) - c_{Q,\eta}\big|.$$

It suffices to show that $\big((1/|Q|)\int_Q|I^*(z)|^\delta dx\big)^{1/\delta}$ is to be controlled by the right side of the above inequality.

Note $\mathcal{W}_{\vec{b},S,\eta}(\vec{f})(z) = \int_{\mathbb{R}^{nm}}K_\eta(z,\vec{y})\prod_{(i,j)\in S}(b_i(z)-b_i(y_j))\prod_{j=1}^m f_j(y_j)d\vec{y}$. For the product $\prod_{(i,j)\in S}(b_i(z)-b_i(y_j))$, apply Lemma 3.2 by viewing $b_i(z)$ as x_{i0}, $b_i(y_j)$ as x_{ij}, and letting $\lambda_i = (b_i)_Q$ be the average of b_i on Q to get

$$\begin{aligned}&\prod_{(i,j)\in S}(b_i(z)-b_i(y_j))\\ &= \prod_{(i,j)\in S}((b_i)_Q - b_i(y_j)) + \sum_{D\subset S}(-1)^{|D^c|+1}\prod_{(i,j)\in S\setminus D}(b_i(z)-(b_i)_Q)\prod_{(i,j)\in D}(b_i(z)-b_i(y_j)).\end{aligned}$$

Then we have

$$I^*(z) \le \sup_{\eta>0}\left|\int_{\mathbb{R}^{nm}}K_\eta(z,\vec{y})\prod_{(i,j)\in S}((b_i)_Q - b_i(y_j))\prod_{j=1}^m f_j(y_j)d\vec{y} - c_{Q,\eta}\right|$$

$$+\sum_{D\subset S}\prod_{(i,j)\in S\setminus D}|(b_i(z)-(b_i)_Q)|\cdot K^*_{\vec{b},D}(\vec{f})(z),$$

with $K^*_{\vec{b},D}(\vec{f})(z)$ is defined by $\sup_{\eta>0}\big|\int_{\mathbb{R}^{nm}}K_\eta(z,\vec{y})\prod_{(i,j)\in D}(b_i(z)-b_i(y_j))\cdot\prod_{j=1}^m f_j(y_j)d\vec{y}\big|$.

For the first term of the right-hand side of the above inequality, split each f_j as $f_j=f_j\chi_Q+f_j\chi_{Q^c}=f_j^0+f_j^\infty$ and write

$$\begin{aligned}\prod_{j=1}^m f_j(y_j)=\prod_{j=1}^m(f_j^0+f_j^\infty)&=\sum_{\alpha_1,\dots,\alpha_m\in\{0,\infty\}}\prod_{j=1}^m f_j^{\alpha_j}(y_j)\\&=\prod_{j=1}^m f_j^0(y_j)+\sum_{\alpha_1,\dots,\alpha_m\in\{0,\infty\},\exists\alpha_j,\alpha_j=\infty}\prod_{j=1}^m f_j^{\alpha_j}(y_j)=\vec{f}^{\vec{0}}+\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}\vec{f}^{\vec{\alpha}},\end{aligned}\tag{3.3}$$

where $\vec{\alpha}=(\alpha_1,\dots,\alpha_m)$, $\alpha_i=0$ or ∞, $\vec{f}^{\vec{\alpha}}=\prod_{j=1}^m f_j^{\alpha_j}(y_i)$.

Set $c_{Q,\eta}=\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}\int_{\mathbb{R}^{nm}}K_\eta(x,\vec{y})\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\prod_{j=1}^m f_j(y_j)d\vec{y}$. By (3.3),

$$I^*(z)\le I^*_{\vec{0}}(z)+\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}I^*_{\vec{\alpha}}(z)+\sum_{D\subset S}\prod_{(i,j)\in S\setminus D}|(b_i(z)-(b_i)_Q)|\cdot\big|K^*_{\vec{b},D}(\vec{f})(z)\big|,$$

with $I^*_{\vec{0}}(z)$ defined by $\sup_{\eta>0}\big|\int_{\mathbb{R}^{nm}}K_\eta(z,\vec{y})\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\prod_{j=1}^m f_j^0(y_j)d\vec{y}\big|$ and $I^*_{\vec{\alpha}}(z)$ defined by $\sup_{\eta>0}\big|\int_{\mathbb{R}^{nm}}(K_\eta(z,\vec{y})-K_\eta(x,\vec{y}))\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\cdot\prod_{j=1}^m f_j^{\alpha_j}(y_j)d\vec{y}\big|$.

Thus

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q|I^*(z)|^\delta dz\right)^{1/\delta}\\&\le C\left(\frac{1}{|Q|}\int_Q\big|I^*_{\vec{0}}(z)\big|^\delta(z)dz\right)^{1/\delta}+C\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}\left(\frac{1}{|Q|}\int_Q|I^*_{\vec{\alpha}}(z)|^\delta(z)dz\right)^{1/\delta}\\&\quad+C\sum_{D\subset S}\left(\frac{1}{|Q|}\int_Q\prod_{(i,j)\in S\setminus D}|(b_i(z)-(b_i)_Q)|^\delta\cdot\big|W^*_{\vec{b},D}(\vec{f})(z)\big|^\delta dz\right)^{1/\delta}\\&=:CI^*_{\vec{0}}+C\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}I^*_{\vec{\alpha}}+C\sum_{D\subset S}I^*_D.\end{aligned}\tag{3.4}$$

By Hölder's inequality,

$$\begin{aligned}I^*_D&\le\prod_{(i,j)\in S\setminus D}\left(\frac{1}{|Q|}\int_Q|b_i(z)-(b_i)_Q|^{\delta_{ij}}dz\right)^{1/\delta_{ij}}\left(\frac{1}{|Q|}\int_Q\big|W^*_{\vec{b},D}(\vec{f})(z)\big|^{\delta_0}dz\right)^{1/\delta_0}\\&\le C\prod_{(i,j)\in S\setminus D}\|b_i\|_{BMO}M_{\delta_0}\big(W^*_{\vec{b},D}(\vec{f})\big)(x),\end{aligned}\tag{3.5}$$

where $\delta_{ij} \geq 1$, $\delta_0 \geq 0$ and $\sum_{(i,j)\in D^c}(1/\delta_{ij}) + (1/\delta_0) = 1/\delta$.

By the weak type endpoint boundedness of W^* and the generalized Hölder's inequality (2.3),

$$\begin{aligned} I_{\vec{0}}^* &\leq C\Big\| W^*\Big(f_1 \prod_{(i,1)\in S}((b_i)_Q - b_i(y_1)), \dots, f_m \prod_{(i,m)\in S}((b_i)_Q - b_i(y_m))\Big)\Big\|_{L^{1/m,\infty}(Q,dx/|Q|)} \\ &\leq C\|W^*\|_{L^1\times\cdots\times L^1\to L^{1/m,\infty}} \prod_{j=1}^m \frac{1}{|Q|}\int_Q |f_j| \prod_{(i,j)\in S}|(b_i)_Q - b_i(y_j)|dy_j \\ &\leq C\|W^*\|\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})(x) \prod_{(i,j)\in S}\|b_i\|_{Osc_{expL^{r_{ij}}}}. \end{aligned} \tag{3.6}$$

We are now in a position to estimate $I_{\vec{\alpha}}^*$ with $\vec{\alpha} \neq \vec{0}$. Without loss of generality, we assume that $\alpha_{j_1} = \cdots = \alpha_{j_l} = 0$ and $\alpha_j = \infty$ if $j \notin \{j_1, \dots, j_l\}$, $0 \leq l < m$. By (2.1), we have

$$\begin{aligned} I_{\vec{\alpha}}^* &\leq A_\varepsilon \prod_{j\in\{j_1,\dots,j_l\}} \int_Q |f_j(y_j)| \prod_{(i,j)\in S}|(b_i)_Q - b_i(y_j)|dy_j \\ &\quad\times \sum_{k=1}^\infty \frac{|Q|^{\varepsilon/n}}{(3^k|Q^{1/n}|)^{nm+\varepsilon}} \int_{3^kQ} \prod_{j\notin\{j_1,\dots,j_l\}} \Big(|f_j(y_j)| \prod_{(i,j)\in S}|(b_i)_Q - b_i(y_j)|\Big)dy_j \\ &\leq A_\varepsilon \sum_{k=1}^\infty \frac{1}{3^{k\varepsilon}} \prod_{j=1}^m \frac{1}{|3^kQ|}\int_{3^kQ}|f_j(y_j)| \prod_{(i,j)\in S}|b_i(y_j) - (b_i)_Q|dy_j. \end{aligned} \tag{3.7}$$

Applying the generalized Hölder's inequality (2.3), and noting that $\sum_{k=1}^\infty (k^{|S|}/3^{k\varepsilon}) \leq 2\int_0^\infty (x^{|S|}/3^{\varepsilon x})dx = (2/(\varepsilon \ln 3)^{|S|+1})\int_1^\infty ((\ln y)^{|S|}/y^2)dy$, one obtain

$$\begin{aligned} I_{\vec{\alpha}}^* &\leq A_\varepsilon \sum_{k=1}^\infty \frac{1}{3^{k\varepsilon}} \prod_{j=1}^m \|f_j\|_{L(\log L)^{1/r_j},3^kQ} \prod_{(i,j)\in S}\|b_i - (b_i)_Q\|_{expL^{r_{ij}},3^kQ} \\ &\leq A_\varepsilon \sum_{k=1}^\infty \frac{k^{|S|}}{3^{k\varepsilon}} \prod_{j=1}^m \|f_j\|_{L(\log L)^{1/r_j},3^kQ} \prod_{(i,j)\in S}\|b_i - (b_i)_{3^kQ}\|_{expL^{r_{ij}},3^kQ} \\ &\leq \frac{C}{\varepsilon^{|S|+1}} A_\varepsilon \mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})(x) \prod_{(i,j)\in S}\|b_i\|_{Osc_{expL^{r_{ij}}}}. \end{aligned} \tag{3.8}$$

By (3.4), (3.5), (3.6) and (3.8), Lemma 3.1 is proved. □

REMARK 3.1. By the proof of Lemma 3.1, it is easy to show that the same estimate holds for $M_\delta^\#(W_{\vec{b},S}^{*,+}\vec{f})(x)$ with

$$W_{\vec{b},S}^{*,+}(\vec{f})(x) = \sup_{\eta>0} \int_{\mathbb{R}^{nm}} \prod_{(i,j)\in S} |b_i(x) - b_i(y_j)| K_\eta(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j(y_j) d\vec{y},$$

when $K(x, \vec{y})$ and $f_j(y_j)$ $(1 \le j \le m)$ are all positive functions. We will use this fact in the next section.

4. Weighted estimates.

We first show how to apply Lemma 3.1 to prove these estimates. As in [**31**] and also [**37**], let $u, v \in C^\infty([0,\infty))$ such that $|u'(t)| \le Ct^{-1}$, $v'(t) \le Ct^{-1}$ and satisfy

$$\chi_{[2,\infty)} \le u(t) \le \chi_{[1,\infty)}, \ \chi_{[1,2)} \le v(t) \le \chi_{[1/2,3)}.$$

Denote $\mathcal{U}_\eta(x, y_1, \ldots, y_m) = K(x, y_1, \ldots, y_m) u\big(\sqrt{|x-y_1|^2 + \cdots + |x-y_m|^2}/\eta\big)$ and $\mathcal{V}_\eta(x, y_1, \ldots, y_m) = \big|K(x, y_1, \ldots, y_m) v\big(\sqrt{|x-y_1|^2 + \cdots + |x-y_m|^2}/\eta\big)\big|$. Define

$$\mathcal{U}_{\vec{b},S}^*(\vec{f})(x) = \sup_{\eta>0} \left| \int_{(\mathbb{R}^n)^m} \prod_{(i,j)\in S} (b_i(x) - b_i(y_j)) \mathcal{U}_\eta(x, y_1, \ldots, y_m) \prod_{i=1}^m f_i(y_i) d\vec{y} \right|,$$

$$\mathcal{V}_{\vec{b},S}^*(\vec{f})(x) = \sup_{\eta>0} \int_{(\mathbb{R}^n)^m} \left| \prod_{(i,j)\in S} (b_i(x) - b_i(y_j)) \mathcal{V}_\eta(x, y_1, \ldots, y_m) \prod_{i=1}^m f_i(y_i) \right| d\vec{y}.$$

If $S = \emptyset$, $\mathcal{U}_{\vec{b},S}^*(\vec{f})(x)$ and $\mathcal{V}_{\vec{b},S}^*(\vec{f})(x)$ are defined in a similar way as before.

By arguments similar to [**31**], $\mathcal{U}_\eta(x, y_1, \ldots, y_m)$ and $\mathcal{V}_\eta(x, y_1, \ldots, y_m)$ satisfy (2.1) and (2.2) with A_ε replaced by $2A_\varepsilon$ uniformly in η. It is clear that for any finite set S, $T_{\vec{b},S}^* \vec{f}(x) \le \mathcal{U}_{\vec{b},S}^*(\vec{f})(x) + \mathcal{V}_{\vec{b},S}^*(\vec{f})(x)$, which also implies $\|T^*\| \le \|\mathcal{U}^*\| + \|\mathcal{V}^*\|$.

Thus instead of estimating $T_{\vec{b},S}^* \vec{f}(x)$ directly, it suffices to estimate $\mathcal{U}_{\vec{b},S}^*(\vec{f})$ and $\mathcal{V}_{\vec{b},S}^*(\vec{f})$ respectively. The key point would be a Fefferman–Stein function estimate. The one for $M_\delta^\#\big(\mathcal{U}_{\vec{b},S}^* \vec{f}\big)$ follows from Lemma 3.1. For $M_\delta^\#\big(\mathcal{V}_{\vec{b},S}^* \vec{f}\big)$, we only need to consider positive functions f_j. It will then suffice to estimate $M_\delta^\#\big(\mathcal{V}_{\vec{b},S}^{*,+} \vec{f}\big)$ with $\mathcal{V}_{\vec{b},S}^{*,+}(f_1, \ldots, f_m)$ defined by

$$\mathcal{V}_{\vec{b},S}^*(\vec{f})(x) = \sup_{\eta>0} \int_{(\mathbb{R}^n)^m} \prod_{(i,j)\in S} \big|b_i(x) - b_i(y_j)\big| \mathcal{V}_\eta(x, y_1, \ldots, y_m) \prod_{i=1}^m f_i(y_i) d\vec{y}.$$

However by Remark 3.1, similar estimate for $M_\delta^\#\big(\mathcal{V}_{\vec{b},S,q}^{*,+} \vec{f}\big)$ also holds. For simplicity, we will only prove these estimates for $\mathcal{U}_{\vec{b},S}^*(\vec{f})$.

Proof of Theorem 1.1. Assume $\|\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})\|_{L^p(\omega)} < \infty$. Admit first the following:

CLAIM A. *There exists some δ with $0<\delta<1/m$ such that for any finite subset D of $Z^+\times\{1,\dots,m\}$ and any bounded and compact supported functions f_i $(i=1,\dots,m)$, $\|M_\delta(\mathcal{U}^*{}_{\vec{b},D}(\vec{f}))\|_{L^p(\omega)}<\infty$.*

By Lemma 2.1 and Claim A, it suffices to show that there exists δ_0 with $0<\delta_0<\delta<\infty$ such that

$$\begin{aligned}&\big\|M^{\#}_{\delta_0}(\mathcal{U}^*_{\vec{b},S}(\vec{f}))\big\|_{L^p(\omega)}\\&\quad\le C\big(\|\mathcal{U}^*\|[\omega]_{A_\infty}^{|S|}+\|K\|_{|S|+1}\big)\prod_{(i,j)\in S}\|b_i\|_{Osc_{expL^{r_{ij}}}}\big\|\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})\big\|_{L^p(\omega)}.\end{aligned}\tag{4.1}$$

We prove this by induction. Consider first the case $|S|=1$ and let $S=(1,j)$ for some $1\le j\le m$. By Lemma 3.1, for any δ_1 with $\delta_0<\delta_1<\delta$, we have

$$\begin{aligned}&M^{\#}_{\delta_0}\big(\mathcal{U}^*_{\vec{b},S}\vec{f}\big)(x)\\&\quad\le C\|\mathcal{U}^*\|_2\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})(x)\|b_1\|_{Osc_{expL^{r_{1j}}}}+CM_{\delta_1}(\mathcal{U}^*(\vec{f}))(x)\|b_1\|_{BMO}.\end{aligned}$$

By Lemma 2.1 and Claim A, using the estimate $M^{\#}_{\delta_1}(\mathcal{U}^*\vec{f})(x)\le C\|\mathcal{U}^*\|_1\mathcal{M}(\vec{f})(x)$, we have

$$\begin{aligned}&\big\|M^{\#}_{\delta_0}(\mathcal{U}^*_{\vec{b},S}\vec{f})\big\|_{L^p(\omega)}\\&\quad\le C\|\mathcal{U}^*\|_2\big\|\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})\big\|_{L^p(\omega)}\|b_1\|_{Osc_{expL^{r_{1j}}}}+C\|M_{\delta_1}(\mathcal{U}^*(\vec{f}))\|_{L^p(\omega)}\|b_1\|_{BMO}\\&\quad\le C\big(\|\mathcal{U}^*\|[\omega]_{A_\infty}^{|S|}+\|K\|_2\big)\|b_1\|_{Osc_{expL^{r_{1j}}}}\big\|\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})\big\|_{L^p(\omega)},\end{aligned}$$

where $\vec{R}_S=(0,\dots,1/r_{1j},\dots,0)$. This is (4.1) when $|S|=1$.

Now assume (4.1) holds for any S with $0\le|S|\le N-1$, and we are in a position to prove it holds for any S with $0\le|S|=N$. By Lemma 2.1 (and Claim A again) and Lemma 3.1, there exists δ_1 with $\delta_0<\delta_1<\delta$ such that

$$\begin{aligned}\big\|M^{\#}_{\delta_0}(\mathcal{U}^*_{\vec{b},S}\vec{f})\big\|_{L^p(\omega)}&\le C\|\mathcal{U}^*\|_{|S|+1}\prod_{(i,j)\in S}\|b_i\|_{Osc_{expL^{r_{ij}}}}\big\|\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})\big\|_{L^p(\omega)}\\&\quad+C\sum_{D\subset S}\prod_{(i,j)\in D^c}\|b_i\|_{BMO}\big\|M_{\delta_1}(\mathcal{U}^*{}_{\vec{b},D}(\vec{f}))\big\|_{L^p(\omega)}\\&\le C\|\mathcal{U}^*\|_{|S|+1}\prod_{(i,j)\in S}\|b_i\|_{Osc_{expL^{r_{ij}}}}\big\|\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})\big\|_{L^p(\omega)}\\&\quad+C[\omega]_{A_\infty}\sum_{D\subset S}\prod_{(i,j)\in D^c}\|b_i\|_{BMO}\big\|M^{\#}_{\delta_1}(\mathcal{U}^*{}_{\vec{b},D}(\vec{f}))\big\|_{L^p(\omega)}.\end{aligned}\tag{4.2}$$

As $D\subset S$, $|D|\le N-1$, by assumption, we have

$$\big\|M^{\#}_{\delta_1}(\mathcal{U}^*{}_{\vec{b},D}(\vec{f}))\big\|_{L^p(\omega)}$$
$$\leq C\big(\|\mathcal{U}^*\|[\omega]^{|D|}_{A_\infty}+\|K\|_{|D|+1}\big)\prod_{(i,j)\in D}\|b_i\|_{Osc_{expL^{r_{ij}}}}\big\|\mathcal{M}_{L(\log L)^{\vec{R}_D}}(\vec{f})\big\|_{L^p(\omega)}.$$

Combine this with (4.2),

$$\big\|M^{\#}_{\delta_0}(\mathcal{U}^*_{\vec{b},S}\vec{f})\big\|_{L^p(\omega)}$$
$$\leq C\big(\|\mathcal{U}^*\|_{|S|+1}+[\omega]_{A_\infty}\|K\|_{|S|}+[\omega]^{|S|}_{A_\infty}\|\mathcal{U}^*\|\big)$$
$$\times\prod_{(i,j)\in S}\|b_i\|_{Osc_{expL^{r_{ij}}}}\big\|\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})\big\|_{L^p(\omega)}$$
$$\leq C\big(\|K\|_{|S|+1}+[\omega]^{|S|}_{A_\infty}\|\mathcal{U}^*\|\big)\prod_{(i,j)\in S}\|b_i\|_{Osc_{expL^{r_{ij}}}}\big\|\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})\big\|_{L^p(\omega)}.$$

This is exactly (4.1).

Now we are left to prove Claim A when $\big\|\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})\big\|_{L^p(\omega)}<\infty$. As $\omega\in A_\infty$, there exists p_0 with $\max\{pm,1\}<p_0<\infty$ such that $\omega\in A_{p_0}$. We have

$$\big\|M_\delta(\mathcal{U}^*{}_{\vec{b},D}(\vec{f}))\big\|_{L^p(\omega)}=\big\|M_{p/p_0}(\mathcal{U}^*_{\vec{b},D}(\vec{f}))\big\|_{L^p(\omega)}=\big\|M(\mathcal{U}^*_{\vec{b},D}(\vec{f})^{p/p_0})\big\|^{p_0/p}_{L^{p_0}(\omega)}$$
$$\leq C\big\|\mathcal{U}^*_{\vec{b},D}(\vec{f})^{p/p_0}\big\|^{p_0/p}_{L^{p_0}(\omega)}=C\big\|\mathcal{U}^*_{\vec{b},D}(\vec{f})\big\|_{L^p(\omega)}.$$

It is sufficient to show for any finite subset D of $Z^+\times\{1,\dots,m\}$, $\|\mathcal{U}^*_{\vec{b},D}(\vec{f})\|_{L^p(\omega)}<\infty$. Note that if S is an empty set, $(\mathcal{U}^*_{\vec{b},D}(\vec{f}))(x)=\mathcal{U}^*(\vec{f})(x)$. If $0\leq|D|\leq 1$, $\|\mathcal{U}^*_{\vec{b},D}(\vec{f})\|_{L^p(\omega)}$ has been shown to be finite [**15**]. For general D, the same sketch as in the case $|D|=1$ applies too. □

The proof of the Corollary 1.2 is almost the same as the argument of Theorem 3.18 in [**15**] and we omit its proof.

Proof of Theorem 1.3. We may assume the right-hand side of (1.8) is finite, since otherwise there is nothing needed to prove. As in [**15**], we may also assume first that ω is bounded. Then the Monotone Convergence Theorem will justify the theorem by first proving (1.8) for $\omega_r=\min\{\omega,r\}$ and taking the limit $r\to\infty$.

We assume momentarily that the following:

Claim B. *For some δ with $0<\delta<1/m$ and for each $D\subseteq S$, it holds that*

$$\sup_{\lambda>0}\varphi\bigg(\lambda\prod_{(i,j)\in S/D}\|b_i\|_{BMO}\bigg)\omega\big(y\in\mathbb{R}^n: M_{\delta_0}(\mathcal{U}^*_{\vec{b},D}(\vec{f}))(y)>\lambda^m\big)<\infty.$$

As $\varphi:(0,\infty)\to(0,\infty)$ is doubling, by Lemma 2.1 and Lemma 3.1, for any δ and δ_0 with

$0<\delta<\delta_0<1/m$,

$$\begin{aligned}
&\sup_{\lambda>0}\varphi(\lambda)\omega\big(\{y\in\mathbb{R}^n:|\mathcal{U}^*_{\vec b,S}(\vec f)(y)|>\lambda^m\}\big)\\
&\quad\le C\sup_{\lambda>0}\varphi(\lambda)\omega\big(y\in\mathbb{R}^n:M^{\#}_{\delta}\mathcal{U}^*_{\vec b,S}(\vec f)(y)>\lambda^m\big)\\
&\quad\le C\sup_{\lambda>0}\varphi\bigg(\lambda\ \|\mathcal{U}^*\|^{1/m}_{|S|+1}\prod_{(i,j)\in S}\|b_i\|^{1/m}_{Osc_{expL^{r_{ij}}}}\bigg)\omega\big(y\in\mathbb{R}^n:\mathcal{M}_{L(\log L)^{\vec R_S}}(\vec f)(y)>\lambda^m\big)\\
&\qquad+C\sum_{D_1\subset S}\sup_{\lambda>0}\varphi\bigg(\lambda\prod_{(i,j)\in S/D_1}\|b_i\|^{1/m}_{BMO}\bigg)\omega\big(y\in\mathbb{R}^n:M_{\delta_0}(\mathcal{U}^*_{\vec b,D}(\vec f))(y)>\lambda^m\big).
\end{aligned}$$

For the second term in the right side of the last inequality, we can apply Lemma 2.1 by Claim B and then Lemma 3.1 to control it by

$$\begin{aligned}
&C\sup_{\lambda>0}\varphi\bigg(\lambda\|\mathcal{U}^*\|^{1/m}_{|S|+1}\prod_{(i,j)\in S}\|b_i\|^{1/m}_{Osc_{expL^{r_{ij}}}}\bigg)\omega\big(y\in\mathbb{R}^n:\mathcal{M}_{L(\log L)^{\vec R_S}}(\vec f)(y)>\lambda^m\big)\\
&\quad+C\sum_{D_1\subset S}\sum_{D_2\subset D_1}\sup_{\lambda>0}\varphi\bigg(\lambda\prod_{(i,j)\in S/D_2}\|b_i\|^{1/m}_{BMO}\bigg)\omega\big(y\in\mathbb{R}^n:CM_{\delta_0}(\mathcal{U}^*_{\vec b,D_2}(\vec f))(x)>\lambda^m\big).
\end{aligned}$$

For every family of subsets $D_1\subset S$, every family of subsets $D_{k+1}\subset D_k$, $1\le k\le|S|$, we continue to apply Lemma 2.1 to decompose these subsets until $|D_k|=0$. This can be done in finite steps as every time we use Lemma 2.1, we get a strictly proper subset.

Then we will obtain

$$\begin{aligned}
&\sup_{\lambda>0}\varphi(\lambda)\omega\big(\{y\in\mathbb{R}^n:|\mathcal{U}^*_{\vec b,S}(\vec f)(y)|>\lambda^m\}\big)\\
&\quad\le C\omega\big(y\in\mathbb{R}^n:\mathcal{M}_{L(\log L)^{\vec R_S}}(\vec f)(y)>\lambda^m\big)\sum_{D_1\subset S}\cdots\sum_{D_{|S|}\subset D_{|S|-1}}\\
&\qquad\sup_{\lambda>0}\varphi\bigg(\lambda\|\mathcal{U}^*\|^{1/m}_{|S|+1}\prod_{(i,j)\in D_1^c}\|b_i\|^{1/m}Osc_{expL^{r_{ij}}}\prod_{k=1}^{|S|-1}\prod_{(i,j)\in D_k/D_{k+1}}\|b_i\|^{1/m}_{Osc_{expL^{r_{ij}}}}\bigg)\\
&\quad=C\sup_{\lambda>0}\varphi\bigg(\lambda\|\mathcal{U}^*\|^{1/m}_{|S|+1}\prod_{(i,j)\in S}\|b_i\|^{1/m}_{Osc_{expL^{r_{ij}}}}\bigg)\omega\big(y\in\mathbb{R}^n:\mathcal{M}_{L(\log L)^{\vec R_S}}(\vec f)(y)>\lambda^m\big).
\end{aligned}$$

It is now enough to prove Claim B. If $0\le|D|\le1$, this again have been shown to be finite in [**15**], which also applies for the case of general set D. We omit the details and will specify a little more in the following section. □

PROOF OF THEOREM 1.5. The basic idea is taken from the proof of Theorem 3.17 in [**15**]. Assume without loss of generality that $t=1$. Define the open set

$$\Omega=\big\{x\in\mathbb{R}^n:\mathcal{M}_{L(\log L)^{\vec{\alpha}}}(\vec{f})(x)>1\big\}.$$

Assume moreover it is not empty and for any compact sbuset $F\subset\Omega$, we can cover it by a finite family of cubes Q_k with finite overlap for which

$$1<\prod_{j=1}^m\|f_j\|_{\Phi_j,Q_k}.$$

Denote $\Phi_j(t)=t(1+\log^+t)^{\alpha_j}$. Note that $\Phi_j(t)\geq t$, and

$$\begin{aligned}\Phi_{j_1}(t)\circ\Phi_{j_2}(t)&=t(1+\log^+t)^{\alpha_{j_2}}\big(1+\log^+(t(1+\log^+t))^{\alpha_{j_2}}\big)^{\alpha_{j_1}}\\&\leq Ct(1+\log^+t)^{\alpha_{j_2}}(1+\log^+t)^{\alpha_{j_1}}=C2(1+\log^+t)^{\alpha_{j_1}+\alpha_{j_2}}.\end{aligned}$$

We have for any $j\in\{1,\dots,m\}$ and any set $\beta\in B_{\{1,\dots,m\}}$,

$$\Phi_{\beta,j}=\Phi_{\beta(1)}\circ\cdots\circ\Phi_{\beta(j)}\leq\Phi_{\beta,m}\leq\Phi(t).$$

By Lemma 2.2,

$$\begin{aligned}\prod_{j=1}^m\|f_j\|_{\Phi_j,Q_k}&\leq C\max\left\{\prod_{j=1}^m\frac{1}{|Q_k|}\int_E\Phi_{\beta,j}(|f_{\beta(j)}(x)|)dx:\beta\in B_{\{1,\dots,m\}}\right\}\\&\leq C\prod_{j=1}^m\frac{1}{|Q_k|}\int_{Q_k}\Phi(f_j(x))dx.\end{aligned}$$

Then we have

$$\begin{aligned}v_\omega(F)&\leq\sum_k v_\omega(Q_k)\leq C[\vec{\omega}]_{A_{\vec{1}}}^{1/m}\sum_k\prod_{j=1}^m\inf_{Q_k}\omega_j^{1/m}|Q_k|^{1/m}\Big(\prod_{j=1}^m\frac{1}{|Q_k|}\int_{Q_k}\Phi(f_j(x))dx\Big)^{1/m}\\&\leq C[\vec{\omega}]_{A_{\vec{1}}}^{1/m}\sum_k\Big(\prod_{j=1}^m\int_{Q_k}\Phi(f_j(x))\omega_jdx\Big)^{1/m}\\&\leq C[\vec{\omega}]_{A_{\vec{1}}}^{1/m}\prod_{j=1}^m\Big(\sum_k\int_{Q_k}\Phi(f_j(x))\omega_jdx\Big)^{1/m}\\&\leq C[\vec{\omega}]_{A_{\vec{1}}}^{1/m}\prod_{j=1}^m\Big(\int_{\Omega}\Phi(f_j(x))\omega_jdx\Big)^{1/m},\end{aligned}$$

where the second last inequality is due to Hölder's inequality, and the last inequality is due to the finite overlap of the family of sets Q_k.

As the set F is arbitrary in Ω, the conclusion is obtained. □

PROOF OF THEOREM 1.4. By homogeneity, we only need to prove (1.9) when $t = 1$. It is easily checked that $1/(\Phi(1/t))$ is doubling and controlled by $g(t) = Ct$ for some $C > 0$. By Theorem 1.3 and Theorem 1.5,

$$\begin{aligned}
&v_\omega\{x \in \mathbb{R}^n : |\mathcal{U}^*_{\vec{b},S}(\vec{f})(x)| > 1\} \\
&\le C \sup_{t>0} \frac{1}{\Phi(1/t)} v_\omega\{x \in \mathbb{R}^n : M_\delta(\mathcal{U}^*_{\vec{b},S}(\vec{f}))(x) > t^m\} \\
&\le C \sup_{t>0} \frac{1}{\Phi(1/t)} v_\omega\left(y \in \mathbb{R}^n : \mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})(y) > \frac{t^m}{\prod_{(i,j)\in S} \|b_i\|_{Osc_{expL^{r_{ij}}}}}\right) \\
&\le C \sup_{t>0} \frac{1}{\Phi(1/t)} \prod_{j=1}^m \left(\int_{\mathbb{R}^n} \Phi\left(\frac{\prod_{i:(i,j)\in S} \|b_i\|_{Osc_{expL^{r_{ij}}}} |f_j(x)|}{t}\right)\omega_j(x)dx\right)^{1/m}.
\end{aligned}$$

Considering the multiplicative of Φ, this implies the theorem.

To show the sharpness of Theorem 1.4, take $n = 1$, $f_j = \chi_{(0,1)}$ and $b_i(x) = \log|1+x|$. This example is due to Pérez [**25**], where he considered the sharpness for the operator in (1.4). The general case follows in a similar way. □

5. Vector-valued extension.

The above theorems can be extended to the vector-valued case.

DEFINITION 5.1. For any $k \in \mathrm{N}^+$, let T_k be an m-linear Calderón–Zygmund operator with kernel K_k and S_k be an finite subset of $Z^+ \times \{1, \dots, m\}$. For $0 < q < \infty$, the vector-valued maximal commutator $T^*_{\vec{b},\vec{S},q}$ is defined by

$$\begin{aligned}
T^*_{\vec{b},\vec{S},q}(\vec{f})(x) = \left\|\{T^*_{k,\vec{b},S_k}(\vec{f_k})(x)\}\right\|_{l^q} &= \left(\sum_{k=1}^\infty \left|T^*_{k,\vec{b},S_k}(\vec{f_k})(x)\right|^q\right)^{1/q} \\
&= \left(\sum_{k=1}^\infty \left|T^*_{k,\vec{b},S_k}(f_{1k}, \dots, f_{mk})(x)\right|^q\right)^{1/q}, \qquad (5.1)
\end{aligned}$$

where

$$\begin{aligned}
&T^*_{k,\vec{b},S}(f_{1k}, \dots, f_{mk})(x) \\
&= \sup_{\delta>0} \left| \int_{(\sum_{j=1}^m |x-y_j|^2)^{1/2}>\delta} \prod_{(i,j)\in S_k} (b_i(x) - b_i(y_j))K_k(x, y_1, \dots, y_m) \prod_{j=1}^m f_{jk}(y_j)d\vec{y}\right|.
\end{aligned}$$

Again we write $T^*_{\vec{b},\emptyset,q} = T^*_q$. We can define $T_{\vec{b},S,q}(\vec{f})$ in a similar way.

When $S_{k_1} = S_{k_2} = S$ and $K_{k_1} = K_{k_2} = K$ for any $k_1, k_2 \in N^+$, $m = 1$, and $|S| = 1$, Pérez and Trujillo-González [**28**] gave the weighted strong and endpoint boundedness of

$T_{\vec{b},S,q}(\vec{f})$. Grafakos and Martell [**10**] gave the weighted strong boundedness in the case of $|S|=0$ for general $m\geq 1$. The endpoint estimate was obtained by Cruz-Uribe, Martell and Pérez in [**8**], also under the case of $|S|=0$. In [**34**], Tang considered

$$T_{\vec{b},S_k}(f_{1k},\ldots,f_{mk})(x)=\int_{\mathbb{R}^{nm}}\prod_{j=1}^{l}(b_j(x)-b_j(y_j))K(x,y_1,\ldots,y_m)\prod_{j=1}^{m}f_{jk}(y_j)d\vec{y},$$

where $1\leq l\leq m$, and gave some weighted and endpoint estimates. In a recent paper [**33**], Si and Xue considered the vector-valued maximal operators

$$T^*_{\prod b,q}(\vec{f})(x)=\left(\sum_{k=1}^{\infty}|T^*_{\prod b,S}(\vec{f_k})(x)|^q\right)^{1/q}=\left(\sum_{k=1}^{\infty}|T^*_{\prod b,S}(f_{1k},\ldots,f_{mk})(x)|^q\right)^{1/q}.$$

Based on an estimate for the sharp Feffermann–Stein function of this vector-valued commutator, we can obtain the following theorems. Similar estimates for $T_{\vec{b},S,q}(\vec{f})$ will follow along the same line.

For any $k\in N^+$, let R_k be a map from S_k to the set of positive numbers which are all bigger than one. Denote $r_{ij,k}=R_k(i,j)$, $1/r_{j,k}=\sum_{i:(i,j)\in S_k}(1/r_{ij,k})$ and $\vec{R}_{S_k}=(1/r_{1,k},\ldots,1/r_{m,k})$.

Theorem 5.1. *Let $0<p,q<\infty$, $0<s\leq\infty$ and $\omega\in A_\infty$. Let $b_i\in Osc_{expL^{r_{ij,k}}}$. Assume $|S|=\sup_{k\geq 1}\{|S_k|\}<\infty$. Then there exists a constant C such that*

$$\big\|T^*_{\vec{b},S,q}(\vec{f})\big\|_{L^{p,s}(\omega)}\leq C\Bigg\|\Bigg(\sum_{k=1}^{\infty}\Bigg(\big(\|T^*_k\|[\omega]^{|S_k|}_{A_\infty}+\|K_k\|_{|S_k|+1}\big)[\omega]_{A_\infty}\prod_{(i,j)\in S_k}\|b_i\|_{Osc_{expL^{r_{ij,k}}}}\times\mathcal{M}_{L(\log L)^{R_{\vec{S}_k}}}(f_{1k},\ldots,f_{mk})\Bigg)^q\Bigg)^{1/q}\Bigg\|_{L^{p,s}(\omega)}$$

for any bounded and compact supported functions f_{jk}, $j=1,\ldots,m$, $k\in N^+$.

Theorem 5.1 follows from the extrapolation Lemma 2.3 proved in [**8**]. Although a strong weighted boundedness involving $A_{\vec{p}}$ is expectable by studying the weighed bounds of vector-valued maximal operator $\mathcal{M}$, we do not know how to obtain the weak endpoint estimate via such extrapolation for $|S|>0$. However, if we take $S_{k_1}=S_{k_2}=S$ and $K_{k_1}=K_{k_2}=K$ for any $k_1,k_2\in N^+$, we can get both the strong and weak type endpoint weighted estimates.

Theorem 5.2. *Let $0<p<\infty$, $\omega\in A_\infty$, $1/q=\sum_{j=1}^{m}(1/q_j)$ with $1<q_j<\infty$. Then there exists a constant $C>0$ depending on T such that*

$$\big\|T^*_{\vec{b},S,q}\vec{f}\big\|_{L^p(\omega)}\leq C[\omega]^{|S|}_{A_\infty}\prod_{(i,j)\in S}\|b_i\|_{OscL^{r_{ij}}}\big\|\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f_{\vec{q}}})\big\|_{L^p(\omega)}. \tag{5.2}$$

Corollary 5.3. *Let $1 < p_j < \infty$, $1 < q_j < \infty$ for $j = 1, \dots, m$ with $1/p = \sum_{j=1}^m (1/p_j)$ and $1/q = \sum_{j=1}^m (1/q_j)$. Let $\vec{\omega} \in A_{\vec{p}}$. Then there exists a constant C depending on $\vec{\omega}$ and T such that*

$$\|T^*_{\vec{b},S,q}(\vec{f})\|_{L^p(v_{\vec{\omega}})} \le C \prod_{(i,j)\in S} \|b_i\|_{Osc_{expL^{r_{ij}}}} \prod_{j=1}^m \Big\| \Big(\sum_{k=1}^\infty |f_{jk}|^{q_j} \Big)^{1/q_j} \Big\|_{L^{p_j}(\omega_j)}.$$

Theorem 5.4. *Let $\omega \in A_\infty$ and $1/q = \sum_{j=1}^m (1/q_j)$ with $1 < q_j < \infty$. Let R be a map from S to the set of positive numbers that are bigger than 1. Let $\varphi : (0,\infty) \to (0,\infty)$ be doubling and for some constant C, $\varphi(t) < Ct$ for any $t \in (0,\infty)$. Then there exists a constant $C > 0$,depending on T and the A_∞ constant of ω, such that*

$$\begin{aligned} &\sup_{\lambda>0} \varphi(\lambda)\omega\big(\{y \in \mathbb{R}^n : |T^*_{\vec{b},S,q}(\vec{f})(y)| > \lambda^m\}\big) \\ &\quad \le C \sup_{\lambda>0} \varphi(\lambda)\omega\bigg(y \in \mathbb{R}^n : \mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f_{\vec{q}}})(y) > \frac{\lambda^m}{\prod_{(i,j)\in S} \|b_i\|_{Osc_{expL^{r_{ij}}}}} \bigg), \end{aligned} \tag{5.3}$$

for any bounded and uniform compact supported functions f_{jk}, $j = 1, \dots, m$, $k \in N^+$.

The following theorem to Theorem 5.4 is Theorem 1.4 to Theorem 1.3. We state it without specifying the proof.

Theorem 5.5. *Let $(\omega_1, \dots, \omega_m) \in A_{(1,\dots,1)}$ and $1/q = \sum_{j=1}^m (1/q_j)$ with $1 < q_j < \infty$. Let $b_i \in Osc_{expL^{r_{ij}}}$, $r_{ij} \ge 1$, $i = 1, \dots, m$. $1/r_j = \sum_{(i,j)\in S} (1/r_{ij})$. Denote $\Phi_j(t) = t(1 + \log^+ t)^{1/r_j}$ and $\Phi(t) = t(1 + \log^+ t)^{\sum_{j=1}^m (1/r_j)}$. Then there exists a constant C depending on $\vec{\omega}$ and T such that for any bounded and uniform compact supported functions f_{jk}, $j = 1, \dots, m$, $k \in N^+$,*

$$\begin{aligned} &v_\omega\{x \in \mathbb{R}^n : |T^*_{\vec{b},S,q}(\vec{f})(x)| > t^m\} \\ &\quad \le C \prod_{j=1}^m \bigg(\int_{\mathbb{R}^n} \Phi\bigg(\frac{\big(\sum_k |f_{jk}|^{q_j}\big)^{1/q_j} \prod_{i:(i,j)\in S} \|b_i\|_{Osc_{expL^{r_{ij}}}}}{t} \bigg) \omega_j(x)dx \bigg)^{1/m}. \end{aligned} \tag{5.4}$$

Moreover, when $r_{ij} \equiv 1$ for any $(i,j) \in S$, this result is sharp in the sense that it does not hold for any $\Phi(t) = t(1 + \log^+ t)^\alpha$ with $\alpha < \sum_{j=1}^m (1/r_j)$.

To prove Theorem 5.2 and Theorem 5.4, define $\mathcal{U}^*_{\vec{b},S}(\vec{f})(x)$ and $\mathcal{V}^*_{\vec{b},S}(\vec{f})(x)$ as in Section 4. Define the vector-valued operator $\mathcal{U}^*_{\vec{b},S,q}$ by

$$\begin{aligned} \mathcal{U}^*_{\vec{b},S,q}(\vec{f})(x) &= \big\|\{\mathcal{U}^*_{\vec{b},S}(\vec{f_k})(x)\}\big\|_{l^q} = \bigg(\sum_{k=1}^\infty \big|T^*_{\vec{b},S}(\vec{f_k})(x)\big|^q \bigg)^{1/q} \\ &= \bigg(\sum_{k=1}^\infty \big|\mathcal{U}^*_{\vec{b},S}(f_{1k}, \dots, f_{mk})(x)\big|^q \bigg)^{1/q} \end{aligned}$$

and $\mathcal{V}^*_{\vec{b},S,q}(\vec{f}\,)$ in a similar way.

As $T^*_{\vec{b},S}\vec{f}(x) \le \mathcal{U}^*_{\vec{b},S}(\vec{f}\,)(x) + \mathcal{V}^*_{\vec{b},S}(\vec{f}\,)(x)$, we have

$$T^*_{\vec{b},S,q}\vec{f}(x) \le \min\{1, 2^{(1-q)/q}\}\big(\mathcal{U}^*_{\vec{b},S,q}(\vec{f}\,)(x) + \mathcal{V}^*_{\vec{b},S,q}(\vec{f}\,)(x)\big).$$

As mentioned before, the following lemma will be crucial in this part.

Lemma 5.6. *Let $0 < \delta < 1/m$. For any number δ_0, $\delta < \delta_0 < \infty$, there exists a constant C depending on $\mathcal{U}^*$ such that for any bounded and compact supported functions f_i $(i = 1, \ldots, m)$,*

$$\begin{aligned} M^{\#}_{\delta}\big(\mathcal{U}^*_{\vec{b},S,q}\vec{f}\,\big)(x) &\le C \prod_{(i,j)\in S} \|b_i\|_{OscL^{r_{ij}}} \mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f_{\vec{q}}})(x) \\ &\quad + C \sum_{D\subset S} \prod_{(i,j)\in S\setminus D} \|b_i\|_{BMO} M_{\delta_0}\big(\mathcal{U}^*_{\vec{b},D,q}(\vec{f}\,)\big)(x). \qquad (5.5) \end{aligned}$$

The same inequality also holds for $M^{\#}_{\delta}(\mathcal{V}^*_{\vec{b},S,q}\vec{f}\,)(x)$.

The proof of Lemma 5.6 will be similar to the one of Lemma 3.1, although the vector-valued operator adds more terms and with that, more complexities. The key for tackling the new complexities is a very careful application of Hölder's inequality and Minkowski's inequality. For reader's convenience, we give a rather completed proof.

Proof of Lemma 5.6. We are only going to estimate $M^{\#}_{\delta}(\mathcal{U}^*_{\vec{b},S,q}\vec{f}\,)(x)$. The estimate of $M^{\#}_{\delta}(\mathcal{V}^*_{\vec{b},S}\vec{f}\,)(x)$ is almost the same.

By definition of the Fefferman–Stein function, our aim is to show that, for any point $x \in \mathbb{R}^n$, any cube Q containing x and $0 < \delta < \delta_0 < 1/m$, there exists a constant c_Q such that for any bounded and compact supported functions f_i $(i = 1, \ldots, m)$, $\big((1/|Q|)\int_Q \big||\mathcal{U}^*_{\vec{b},S,q}(\vec{f}\,)(z)|^\delta - |c^*_Q|^\delta\big|dx\big)^{1/\delta}$ is bounded by the right side of the above inequality.

Let $c^*_Q = (\sum_{k=1}^{\infty}|c^*_{Q,k}|^q)^{1/q} = \|\{c^*_{Q,k}\}\|_{l^q}$, where $c^*_{Q,k} = \sup_{\eta>0}|c_{Q,k,\eta}|$. As $\delta < 1/m \le q$, and by $\big|\|f\|_{L^\infty} - \|g\|_{L^\infty}\big| \le \|f-g\|_{L^\infty}$, we have

$$\begin{aligned} \big|\|\{\mathcal{U}^*_{\vec{b},S}(\vec{f_k})(z)\}\|^\delta_{l^q} - \|\{c^*_{Q,k}\}\|^\delta_{l^q}\big| &\le \big\|\{\mathcal{U}^*_{\vec{b},S}(\vec{f_k})(z) - c^*_{Q,k}\}\big\|^\delta_{l^q} \\ &\le \Big\|\Big\{\sup_{\eta>0}\big|\mathcal{U}_{\vec{b},S,\eta}(\vec{f_k})(z) - c_{Q,k,\eta}\big|\Big\}\Big\|^\delta_{l^q} = \|\{I^*_k(z)\}\|^\delta_{l^q}, \end{aligned}$$

where $I^*_k(z) = \sup_{\eta>0}\big|\mathcal{U}_{\vec{b},S,\eta}(\vec{f_k})(z) - c_{Q,k,\eta}\big|$.

By setting $c_{Q,k,\eta} = \sum_{\vec{\alpha},\vec{\alpha}\ne\vec{0}}\int_{\mathbb{R}^{nm}}\mathcal{U}_\eta(x,\vec{y}\,)\prod_{(i,j)\in S}((b_i)_Q - b_i(y_j))\prod_{j=1}^m f_{jk}(y_j)d\vec{y}$, applying Lemma 3.1 and splitting $\prod_{j=1}^m f_{jk}(y_j)$ as (3.3), we have

$$I_k^*(z) \le \sup_{\eta>0}\left|\int_{\mathbb{R}^{nm}} \mathcal{U}_\eta(z,\vec{y}) \prod_{(i,j)\in S} ((b_i)_Q - b_i(y_j)) \prod_{j=1}^m f_{jk}^0(y_j) d\vec{y}\right|$$

$$+ \sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}} \sup_{\eta>0}\left|\int_{\mathbb{R}^{nm}} (\mathcal{U}_\eta(z,\vec{y}) - \mathcal{U}_\eta(x,\vec{y})) \prod_{(i,j)\in S} ((b_i)_Q - b_i(y_j)) \prod_{j=1}^m f_{jk}^{\alpha_j}(y_j) d\vec{y}\right|$$

$$+ \sum_{D\subset S} \prod_{(i,j)\in S\setminus D} |(b_i(z) - (b_i)_Q)|$$

$$\times \sup_{\eta>0}\left|\int_{\mathbb{R}^{nm}} \mathcal{U}_\eta(z,\vec{y})) \prod_{(i,j)\in D} (b_i(z) - b_i(y_j)) \prod_{j=1}^m f_j(y_{jk}) d\vec{y}\right|$$

$$=: I_{\vec{0},k}^*(z) + \sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}} I_{\vec{\alpha},k}^*(z) + \sum_{D\subset S} \prod_{(i,j)\in S\setminus D} |(b_i(z) - (b_i)_Q)| \cdot \left|\mathcal{U}_{\vec{b},D}^*(\vec{f_k})(z)\right|.$$

Therefore

$$\|\{I_k^*(z)\}\|_{l^q} \le C\left(\frac{1}{|Q|}\int_Q \|\{I_{\vec{0},k}^*(z)\}\|_{l^q}^\delta dz\right)^{1/\delta} + C\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}\left(\frac{1}{|Q|}\int_Q \|\{I_{\vec{\alpha},k}^*(z)\}\|_{l^q}^\delta dz\right)^{1/\delta}$$

$$+ C\sum_{D\subset S}\left(\frac{1}{|Q|}\int_Q \prod_{(i,j)\in S\setminus D} |(b_i(z) - (b_i)_Q)|^\delta \cdot \left\|\{\mathcal{U}_{\vec{b},D}^*(\vec{f_k})(z)\}\right\|_{l^q}^\delta dz\right)^{1/\delta}$$

$$=: CI_{\vec{0}}^* + C\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}} I_{\vec{\alpha}}^* + C\sum_{D\subset S} I_D^*. \tag{5.6}$$

As $0 < \delta < 1/m$, by Kolmogrov's inequality and Corollary 3.3 in [**8**] which proved an endpoint weighted boundedness of T^*, we have

$$I_{\vec{0}}^* \le C\left\|\mathcal{U}_q^*\left(f_1^0 \prod_{i:(i,1)\in S} ((b_i)_Q - b_i(y_1)), \dots, f_m^0 \prod_{i:(i,m)\in S} ((b_i)_Q - b_i(y_m))\right)\right\|_{L^{1/m,\infty}(Q,dx/|Q|)}$$

$$\le C\prod_{j=1}^m \left\|\left(\sum_{k=1}^\infty\left(\prod_{i:(i,j)\in S} |(b_i)_Q - b_i(y_1)| \cdot |f_{jk}^0|\right)^{q_j}\right)^{1/q_j}\right\|_{L^1(\mathbb{R}^n,dx/|Q|)}$$

$$\le C\prod_{(i,j)\in S} \|b_i\|_{Osc_{expL^{r_{ij}}}} \mathcal{M}_{L(\log L)^{\vec{\alpha}}}(\vec{f_{\vec{q}}})(x). \tag{5.7}$$

For any $\vec{\alpha} \neq \vec{0}$, by (3.7) and the fact that $\mathcal{U}_\eta(x, y_1, \dots, y_m)$ of $\mathcal{U}_\eta(\vec{f})$ satisfies (2.1) uniformly, we have

$$I_{\vec{\alpha},k}^*(z) \le C\sum_{t=1}^\infty \frac{1}{3^{t\varepsilon}} \prod_{j=1}^m \frac{1}{|3^t Q|}\int_{3^t Q} |f_{jk}(y_j)| \prod_{i:(i,j)\in S} |b_i(y_j) - (b_i)_Q| dy_j.$$

Then

$$I_{\vec{\alpha}}^{*\,q} \leq C\sum_{k=1}^{\infty}\left(\sum_{t=1}^{\infty}\frac{1}{3^{t\varepsilon}}\prod_{j=1}^{m}\frac{1}{|3^tQ|}\int_{3^tQ}|f_{jk}(y_j)|\prod_{i:(i,j)\in S}|b_i(y_j)-(b_i)_Q|dy_j\right)^q$$

$$\leq C\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\left(\frac{1}{2^{t\varepsilon}}\prod_{j=1}^{m}\frac{1}{|3^tQ|}\int_{3^tQ}|f_{jk}(y_j)|\prod_{i:(i,j)\in S}|b_i(y_j)-(b_i)_Q|dy_j\right)^q,$$

by Hölder's inequality ($1 \leq q \leq \infty$) or Minkowski's inequality ($0 < q < 1$). Apply Hölder's inequality again, we can control $I_{\vec{\alpha}}^{*\,q}$ further by

$$C\sum_{t=1}^{\infty}\frac{1}{2^{t\varepsilon q}}\prod_{j=1}^{m}\left(\frac{1}{|3^tQ|}\int_{3^tQ}\left(\sum_{k=1}^{\infty}|f_{jk}(y_j)|^{q_j}\right)^{1/q_j}\prod_{i:(i,j)\in S}|b_i(y_j)-(b_i)_Q|dy_j\right)^q.$$

By the generalized Hölder's inequality (2.3),

$$\begin{aligned}I_{\vec{\alpha}}^{*\,q} &\leq C\sum_{t=1}^{\infty}\frac{1}{2^{t\varepsilon q}}\prod_{j=1}^{m}\left\|\left(\sum_{k=1}^{\infty}|f_{jk}(y_j)|^{q_j}\right)^{1/q_j}\right\|^q_{L(\log L)^{1/r_j},3^tQ}\\ &\quad\times\prod_{i:(i,j)\in S}\|b_i-(b_i)_{3^tQ}\|^q_{expL^{r_{ij}},3^tQ}\\ &\leq C\prod_{(i,j)\in S}\|b_i\|^q_{OscL^{r_{ij}}}\mathcal{M}^q_{L(\log L)^{\vec{\alpha}}}(\vec{f_{\vec{q}}})(x).\end{aligned}\tag{5.8}$$

We now estimate I_D^* for $D \subseteq S$. By Hölder's inequality,

$$\begin{aligned}I_D^* &\leq \prod_{(i,j)\in S\setminus D}\left(\frac{1}{|Q|}\int_Q|(b_i(z)-(b_i)_Q)|^{\delta_{ij}}\right)^{1/\delta_{ij}}\left(\frac{1}{|Q|}\int_Q\left\|\{\mathcal{U}^*_{\vec{b},D}(\vec{f_k})(z)\}\right\|^{\delta_0}_{l^q}dz\right)^{1/\delta_0}\\ &\leq C\prod_{(i,j)\in S\setminus D}\|b_i\|_{BMO}M_{\delta_0}\big(\mathcal{U}^*_{\vec{b},D,q}(\vec{f}\,)\big)(x).\end{aligned}\tag{5.9}$$

Combine (5.7), (5.8) and (5.9), we finish the proof of this lemma by (5.6). □

Theorem 5.2 can be directly obtained by the above lemma, we omit the proofs and proceed to the proof of Theorem 5.4.

Proof of Theorem 5.4. As $T^*_{\vec{b},S,q}\vec{f}(x) \leq \min\{1,2^{(1-q)/q}\}\big(\mathcal{U}^*_{\vec{b},S,q}(\vec{f}\,)(x)+\mathcal{V}^*_{\vec{b},S,q}(\vec{f}\,)(x)\big)$, we only need to prove this theorem for $\mathcal{U}^*_{\vec{b},S,q}(\vec{f}\,)$ and $\mathcal{V}^*_{\vec{b},S,q}(\vec{f}\,)$. The proof is almost identical with that of Theorem 1.4, where we use Lemma 5.6 instead of Lemma 3.1. We omit the details. The only place we need to pay attention is that this time, to apply Lemma 2.1, we need to show for any subset $D \subset S$, b_i bounded, and any bounded f_j with compact support,

$$\sup_{\lambda>0}\varphi(\lambda)\omega\big\{y\in\mathbb{R}^n:\big|T^*_{\vec{b},D,q}(\vec{f})(y)\big|>\lambda^m\big\}<\infty, \tag{5.10}$$

whenever

$$\sup_{\lambda>0}\varphi(\lambda)\omega\bigg(y\in\mathbb{R}^n:\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f}_{T,\vec{q}})(y)>\frac{\lambda^m}{\prod_{(i,j)\in S}\|b_i\|_{Osc_{expL^{r_{ij}}}}}\bigg)<\infty.$$

By the uniform compactness of f_{jk}, assume that $\operatorname{supp} f_{jk}\subset B(0,R)$.

$$\begin{aligned}&\sup_{\lambda>0}\varphi(\lambda)\omega\big\{y\in\mathbb{R}^n:\big|T^*_{\vec{b},D,q}(\vec{f})(y)\big|>\lambda^m\big\}\\&\quad\le\sup_{\lambda>0}\varphi(\lambda)\omega\big\{y\in B(0,2R):\big|T^*_{\vec{b},D,q}(\vec{f})(y)\big|>\lambda^m\big\}\\&\qquad+\sup_{\lambda>0}\varphi(\lambda)\omega\big\{y\in\mathbb{R}^n/B(0,2R):\big|T^*_{\vec{b},D,q}(\vec{f})(y)\big|>\lambda^m\big\}=:I+II.\end{aligned}$$

We can assume ω is bounded. By the fact that $\varphi(t)\le Ct$ and Hölder's inequality, let $1<p<\infty$,

$$\begin{aligned}I&\le C\sup_{\lambda>0}\lambda\big|\big\{y\in B(0,2R):\big|T^*_{\vec{b},D,q}(\vec{f})(y)\big|>\lambda^m\big\}\big|\\&\le C\int_{B(0,2R)}\big|T^*_{\vec{b},D,q}(\vec{f})(y)\big|^{1/m}dy\le CR^{(1-(1/p))n}\bigg(\int_{\mathbb{R}^n}\big|T^*_{\vec{b},D,q}(\vec{f})(y)\big|^{p/m}\bigg)^{1/p}.\end{aligned}\tag{5.11}$$

By Corollary 5.3, the last term is finite. For the term II, since $y\notin B(0,2R)$,

$$\begin{aligned}T^*_{\vec{b},D,q}(\vec{f})(y)&\le\bigg(\sum_{k=1}^{\infty}\bigg(\int_{B(0,R)^m}\frac{A_k\prod_{(i,j)\in S}|b_i(y)-b_i(y_j)|}{\prod_{j=1}^m|y-y_j|^n}\prod_{j=1}^m|f_{jk(y_j)}|d\vec{y}\bigg)^q\bigg)^{1/q}\\&\le C\prod_{(i,j)\in S}\|b_i\|_{L^\infty}\prod_{j=1}^m\frac{1}{|y|^n}\int_{B(0,|y|)}\bigg(\sum_{k=1}^{\infty}|A_k|^{q_j/m}|f_{jk}|^{q_j}\bigg)^{1/q_j}dy_j\\&\le C\prod_{(i,j)\in S}\|b_i\|_{L^\infty}\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f}_{T,\vec{q}})(y).\end{aligned}$$

Then

$$\begin{aligned}II&\le\sup_{\lambda>0}\varphi(\lambda)\omega\bigg\{y\in\mathbb{R}^n:\bigg|C\prod_{(i,j)\in S}\|b_i\|_{L^\infty}\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f}_{T,\vec{q}})(y)\bigg|>\lambda^m\bigg\}\\&\le C\sup_{\lambda>0}\varphi(\lambda)\omega\bigg(y\in\mathbb{R}^n:\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f}_{T,\vec{q}})(y)>\frac{\lambda^m}{\prod_{(i,j)\in S}\|b_i\|_{Osc_{expL^{r_{ij}}}}}\bigg)<\infty.\end{aligned}\tag{5.12}$$

By (5.11) and (5.12), we get (5.10). Thus we finish the proof. □

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