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Jacobian fibrations on the singular $K3$ surface of discriminant 3

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Abstract. In this paper we give the Weierstrass equations and the generators of Mordell–Weil groups for Jacobian fibrations on the singular $K3$ surface of discriminant 3.

1. Introduction.

A $K3$ surface defined over the complex number field whose Picard number equals to maximum possible number 20 is called a *singular* $K3$ *surface*. Shioda and Inose [**11**] showed that the map which associates a singular $K3$ surface X with its transcendental lattice T_X is a bijective correspondence from the set of singular $K3$ surfaces onto the set of equivalence classes of positive-definite even integral lattice of rank two with respect to $SL_2(\mathbb{Z})$. The discriminant of a singular $K3$ surface X is the determinant of the Gram matrix of the transcendental lattice T_X.

In this paper we study Jacobian fibrations, i.e., elliptic fibrations with a section, on the singular $K3$ surface X_3 of discriminant 3, which corresponds to the lattice defined by $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ and is uniquely determined up to isomorphism. Jacobian fibrations on X_3 were classified by Nishiyama [**8**]. He classified all configurations of singular fibers of Jacobian fibrations on X_3 into 6 classes and determined their Mordell–Weil groups. We give a Weierstrass model of a fibration in each class. More precisely, we state our main theorem.

Theorem 1. *Let X_3 be the singular $K3$ surface of discriminant 3. For a Jacobian fibration in each class of Nishiyama's list* [**8**, Table 1.1], *an elliptic parameter u_i, a Weierstrass equation and the generators of the Mordell–Weil group are given by Table* 1.

An *elliptic parameter* of a Jacobian fibration $\pi : X_3 \to \mathbb{P}^1$ is the pull-back $\pi^*(u_i)$ of the affine coordinate u of $\mathbb{P}^1$. We also denote it by u, and regard u as a rational function on X_3. The generic fiber of π defines an elliptic curve E over the rational function field $\mathbb{C}(u)$. Therefore, it may be defined by a Weierstrass equation, which is called a *Weierstrass equation for the Jacobian fibration* π. It is well known that the set of sections of π forms an abelian group that is isomorphic to the Mordell–Weil group $E(\mathbb{C}(u))$. It is also called the *Mordell–Weil group of the Jacobian fibration* π.

We explain about Table 1. The first column shows the name of each Jacobian fibrations following Nishiyama's notation. The second column shows the configuration of singular fibers. Here, for example, by $2\,\mathrm{II}^* + \mathrm{IV}$ means that the surface has two singular

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Table 1. Classification of Jacobian fibrations on X_3.

No.	sing. fibs	MWG	u_i	equation and rational points
1	$2\,\mathrm{II}^* + \mathrm{IV}$	0	$\dfrac{2(y_2+1)}{(y_1-1)^2}$	$Y^2 = X^3 + u_1^5(u_1-1)^2$ O
2	$\mathrm{I}_{12}^* + \mathrm{I}_3 + 3\,\mathrm{I}_1$	$\mathbb{Z}/2\mathbb{Z}$	$\dfrac{2t^2}{(y_2+1)(y_1^2+2y_1+2y_2-1)}$	$Y^2 = X^3 - 2u_2(u_2^3-2)X^2 + u_2^8 X$ $O,\ (0,0)$
3	$\mathrm{III}^* + \mathrm{I}_6^* + 3\,\mathrm{I}_1$	$\left\langle \dfrac{3}{2} \right\rangle \oplus \mathbb{Z}/2\mathbb{Z}$	$\dfrac{t}{y_1^2-1}$	$Y^2 = X^3 + 4u_3^3X^2 - 4u_3^3X$ 2-tor.: $O,\ (0,0)$ free gen. : $(1,-1)$
4	$\mathrm{I}_{18} + 6\,\mathrm{I}_1$	$\left\langle \dfrac{3}{2} \right\rangle \oplus \mathbb{Z}/3\mathbb{Z}$	$\dfrac{t}{y_1+y_2}$	$Y^2 = X^3 + (X-u_4^6)^2$ 3-tor. : $O,\ (0,\pm u_4^6)$ free gen. : $(2u_4^3, 2u_4^3+u_4^6)$
5	$3\,\mathrm{IV}^*$	$\mathbb{Z}/3\mathbb{Z}$	y_1	$Y^2 = X^3 + (u_5^2-1)^4$ $O,\ (0,\pm(u_5^2-1)^2)$
6	$\mathrm{I}_3^* + \mathrm{I}_{12} + 3\,\mathrm{I}_1$	$\mathbb{Z}/4\mathbb{Z}$	t	$Y^2 = X^3 - 2(u_6^3-2)X^2 + u_6^6X$ $O,\ (0,0),\ (u_6^3, \pm 2u_6^3)$

fibers of type II^* and a singular fiber of type IV (Kodaira's notation [**4**]). The third column shows the Mordell–Weil group (MWG) of the fibration. The fourth column shows an elliptic parameter u_i of the fibration under the singular affine model (2.6) of X_3. The index i is the name of the fibration. The last column shows a Weierstrass equation and rational points corresponding to Mordell–Weil generator of the fibration, where O is the rational point corresponding to the zero of MWG. We will give an outline of a way to get these data in the next section after we fix the notation.

Recently, Braun, Kimura and Watari [**2**] showed that Nishiyama's list also gives the classification of Jacobian fibrations on X_3 modulo isomorphism. Thus, our and their results answer completely a question of Kuwata and Shioda [**7**].

2. Notation.

The singular $K3$ surface X_3 is known as *a generalized Kummer surface* constructed as follows. Let C_ω be the complex elliptic curve with the fundamental periods 1 and $\omega = e^{2\pi\sqrt{-1}/3}$. Let σ be an automorphism of $C_\omega \times C_\omega$ defined by $\sigma(z_1, z_2) \mapsto (\omega z_1, \omega^2 z_2)$. Then the minimal resolution of the quotient $C_\omega \times C_\omega/\langle\sigma\rangle$ is isomorphic to the singular $K3$ surface X_3 (see [**11**, Lemma 5.1]). The automorphism σ has 9 fixed points $(v_i, v_j)\,(1 \le i, j \le 3)$, where $\{v_i\}$ are the fixed points of the automorphism σ_1 of C_ω defined by $\sigma_1(z) = \omega z$. These 9 points (v_i, v_j) correspond to the singular points p_{ij} of the quotient $C_\omega \times C_\omega/\langle\sigma\rangle$. The minimal resolution X_3 of $C_\omega \times C_\omega/\langle\sigma\rangle$ is obtained by replacing each p_{ij} by 2 non-singular rational curves $E_{i,j}$ and $E'_{i,j}$ with $E_{i,j} \cdot E'_{i,j} = 1$. Moreover, X_3 contains 6 non-singular rational curves, i.e. the image F_i (or G_j) of $\{v_i\} \times C_\omega$ (or $C_\omega \times \{v_j\}$) in X_3. We have the following intersection numbers.

$$F_i^2 = G_i^2 = E_{i,j}^2 = E_{i,j}'^2 = -2, \quad F_i \cdot E_{j,k} = G_i \cdot E'_{j,k} = F_i \cdot G_j = 0,$$
$$E_{i,j} \cdot E'_{k,l} = \delta_{i,k} \cdot \delta_{j,l}, \quad F_i \cdot E'_{j,k} = G_i \cdot E_{k,j} = \delta_{i,j}. \tag{2.1}$$

These 24 curves on X_3 form the configuration of Figure 1.

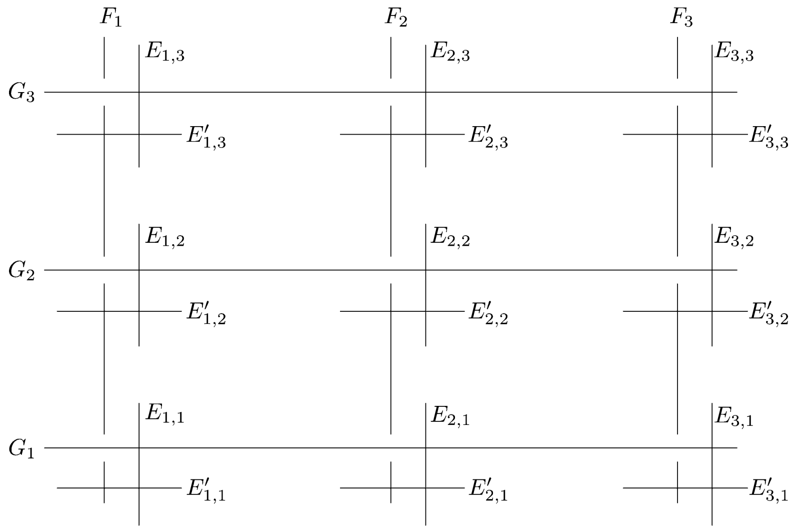


Figure 1. (-2)-curves.

It is well known that the elliptic curve C_ω has the following Weierstrass form

$$C_\omega : y^2 = x^3 + 1. \tag{2.2}$$

We denote each factor of $C_\omega \times C_\omega$ by

$$C_\omega^1 : y_1^2 = x_1^3 + 1, \quad C_\omega^2 : y_2^2 = x_2^3 + 1. \tag{2.3}$$

Then the automorphism σ is written by

$$\begin{aligned} \sigma : C_\omega^1 \times C_\omega^2 &\to C_\omega^1 \times C_\omega^2 \\ (x_1, y_1, x_2, y_2) &\mapsto (\omega x_1, y_1, \omega^2 x_2, y_2). \end{aligned} \tag{2.4}$$

The function field $\mathbb{C}(X_3)$ is equal to the invariant subfield of the function field $\mathbb{C}(C_\omega^1 \times C_\omega^2) = \mathbb{C}(x_1, x_2, y_1, y_2)$ under the automorphism σ. Then we have

$$\mathbb{C}(X_3) = \mathbb{C}(y_1, y_2, t), \quad t = x_1 x_2, \tag{2.5}$$

where y_1, y_2, and t are naturally regarded as functions on X_3 with the relation

$$t^3 = (y_1^2 - 1)(y_2^2 - 1). \tag{2.6}$$

This gives a singular affine model of X_3. We start from the equation to obtain a Weierstrass form for each Jacobian fibration on X_3. Under the above notation, we see that the divisors of typical functions are as follows.

$$
\begin{aligned}
(y_1 - 1) &= 3F_2 + 2\big(E'_{2,1} + E'_{2,2} + E'_{2,3}\big) + E_{2,1} + E_{2,2} + E_{2,3} \\
&\quad - \big(3F_1 + 2(E'_{1,1} + E'_{1,2} + E'_{1,3}) + E_{1,1} + E_{1,2} + E_{1,3}\big) \\
(y_1 + 1) &= 3F_3 + 2\big(E'_{3,1} + E'_{3,2} + E'_{3,3}\big) + E_{3,1} + E_{3,2} + E_{3,3} \\
&\quad - \big(3F_1 + 2(E'_{1,1} + E'_{1,2} + E'_{1,3}) + E_{1,1} + E_{1,2} + E_{1,3}\big) \\
(y_2 - 1) &= 3G_2 + 2\big(E_{1,2} + E_{2,2} + E_{3,2}\big) + E'_{1,2} + E'_{2,2} + E'_{3,2} \\
&\quad - \big(3G_1 + 2(E_{1,1} + E_{2,1} + E_{3,1}) + E'_{1,1} + E'_{2,1} + E'_{3,1}\big) \\
(y_2 + 1) &= 3G_3 + 2\big(E_{1,3} + E_{2,3} + E_{3,3}\big) + E'_{1,3} + E'_{2,3} + E'_{3,3} \\
&\quad - \big(3G_1 + 2(E_{1,1} + E_{2,1} + E_{3,1}) + E'_{1,1} + E'_{2,1} + E'_{3,1}\big) \\
(t) &= F_2 + E'_{2,3} + E_{2,3} + G_3 + E_{3,3} + E'_{3,3} + F_3 + E'_{3,2} + E_{3,2} + G_2 + E_{2,2} + E'_{2,2} \\
&\quad - \big(E_{2,1} + E_{3,1} + 2(G_1 + E_{1,1} + E'_{1,1} + F_1) + E'_{1,2} + E'_{1,3}\big).
\end{aligned}
\tag{2.7}
$$

For a Jacobian fibration in each class of Table 1, we compute a Weierstrass equation by using the following two methods.

The first method is the elimination method. Theoretically, constructing a Jacobian fibration on a $K3$ surface is done by finding a divisor that has the same type as a singular fiber in the Kodaira's list (see [**4**]). In practice, however, we need to find two divisors, one for the fiber at $u = 0$, and the other for the fiber at $u = \infty$, to write down an actual elliptic parameter u. Once an elliptic parameter is found, we want to find a change of variables that converts the defining equation to a Weierstrass form. Since an elliptic parameter u is a rational function, we can write $u = f/g$ for some $f, g \in \mathbb{C}[t, y_1, y_2]$. Thus, we can eliminate one variable from the equations (2.6) and $gu - f = 0$. If such an equation can be converted to the form $y^2 =$ (quartic polynomial) by a simple change of coordinates, we can transform it to a Weierstrass form by using a standard algorithm (see for example [**1**] or [**3**]). We use this method to compute Weierstrass equations for Fibrations 1, 3, 5 and 6 in Sections 3–6.

For Fibrations 2 and 4, it is difficult to find such two divisors described as above. Thus, we use the other method for them, which is called 2-*neighbor step* by Noam Elkies. This is a technique to transform a Weierstrass equation for a Jacobian fibration to another for a distinct Jacobian fibration. Using this, we obtain a Weierstrass equation for Fibration 4 from Fibration 3 in Section 7. Moreover, we can transform it to a Weierstrass equation for Fibration 2 in Section 8.

Every Jacobian fibration except for Fibration 1 has nontrivial Mordell–Weil group. In each case, we can easily write down the torsion part of the Mordell–Weil group as rational points of the elliptic curve defined over $\mathbb{C}(u)$ by the Weierstrass equation. To determine the free generators of Fibrations 3 and 4, we compute the height paring by using the method in [**10**] from the intersection numbers (2.1) and establish some changes of variables.

3. Fibration 1.

An elliptic parameter for Fibration 1 is given by

$$u_1 = \frac{2(y_1+1)}{(y_1-1)^2}. \tag{3.1}$$

The divisor of u_1 is given by

$$\begin{aligned}(u_1) = {} & E'_{3,3} + 2E_{3,3} + 3G_3 + 4E_{1,3} + 5E'_{1,3} + 6F_1 + 3E'_{1,1} + 4E'_{1,2} + 2E_{1,2} \\ & - \left(E'_{3,1} + 2E_{3,1} + 3G_1 + 4E_{2,1} + 5E'_{2,1} + 6F_2 + 3E'_{2,3} + 4E'_{2,2} + 2E_{2,2}\right). \end{aligned} \tag{3.2}$$

The zero divisor $(u_1)_0$ (the bold lines in Figure 2) and the polar divisor $(u_1)_\infty$ (the thin lines in Figure 2) are the singular fibers both of type II*.

Eliminating the variable y_2 from (2.6) and (3.1), we obtain the following equation

$$4t^3 = u_1(y_1+1)(y_1-1)^3(u_1y_1^2 - 2u_1y_1 + u_1 - 4), \tag{3.3}$$

which defines a plane curve over $\mathbb{C}(u_1)$ with a singularity at $(t, y_1) = (0, 1)$. Blowing up by $t = v(y_1 - 1)$, we have the following equation

$$4v^3 = u_1(y_1+1)(u_1y_1^2 - 2u_1y_1 + u_1 - 4), \tag{3.4}$$

which defines a nonsingular plane cubic curve over $\mathbb{C}(u_1)$ with a rational point $(v, y_1) = (0, -1)$. Then we can convert it into a Weierstrass form (see [**1**] or [**3**]). Since the rational point $(v, y_1) = (0, -1)$ corresponds to the divisor F_3 (the dotted line in Figure 2), choosing it as the zero section of the group structure, we obtain the Weierstrass equation for Fibration 1

$$Y^2 = X^3 + u_1^5(u_1-1)^2, \tag{3.5}$$

where the change of variables is given by

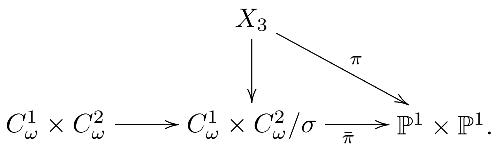
$$X = \frac{\sqrt[3]{4}(u_1-1)u_1t}{(y_1^2-1)}, \quad Y = -\frac{u_1^2(u_1-1)(u_1y_1-u_1+2)}{y_1+1}. \tag{3.6}$$

Besides the two singular fibers of type II* at $u_1 = 0$ and ∞, there is one singular fiber of type IV at $u_1 = 1$. It is the divisor $E_{3,2} + E'_{3,2} + Q_1$ (the long dashed dotted lines in Figure 2), where Q_1 is a (-2)-curve on X_3 arising from a curve on $\mathbb{P}^1 \times \mathbb{P}^1$ below.

Let $p_j : C^j_\omega \to \mathbb{P}^1$ $(j = 1, 2)$ be the projection given by

$$\begin{array}{cccc} p_j : & C^j_\omega & \to & \mathbb{P}^1 \\ & (x_j : y_j : z_j) & \mapsto & \begin{cases} (y_j : z_j) & \text{if } z_j \neq 0 \\ (1:0) & \text{if } z_j = 0. \end{cases} \end{array} \tag{3.7}$$

Then the map $p_1 \times p_2 : C^1_\omega \times C^2_\omega \to \mathbb{P}^1 \times \mathbb{P}^1$ factors through $\bar{\pi} : C^1_\omega \times C^2_\omega/\sigma \to \mathbb{P}^1 \times \mathbb{P}^1$. Let π be the morphism of degree three from X_3 to $\mathbb{P}^1 \times \mathbb{P}^1$ that makes the following diagram commutative:



It is easy to verify that the equation $u_1 = 1$ means

$$y_1^2 - 2y_1 - 2y_2 - 1 = 0 \tag{3.8}$$

from (3.1). This equation defines a curve on $\mathbb{P}^1 \times \mathbb{P}^1$. Then it lifts to the (-2)-curve Q_1 on X_3 via the map π.

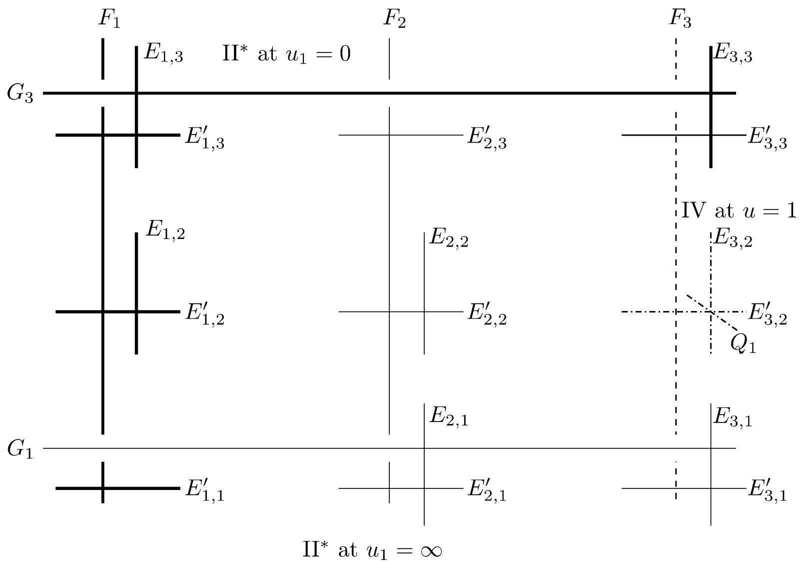


Figure 2. Fibration 1.

4. Fibration 3.

An elliptic parameter for Fibration 3 is given by

$$u_3 = \frac{t}{y_1^2 - 1}. \tag{4.1}$$

The divisor of u_3 is given by

$$\begin{aligned}(u_3) = {} & G_2 + 2E_{1,2} + 3E'_{1,2} + 4F_1 + 3E'_{1,1} + 2E_{1,3} + G_3 + 3E'_{1,2} \\ & - \left(E'_{2,2} + E'_{2,3} + 2(F_2 + E'_{2,1} + E_{2,1} + G_1 + E_{3,1} + E'_{3,1} + F_3) + E'_{3,2} + E'_{3,3}\right),\end{aligned} \tag{4.2}$$

which is indicated in Figure 3. The zero divisor $(u_3)_0$ is the singular fiber of type III^* (the bold lines) and the polar divisor $(u_3)_\infty$ is the singular fiber of type I_6^* (the thin lines). The curves $E_{2,2}, E_{2,3}, E_{3,2}$ and $E_{3,3}$ (the dotted lines) are all the sections.

Eliminating the variable t from (2.6) and (4.1), we have the following equation

$$y_2^2 = u_3^3(y_1^2 - 1)^2 + 1, \tag{4.3}$$

which has a rational point $(y_1, y_2) = (1, 1)$ corresponding to the curve $E_{2,2}$. Thus, choosing $E_{2,2}$ as the zero section of the group structure, we obtain the Weierstrass equation for Fibration 3

$$Y^2 = X^3 + 4u_3^3 X^2 - 4u_3^3 X, \tag{4.4}$$

where the change of variables is given by

$$X = \frac{2(y_2 + 1)}{(y_1 - 1)^2}, \quad Y = \frac{4(u_3^3(y_1 + 1)(y_1 - 1)^2 + y_2 + 1)}{(y_1 - 1)^3}. \tag{4.5}$$

Besides the above two singular fibers of types III^* and I_6^*, the fibration has three I_1 fibers at $u_3 = -1, -\omega$ and $-\omega^2$.

The 2-torsion rational point $(X, Y) = (0, 0)$ corresponds to the curve $E_{3,3}$. The rational point $(X, Y) = (1, -1)$ corresponds to the curve $E_{3,2}$ of height $\langle E_{3,2}, E_{3,2} \rangle = 3/2$, which is a generator of the Mordell–Weil lattice of the fibration. The curve $E_{2,3}$ is another free section corresponding to the rational point $(1, 1)$ with the relation $E_{2,3} = -E_{3,2}$ in the Mordell–Weil group.

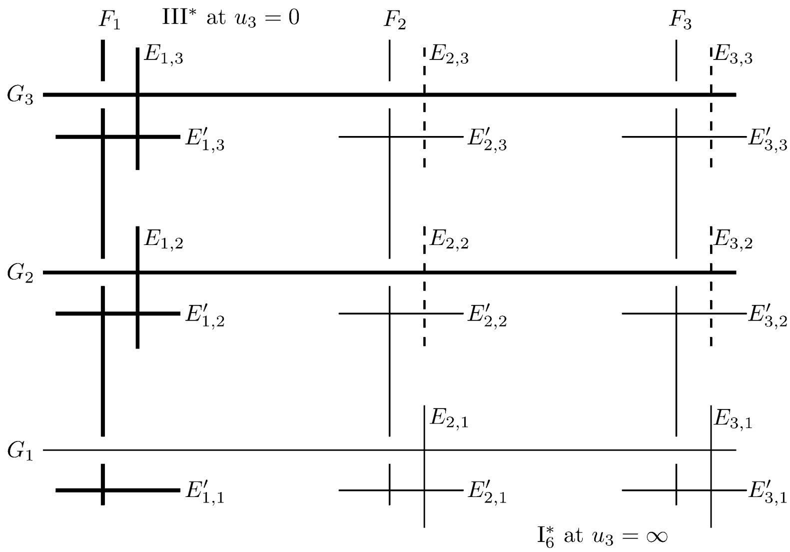


Figure 3. Fibration 3.

5. Fibration 5.

An elliptic parameter for Fibration 5 is given by

$$u_5 = y_1. \tag{5.1}$$

It is clear that this elliptic parameter defines a fibration having three singular fibers all of types IV^* at $u_5 = 1, -1$ and ∞ (the bold lines in Figure 4) from (2.7). Furthermore the fibration is induced by the composition of the first projection $C^1_\omega \times C^2_\omega \to C^1_\omega$ and the covering map of degree three $p_1 : C^1_\omega \to \mathbb{P}^1$ in (3.7).

The following simple coordinate change

$$X = (u_5^2 - 1)t, \quad Y = (u_5^2 - 1)^2 y_2 \tag{5.2}$$

converts the equation (2.6) into the Weierstrass equation for Fibration 5

$$Y^2 = X^3 + (u_5^2 - 1)^4. \tag{5.3}$$

The curve G_1, G_2 and G_3 correspond to the zero section, 3-torsion rational points $(0, (u_5^2 - 1)^2)$ and $(0, -(u_5^2 - 1)^2)$, respectively (the dotted lines in Figure 4).

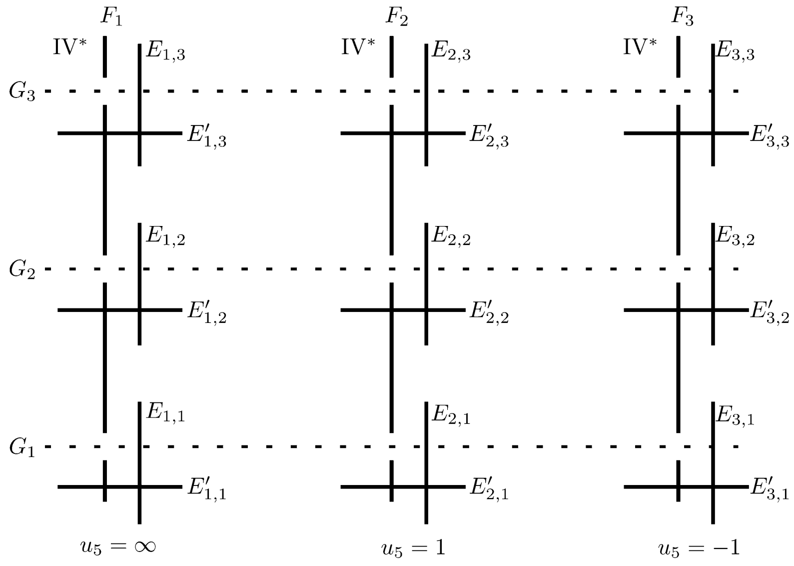


Figure 4. Fibration 5.

6. Fibration 6.

An elliptic parameter for Fibration 6 is given by

$$u_6 = t. \tag{6.1}$$

Since we gave the divisor of t in (2.7), we know that the zero divisor $(u_6)_0$ is the singular fiber of type I_{12} (the bold lines in Figure 5) and the polar divisor $(u_6)_\infty$ is the singular fiber of type I_3^* (the thin lines in Figure 5). The curves $E_{1,2}, E_{1,3}, E'_{2,1}$ and $E'_{3,1}$ (the dotted lines in Figure 5) are all the sections. Choosing $E_{1,2}$ as the zero section of the group structure, we obtain the Weierstrass equation for Fibration 6

$$Y^2 = X^3 - 2(u_6^3 - 2)X^2 - u_6^6 X, \tag{6.2}$$

where the change of variables is given by

$$X = \frac{t^3(y_2+1)}{y_2-1}, \quad Y = \frac{2t^3 y_1(y_2+1)}{y_2-1}. \tag{6.3}$$

Besides the two singular fibers of type I_{12} at $u_6 = 0$ and of type I_3^* at $u_6 = \infty$, there are three I_1 fibers at $u_6 = 1, \omega$ and ω^2. The Mordell–Weil group of the fibration is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. The curve $E_{1,3}$ corresponds to the rational point $(0,0)$ of order two, and remaining curves $E'_{2,1}$ and $E'_{3,1}$ correspond to the rational points $(u_6^3, 2u_6^3)$, $(u_6^3, -2u_6^3)$ of order four, respectively.

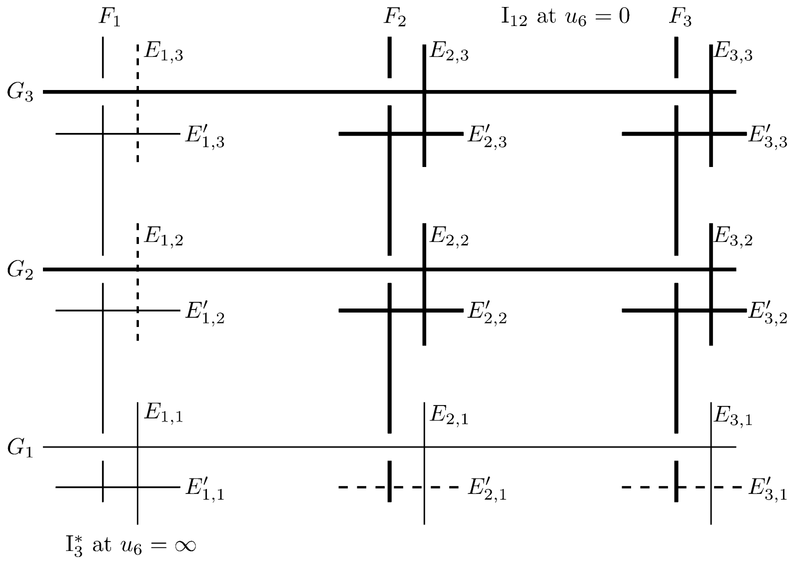


Figure 5. Fibration 6.

7. Fibration 4.

To obtain the Weierstrass equation for Fibration 4, we use a 2-neighbor step from Fibration 3. For more detail about 2-*neighbor step*, we refer to [**5**], [**9**], [**12**].

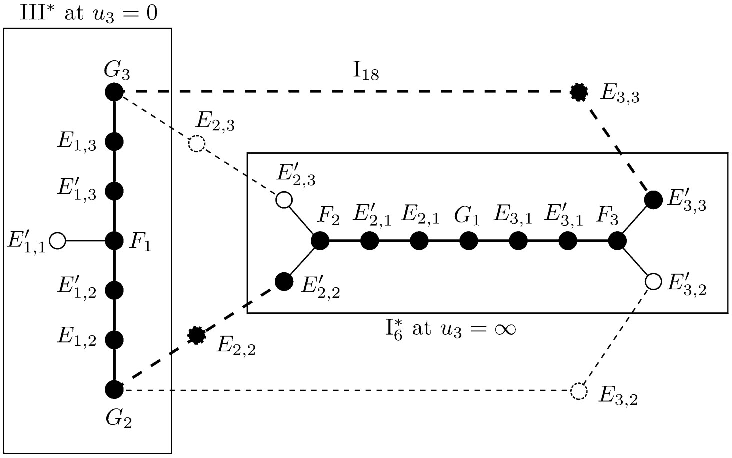


Figure 6. 2-neighbor from Fibration 3 to Fibration 4.

We compute explicitly the elements of $\mathcal{O}_{X_3}(F)$ where

$$\begin{aligned} F = E_{2,2} + G_2 + E_{1,2} + E'_{1,2} + F_1 + E'_{1,3} + E_{1,3} + G_3 + E_{3,3} + E'_{3,3} + F_3 \\ + E'_{3,1} + E_{3,1} + G_1 + E_{2,1} + E'_{2,1} + F_2 + E'_{2,2} \end{aligned} \tag{7.1}$$

is the class of the fiber of type I_{18} we are considering. The linear space $\mathcal{O}_{X_3}(F)$ is 2-dimensional, and the ratio of two linearly independent elements is an elliptic parameter for X_3. Since 1 is an element of $\mathcal{O}_{X_3}$, we may find a non-constant element of $\mathcal{O}_{X_3}(F)$. Then it will be an elliptic parameter of Fibration 4. Let us $u'_4 \in \mathcal{O}_{X_3}(F)$ be a non-constant. The function u'_4 has a simple pole along $E_{2,2}$ and $E_{3,3}$, which are the zero section and 2-torsion of Fibration 3. Also, it has a simple pole along G_2, the identity component of the fiber at $u_3 = 0$, a simple pole along $E'_{3,3}$, the identity component of the fiber at $u_3 = \infty$. Therefore we can put

$$u'_4 = \frac{\frac{Y}{X} + A_0 + A_1 u_3 + A_2 u_3^2}{u_3}, \tag{7.2}$$

where the variables u_3, X, Y are given by (4.1) and (4.5). Assume $A_1 = 0$, since 1 is an element of $\mathcal{O}_{X_3}(F)$. To obtain the coefficients A_0 and A_2, we look at the order of vanishing along the non-identity components of fibers at $u_3 = \infty$. The function u'_4 does not have any pole along $E'_{3,2}$, which intersects with the section $E_{3,2}$ of the fibration 3 at $u_3 = \infty$. Hence u'_4 has no pole at $(X, Y, u_3) = (1, -1, \infty)$, and that gives us $A_2 = 0$. Similarly, the component $E'_{2,3}$, which intersects with the section $E_{2,3}$, gives us $A_0 = 0$. Consequently, we have a new elliptic parameter

$$u_4' = \frac{Y}{u_3 X}, \tag{7.3}$$

where the variables u_3, X, Y are given by (4.1) and (4.5). Solving for Y and substituting into the Weierstrass equation (4.4), after suitable coordinate changes we have the following

$$y^2 = x^3 + \frac{1}{4}\left({u_4'}^2 x - 16\right)^2. \tag{7.4}$$

Although this is a Weierstrass equation for Fibration 4, for latter calculations, we put

$$u_4{}' = \frac{2}{u_4},\ x = \frac{2^2 X}{u_4^4},\ y = \frac{2^3 Y}{u_4^6} \tag{7.5}$$

and obtain another Weierstrass equation for Fibration 4

$$Y^2 = X^3 + (X - u_4^6)^2. \tag{7.6}$$

The change of variables is given by

$$u_4 = \frac{t}{y_1 + y_2},\ X = \frac{(y_1^2 - 1)t^3}{(y_1 + y_2)^4},\ Y = \frac{(y_1^2 y_2 + 2y_1 + y_2)t^6}{(y_2^2 - 1)(y_1 + y_2)^6}. \tag{7.7}$$

The fibration has singular fibers of type I_{18} at $u_4 = 0$ and of type I_1 at the zeros of $27u_4^6 + 4 = 0$. The zero section corresponds to the divisor $E'_{1,1}$. The 3-torsion rational points $(0, u_4^6)$ and $(0, -u_4^6)$ correspond to the divisors $E'_{3,2}$ and $E'_{2,3}$, respectively. The free rational points $(2u_4^3, u_4^4 + 2u_4^3)$ and $(-2u_4^6, u_4^3 - 2u_4^3)$ correspond to the divisors $E_{3,2}$

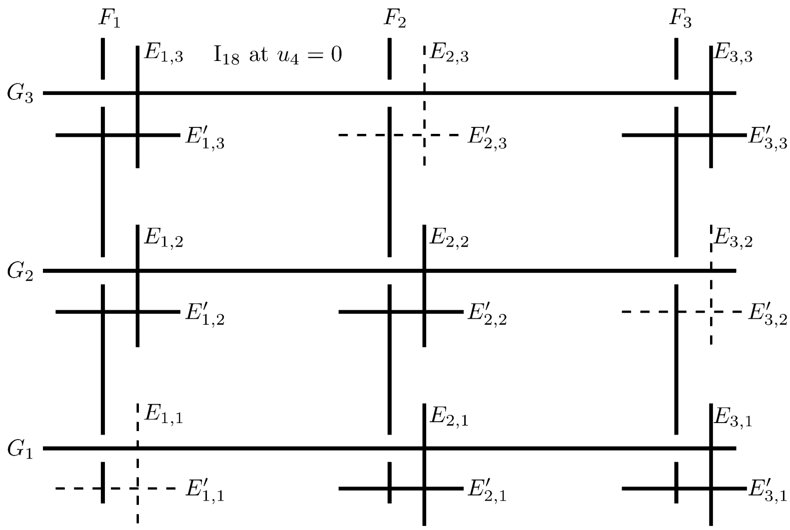


Figure 7. Fibration 4.

and $E_{2,3}$, respectively with the relation $E_{2,3} + E_{3,2} = E'_{2,3}$ in the Mordell–Weil group. Since the height of $E_{2,3}$ is equal to $3/2$, $E_{2,3}$ generates the Mordell–Weil lattice of the fibration.

8. Fibration 2.

We obtain the following elliptic parameter u'_2 for Fibration 2 by a 2-neighbor step from Fibration 4 (see Figure 8).

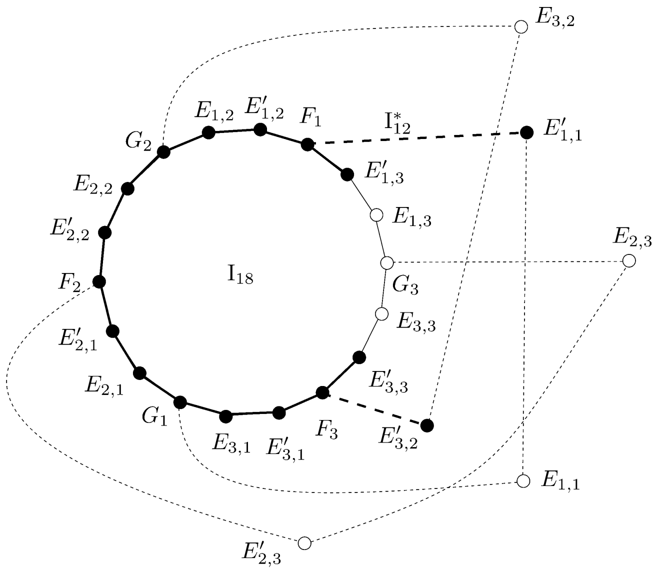


Figure 8. 2-neighbor from Fibration 4 to Fibration 2.

$$u'_2 = \frac{u_4^6 + X + Y}{u_4^2 X} \tag{8.1}$$

The variables u_4, X, Y are given by (7.7). Then we get the following Weierstrass equation for Fibration 2.

$$y^2 = x^3 + 2\big({u'_2}^3 - 4\big)x^2 + 16x. \tag{8.2}$$

We put

$$u'_2 = \frac{2}{u_2},\ x = \frac{2^2 X}{u_2^4},\ y = \frac{2^3 Y}{u_2^6} \tag{8.3}$$

and obtain another Weierstrass equation for Fibration 4.

$$Y^2 = X^3 - 2(u_2^3 - 2)X^2 - u_2^8 X. \tag{8.4}$$

The change of variables is given by

$$\begin{aligned} u_2 &= \frac{2t^2}{(y_2+1)(y_1^2+2y_1+2y_2-1)}, \\ X &= -\frac{32(y_1-1)^2(y_2-1)^3t^2}{(y_2+1)^2(y_1^2+2y_1+2y_2-1)^4}, \\ Y &= -\frac{128(y_1-1)^3(y_2-1)^4(y_1+1)(y_1+y_2)}{(y_2+1)^2(y_1^2+2y_1+2y_2-1)^5}. \end{aligned} \tag{8.5}$$

The zero divisor $(u_4)_0$ is the singular fiber of type I_{12}^* (the bold lines in Figure 9). The polar divisor $(u_4)_\infty = G_3 + E_{2,3} + Q_2$ is the singular fiber of type I_3 (the thin lines in Figure 9), where the divisor Q_2 is the lifting of the curve $y_1^2+2y_1+2y_2-1=0$ on $\mathbb{P}^1\times\mathbb{P}^1$ by the map π in Section 3. Besides these two singular fibers, there are three I_1 fibers at $u_2 = 1, \omega$ and ω^2. The zero section corresponds to the divisor $E_{1,3}$. The 2-torsion rational point $(0,0)$ corresponds to the divisor $E_{3,3}$.

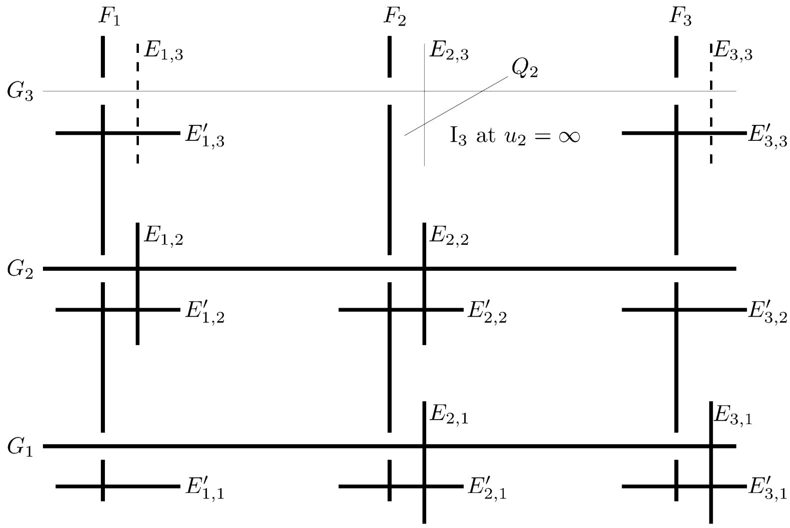


Figure 9. Fibration 2.

Remark 2. We give a Weierstrass equation for Fibration 6 in Section 6. Comparing the equations (8.4) and (6.2), we know easily that Fibration 2 is a quadratic twist of Fibration 6. This is the reason why we adopt the equation (8.4) as the Weierstrass equation for Fibration 2 rather than the equation (8.2).

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