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Reflection principles for ω_2 and the semi-stationary reflection principle

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Abstract. Starting from a model with a weakly compact cardinal, we construct a model in which the weak stationary reflection principle for ω_2 holds but the Fodor-type reflection principle for ω_2 fails. So the stationary reflection principle for ω_2 fails in this model. We also construct a model in which the semi-stationary reflection principle holds but the Fodor-type reflection principle for ω_2 fails.

1. Introduction.

Various *reflection principles* are known and studied widely. First we review several reflection principles.

We recall some basic definitions. Let X be an uncountable set. A set $C \subseteq [X]^\omega$ is *club in* $[X]^\omega$ if:

(1) For every $x \in [X]^\omega$ there is $y \in C$ with $x \subseteq y$.
(2) For every $\alpha < \omega_1$ and every $\subseteq$-increasing sequence $\langle x_i : i < \alpha \rangle$ in C, we have $\bigcup_{i<\alpha} x_i \in C$.

A set $S \subseteq [X]^\omega$ is *stationary in* $[X]^\omega$ if $S \cap C \neq \emptyset$ for every club C in $[X]^\omega$. It is known that $S \subseteq [X]^\omega$ is stationary in $[X]^\omega$ if and only if for every function $f : [X]^{<\omega} \to X$, there is $x \in S$ which is closed under f.

Definition 1.1. Let λ be a cardinal $\geq \omega_2$. WRP(λ) (*the Weak stationary Reflection Principle for* λ) is the assertion that for every stationary $S \subseteq [\lambda]^\omega$, there exists $X \in [\lambda]^{\omega_1}$ such that $\omega_1 \subseteq X$ and $S \cap [X]^\omega$ is stationary in $[X]^\omega$. Let WRP be the assertion that WRP(λ) holds for every $\lambda \geq \omega_2$. When λ is regular, RP(λ) (*the stationary Reflection Principle for* λ) is the assertion that for every stationary $S \subseteq [\lambda]^\omega$, there exists $X \in [\lambda]^{\omega_1}$ such that $\omega_1 \subseteq X$, $\mathrm{cf}(\sup(X)) = \omega_1$, and $S \cap [X]^\omega$ is stationary in $[X]^\omega$. RP is the assertion that RP(λ) holds for every regular $\lambda \geq \omega_2$.

WRP was introduced by Foreman–Magidor–Shelah [**4**] and it has many interesting consequences.

Shelah developed a reflection principle of semi-stationary sets in the study of semiproper forcing notions:

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Definition 1.2 (Shelah [**14**, Chapter XIII, Section 1, 1.1. Definition and 1.5. Definition]). For a set X with $\omega_1 \subseteq X$, a subset $S \subseteq [X]^\omega$ is *semi-stationary in* $[X]^\omega$ if the set $\{x \in [X]^\omega : \exists y \in S\, (y \subseteq x \text{ and } y \cap \omega_1 = x \cap \omega_1)\}$ is stationary in $[X]^\omega$. For a cardinal $\lambda \geq \omega_2$, $\mathrm{SSR}(\lambda)$ (*the Semi-Stationary Reflection principle for* λ) is the assertion that for every semi-stationary $S \subseteq [\lambda]^\omega$, there is $X \in [\lambda]^{\omega_1}$ such that $\omega_1 \subseteq X$ and $S \cap [X]^\omega$ is semi-stationary in $[X]^\omega$. SSR is the assertion that $\mathrm{SSR}(\lambda)$ holds for every $\lambda \geq \omega_2$.

Shelah [**14**] showed that SSR is equivalent to the statement that every ω_1-stationary preserving forcing notion is semiproper, and Doebler–Schindler [**3**] showed that SSR can be characterized by a generalized Chang's conjecture.

We turn to another reflection principle. For a set E of ordinals, a *ladder system on* E is a sequence $\vec{c} = \langle c_\alpha : \alpha \in E\rangle$ such that each $c_\alpha \subseteq \alpha$ is unbounded in α and $\mathrm{ot}(c_\alpha) = \mathrm{cf}(\alpha)$. We will sometime denote c_α as $\vec{c}_\alpha$ to emphasize the sequence $\vec{c}$.

Definition 1.3. Let λ be a regular cardinal $\geq \omega_2$. $\mathrm{FRP}(\lambda)$ (*the Fodor-type Reflection Principle for* λ) is the assertion that for every stationary $E \subseteq \{\alpha < \lambda : \mathrm{cf}(\alpha) = \omega\}$ in λ and every ladder system $\vec{c}$ on E, there is $I \in [\lambda]^{\omega_1}$ such that $\mathrm{cf}(\sup(I)) = \omega_1$, $c_\alpha \subseteq I$ for $\alpha \in E \cap I$, and for every function $f : E \cap I \to I$ with $f(\alpha) \in c_\alpha$, there is γ with $\{\alpha \in I \cap E : f(\alpha) = \gamma\}$ stationary in $\sup(I)$. FRP is the assertion that $\mathrm{FRP}(\lambda)$ holds for every regular $\lambda \geq \omega_2$.

It is known that FRP can be characterized by various reflection phenomenons. For instance:

Fact 1.4 (Fuchino–Juhász–Soukup–Szentmiklóssy–Usuba [**5**], Fuchino–Sakai–Soukup–Usuba [**7**]). *The following are equivalent*:

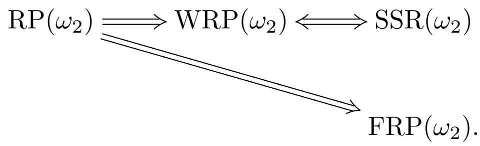
(1) FRP *holds.*
(2) *For every locally compact Hausdorff topological space X, if every subspace of X of size $\leq \omega_1$ is metrizable, then X is metrizable.*
(3) *For every regular $\lambda \geq \omega_2$, stationary $S \subseteq \{\alpha < \lambda : \mathrm{cf}(\alpha) = \omega\}$, and ladder system $\vec{c}$ on S, there is $\beta < \lambda$ such that for every regressive $f : S \cap \beta \to \beta$, there are distinct $\alpha_0, \alpha_1 \in S \cap \beta$ with $(c_{\alpha_0} \setminus f(\alpha_0)) \cap (c_{\alpha_1} \setminus f(\alpha_1)) \neq \emptyset$.*

See Fuchino–Juhász–Soukup–Szentmiklóssy–Usuba [**5**], Fuchino–Rinot [**6**], and Fuchino–Sakai–Soukup–Usuba [**7**] for FRP.

The following implications between our reflection principles are known:

Fact 1.5. (1) ([**5**]) *For every regular $\lambda \geq \omega_2$,* $\mathrm{RP}(\lambda) \Rightarrow \mathrm{FRP}(\lambda)$. *Hence* RP $\Rightarrow$ FRP.
(2) $\mathrm{WRP}(\lambda) \Rightarrow \mathrm{SSR}(\lambda)$ *for every $\lambda \geq \omega_2$. So* WRP $\Rightarrow$ SSR.
(3) (Sakai [**13**]) $\mathrm{WRP}(\omega_2) \iff \mathrm{SSR}(\omega_2)$.
(4) ([**13**]) *It is consistent that* SSR *holds but* $\mathrm{WRP}(\omega_3)$ *fails. So* SSR $\not\Rightarrow$ WRP.
(5) (Baumgartner [**1**], Veličković [**16**]) $\mathrm{ZFC} + \mathrm{WRP}(\omega_2)$ *is equiconsistent with* $\mathrm{ZFC} + \exists$*weakly compact cardinal.*
(6) (Miyamoto [**11**]) $\mathrm{ZFC} + \mathrm{FRP}(\omega_2)$ *is equiconsistent with* $\mathrm{ZFC} + \exists$*Mahlo cardinal.*

Now we have the following diagram:



FRP(ω_2) does not imply WRP(ω_2) by (5) and (6) of Fact 1.5, in fact FRP does not imply WRP(ω_2); Todorčević showed that WRP(ω_2) implies $2^\omega \leq \omega_2$ (e.g., see Todorčević [**15**]) but FRP is consistent with arbitrary large continuum ([**5**]). On the other hand, Kruger [**10**] showed that WRP(ω_2) does not imply RP(ω_2): Starting from a model with a κ^+-supercompact κ, he constructed a model in which WRP(ω_2) holds but RP(ω_2) fails.

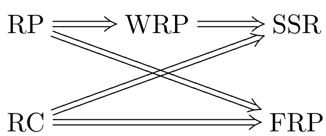
In this paper, we give a simpler construction of a model of WRP(ω_2)+$\neg$ RP(ω_2) than Krueger's one. Moreover we reduce the large cardinal assumption of a κ^+-supercompact cardinal κ to that of a weakly compact cardinal κ, which is optimal by Fact 1.5 (5), and obtain a model in which WRP(ω_2) holds but FRP(ω_2) fails (so RP(ω_2) also fails by Fact 1.5 (1)).

Theorem 1.6. *Suppose "*ZFC + $\exists$*weakly compact cardinal" is consistent. Then so is "*ZFC + WRP(ω_2) + $\neg$ FRP(ω_2) *(so* $\neg$ RP(ω_2))*".*

We also prove that even SSR does not imply FRP(ω_2), so does not RP(ω_2):

Theorem 1.7. *Suppose "*ZFC + $\exists$*supercompact cardinal" is consistent. Then so is "*ZFC + SSR +$\neg$ FRP(ω_2)*".*

By the way, we also mention RC (*Rado's Conjecture*). A tree T is *special* if T is a countable union of antichains of T. RC is the assertion that every uncountable non-special tree has a non-special subtree of size ω_1. See Todorčević's article [**15**] for RC and related topics. Recently Doebler [**2**] proved that RC implies SSR, and Fuchino–Sakai–Perez–Usuba [**8**] showed that RC implies FRP and RP(ω_2). So Theorem 1.7 tells us that SSR does not imply RC. In [**8**], they pointed out that RC does not imply WRP(ω_3).



Here we present some basic definitions, notations, and facts which will be used later sections.

For $i < 2$, let $S^2_i = \{\alpha < \omega_2 : \alpha > \omega_1, \mathrm{cf}(\alpha) = \omega_i\}$.

For a set x of ordinals, let $\lim(x) = \{\alpha < \sup(x) : \alpha$ is limit and $x \cap \alpha$ is unbounded in $\alpha\}$.

Definition 1.8. Let $S \subseteq [\omega_2]^\omega$ be stationary in $[\omega_2]^\omega$. Let WRP(S) (RP(S), respectively) be the assertion that for every stationary $S' \subseteq S$, there exists $\alpha < \omega_2$

($\alpha \in S^2_1$, respectively) such that $S' \cap [\alpha]^\omega$ is stationary in $[\alpha]^\omega$.

Note that $\mathrm{WRP}(\omega_2) \iff \mathrm{WRP}([\omega_2]^\omega)$ and $\mathrm{RP}(\omega_2) \iff \mathrm{RP}([\omega_2]^\omega)$. For a stationary $S \subseteq [\omega_2]^\omega$, $\mathrm{WRP}(S)$ holds ($\mathrm{RP}(S)$ holds, respectively) if and only if for every stationary $S' \subseteq S$, the set $\{\alpha < \omega_2 : S' \cap [\alpha]^\omega$ is stationary in $[\alpha]^\omega\}$ ($\{\alpha \in S^2_1 : S' \cap [\alpha]^\omega$ is stationary in $[\alpha]^\omega\}$, respectively) is stationary in ω_2.

We say that a subset $S \subseteq [\omega_2]^\omega$ is *non-reflecting* if $S \cap [\alpha]^\omega$ is non-stationary in $[\alpha]^\omega$ for every $\alpha < \omega_2$.

Definition 1.9. For a sequence $\vec{\pi} = \langle \pi_\alpha : \alpha < \omega_2 \rangle$ of surjections $\pi_\alpha : \omega_1 \to \alpha$, let $\mathcal{C}^{\vec{\pi}}$ be the set of all $x \in [\omega_2]^\omega$ such that:

(1) $\omega_1 \in x$ and $x \cap \omega_1 \in \omega_1$.
(2) $\sup(x) \notin x$.
(3) π_α"$(x \cap \omega_1) = x \cap \alpha$ for every $\alpha \in x$.

$\mathcal{C}^{\vec{\pi}}$ forms a club in $[\omega_2]^\omega$, and we denote it by $\mathcal{C}^*$ if $\vec{\pi}$ is clear from the context.

Definition 1.10. Let $\vec{c}$ be a ladder system on S^2_0. Let $\mathcal{S}^{\vec{c}}$ be the set of all $x \in \mathcal{C}^*$ such that $c_{\sup(x)} \subseteq x$. $\mathcal{S}^{\vec{c}}$ is stationary for every ladder system $\vec{c}$ on S^2_0.

Of course, we should denote $\mathcal{S}^{\vec{c}}$ as $\mathcal{S}^{\vec{\pi},\vec{c}}$. But the choice of $\vec{\pi}$ is not important and we omit $\vec{\pi}$ for simplicity.

The following fact might be well-known, but the author could not find the proof of it. So we will give a proof in Section 7 for the completeness.

Fact 1.11. *Let $\vec{c}$ be a ladder system on S^2_0 and $E \subseteq \omega_1$ stationary in ω_1. Let $T = \{x \in [\omega_2]^\omega : x \cap \omega_1 \in E, x \notin \mathcal{S}^{\vec{c}}\}$. Then T is stationary in $[\omega_2]^\omega$.*

Let θ be a sufficiently large regular cardinal, $M \prec H_\theta$ a countable model, and $\mathbb{P} \in M$ a poset. A condition $p \in \mathbb{P}$ is an $(M, \mathbb{P})$-*generic condition* if $p \Vdash$"$M \cap \mathrm{ON} = M[G] \cap \mathrm{ON}$".

See Shelah [**14**, Chapter V, Section 1] for the following.

Definition 1.12 (Shelah [**14**]). Let $\mathbb{P}$ be a poset and θ a sufficiently large regular cardinal.

(1) For a countable $M \prec H_\theta$ with $\mathbb{P} \in M$, a descending sequence $\langle p_n : n < \omega \rangle$ in $\mathbb{P} \cap M$ is called an $(M, \mathbb{P})$-*generic sequence* if for every dense open set $D \in M$ in $\mathbb{P}$, there is some n with $p_n \in D \cap M$.
(2) Let $\lambda \geq \omega_1$ be a cardinal and T a stationary subset of $[\lambda]^\omega$. A poset $\mathbb{P}$ is said to be T-*complete* if for every countable $M \prec H_\theta$, if $\mathbb{P}, T \in M$ and $M \cap \lambda \in T$ then every $(M, \mathbb{P})$-generic sequence has a lower bound.

Fact 1.13 (Shelah [**14**]). *Let $\lambda \geq \omega_1$ be a cardinal and $T \subseteq [\lambda]^\omega$ stationary.*

(1) *If $\mathbb{P}$ is T-complete, then $\mathbb{P}$ is σ-Baire, and $\mathbb{P}$ preserves the stationarity of all stationary subsets of T.*
(2) *Every countable support iteration of T-complete forcings is also T-complete.*

Now we explain an outline of the proof of Theorem 1.6. First, by forcings, we collapse a weakly compact cardinal κ to ω_2 and add a special ladder system $\vec{c}$ on S^2_0, which implies $\neg\,\mathrm{FRP}(\omega_2)$. Second, we force $\mathrm{WRP}(\mathcal{S}^{\vec{c}})$ by an iteration of club shootings. Finally we check that the weak compactness of κ in the ground model yields that $\mathrm{WRP}([\omega_2]^\omega \setminus \mathcal{S}^{\vec{c}})$ holds in the final model. Theorem 1.7 can be obtained by the same argument with replacing weak compact by supercompact.

2. Non-reflecting ladder system.

In this section, we study a special ladder system of which the existence implies $\neg\,\mathrm{FRP}(\omega_2)$.

Definition 2.1. A ladder system $\vec{c}$ on S^2_0 is said to be *non-reflecting* if for every $\beta \in S^2_1$, there are a club C in β and an injection f on C such that $f(\alpha) \in c_\alpha$ for every $\alpha \in C$.

The following is obvious from the definitions of a non-reflecting ladder system and $\mathcal{S}^{\vec{c}}$:

Lemma 2.2. *Suppose that there is a non-reflecting ladder system $\vec{c}$ on S^2_0.*

(1) *For every $\alpha \in S^2_1$, $\mathcal{S}^{\vec{c}} \cap [\alpha]^\omega$ is non-stationary in $[\alpha]^\omega$. Hence* $\mathrm{RP}(\mathcal{S}^{\vec{c}})$ *fails.*
(2) $\mathrm{FRP}(\omega_2)$ *fails.*

We define a poset which adds a generic non-reflecting ladder system and collapses a regular κ to ω_2.

Definition 2.3. Let κ be a regular cardinal $\geq \omega_2$. Let $E^\kappa_\omega = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega, \alpha > \omega_1\}$ and $E^\kappa_{>\omega} = \{\alpha < \kappa : \mathrm{cf}(\alpha) > \omega, \alpha > \omega_1\}$.

Definition 2.4. Let $\kappa > \omega_1$ be a regular cardinal. $\mathbb{L}_\kappa$ is the poset which consists of all pairs $\langle f, g\rangle$ such that:

(1) f is a function with $\mathrm{dom}(f) \in [E^\kappa_\omega]^\omega$ and for every $\alpha \in \mathrm{dom}(f)$, $f(\alpha) \subseteq \alpha$ is a cofinal subset of α with $\mathrm{ot}(f(\alpha)) = \omega$.
(2) g is a function with $\mathrm{dom}(g) \in [E^\kappa_{>\omega}]^\omega$.
(3) For every $\alpha \in \mathrm{dom}(g)$, $g(\alpha)$ is an injection such that $\mathrm{dom}(g(\alpha)) \in [\mathrm{dom}(f) \cap \alpha]^\omega$ is a closed bounded subset of α, and $g(\alpha)(\beta) \in f(\beta)$ for every $\beta \in \mathrm{dom}(g(\alpha))$.

For $\langle f_0, g_0\rangle, \langle f_1, g_1\rangle \in \mathbb{L}_\kappa$, define $\langle f_0, g_0\rangle \leq \langle f_1, g_1\rangle$ in $\mathbb{L}_\kappa$ if:

(a) $f_0 \supseteq f_1$.
(b) $\mathrm{dom}(g_0) \supseteq \mathrm{dom}(g_1)$.
(c) For every $\alpha \in \mathrm{dom}(g_1)$, $g_0(\alpha) \supseteq g_1(\alpha)$ and $\mathrm{dom}(g_0(\alpha))$ is an end-extension of $\mathrm{dom}(g_1(\alpha))$.

Lemma 2.5. *Suppose $x \in [\kappa]^\omega$. Let D be the set of all $\langle f, g\rangle \in \mathbb{L}_\kappa$ such that $x \cap E^\kappa_\omega \subseteq \mathrm{dom}(f)$, $x \cap E^\kappa_{>\omega} \subseteq \mathrm{dom}(g)$, and* $\max(\mathrm{dom}(g(\alpha)) = \sup(\mathrm{dom}(f) \cap \alpha)$ *for every $\alpha \in \mathrm{dom}(g)$. Then D is dense in $\mathbb{L}_\kappa$.*

Proof. Take $\langle f', g'\rangle \in \mathbb{L}_\kappa$. Take a sufficiently large regular θ and take a countable

$M \prec H_\theta$ containing all relevant objects. Note that for each $\alpha \in M \cap E^\kappa_{>\omega}$, $\sup(M \cap \alpha) \in E^\kappa_\omega$ and $\sup(M \cap \alpha) \notin M$. Then define $\langle f, g \rangle$ as follows:

(1) $\mathrm{dom}(f) = \lim(M \cap \kappa) \cap E^\kappa_\omega$.
(2) $f(\alpha) = f'(\alpha)$ for every $\alpha \in \mathrm{dom}(f')$.
(3) For $\alpha \in \mathrm{dom}(f) \setminus \mathrm{dom}(f')$, let $f(\alpha)$ be a cofinal subset of α with $\mathrm{ot}(f(\alpha)) = \omega$ and $f(\alpha) \nsubseteq M$ (this is possible since $\alpha > \omega_1$ and M is countable).
(4) $\mathrm{dom}(g) = M \cap E^\kappa_{>\omega}$.
(5) For $\alpha \in \mathrm{dom}(g')$, $\mathrm{dom}(g(\alpha)) = \mathrm{dom}(g'(\alpha)) \cup \{\sup(M \cap \alpha)\}$, $g(\alpha)(\beta) = g'(\alpha)(\beta)$ for $\beta \in \mathrm{dom}(g'(\alpha))$, and $g(\alpha)(\sup(M \cap \alpha))$ is an element of $f(\sup(M \cap \alpha)) \setminus M$ (note that $f(\sup(M \cap \alpha)) \nsubseteq M$).
(6) For $\alpha \in \mathrm{dom}(g) \setminus \mathrm{dom}(g')$, $\mathrm{dom}(g(\alpha)) = \{\sup(M \cap \alpha)\}$ and $g(\alpha)(\sup(M \cap \alpha))$ is an element of $f(\sup(M \cap \alpha)) \setminus M$ (as before, $f(\sup(M \cap \alpha)) \nsubseteq M$).

Then it is a routine to check that $\langle f, g \rangle$ is an element of the dense subset and $\langle f, g \rangle \leq \langle f', g' \rangle$. □

LEMMA 2.6. *$\mathbb{L}_\kappa$ has a σ-closed dense subset.*

PROOF. Let $D = \{\langle f, g \rangle \in \mathbb{L}_\kappa : \max(\mathrm{dom}(g(\alpha)) = \sup(\mathrm{dom}(f) \cap \alpha)$ for every $\alpha \in \mathrm{dom}(g)\}$. D is dense by Lemma 2.5. We see that D is σ-closed. Let $\langle \langle f_n, g_n \rangle : n < \omega \rangle$ be a decreasing sequence in D. Define $\langle f, g \rangle$ as follows:

(1) $\mathrm{dom}(f) = \bigcup_{n<\omega} \mathrm{dom}(f_n) \cup \lim(\bigcup_{n<\omega} \mathrm{dom}(f_n))$.
(2) For $\alpha \in \bigcup_{n<\omega} \mathrm{dom}(f_n)$, $f(\alpha) = f_n(\alpha)$ for some $n < \omega$ with $\alpha \in \mathrm{dom}(f_n)$.
(3) For $\alpha \notin \bigcup_{n<\omega} \mathrm{dom}(f_n)$, let $f(\alpha)$ be a cofinal subset of α with $\mathrm{ot}(f(\alpha)) = \omega$ and $f(\alpha) \nsubseteq \bigcup\{f_n(\beta) : \beta \in \mathrm{dom}(f_n), n < \omega\}$ (note that $\alpha > \omega_1$, so we can take such an $f(\alpha)$).
(4) $\mathrm{dom}(g) = \bigcup_{n<\omega} \mathrm{dom}(g_n)$.
(5) For $\alpha \in \bigcup_{n<\omega} \mathrm{dom}(g_n)$, let $d_\alpha = \bigcup\{\mathrm{dom}(g_n(\alpha)) : n < \omega, \alpha \in \mathrm{dom}(g_n)\}$. Then $\mathrm{dom}(g(\alpha)) = d_\alpha \cup \{\sup(d_\alpha)\}$, and for $\beta \in d_\alpha$, let $g(\alpha)(\beta) = g_n(\alpha)(\beta)$, where $n < \omega$ is minimal with $\beta \in \mathrm{dom}(g_n(\alpha))$. If $\sup(d_\alpha) \notin d_\alpha$, then we have $\sup(d_\alpha) \notin \bigcup_{n<\omega} \mathrm{dom}(f_n)$ since each $\langle f_n, g_n \rangle$ is in D, and let $g(\alpha)(\sup(d_\alpha))$ be an element of $f(\sup(d_\alpha)) \setminus \bigcup\{f_n(\beta) : \beta \in \mathrm{dom}(f_n), n < \omega\}$.

One can check that $\langle f, g \rangle$ is a lower bound of the $\langle f_n, g_n \rangle$'s. □

The following is immediate from the definition of $\mathbb{L}_\kappa$ and Lemmas 2.5, 2.6.

LEMMA 2.7. *Let G be $(V, \mathbb{L}_\kappa)$-generic. In $V[G]$, $\omega_1^{V[G]} = \omega_1^V$ and $\mathrm{cf}(\alpha) = \omega_1$ for every $\alpha \in (E^\kappa_{>\omega})^V$. Let $\vec{c} = \bigcup\{f : \exists g\, (\langle f, g \rangle \in G)\}$. Then $\vec{c}$ is a ladder system on $(E^\kappa_\omega)^V$. Define the function h by $\mathrm{dom}(h) = \bigcup\{\mathrm{dom}(g) : \exists f\, (\langle f, g \rangle \in G)\}$, and for $\alpha \in \mathrm{dom}(h)$, $h(\alpha) = \bigcup\{g(\alpha) : \exists f\, (\langle f, g \rangle \in G), \alpha \in \mathrm{dom}(g))\}$. Then h is a function with $\mathrm{dom}(h) = (E^\kappa_{>\omega})^V$, and for every $\alpha \in \mathrm{dom}(h)$, $h(\alpha)$ is an injection, $\mathrm{dom}(h(\alpha))$ is a club in α, and $h(\alpha)(\beta) \in c_\beta$ for every $\beta \in \mathrm{dom}(h_\alpha)$. So if $\kappa = \omega_2^{V[G]}$, then the function h witnesses that $\vec{c}$ is a non-reflecting ladder system on S^2_0.*

LEMMA 2.8. *If $\lambda^\omega < \kappa$ for every $\lambda < \kappa$, then $\mathbb{L}_\kappa$ satisfies the κ-c.c.*

PROOF. Take $\mathcal{F} \subseteq \mathbb{L}_\kappa$ with size κ. By the Δ-system lemma and the pigeonhole principle, we can find $\mathcal{F}' \subseteq \mathcal{F}$ such that:

(1) $|\mathcal{F}'| = \kappa$.
(2) There exists $R_0 \in [\kappa]^\omega$ such that for each distinct $\langle f, g\rangle, \langle f', g'\rangle \in \mathcal{F}'$, $\mathrm{dom}(f) \cap \mathrm{dom}(f') = R_0$ and $f|R_0 = f'|R_0$.
(3) There exists $R_1 \in [\kappa]^\omega$ such that for each distinct $\langle f, g\rangle, \langle f', g'\rangle \in \mathcal{F}'$, $\mathrm{dom}(g) \cap \mathrm{dom}(g') = R_1$ and $g|R_1 = g'|R_1$.

We see that every pair from $\mathcal{F}'$ is compatible. Take distinct $\langle f, g\rangle, \langle f', g'\rangle \in \mathcal{F}'$. Then define $\langle f^*, g^*\rangle$ as follows:

(1) $\mathrm{dom}(f^*) = \mathrm{dom}(f) \cup \mathrm{dom}(f')$, $f^*|\mathrm{dom}(f) = f$ and $f^*|\mathrm{dom}(f') = f'$.
(2) $\mathrm{dom}(g^*) = \mathrm{dom}(g) \cup \mathrm{dom}(g')$, $g^*|\mathrm{dom}(g) = g$ and $g^*|\mathrm{dom}(g') = g'$.

Then we can check that $\langle f^*, g^*\rangle$ is a common extension of $\langle f, g\rangle$ and $\langle f', g'\rangle$. □

3. Destroying the stationarity of non-reflecting subsets.

In this section, we study a poset which destroys the stationarity of non-reflecting subsets of $\mathcal{S}^{\vec{c}}$.

We define a club shooting into $[\omega_2]^\omega$ with countable approximations, which was observed in Sakai [**12**].

DEFINITION 3.1. Let $\mathbb{C}$ be the poset which consists of all functions $p : d(p) \times d(p) \to \omega_1$ such that $d(p) \in [\omega_2]^\omega$. For $p, q \in \mathbb{C}$, let $p \leq q \iff p \supseteq q$.

For $S \subseteq [\omega_2]^\omega$, let $\mathbb{C}(S)$ be the suborder of $\mathbb{C}$ which consists of all $p \in \mathbb{C}$ with the property that $\forall x \subseteq d(p)\, (x \in S \Rightarrow x$ is not closed under $p)$.

LEMMA 3.2. (1) *For every $x \in [\omega_2]^\omega$, the set $\{p \in \mathbb{C}(S) : x \subseteq d(p)\}$ is dense open in $\mathbb{C}(S)$.*
(2) *Let G be $(V, \mathbb{P})$-generic and $f = \bigcup G$. Then f is a function from $(\omega_2)^V \times (\omega_2)^V$ to $(\omega_1)^V$, and there is no $x \in S$ closed under f.*
(3) *$\mathbb{C}(S)$ satisfies $(2^\omega)^+$-c.c.*

PROOF. (1) Take $x \in [\omega_2]^\omega$ and $q \in \mathbb{C}(S)$. Then let $a = d(q) \cup x$ and fix $\alpha \in \omega_1 \setminus a$. Define p as follows: $p : a \times a \to \omega_1$, $p(\beta_0, \beta_1) = q(\beta_0, \beta_1)$ if $\langle \beta_0, \beta_1\rangle \in d(q) \times d(q)$, and $p(\beta_0, \beta_1) = \alpha$ otherwise. Clearly $p \supseteq q$. We have to check that $p \in \mathbb{C}(S)$. Take $x \in S$ and $x \subseteq d(p) = a$. If $x \subseteq d(q)$, then x is not closed under q, and is not under p. Suppose $x \not\subseteq d(q)$. Fix $\beta \in x \setminus d(q)$. Then $p(\beta, \beta) = \alpha \notin d(p)$, so x cannot be closed under p.

(2) follows from (1).

(3) Let $\{p_i : i < (2^\omega)^+\} \subseteq \mathbb{C}(S)$. By the Δ-system lemma, we may assume that $\{d(p_i) : i < (2^\omega)^+\}$ forms a Δ-system with root R. Moreover, by a standard pigeonhole argument, we may assume that $p_i|(R \times R) = p_j|(R \times R)$ for every $i < j < (2^\omega)^+$. We check that for every $i < j < (2^\omega)^+$, p_i is compatible with p_j. Let $a = d(p_i) \cup d(p_j)$, and fix $\alpha \in \omega_1 \setminus a$. Define q as follows: $q : a \times a \to \omega_1$, $q(\beta_0, \beta_1) = p_i(\beta_0, \beta_1)$ if $\langle \beta_0, \beta_1\rangle \in d(p_i) \times d(p_i)$, $q(\beta_0, \beta_1) = p_j(\beta_0, \beta_1)$ if $\langle \beta_0, \beta_1\rangle \in d(p_j) \times d(p_j)$, and $q(\beta_0, \beta_1) = \alpha$ otherwise. This q is well-defined since $p_i|(R \times R) = p_j|(R \times R)$. We see that $q \in \mathbb{C}(S)$,

and then clearly $q \leq p_i, p_j$. Take $x \subseteq d(q) = a$. If $x \subseteq d(p_i)$ then x is not closed under p_i, hence is not q. The case $x \subseteq d(p_j)$ is the same. Suppose $x \nsubseteq d(p_i)$ and $x \nsubseteq d(p_j)$. Pick $\beta_0 \in x \setminus d(p_i)$ and $\beta_1 \in x \setminus d(p_j)$. Then $q(\beta_0, \beta_1) = \alpha \notin a$, hence x cannot be closed under q. □

Now we assume that there is a non-reflecting ladder system $\vec{c}$ on S^2_0. From now on, we will work with a fixed non-reflecting ladder system $\vec{c}$ on S^2_0. Let $\mathcal{S}^* = \mathcal{S}^{\vec{c}}$. $\mathcal{S}^* \cap [\alpha]^\omega$ is non-stationary in $[\alpha]^\omega$ for every $\alpha \in S^2_1$. We show that if $S \subseteq \mathcal{S}^*$ is a non-reflecting subset, then $\mathbb{C}(S)$ has good properties.

LEMMA 3.3. *For $x, y \in \mathcal{S}^*$, if $x \cap \omega_1 = y \cap \omega_1$ and $\sup(x) = \sup(y)$ then $x = y$.*

PROOF. Let $\alpha = \sup(x) = \sup(y)$. We know $c_\alpha \subseteq x \cap y$. For each $\beta \in c_\alpha$, we have $x \cap \beta = \pi_\beta``(x \cap \omega_1) = \pi_\beta``(y \cap \omega_1) = y \cap \beta$. c_α is unbounded in α, so we have $x = y$. □

LEMMA 3.4. *Let θ be a sufficiently large regular cardinal, and $M \prec H_\theta$ a countable model containing all relevant objects. Suppose $M \cap \omega_2 \notin \mathcal{S}^*$.*

(1) *For $x \in \mathcal{S}^*$, if $x \cap \omega_1 < M \cap \omega_1$ and $\sup(x) \in M$ then $x \in M$.*
(2) *For every $x \in \mathcal{S}^*$, if $x \subseteq M \cap \omega_2$ and $x \notin M$, then $x = M \cap \alpha$ for some $\alpha \in M \cap \omega_2$.*
(3) *If $S \in M$ is a non-reflecting subset of $\mathcal{S}^*$, then for every $x \in S$ with $x \subseteq M \cap \omega_2$, it holds that $x \in M$.*

PROOF. First note that $M \cap \omega_2 \in \mathcal{C}^*$, hence we have $c_{\sup(M \cap \omega_2)} \nsubseteq M \cap \omega_2$.

(1) follows from Lemma 3.3.

For (2), we have that $\sup(x) < \sup(M \cap \omega_2)$ since $c_{\sup(x)} \subseteq x$ but $c_{\sup(M \cap \omega_2)} \nsubseteq M \cap \omega_2$. Let $\alpha = \min((M \cap \omega_2) \setminus x) \in M$. We show $x = M \cap \alpha$.

First we see that $\alpha = \sup(x)$. If $\alpha > \sup(x)$, then $\mathrm{cf}(\alpha) > \omega$. Since $\vec{c}$ is a non-reflecting ladder system, there is an injection $f \in M$ such that $\mathrm{dom}(f)$ is a club in α and $f(\beta) \in c_\beta$ for every $\beta \in \mathrm{dom}(f)$. We have $\sup(x) = \sup(M \cap \alpha) \in \mathrm{dom}(f)$. Let $\gamma = f(\sup(x))$. We have $\gamma \in M$ since $x \subseteq M$. Then $\sup(x)$ is definable in M as "the unique $\beta \in \mathrm{dom}(f)$ with $f(\beta) = \gamma$", so $\sup(x) \in M$. This is a contradiction. Therefore we have $\sup(x) = \alpha \in M$. If $x \cap \omega_1 < M \cap \omega_1$, then $x \in M$ by (1). This is a contradiction. So $x \cap \omega_1 = M \cap \omega_1$. It is easy to see that $M \cap \alpha \in \mathcal{S}^*$, hence $x = M \cap \alpha$ by Lemma 3.3.

For (3), take $x \in S$ with $x \subseteq M \cap \omega_2$. If $x \notin M$, then $x = M \cap \alpha$ for some $\alpha \in M \cap \omega_2$ by (2). Since S is non-reflecting, $S \cap [\alpha]^\omega$ is non-stationary. So there is a $\subseteq$-increasing continuous cofinal sequence $\langle x_i : i < \omega_1 \rangle$ in $[\alpha]^\omega$ such that $\langle x_i : i < \omega_1 \rangle \in M$ and $S \cap \{x_i : i < \omega_1\} = \emptyset$. By the elementarity of M, we have $x_{M \cap \omega_1} = M \cap \alpha$. However $M \cap \alpha = x \in S$, this is a contradiction. □

LEMMA 3.5. *Let $S \subseteq \mathcal{S}^*$ be a non-reflecting set. Then $\mathbb{C}(S)$ is $[\omega_2]^\omega \setminus \mathcal{S}^*$-complete.*

PROOF. Take a countable $M \prec H_\theta$ such that $M \cap \omega_2 \in [\omega_2]^\omega \setminus \mathcal{S}^*$ and M contains all relevant objects. Let $\langle p_n : n < \omega \rangle$ be an $(M, \mathbb{C}(S))$-generic sequence. Let $p = \bigcup_{n<\omega} p_n$. We see that $p \in \mathbb{C}(S)$, this completes the proof. Since $\langle p_n : n < \omega \rangle$ is $(M, \mathbb{C}(S))$-generic, p is a function from $(M \cap \omega_2) \times (M \cap \omega_2)$ to ω_1. To see that $p \in \mathbb{C}(S)$, take $x \subseteq d(p) = M \cap \omega_2$ with $x \in S$. Since $M \cap \omega_2 \in [\omega_2]^\omega \setminus \mathcal{S}^*$, we have $x \in M$ by Lemma 3.4. The set

$\{q \in \mathbb{C}(S) : x \subseteq d(q)\}$ is dense open in $\mathbb{C}(S)$ and belongs to M, hence there is n with $x \subseteq d(p_n)$. x is not closed under p_n, hence is not closed under p. □

Next we consider a countable support iteration of club shootings. Let l be an ordinal and $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \eta < \xi \le l\rangle$ be a countable support iteration such that for $\eta < l$, $\Vdash_{\mathbb{P}_\eta}$ "$\dot{\mathbb{Q}}_\eta$ is of the form $\mathbb{C}(\dot{S}_\eta)$ for some non-reflecting subset $\dot{S}_\eta$ of $\mathcal{S}^*$". Then by Fact 1.13, $\mathbb{P}_l$ is $[\omega_2]^\omega \setminus \mathcal{S}^*$-complete, hence is σ-Baire. Under CH, we show that $\mathbb{P}_\xi$ satisfies the ω_2-c.c. and more for every $\xi \le l$.

Lemma 3.6. *Suppose CH. Let l be an ordinal and $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \eta < \xi < l\rangle$ be a countable support iteration such that for $\eta < l$, $\Vdash_{\mathbb{P}_\eta}$ "$\dot{\mathbb{Q}}_\eta$ is of the form $\mathbb{C}(\dot{S}_\eta)$ for some non-reflecting subset $\dot{S}_\eta$ of $\mathcal{S}^*$". Let D be the set of all $p \in \mathbb{P}_\xi$ such that for all $\eta \in \operatorname{supp}(p)$, $p(\eta)$ is the canonical name for some $r \in \mathbb{C}$. Then the following hold:*

(1) *$\mathbb{P}_l$ is $[\omega_2]^\omega \setminus \mathcal{S}^*$-complete.*
(2) *$\mathbb{P}_l$ satisfies the ω_2-c.c.*
(3) *D is dense in $\mathbb{P}_l$.*
(4) *Let $M \prec H_\theta$ be countable such that $M \cap \omega_2 \in [\omega_2]^\omega \setminus \mathcal{S}^*$ and M contains all relevant objects. Let $\langle p_n : n < \omega\rangle$ be an $(M, \mathbb{P}_l)$-generic sequence such that $p_n \in D$ for every n. For $n < \omega$ and $\eta \in \operatorname{supp}(p_n)$, let $r_{n,\eta}$ be the function such that $p_n(\eta)$ is the canonical name for $r_{n,\eta}$. Let $p \in \mathbb{P}_l$ be the function defined by $\operatorname{dom}(p) = l$, $p(\eta) = \emptyset$ for $\eta \notin M \cap l$, and for $\eta \in M \cap l$, $p(\eta)$ is the canonical name for $\bigcup\{r_{n,\eta} : n < \omega, \eta \in \operatorname{supp}(p_n)\}$. Then p is a lower bound of the p_n's.*

Proof. We prove the assertions by induction on l. For each $\xi < l$, suppose the following induction hypotheses:

(a) $\mathbb{P}_\xi$ is $[\omega_2]^\omega \setminus \mathcal{S}^*$-complete.
(b) $\mathbb{P}_\xi$ satisfies the ω_2-c.c.
(c) Let $D_\xi = \{p \in \mathbb{P}_\xi : \forall \eta \in \operatorname{supp}(p)\,(p(\eta)$ is the canonical name for some $r \in \mathbb{C})\}$. Then D_ξ is dense in $\mathbb{P}_\xi$.
(d) Let $M \prec H_\theta$ be countable such that $M \cap \omega_2 \in [\omega_2]^\omega \setminus \mathcal{S}^*$ and M contains all relevant objects. Let $\langle p_n : n < \omega\rangle$ be an $(M, \mathbb{P}_\xi)$-generic sequence such that $p_n \in D_\xi$ for every n. For $n < \omega$ and $\eta \in \operatorname{supp}(p_n)$, let $r_{n,\eta}$ be the function such that $p_n(\eta)$ is the canonical name for $r_{n,\eta}$. Let $p \in \mathbb{P}_\xi$ be the function defined by $\operatorname{dom}(p) = \xi$, $p(\eta) = \emptyset$ for $\eta \notin M \cap \xi$, and for $\eta \in M \cap \xi$, $p(\eta)$ is the canonical name for $\bigcup\{r_{n,\eta} : n < \omega, \eta \in \operatorname{supp}(p_n)\}$. Then p is a lower bound of the p_n's.

The assertion (1) follows from Fact 1.13.

To prove (2)–(4), first suppose l is limit.

To see that (3) holds, take an arbitrary $q \in \mathbb{P}_l$. We will find $p \le q$ with $p \in D$. Take a countable $M \prec H_\theta$ such that $M \cap \omega_2 \in [\omega_2]^\omega \setminus \mathcal{S}^*$ and M contains all relevant objects. Take an $(M, \mathbb{P}_l)$-generic sequence $\langle p_n : n < \omega\rangle$ with $p_0 \le q$. Fix an increasing sequence $\langle \xi_n : n < \omega\rangle$ in $M \cap l$ with $\sup\{\xi_n : n < \omega\} = \sup(M \cap l)$. Since each D_{ξ_n} is dense in $\mathbb{P}_{\xi_n}$, we may assume that $p_n|\xi_n \in D_{\xi_n}$.

For each $n < \omega$ and $\eta \in \operatorname{supp}(p_n) \cap \xi_n$, let $r_{n,\eta}$ be the function such that $p_n(\eta)$ is the canonical name for $r_{n,\eta}$. Then define p as follows: $\operatorname{dom}(p) = l$, $p(\eta) = \emptyset$ if $\eta \notin M \cap l$, and for $\eta \in M \cap l$, $p(\eta)$ is the canonical name for $\bigcup\{r_{n,\eta} : n < \omega, \eta \in \mathrm{supp}(p_n)|\xi_n\}$. By the induction hypothesis (d), $p|\eta$ is a lower bound of the $p_n|\eta$'s for every $\eta \in M \cap l$. Then clearly p is a lower bound of the p_n's and $p \in D$.

The assertion (4) follows from the same argument.

For (2), as Lemma 3.2, apply a standard Δ-system argument with D.

Next we deal with the case that l is successor, say $l = \xi + 1$. By the induction hypothesis (b), $\mathbb{P}_\xi$ is σ-Baire. So the assertion (3) is clear. Note that $\mathbb{P}_\xi$ forces CH and preserves ω_2 by the induction hypotheses. Hence $\mathbb{P}_\xi$ forces that "$\dot{\mathbb{Q}}_\xi$ satisfies the ω_2-c.c." by Lemma 3.2. Then it is immediate that $\mathbb{P}_l = \mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$ satisfies the ω_2-c.c.

For (4), take a countable $M \prec H_\theta$ such that $M \cap \omega_2 \in [\omega_2]^\omega \setminus \mathcal{S}^*$ and M contains all relevant objects. Take an $(M, \mathbb{P}_l)$-generic sequence $\langle p_n : n < \omega\rangle$ such that $p_n \in D$ for every n. Note that $p_n|\xi \in D_\xi$. For $n < \omega$ and $\eta \in \mathrm{supp}(p_n)$, let $r_{n,\eta}$ be the function such that $p_n(\eta)$ is the canonical name for $r_{n,\eta}$. Let $p \in \mathbb{P}_l$ be the function defined by $\mathrm{dom}(p) = l$, $p(\eta) = \emptyset$ for $\eta \notin M \cap l$, and for $\eta \in M \cap l$, $p(\eta)$ is the canonical name for $\bigcup\{r_{n,\eta} : n < \omega, \eta \in \mathrm{supp}(p_n)\}$. The sequence $\langle p_n|\xi : n < \omega\rangle$ is an $(M, \mathbb{P}_\xi)$-generic sequence with $p_n|\xi \in D_\xi$, hence $p|\xi$ is a lower bound of the $p_n|\xi$'s by the induction hypothesis (d). Then $p|\xi$ is an $(M, \mathbb{P}_\xi)$-generic condition. Take an $(V, \mathbb{P}_\xi)$-generic filter G with $p|\xi \in G$ and work in $V[G]$. We have $M[G] \cap \omega_2 = M \cap \omega_2 \in [\omega_2]^\omega \setminus \mathcal{S}^*$ and $\langle (p_n(\xi))_G : n < \omega\rangle$ is an $(M[G], \mathbb{Q}_\xi)$-generic sequence, where $(p_n(\xi))_G$ is the interpretation of the $\mathbb{P}_\xi$-name $p_n(\xi)$ by G. By Lemma 3.5, $\mathbb{Q}_\xi$ is $[\omega_2]^\omega \setminus \mathcal{S}^*$-complete. Hence $\langle (p_n(\xi))_G : n < \omega\rangle$ has a lower bound $\bigcup_{n<\omega}(p_n(\xi))_G \in \mathbb{Q}_\xi$. Now we have $\bigcup_{n<\omega}(p_n(\xi))_G = (p(\xi))_G$. This argument shows that, in V, the condition p is a lower bound of the p_n's. □

By Facts 1.11 and 1.13, we also have the following:

Lemma 3.7. *Under the same assumptions in Lemma* 3.6, $\mathbb{P}_l$ *is* ω_1*-stationary preserving.*

Combining $\mathbb{L}_{\omega_2}$ with an iteration of club shootings above and a standard book-keeping method, we have the following. A similar result was already obtained by Sakai [**12**].

Proposition 3.8. *Suppose GCH. Then there exists an* ω_3*-stage countable support iteration* $\mathbb{P}$ *such that* $\mathbb{P}$ *is* σ*-Baire, satisfies the* ω_2*-c.c., and forces that "there exists a ladder system* $\vec{c}$ *on* S^2_0 *such that* $\mathrm{WRP}(\mathcal{S}^{\vec{c}})$ *holds but* $\mathrm{RP}(\mathcal{S}^{\vec{c}})$ *fails".*

4. Partial strong Chang's conjecture.

Recall that *strong Chang's conjecture* is the assertion that for every sufficiently large cardinal θ, every well-ordering Δ on H_θ, every countable $M \prec \langle H_\theta, \in, \Delta, \ldots\rangle$, and every $\alpha < \omega_2$, there is a countable $N \prec \langle H_\theta, \in, \Delta, \ldots\rangle$ such that $M \subseteq N$, $\sup(N \cap \omega_2) \geq \alpha$, and $M \cap \omega_2$ is a proper initial segment of $N \cap \omega_2$. It is known that SSR implies strong Chang's conjecture, and strong Chang's conjecture is a large cardinal property as implying the usual Chang's conjecture. We prove that if there is some stationary $S \subseteq [\omega_2]^\omega$ such that $\mathrm{WRP}(S)$ holds but $\mathrm{RP}(S)$ fails, then very weak form of strong Chang's conjecture holds. We will use Lemma 4.1 below to prove Theorem 1.7.

Lemma 4.1. *Suppose that there is a stationary set $S \subseteq [\omega_2]^\omega$ such that* $\mathrm{WRP}(S)$ *holds, but $S \cap [\alpha]^\omega$ is non-stationary in $[\alpha]^\omega$ for every $\alpha \in S^2_1$. Let θ be a sufficiently large regular cardinal and Δ a well-ordering on H_θ. Let $\mathcal{M} = \langle H_\theta, \in, \Delta, S, \ldots \rangle$. Then there is an expansion $\mathcal{M}'$ of $\mathcal{M}$ such that for every countable $M \prec \mathcal{M}'$, if $M \cap \omega_2 \in S$ then there is a countable $N \prec \mathcal{M}'$ such that $M \subseteq N$, $M \cap \omega_2 = N \cap \sup(M \cap \omega_2)$ and* $\sup(M \cap \omega_2) \in N$*, hence $M \cap \omega_2$ is a proper initial segment of $N \cap \omega_2$.*

Proof. For $x \subseteq H_\theta$, let $Sk^{\mathcal{M}}(x)$ denote the Skolem hull of x under the structure $\mathcal{M}$. Let X be the set of all countable $M \prec \mathcal{M}$ such that $M \cap \omega_2 \in S$ and $M \cap \omega_2 \neq Sk^{\mathcal{M}}(M \cup \{\sup(M \cap \omega_2)\}) \cap \sup(M \cap \omega_2)$. It is enough to see that X is non-stationary in $[H_\theta]^\omega$; Fix a club $D \subseteq [H_\theta]^\omega$ with $X \cap D = \emptyset$, and expand $\mathcal{M}$ to $\mathcal{M}' = \langle H_\theta, \in, \Delta, S, D, \ldots \rangle$. Then for every countable $M \prec \mathcal{M}'$, we have $M \in D$, hence $M \cap \omega_2 = Sk^{\mathcal{M}}(M \cup \{\sup(M \cap \omega_2)\}) \cap \sup(M \cap \omega_2)$. Now we know $Sk^{\mathcal{M}}(M \cup \{\sup(M \cap \omega_2)\}) \cap \omega_2 = Sk^{\mathcal{M}'}(M \cup \{\sup(M \cap \omega_2)\}) \cap \omega_2$ (e.g., see Lemma 24 in [**4**]), where $Sk^{\mathcal{M}'}(M \cup \{\sup(M \cap \omega_2)\})$ is the Skolem hull of $M \cup \{\sup(M \cap \omega_2)\}$ under the structure $\mathcal{M}'$. Let $N = Sk^{\mathcal{M}'}(M \cup \{\sup(M \cap \omega_2)\}) \prec \mathcal{M}'$. Then $M \subseteq N$, $N \cap \sup(M \cap \omega_2) = M \cap \omega_2$, and $\sup(M \cap \omega_2) \in N$. So the structure $\mathcal{M}'$ is as required.

Suppose to the contrary that X is stationary in $[H_\theta]^\omega$. By Fodor's lemma, we can find a Skolem term t and $x_0, \ldots, x_n \in H_\theta$ such that the set $Y = \{M \in X : x_0, \ldots, x_n \in M, t(x_0, \ldots, x_n, \sup(M \cap \omega_2)) < \sup(M \cap \omega_2)$ but not in $M \cap \omega_2\}$ is stationary in $[H_\theta]^\omega$. Let $Z = \{M \cap \omega_2 : M \in Y\}$. Z is a stationary subset of S. Since $\mathrm{WRP}(S)$ holds and $S \cap [\alpha]^\omega$ is non-stationary for every $\alpha \in S^2_1$, we can find $\alpha \in S^2_0$ such that $Z \cap [\alpha]^\omega$ is stationary in $[\alpha]^\omega$. Pick $x \in Z \cap [\alpha]^\omega$ with $\sup(x) = \alpha$, and take $M \in Y$ with $x = M \cap \omega_2$. We have $\sup(M \cap \omega_2) = \sup(x) = \alpha$, so $t(x_0, \ldots, x_n, \alpha) < \alpha$ but $t(x_0, \ldots, x_n, \alpha)$ is not in x. However, since $\mathrm{cf}(\alpha) = \omega$ and $Z \cap [\alpha]^\omega$ is stationary, we can find $x \in Z \cap [\alpha]^\omega$ such that $\sup(x) = \alpha$ and $t(x_0, \ldots, x_n, \alpha) \in x$. This is a contradiction. □

5. Proof of Theorem 1.6.

We start the proof of Theorem 1.6. Suppose that κ is weakly compact and GCH holds. Let $\mathbb{L} = \mathbb{L}_\kappa$. Take a $(V, \mathbb{L})$-generic G and work in $V[G]$. Note that $\kappa = \omega_2$ and GCH holds in $V[G]$. Let $\vec{c}$ be a non-reflecting ladder system on S^2_0 induced by G. Let $\mathcal{S}^* = \mathcal{S}^{\vec{c}}$. $\mathcal{S}^*$ is stationary in $[\kappa]^\omega$ and $\mathcal{S}^* \cap [\alpha]^\omega$ is non-stationary for every $\alpha \in S^2_1$. Finally, choose a κ^+-stage countable support iteration of $([\kappa]^\omega \setminus \mathcal{S}^*)$-complete club shootings $\mathbb{P}_{\kappa^+}$ from Section 3 such that $\mathbb{P}_{\kappa^+}$ forces $\mathrm{WRP}(\mathcal{S}^*)$ holds. The poset $\mathbb{P}_{\kappa^+}$ is σ-Baire and satisfies the κ-c.c. Take a $(V[G], \mathbb{P}_{\kappa^+})$-generic H. In $V[G][H]$, we have that $\mathrm{FRP}(\omega_2)$ fails, $\mathcal{S}^*$ is stationary, and $\mathrm{WRP}(\mathcal{S}^*)$ holds. Thus, in order to see that $\mathrm{WRP}(\omega_2)$ holds, it is enough to show that $\mathrm{WRP}([\kappa]^\omega \setminus \mathcal{S}^*)$ holds.

We show the following weak but sufficient assertion:

Lemma 5.1. *For $\xi < \kappa^+$, let $H_\xi = H \cap \mathbb{P}_\xi$ be the $(V[G], \mathbb{P}_\xi)$-generic filter induced by H. Then in $V[G][H_\xi]$,* $\mathrm{WRP}([\kappa]^\omega \setminus \mathcal{S}^*)$ *holds.*

The theorem follows from this lemma. Let $T \subseteq [\kappa]^\omega \setminus \mathcal{S}^*$ be stationary. Since $|T| = \kappa = \omega_2$ and $\mathbb{P}_{\kappa^+}$ satisfies the κ-c.c., there is some $\xi < \kappa^+$ such that $T \in V[G][H_\xi]$. By the lemma, there is some $\alpha < \omega_2$ such that $T \cap [\alpha]^\omega$ is stationary in $V[G][H_\xi]$. The tail poset $\mathbb{P}_{\xi,\kappa^+}$ is also $([\kappa]^\omega \setminus \mathcal{S}^*)$-complete. Because of Lemma 3.7, the tail poset $\mathbb{P}_{\xi,\kappa^+}$ preserves the stationarity of $T \cap [\alpha]^\omega$, hence $T \cap [\alpha]^\omega$ is stationary in $V[G][H]$.

To show the lemma, fix $\xi < \kappa^+$. Return to V. Let $\mathbb{Q} = \mathbb{L} * \mathbb{P}_\xi$. Fix a sufficiently large regular θ and choose $M \prec H_\theta$ such that $|M| = \kappa \subseteq M$, ${}^{<\kappa}M \subseteq M$, and M contains all relevant objects. Now, because κ is weakly compact, we can find a transitive model N of ZFC$^-$ and an elementary embedding $j : M \to N$ such that the critical point of j is κ and ${}^{<\kappa}N \subseteq N$. We notice that $\mathbb{Q} \in N$, because $|\mathbb{Q}| = \kappa$ and $\mathcal{P}(\kappa)^M \subseteq \mathcal{P}(\kappa)^N$.

Consider the map $i = j|\mathbb{Q}$ from $\mathbb{Q}$ to $j(\mathbb{Q})$.

Claim 5.2. *i is a complete embedding.*

Proof. Since $\mathbb{Q}$ satisfies the κ-c.c. and has size κ, every maximal antichain of $\mathbb{Q}$ lies in M. Moreover, for every maximal antichain A of $\mathbb{Q}$, we know $i``A = j(A)$. This shows that $i``A$ is maximal in $j(\mathbb{Q})$. □

We work in $V[G][H_\xi]$. In $V[G][H_\xi]$, it is known that $M[G][H_\xi] \prec H_\theta^{V[G][H_\xi]}$. Moreover, since M and N are closed under $< \kappa$-sequences and $\mathbb{Q}$ satisfies the κ-c.c., $M[G][H_\xi]$ and $N[G][H_\xi]$ are closed under $< \kappa$-sequences in $V[G][H_\xi]$.

Now we consider the quotient poset $\mathbb{R} = j(\mathbb{Q})/G * H_\xi$, where $\mathbb{R} = j(\mathbb{Q})/G * H_\xi$ is the suborder of $j(\mathbb{Q})$ consisting of all $q \in j(\mathbb{Q})$ which is compatible with $i(q')$ for every $q' \in G * H_\xi$. We can identify $\mathbb{R}$ with the forcing product $j(\mathbb{L})/G * j(\mathbb{P}_\xi)/H_\xi$.

Claim 5.3. (1) *$\mathbb{R}$ is σ-Baire in $V[G][H_\xi]$.*

(2) *Let $\lambda \geq \kappa$ be a cardinal and $T \subseteq [\lambda]^\omega$ stationary in $[\lambda]^\omega$ such that $x \cap \kappa \notin \mathcal{S}^*$ for every $x \in T$. Then $\mathbb{R}$ preserves the stationarity of T.*

For (2), we need only the special case $\lambda = \kappa$ of Claim 5.3 in this section, but we will the case $\lambda > \kappa$ in the next section.

If this Claim 5.3 is verified, we can prove that WRP$([\kappa]^\omega \setminus \mathcal{S}^*)$ holds in $V[G][H_\xi]$ as follows: Fix a stationary $T \subseteq [\kappa]^\omega \setminus \mathcal{S}^*$. We may assume that $T \in M[G][H_\xi]$. Take a $(V[G][H_\xi], j(\mathbb{Q})/G * H_\xi)$-generic $j(G) * j(H_\xi)$ and work in $V[G * H_\xi][j(G) * j(H_\xi)] = V[j(G) * j(H_\xi)]$. Then $j : M \to N$ can be extended to $j : M[G][H_\xi] \to N[j(G)][j(H_\xi)]$. Now $\mathbb{R}$ is σ-Baire, hence $j(\mathbb{Q})$ is σ-Baire in V. Since N is closed under $< \kappa$-sequences and $j(\mathbb{Q})$ is σ-Baire in V, $N[j(G)][j(H_\xi)]$ is closed under ω-sequences in $V[j(G) * j(H_\xi)]$, so $[\kappa]^\omega = ([\kappa]^\omega)^{N[j(G)][j(H_\xi)]}$. Consider $j(T) \cap [\kappa]^\omega$. We know that $j(T) \cap [\kappa]^\omega = T \in N[j(G)][j(H_\xi)]$ and T is stationary in $[\kappa]^\omega$ by Claim 5.3. So $N[j(G)][j(H_\xi)]$ satisfies the statement that there is some $\alpha < j(\kappa)$ such that $j(T) \cap [\alpha]^\omega$ is stationary in $[\alpha]^\omega$. By the elementarity of j, $M[G][H_\xi]$ satisfies the statement that there is some $\alpha < \kappa$ such that $T \cap [\alpha]^\omega$ is stationary in $[\alpha]^\omega$. Since $M[G][H_\xi] \prec H_\theta^{V[G][H_\xi]}$, $T \cap [\alpha]^\omega$ is in fact stationary in $V[G][H_\xi]$.

Now we start the proof of Claim 5.3. Fix a cardinal $\lambda \geq \kappa$ and a stationary set $T \subseteq [\lambda]^\omega$ in $[\lambda]^\omega$ such that $x \cap \kappa \notin \mathcal{S}^*$ for every $x \in T$. First we see that $j(\mathbb{L})/G$ is σ-Baire in $V[G][H_\xi]$ and preserves the stationarity of T. The following is straightforward.

Subclaim 5.4. *For $\langle f, g\rangle \in j(\mathbb{L})$, $\langle f|\kappa, g|\kappa\rangle \in \mathbb{L}$, and $\langle f, g\rangle \in j(\mathbb{L})/G$ if and only if $\langle f|\kappa, g|\kappa\rangle \in G$.*

Let D be a σ-closed dense subset of $\mathbb{L}$ from Lemma 2.6. We may assume that $D \in M$. $j(D)$ is dense in $j(\mathbb{L})$, hence $j(D) \cap (j(\mathbb{L})/G)$ is dense in $j(\mathbb{L})/G$. Take $p \in j(\mathbb{L})/G$. Fix another sufficiently large regular cardinal $\chi > \theta$ and take a countable $\overline{M} \prec H_\chi$ containing all relevant objects and $\overline{M} \cap \lambda \in T$. We may assume that $\overline{M} \cap (H_\chi)^V \prec (H_\chi)^V$. The following subclaim immediately shows that $j(\mathbb{L})/G$ is σ-Baire and preserves the stationarity of T.

Subclaim 5.5. *For every $(\overline{M}, j(\mathbb{L})/G)$-generic sequence $\langle p_n : n < \omega \rangle$ with $p_0 \leq p$, there is an $(\overline{M}, j(\mathbb{L})/G)$-generic condition $\overline{p}$ such that, $\overline{p}$ is a lower bound of the p_n's, and, letting $\overline{p} = \langle f, g \rangle$, $\sup(\overline{M} \cap j(\kappa)) \in \mathrm{dom}(f)$ and $f(\sup(\overline{M} \cap j(\kappa))) \nsubseteq \overline{M} \cap j(\kappa)$.*

Proof of Subclaim. Take an $(\overline{M}, j(\mathbb{L})/G)$-generic sequence $\langle p_n : n < \omega \rangle$ in $j(\mathbb{L})/G$ with $p_0 \leq p$. We know that $\langle p_n : n < \omega \rangle$ is an $(\overline{M} \cap (H_\chi)^V, j(\mathbb{L}))$-generic sequence. Let $p_n = \langle f_n, g_n \rangle$. Since $\kappa \in \overline{M}$, we may assume that $\min(j(E^\kappa_\omega) \setminus \kappa) \in \mathrm{dom}(f_n)$ for every $n < \omega$. We also may assume that $\langle f_n, g_n \rangle \in j(D)$, this means that $\max(\mathrm{dom}(g_n(\alpha))) > \kappa$ for every $\alpha \in \mathrm{dom}(g_n) \setminus \kappa$. Take a lower bound $\langle f', g' \rangle \in G$ of the $\langle f_n|\kappa, g_n|\kappa \rangle$'s with $\sup(\overline{M} \cap \kappa) \in \mathrm{dom}(f')$. For $n < \omega$, let $f^*_n = f_n|[\kappa, j(\kappa))$ and $g^*_n = g_n|[\kappa, j(\kappa))$. Then we define $\overline{p} = \langle f, g \rangle \in j(\mathbb{L}_\kappa)$ as in Lemma 2.6 with the following modifications:

(1) $f|\kappa = f'$.
(2) $\mathrm{dom}(f)|[\kappa, j(\kappa)) = \bigcup_{n<\omega} \mathrm{dom}(f^*_n) \cup \lim(\bigcup_{n<\omega} \mathrm{dom}(f^*_n)) \cup \{\sup(\overline{M} \cap j(\kappa))\}$.
(3) For $\alpha \in \bigcup_{n<\omega} \mathrm{dom}(f^*_n)$, $f(\alpha) = f^*_n(\alpha)$ for some $n < \omega$ with $\alpha \in \mathrm{dom}(f^*_n)$.
(4) For $\alpha \notin \bigcup_{n<\omega} \mathrm{dom}(f^*_n)$ (note that $\alpha > \kappa$), $f(\alpha)$ is a cofinal subset of α with $\mathrm{ot}(f(\alpha)) = \omega$ and $f(\alpha) \nsubseteq \bigcup\{f^*_n(\beta) : \beta \in \mathrm{dom}(f^*_n), n < \omega\}$, and if $\alpha = \sup(\overline{M} \cap j(\kappa))$, we require that $f(\alpha) \nsubseteq \overline{M}$.
(5) $g|\kappa = g'$.
(6) $\mathrm{dom}(g)|[\kappa, j(\kappa)) = \bigcup_{n<\omega} \mathrm{dom}(g^*_n)$.
(7) For $\alpha \in \bigcup_{n<\omega} \mathrm{dom}(g^*_n)$, let $d_\alpha = \bigcup\{\mathrm{dom}(g^*_n(\alpha)) : n < \omega, \alpha \in \mathrm{dom}(g_n)\}$. Then $\mathrm{dom}(g(\alpha)) = d_\alpha \cup \{\sup(d_\alpha)\}$. For $\beta \in d_\alpha$, let $g(\alpha)(\beta) = g^*_n(\alpha)(\beta)$, where $n < \omega$ is minimal with $\beta \in \mathrm{dom}(g^*_n(\alpha))$. When $\sup(d_\alpha) \notin d_\alpha$, we consider the following two cases to decide the value of $g(\alpha)(\sup(d_\alpha))$:
   (a) If $\sup(d_\alpha) \notin d_\alpha$ and $\alpha > \kappa$, then $\sup(d_\alpha) > \kappa$, and the value of $f(\sup(d_\alpha))$ was assigned as in (4). Let $g(\alpha)(\sup(d_\alpha))$ be an element of $f(\sup(d_\alpha)) \setminus \bigcup\{f^*_n(\beta) : \beta \in \mathrm{dom}(f^*_n), n < \omega\}$.
   (b) If $\sup(d_\alpha) \notin d_\alpha$ and $\alpha = \kappa$, then $\sup(d_\alpha) = \sup(\overline{M} \cap \kappa) < \kappa$, and $f(\sup(d_\alpha)) = f'(\sup(d_\alpha)) = \vec{c}_{\sup(d_\alpha)}$. So the value of $f(\sup(d_\alpha))$ was already assigned by G. But since $\overline{M} \cap \kappa \notin \mathcal{S}^*$, we have that $f(\sup(d_\alpha)) = \vec{c}_{\sup(d_\alpha))} \nsubseteq \overline{M} \cap \kappa$. Thus we can take $\gamma \in f(\sup(d_\alpha)) \setminus \overline{M} \cap \kappa$, and put $g(\alpha)(\sup(d_\alpha)) = \gamma$.

Since N is closed under $< \kappa$-sequences in V and $V[G][H_\xi]$ is a σ-Baire forcing extension of V, N is closed under ω-sequences in $V[G][H_\xi]$. So we have $\langle \langle f_n, g_n \rangle : n < \omega \rangle \in N$, and $\langle f, g \rangle \in N$. Then it is straightforward to check that $\langle f, g \rangle \in j(\mathbb{L})/G$ and is a lower bound of the $\langle f_n, g_n \rangle$'s. By the choice of the $\langle f_n, g_n \rangle$'s, $\langle f, g \rangle$ is a generic condition for $\overline{M}$ below p. □

Now take an arbitrary $(V[G][H_\xi], j(\mathbb{L})/G)$-generic $j(G)$ and work in $V[G][H_\xi][j(G)]$. We know that T is stationary in $V[G][H_\xi][j(G)]$. Next we see that $j(\mathbb{P}_\xi)/H_\xi$ is σ-

Baire and preserves the stationarity of T, which completes the proof of Claim 5.3. In $V[G][H_\xi][j(G)]$, $j : M \to N$ can be extended to $j : M[G] \to N[j(G)]$. Observe that the following:

(1) The cofinality of κ is collapsed to ω_1.
(2) There is a club $C \subseteq \kappa$ and an injection f on C such that $f(\alpha) \in c_\alpha = j(\vec{c})_\alpha$ for every $\alpha \in (E^\kappa_\omega)^V$. In particular $\mathcal{S}^*$ is non-stationary in $[\kappa]^\omega$.
(3) $N[j(G)]$ and $N[j(G)][H_\xi]$ are still closed under ω-sequences in $V[G][H_\xi][j(G)]$.
(4) $j(\mathcal{S}^*) \cap [\kappa]^\omega = \mathcal{S}^*$.
(5) By Subclaim 5.5, for every $\mu > \lambda + j(\kappa)$, the set $\{x \in [\mu]^\omega : x \cap \lambda \in T, x \cap j(\kappa) \notin j(\mathcal{S}^*)\}$ is stationary in $[\mu]^\omega$.

Now, by Lemma 3.6, we can identify $\mathbb{P}_\xi$ with a poset which consists of functions p with $\mathrm{dom}(p) \in [\xi]^\omega$ and $p(\eta) \in \mathbb{C}$ for every $\eta \in \mathrm{dom}(p)$. Where we identify the domain of p with its support.

For $p \in j(\mathbb{P}_\xi)$ (so p is a function with $\mathrm{dom}(p) \in [j(\xi)]^\omega$), let $\hat{p}$ be the function defined by $\mathrm{dom}(\hat{p}) = j^{-1}$“$(\mathrm{dom}(p))$, and $\hat{p}(\alpha) = p(j(\alpha)) | \kappa \times \kappa$ for $\alpha \in \mathrm{dom}(\hat{p})$.

Subclaim 5.6. *Let $p \in j(\mathbb{P}_\xi)$. Then $\hat{p} \in \mathbb{P}_\xi$, and $p \in j(\mathbb{P}_\xi)/H_\xi$ if and only if $\hat{p} \in H_\xi$.*

Proof. We see only $\hat{p} \in \mathbb{P}_\xi$, the rest is straightforward.

If $\hat{p} \notin \mathbb{P}_\xi$, then there are $\eta \in \mathrm{dom}(\hat{p})$ and $x \in \mathcal{S}^*$ such that $\hat{p}|\eta \in \mathbb{P}_\eta$ but there is some $q \leq \hat{p}|\eta$ with $q \Vdash_{\mathbb{P}_\eta}$ “$x \subseteq d(\hat{p}(\eta))$, $x \in \dot{S}_\eta$ and x is closed under $\hat{p}(\eta)$”. Consider $j(\hat{p}|\eta+1)$. We have $j(\hat{p}|\eta+1) = j(\hat{p}|\eta)^\frown\langle j(\hat{p}(\eta))\rangle$. Since $\hat{p}|\eta \in \mathbb{P}_\eta$, we have $j(\hat{p}|\eta) \in j(\mathbb{P}_\eta)$, and $p|j(\eta) \leq j(\hat{p}|\eta)$ by the definition of $\hat{p}$. On the other hand, $j(q)$ is compatible with $p|j(\eta)$; For every $\zeta \in \mathrm{dom}(j(q)) \cap \mathrm{dom}(p|j(\eta))$, we have that $j(q)(\zeta)$ is compatible with $p(\zeta)$ as a function since $q \leq \hat{p}$. Then we can construct a natural common extension of $j(q)$ and $p|j(\eta)$ using the argument in the proof of Lemma 3.2.

We know $j(q) \Vdash$“$j(x) \subseteq j(d(\hat{p}(\eta)))$, $j(x) \in j(\dot{S}_\eta)$ and $j(x)$ is closed under $j(\hat{p}(\eta))$”. Since $j(q)$ is compatible with $p|j(\eta)$, there is $r \leq j(q), p|j(\eta)$ which forces that statement. Now, $j(x) = x$, $j(\hat{p}(\eta)) = \hat{p}(\eta) = p(j(\eta))|\kappa \times \kappa$, and $j(d(\hat{p}(\eta))) = d(\hat{p}(\eta)) = d(p(j(\eta))) \cap \kappa$. Hence $r \Vdash$“$p(j(\eta)) \notin j(\mathbb{C}(\dot{S}_\eta))$”, this is a contradiction. □

To show that $j(\mathbb{P}_\xi)/H_\xi$ is σ-Baire and preserves the stationarity of T, take $p \in j(\mathbb{P}_\xi)/H_\xi$, a large regular χ, and a countable $\overline{M} \prec H_\chi$ such that $\overline{M} \cap \lambda \in T$ and $\overline{M}$ contains all relevant objects. Take an $(\overline{M}, j(\mathbb{P}_\xi)/H_\xi)$-generic sequence $\langle p_n : n < \omega\rangle$ such that $p_0 \leq p$. We will find an $(\overline{M}, j(\mathbb{P}_\xi)/H_\xi)$-generic condition which is a lower bound of the p_n's. This shows that $j(\mathbb{P}_\xi)/H_\xi$ is σ-Baire and preserves the stationarity of T.

By Subclaim 5.5, we can require that $\overline{M} \cap j(\kappa) \notin j(\mathcal{S}^*)$. Fix $\mu \in N[j(G)]$ such that μ is regular in $N[j(G)]$ and sufficiently larger than $j(\kappa)$ in $N[j(G)]$. We may assume that $(H_\mu)^{N[j(G)]} \in \overline{M}$. Let $M' = \overline{M} \cap (H_\mu)^{N[j(G)]}$. Then $M' \in N[j(G)]$, $M' \prec (H_\mu)^{N[j(G)]}$, and $\overline{M} \cap j(\kappa) = M' \cap j(\kappa) \notin j(\mathcal{S}^*)$. Then $\langle p_n : n < \omega\rangle \in N[j(G)]$ and is an $(M', j(\mathbb{P}_\xi))$-generic sequence. Applying Lemma 3.6 to M' in $N[j(G)]$, the p_n's have a lower bound p^* in $j(\mathbb{P}_\xi)$ defined by:

(1) $\mathrm{dom}(p^*) = \bigcup_{n<\omega} \mathrm{dom}(p_n)$.

(2) For every $\eta \in \mathrm{dom}(p^*)$, $p^*(\eta) = \bigcup\{p_n(\eta) : n < \omega, \eta \in \mathrm{dom}(p_n)\}$.

Then, since $\hat{p_n} \in H_\xi$ for every $n < \omega$, it is easy to see that $\hat{p^*} \in H_\xi$, so $p^* \in j(\mathbb{P}_\xi)/H_\xi$. This completes the proof of Claim 5.3, hence so does the proof of Theorem 1.6.

REMARK 5.7. $2^{\omega_1} > \omega_2$ holds in the final model, and this cardinal arithmetic is necessary; Koenig–Larson–Yoshinobu [**9**] showed that if $2^{\omega_1} = \omega_2$, then $\mathrm{WRP}(\omega_2)$ is equivalent to $\mathrm{RP}(\omega_2)$.

6. Proof of Theorem 1.7.

Suppose GCH, and let κ be a supercompact cardinal. Let $\mathbb{L}_\kappa * \mathbb{P}_{\kappa^+}$ be the poset used in the previous section. So it forces that $\kappa = \omega_2$, $\mathrm{WRP}(\omega_2)$, but $\neg\,\mathrm{FRP}(\omega_2)$. We see that this poset also forces SSR.

To see this, fix a $(V, \mathbb{L}_\kappa)$-generic G and a $(V[G], \mathbb{P}_{\kappa^+})$-generic H. Work in $V[G][H]$. Fix $\lambda > \kappa$ and we check that $\mathrm{SSR}(\lambda)$ holds in $V[G][H]$. Take a semi-stationary set $S \subseteq [\lambda]^\omega$. Let $T = \{x \in [\lambda]^\omega : \exists y \in S\,(y \subseteq x \text{ and } y \cap \omega_1 = x \cap \omega_1)\}$. T is stationary in $[\lambda]^\omega$.

As before, let $\vec{c}$ be a non-reflecting ladder system induced by G and let $\mathcal{S}^* = \mathcal{S}^{\vec{c}}$.

First suppose that $\{x \in T : x \cap \kappa \notin \mathcal{S}^*\}$ is stationary in $[\lambda]^\omega$. In V, take a λ-supercompact embedding $j : V \to N$ with critical point κ. Consider $j(\mathbb{L}_\kappa * \mathbb{P}_{\kappa^+})/G * H$. Then, by the same argument used in the proof of Claim 5.3, we can prove that $j(\mathbb{L}_\kappa * \mathbb{P}_{\kappa^+})/G * H$ preserves the stationarity of T. For a $(V[G][H], j(\mathbb{L}_\kappa * \mathbb{P}_{\kappa^+})/G * H)$-generic $j(G) * j(H)$, we can extend $j : V \to N$ to $j : V[G][H] \to N[j(G)][j(H)]$. Since T is stationary in $V[j(G) * j(H)]$, $j(T) \cap [j``\lambda]^\omega$ is also stationary in $[j``\lambda]^\omega$. Then in $N[j(G)][j(H)]$, $j``\lambda$ witnesses that the statement that there is $X \in [j(\lambda)]^{\omega_1}$ such that $\omega_1 \subseteq X$ and $j(T) \cap [X]^\omega$ is stationary in $[X]^\omega$. By the elementarity of j, in $V[G][H]$, we have that there is $X \in [\lambda]^{\omega_1}$ such that $\omega_1 \subseteq X$ and $T \cap [X]^\omega$ is stationary in $[X]^\omega$. Then clearly $S \cap [X]^\omega$ is semi-stationary in $[X]^\omega$.

Hence it is enough to check that $\{x \in T : x \cap \kappa \notin \mathcal{S}^*\}$ must be stationary in $[\lambda]^\omega$. If $\{x \in T : x \cap \kappa \in \mathcal{S}^*\}$ is non-stationary, we are done. Suppose it is stationary. Take a club D in $[\lambda]^\omega$. We will find $x \in T \cap D$ with $x \cap \kappa \notin \mathcal{S}^*$.

Fix a sufficiently large regular cardinal θ and a well-ordering Δ on H_θ. Let $\mathcal{M} = \langle H_\theta, \in, \Delta, \mathcal{C}^*, \mathcal{S}^*, T, D, \ldots\rangle$. By Lemma 4.1, we may assume that for every countable $M \prec \mathcal{M}$, if $M \cap \kappa \in \mathcal{S}^*$, then there is a countable $N \prec \mathcal{M}$ such that $M \subseteq N$, $M \cap \kappa = N \cap \sup(M \cap \kappa)$, and $\sup(M \cap \kappa) \in N$.

Now take a countable $M_0 \prec \mathcal{M}$ with $M_0 \cap \lambda \in T$. We know $M_0 \cap \lambda \in T \cap D$. If $M_0 \cap \kappa \notin \mathcal{S}^*$, we are done. Suppose $M_0 \cap \kappa \in \mathcal{S}^*$. Then there is a countable $M_1 \prec \mathcal{M}$ such that $M_0 \subseteq M_1$, $M_0 \cap \kappa = M_1 \cap \sup(M_0 \cap \kappa)$, and $\sup(M_0 \cap \kappa) \in M_1$. Note that $M_1 \cap \lambda \in T \cap D$. If $M_1 \cap \kappa \notin \mathcal{S}^*$, we are done. Otherwise, take a countable $M_2 \prec \mathcal{M}$ as before for M_1. We repeat this procedure. Now suppose $i < \omega_1$ and $\langle M_j : j < i\rangle$ was chosen so that:

(1) $M_j \prec \mathcal{M}$ is countable with $M_j \cap \lambda \in T \cap D$ and $M_j \cap \kappa \in \mathcal{S}^*$.
(2) For $j < k < i$, $M_j \subseteq M_k$, $M_j \cap \kappa = M_k \cap \sup(M_j \cap \kappa)$, and $\sup(M_j \cap \kappa) \in M_k$.
(3) If j is limit, then $M_j = \bigcup_{k<j} M_k$.

If i is limit, then let $M_i = \bigcup_{j<i} M_j$. If i is successor, since $M_{i-1} \cap \kappa \in \mathcal{S}^*$, we can take a countable $M_i \prec \mathcal{M}$ such that $M_{i-1} \subseteq M_i$, $M_{i-1} \cap \kappa = M_i \cap \sup(M_{i-1} \cap \kappa)$, and $\sup(M_{i-1} \cap \kappa) \in M_i$. We know that $M_i \cap \lambda \in T \cap D$, and if $M_i \cap \kappa \notin \mathcal{S}^*$, then we stop this construction and $M_i \cap \lambda$ is as required. So suppose to the contrary that we can take $\langle M_i : i < \omega_1 \rangle$. Let $M = \bigcup_{i<\omega_1} M_i$. Then $\mathrm{ot}(M \cap \kappa) = \omega_1$, and $M \cap \sup(M_i \cap \kappa) = M_i \cap \kappa$. By the choice of the M_i's, we have that $\{\sup(M_i \cap \kappa) : i < \omega_1\}$ is a club in $\sup(M \cap \kappa)$. Let $E = \{i < \omega_1 : i = \mathrm{ot}(M_i \cap \kappa)\}$. E is a club in ω_1. Since $\vec{c}$ is a non-reflecting ladder system, there is an injection f such that $\mathrm{dom}(f)$ is a club in $\sup(M \cap \kappa)$ and $f(\alpha) \in c_\alpha$ for every $\alpha \in \mathrm{dom}(f)$. Note that $f(\sup(M_i \cap \kappa)) \in c_{\sup(M_i \cap \kappa)} \subseteq M_i \cap \kappa = M \cap \sup(M_i \cap \kappa)$ for every $i \in E$. Let E' be the set of all $i \in E$ with $\sup(M_i \cap \kappa) \in \mathrm{dom}(f)$, and let $g : E' \to \omega_1$ by $g(i) = \beta \iff f(\sup(M_i \cap \kappa))$ is the β-th element of $M \cap \kappa$. g is regressive. Thus we can find β_0 such that $\{i \in E : g(i) = \beta_0\}$ is stationary. So there are $i < j$ such that $\sup(M_i \cap \kappa), \sup(M_j \cap \kappa) \in \mathrm{dom}(f)$ but $f(\sup(M_i \cap \kappa)) = f(\sup(M_j \cap \kappa))$, this is a contradiction.

Therefore we have that there is some i such that $M_i \cap \lambda \in T \cap D$ but $M_i \cap \kappa \notin \mathcal{S}^*$, so $M_i \cap \lambda$ is as required. Now we complete the proof of Theorem 1.7.

Question 6.1. We have known that SSR does not imply $\mathrm{RP}(\omega_2)$ nor $\mathrm{FRP}(\omega_2)$. But the following are unknown:

(1) Does WRP imply $\mathrm{FRP}(\omega_2)$ or FRP?
(2) Does WRP imply $\mathrm{RP}(\omega_2)$ or RP?

7. Appendix.

In this section we give a proof of Fact 1.11. We prove a slightly stronger result.

Proposition 7.1. *Let $\kappa \geq \omega_2$ be regular, $S \subseteq \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ stationary in κ, and $E \subseteq \omega_1$ stationary in ω_1. Let $\vec{c}$ be a ladder system on S. Then the set $\{x \in [\kappa]^\omega : x \cap \omega_1 \in E, \sup(x) \in S, c_{\sup(x)} \nsubseteq x\}$ is stationary in $[\kappa]^\omega$.*

To prove this, fix a function $f : [\kappa]^{<\omega} \to \kappa$. We will find $x \in [\kappa]^\omega$ such that $x \cap \omega_1 \in E$, $\sup(x) \in S$, $c_{\sup(x)} \nsubseteq x$, and x is closed under f.

For $x \in [\kappa]^\omega$, let $C_f(x)$ be the closure of x under f.

For $i < \omega_1$, we consider the following two players game Γ_i of length ω, which is a variant of Veličković's game in [**16**]:

ONE	α_0, β_0		α_1, β_1		$\cdots$
TWO		γ_0, δ_0		γ_1, δ_1	$\cdots$

Where $\alpha_n < \beta_n < \gamma_n < \delta_n < \alpha_{n+1} < \kappa$. For a play $\langle \alpha_n, \beta_n, \gamma_n, \delta_n : n < \omega \rangle$, ONE *wins* if $C_f(\{\alpha_n : n < \omega\} \cup i) \cap \omega_1 = i$ and $C_f(\{\alpha_n : n < \omega\} \cup i) \cap \bigcup_{n<\omega} [\gamma_n, \delta_n] = \emptyset$, otherwise TWO *wins*. Clearly the game Γ_i is open for TWO, so it is a determined game.

Lemma 7.2. *Let $E_0 = \{i < \omega_1 :$ player ONE has a winning strategy in $\Gamma_i\}$. Then E_0 contains a club in ω_1.*

PROOF. Suppose to the contrary that $\omega_1 \setminus E_0$ is stationary in ω_1. For each $i \in \omega_1 \setminus E_0$, TWO has a winning strategy σ_i in Γ_i. Fix a sufficiently large regular θ. Take elementary submodels $M_n \prec H_\theta$ for $n < \omega$ such that:

(1) M_n contains all relevant objects.
(2) $|M_n| < \kappa$, $M_n \cap \kappa \in \kappa$, and $\mathrm{cf}(M_n \cap \kappa) > \omega$.
(3) $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ and $M_0 \cap \kappa < M_1 \cap \kappa < M_2 \cap \kappa < \cdots$.

Fix $\alpha_0 \in M_0 \cap \kappa$ and $\alpha_{n+1} \in (M_{n+1} \cap \kappa) \setminus M_n$ for $n < \omega$. Since $\omega_1 \setminus E_0$ is stationary, there is $i \in \omega_1 \setminus E_0$ such that $C_f(\{\alpha_n : n < \omega\} \cup i) \cap \omega_1 = i$. Let $x = C_f(\{\alpha_n : n < \omega\} \cup i)$. We have $\sup(x) = \sup\{M_n \cap \kappa : n < \omega\}$. Since $\mathrm{cf}(M_n \cap \kappa) > \omega$ but x is countable, there is $\beta_n \in M_n \cap \kappa$ with $\sup(x \cap M_n \cap \kappa) < \beta_n < M_n \cap \kappa$. Note that $i \in M_0$, $\alpha_n < \beta_n < M_n \cap \kappa \le \min(x \setminus (M_n \cap \kappa)) \le \alpha_{n+1}$, and $\langle \alpha_j, \beta_j : j \le n\rangle \in M_n$ for $n < \omega$. Let $\langle \gamma_n, \delta_n \rangle = \sigma_i(\langle \alpha_j, \beta_j : j \le n \rangle) \in M_n$. Since $\gamma_n, \delta_n \in M_n$, we have $\gamma_n < \delta_n < M_n \cap \kappa < \alpha_{n+1}$. Hence $\langle \alpha_n, \beta_n, \gamma_n, \delta_n : n < \omega\rangle$ is a play in Γ_i such that $\langle \gamma_n, \delta_n \rangle = \sigma_i(\langle \alpha_j, \beta_j : j \le n\rangle)$. σ_i is a winning strategy of TWO, thus $x \cap [\gamma_n, \delta_n] \neq \emptyset$ for some $n < \omega$. But $\sup(x \cap M_n \cap \kappa) < \beta_n < \gamma_n < \delta_n < M_n \cap \kappa \le \min(x \setminus M_n \cap \kappa)$, this is a contradiction. □

Now we construct $x \in [\kappa]^\omega$ such that $x \cap \omega_1 \in E$, $\sup(x) \in S$, $c_{\sup(x)} \nsubseteq x$, and x is closed under f. Take a countable $M \prec H_\theta$ such that $\sup(M \cap \kappa) \in S$ and M contains all relevant objects. Fix an increasing sequence $\langle \eta_n : n < \omega \rangle$ with limit $\sup(M \cap \kappa)$. By Lemma 7.2, there is $i \in E \cap M$ such that ONE has a winning strategy $\sigma \in M$ in Γ_i. Then define a sequence $\langle \alpha_n, \beta_n, \delta_n, \gamma_n : n < \omega\rangle$ in M as follows: First let $\langle \alpha_0, \beta_0\rangle = \sigma(\emptyset) \in M$. Then take $\gamma_0, \delta_0 \in M \cap \kappa$ such that $\beta_0, \eta_0 < \gamma_0$ and $c_{\sup(M\cap\kappa)} \cap [\gamma_0, \delta_0] \neq \emptyset$. Put $\langle \alpha_1, \beta_1 \rangle = \sigma(\langle \gamma_0, \delta_0\rangle)$, and take $\gamma_1, \delta_1 \in M \cap \kappa$ such that $\beta_1, \eta_1 < \gamma_1$ and $c_{\sup(M\cap\kappa)} \cap [\gamma_1, \delta_1] \neq \emptyset$. Repeat this procedure. Let $x = C_f(\{\alpha_n : n < \omega\} \cup i)$. Then $x \cap \omega_1 = i \in E$, $\sup(x) = \sup(M \cap \kappa) \in S$, and $c_{\sup(M\cap\kappa)} \nsubseteq x$.

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