J. Math. Soc. Japan
Vol. 68, No. 2 (2016) pp. 797–805
doi: 10.2969/jmsj/06820797

Coefficient multipliers of H^1 into ℓ^q associated with Laguerre expansions

By Yehao Shi and Zhongkai Li

(Received June 24, 2014)

Abstract. The purpose of the paper is to study coefficient multipliers of the Hardy space $H^1([0,\infty))$ associated with Laguerre expansions. As a consequence, a Paley type inequality is obtained.

1. Introduction and results.

If $\alpha > -1$, the Laguerre function $\mathcal{L}_n^{(\alpha)}(x)$ is defined by

$$\mathcal{L}_n^{(\alpha)}(x) = \tau_n^\alpha L_n^{(\alpha)}(x) e^{-x/2} x^{\alpha/2}, \tag{1}$$

where $\tau_n^\alpha = (\Gamma(n+1)/\Gamma(n+\alpha+1))^{1/2}$ and $L_n^{(\alpha)}(x)$ $(n \geq 0)$ are the Laguerre polynomials determined by the orthogonal relation (see [**16**, (5.1.1)])

$$\int_0^\infty e^{-x} x^\alpha L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) dx = (\tau_n^\alpha)^{-2} \delta_{mn}.$$

The system $\{\mathcal{L}_n^{(\alpha)}(x)\}_{n=0}^\infty$ is a complete orthonormal system on the interval $[0, +\infty)$ with respect to the Lebesgue measure. For a function $f \in L^p([0,\infty))$, $1 \leq p \leq \infty$, its Laguerre expansion is

$$f \sim \sum_{n=0}^\infty c_n^{(\alpha)}(f) \mathcal{L}_n^{(\alpha)}(x), \qquad c_n^{(\alpha)}(f) = \int_0^\infty f(t) \mathcal{L}_n^{(\alpha)}(t) dt. \tag{2}$$

$H^1(\mathbb{R})$ is the real Hardy space of the boundary values $f(x) = \Re F(x)$ of the real parts $\Re F(z)$ of functions $F(z)$, where $F(z)$ is an element of the Hardy space $H^1(\mathbb{R}^2_+)$, that is, $F(z)$ is analytic on the upper half plane $\mathbb{R}^2_+ = \{z = x + iy; y > 0\}$ with the norm

$$\|f\|_{H^1(\mathbb{R})} = \|F\|_{H^1(\mathbb{R}^2_+)} = \sup_{y>0} \int_{-\infty}^\infty |F(x+iy)| dx.$$

In the present paper, we shall study the coefficient multipliers associated with Laguerre

---

2010 *Mathematics Subject Classification.* Primary 42B30; Secondary 42C10, 42A45.

Key Words and Phrases. Hardy space, multipliers, Laguerre expansion, Paley inequality.

This research was supported by the National Natural Science Foundation of China (No. 11371258, 11571261).

expansions on the space

$$H^1([0,\infty)) = \{f \in H^1(\mathbb{R}) : \mathrm{supp} f \subset [0,\infty)\}.$$

Our main theorem is as follows:

Theorem 1.1. *Let $\alpha \geq 0$ and $2 \leq q < \infty$. If a sequence $\{\lambda_n\}_{n=0}^{\infty}$ satisfies the condition*

$$\sum_{k=n}^{2n} |\lambda_k|^q = O(1), \qquad \text{as } \; n \to \infty, \tag{3}$$

then for all $f \in H^1([0,+\infty))$, the coefficients $c_n^{(\alpha)}(f)$ of its Laguerre expansion (2) *satisfy*

$$\sum_{n=0}^{\infty} |\lambda_n c_n^{(\alpha)}(f)|^q \leq c\|f\|^q_{H^1([0,\infty))}, \tag{4}$$

where c is a constant independent of f.

An interesting application of Theorem 1.1 is the Paley type inequality for Laguerre expansions, which is stated in the following corollary.

Corollary 1.2. *Let $\alpha \geq 0$. If $\{n_k\}$ is a Hadamard sequence satisfying $n_{k+1}/n_k \geq \rho > 1$ $(k = 1, 2, \dots)$, then for all $f \in H^1([0,\infty))$, the coefficients $c_n^{(\alpha)}(f)$ of its Laguerre expansion* (2) *satisfy*

$$\sum_{k=1}^{\infty} |c_{n_k}^{(\alpha)}(f)|^2 \leq c\|f\|^2_{H^1([0,\infty))}, \tag{5}$$

where c is a constant independent of f.

A function F analytic in the unit disk $\mathbb{D}$ is said to be in the Hardy space $H^p(\mathbb{D})$, $0 < p < \infty$, if $\|F\|_{H^p} := \sup_{0\leq r<1} M_p(F;r) < \infty$, where $M_p(F;r) = \big\{(1/2\pi)\int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta\big\}^{1/p}$. Denote by ℓ^q the sequence space $\ell^q = \big\{\{a_k\} : \|\{a_k\}\|_q = \big(\sum_{k=0}^{\infty} |a_k|^q\big)^{1/q} < \infty\big\}$ for $0 < q < \infty$, and ℓ^∞ the set of bounded sequences. A sequence $\{\lambda_n\}$ is said to be a multiplier of $H^p(\mathbb{D})$ into the sequence spaces ℓ^q provided $\{\lambda_n c_n\} \in \ell^q$ whenever $\sum_{n=0}^{\infty} c_n z^n \in H^p(\mathbb{D})$. Similarly, a sequence $\{\lambda_n\}_{n=0}^{\infty}$ is a multiplier of $H^1([0,\infty))$ into ℓ^q associated with Laguerre expansions if (4) holds.

Coefficient multipliers of the Hardy spaces $H^p(\mathbb{D})$ into ℓ^q are characterized in Duren and Shields [**4**]. According to [**4**, pp. 72–73], the sequence $\{\lambda_n\}$ is a multiplier of $H^1(\mathbb{D})$ into ℓ^q for $2 \leq q < \infty$ if and only if $\sum_{n=N}^{2N} |\lambda_n|^q = O(1)$. It is very remarkable that the sufficient condition for coefficient multipliers of H^1 into $\ell^q (2 \leq q < \infty)$ associated with Laguerre expansions coincides with that of Taylor expansions. For a survey on multipliers from $H^p(\mathbb{D})$ to ℓ^q for various p and q, we may refer to [**12**]. The original proofs of classical theorems on coefficient multipliers depend strongly on the complexvariable structures of analytic functions, but this does not suit for other eigenfunction expansions. Recently, by means of real-variable methods in harmonic analysis (see the book [**15**]), [**8**], [**9**], [**17**] proved a series of theorems on coefficient multipliers of the Hardy spaces H^p associated with three orthogonal systems of functions, such as exponential Jacobi functions, generalized Hermite functions, and Laguerre functions. In particular, some Paley-type inequalities and Hardy-type inequalities in each case were described. In our previous paper [**17**], we studied coefficient multipliers of the Hardy spaces $H^p([0,\infty))$ $(0<p<1)$ associated with Laguerre expansions, which is based on the duality relation of the Hardy space $H^p(\mathbb{R})$ and the Lipschitz space $\Lambda_{p^{-1}-1}(\mathbb{R})$. Analogs of the Hardy inequality and Paley inequality in the context of eigenfunction expansions were studied by several authors (cf. [**2**], [**3**], [**5**], [**6**], [**7**], [**13**], [**14**], [**18**]).

Throughout the paper, $A = O(B)$ or $A \lesssim B$ means that $A \le cB$ for some positive constant c independent of variables, functions, n, k, etc., but possibly dependent of some fixed parameters and fixed m. $\mathbb{N}_0 = \{0,1,2,\dots\}$ denotes the set of all nonnegative integers. If n^a or k^a appears in some estimates, then it will be understood as the constant 1 for $n=0$ or $k=0$, regardless of whether a is positive or negative.

2. Preliminieries.

In order to apply the duality of $H^1(\mathbb{R})$ and $BMO(\mathbb{R})$, we must extend $\mathcal{L}_n^{(\alpha)}(x)$ from the half line $\mathbb{R}_+$ to the whole line $\mathbb{R}$ in the same way as [**17**]. If $\alpha/2>0$ is not an integer, then we define

$$\tilde{\mathcal{L}}_n^{(\alpha)}(x) = \begin{cases} \mathcal{L}_n^{(\alpha)}(x), & \text{for } x>0; \\ 0, & \text{for } x \le 0. \end{cases} \tag{6}$$

If $\alpha/2 \ge 0$ is an integer, we shall use the function

$$\psi(x) = \begin{cases} 1, & \text{for } x \ge 0; \\ (1-e^{1/x}) \exp\left(-\dfrac{e^{1/x}}{x+1}\right), & \text{for } -1<x<0; \\ 0, & \text{for } x \le -1. \end{cases}$$

It is clear that $\psi(x) \in C(\mathbb{R})$. Furthermore, for $k \ge 1$, the k-th derivative $\psi^{(k)}(x)$ of $\psi(x)$ satisfies $\lim\limits_{x\to -1+0} \psi^{(k)}(x) = \lim\limits_{x \to 0-0} \psi^{(k)}(x) = 0$ by routine evaluations, which implies that $\psi(x) \in C^\infty(\mathbb{R})$ and $|\psi^{(k)}(x)| \le c$, where c is a constant independent of x.

We define, for even integer $\alpha \ge 0$,

$$\tilde{\mathcal{L}}_n^{(\alpha)}(x) = \psi(nx)\mathcal{L}_n^{(\alpha)}(x). \tag{7}$$

We see that the coefficients $c_n^{(\alpha)}(f)$ are independent of the choice of an extension $\tilde{\mathcal{L}}_n^{(\alpha)}(x)$.

The estimations of the higher order derivatives for Laguerre functions in [**14**, Lemma 1 and Lemma 2] are valid for $\tilde{\mathcal{L}}_n^{(\alpha)}(x)$ instead of $\mathcal{L}_n^{(\alpha)}(x)$ on the whole line $\mathbb{R}$.

LEMMA 2.1 ([**17**, Corollary 2.4]). *Let $\alpha \geq 0$ and $M = [\alpha/2]$. Then for $x \in \mathbb{R}$,*

(i) *if $\alpha/2$ is not an integer,*

$$\left|(\tilde{\mathcal{L}}_n^{(\alpha)})^{(m)}(x)\right| \lesssim n^m, \quad m \leq M; \tag{8}$$

(ii) *if $\alpha/2$ is not an integer,*

$$\left|(\tilde{\mathcal{L}}_n^{(\alpha)})^{(M)}(x+h) - (\tilde{\mathcal{L}}_n^{(\alpha)})^{(M)}(x)\right| \lesssim n^{\alpha/2}|h|^\delta, \quad \alpha/2 = M + \delta,\ 0 < \delta < 1; \tag{9}$$

(iii) *if $\alpha/2$ is an integer, (8) is true for all $m \in \mathbb{N}_0$.*

For given $\alpha > -1$ and $\tau > 0$, a precise estimate of the Laguerre polynomials is given by (see [**1**], [**10**], [**11**])

$$\left|L_n^{(\alpha)}(x)\right| \lesssim e^{x/2} n^{\alpha/2} (\nu^{-1} + x)^{-\alpha/2-1/4} (\nu^{1/3} + |x - \nu|)^{-1/4} \Phi_n^{(\alpha)}(x), \tag{10}$$

where $\nu = 4n + 2\alpha + 2$, and

$$\Phi_n^{(\alpha)}(x) = \begin{cases} 1, & \text{for } 0 \leq x \leq \nu; \\ \exp\left(\dfrac{-\eta|x-\nu|^{3/2}}{\nu^{1/2}}\right), & \text{for } \nu \leq x \leq (1+\tau)\nu; \\ e^{-\xi x}, & \text{for } (1+\tau)\nu \leq x \end{cases}$$

for some given positive constants $\eta = \eta(\alpha, \tau)$ and $\xi = \xi(\alpha, \tau)$. The unified and simplified form as (10) is stated in [**8**], which prefer to use $4n$ instead of ν for convenience in subsequent applications.

LEMMA 2.2 ([**8**, Lemma 2.1]). *For given $\alpha > -1$ and $\tau > 0$, there exist positive constants η and ξ such that*

$$\left|L_n^{(\alpha)}(x)\right| \lesssim e^{x/2} n^{\alpha/2} x^{-\alpha/2} \mathscr{M}_n^{(\alpha)}(x) \tag{11}$$

holds for all $x > 0$ and $n \geq 0$, where

$$\mathscr{M}_n^{(\alpha)}(x) = x^{\alpha/2}(n^{-1} + x)^{-\alpha/2-1/4}\left(n^{1/3} + |x - 4n|\right)^{-1/4} \Phi_n(x), \tag{12}$$

and

$$\Phi_n(x) = \begin{cases} 1, & \text{for } 0 \leq x \leq 4n; \\ \exp\left(\dfrac{-\eta|x-4n|^{3/2}}{n^{1/2}}\right), & \text{for } 4n \leq x \leq (1+\tau)4n; \\ e^{-\xi x}, & \text{for } (1+\tau)4n \leq x. \end{cases}$$

A direct consequence of (12) is

$$\left|x^{1/4}\mathscr{M}_n^{(\alpha)}(x)\right| \lesssim n^{-1/12},$$

and $|x^{1/4}\mathscr{M}_n^{(\alpha)}(x)|$ attains this bound near the point $x = 4n$. But in the most part of x it has a much smaller bound as a multiple of $n^{-1/4}$.

To establish the main result of the paper, the next lemma is fundamental.

LEMMA 2.3. *Let $\alpha \geq 0$. For any interval $I \subset \mathbb{R}$ and for all $j \leq k$, $j, k \in \mathbb{N}_0$, one has*

$$\left|\int_I \tilde{\mathcal{L}}_k^{(\alpha)}(x)\tilde{\mathcal{L}}_j^{(\alpha)}(x)dx\right| \lesssim \left(\frac{j}{k}\right)^{1/4}|I| + \frac{1}{k^{1/4}j^{3/4}}. \tag{13}$$

PROOF. If $k/2 \leq j \leq k$, then by Lemma 2.1, $\left|\int_I \tilde{\mathcal{L}}_k^{(\alpha)}(x)\tilde{\mathcal{L}}_j^{(\alpha)}(x)dx\right| \lesssim |I|$ for all $\alpha \geq 0$. In what follows, we assume that $j \leq k/2$.

If $\alpha/2 > 0$ is not an integer, for any interval $I \subseteq \mathbb{R}$, by (6) we have

$$\int_I \tilde{\mathcal{L}}_k^{(\alpha)}(x)\tilde{\mathcal{L}}_j^{(\alpha)}(x)dx = \int_{I\cap[0,\infty)} \tilde{\mathcal{L}}_k^{(\alpha)}(x)\tilde{\mathcal{L}}_j^{(\alpha)}(x)dx.$$

If $\alpha/2 \in \mathbb{N}_0$, for any interval $I \subseteq \mathbb{R}$, since $\tilde{\mathcal{L}}_k^{(\alpha)}(x) = 0$ for $x \leq -k^{-1}$ by (7), therefore

$$\int_I \tilde{\mathcal{L}}_k^{(\alpha)}(x)\tilde{\mathcal{L}}_j^{(\alpha)}(x)dx = \int_{I\cap[-k^{-1},0)} \tilde{\mathcal{L}}_k^{(\alpha)}(x)\tilde{\mathcal{L}}_j^{(\alpha)}(x)dx + \int_{I\cap[0,\infty)} \tilde{\mathcal{L}}_k^{(\alpha)}(x)\tilde{\mathcal{L}}_j^{(\alpha)}(x)dx.$$

By Lemma 2.1, the first term on the right hand side above is dominated by ck^{-1}. Since $j \leq k/2$, it is easy to see $k^{-1} \lesssim j^{-3/4}k^{-1/4}$, which yields the required estimate. It remains to estimate $\int_{I\cap[0,\infty)} \tilde{\mathcal{L}}_k^{(\alpha)}(x)\tilde{\mathcal{L}}_j^{(\alpha)}(x)dx$ for all $\alpha \geq 0$. In proving (13), we may assume $I \subseteq [0,\infty)$. Otherwise we can divide I by 0 into two parts if it contains 0 as an interior point. By (6) and (7), $\tilde{\mathcal{L}}_n^{(\alpha)}(x) = \mathcal{L}_n^{(\alpha)}(x)$ for all $x > 0$. In view of (1) and Lemma 2.2, since $\tau_n^\alpha = O(n^{-\alpha/2})$, it follows that $|\mathcal{L}_n^{(\alpha)}(x)| \lesssim \mathscr{M}_n^{(\alpha)}(x)$ with $x > 0$. We have

$$\left|\int_I \tilde{\mathcal{L}}_k^{(\alpha)}(x)\tilde{\mathcal{L}}_j^{(\alpha)}(x)dx\right| \lesssim \left|\int_I \mathscr{M}_k^{(\alpha)}(x)\mathscr{M}_j^{(\alpha)}(x)dx\right|. \tag{14}$$

Dividing the last integral above into five parts, we write

$$\int_I \mathscr{M}_k^{(\alpha)}(x)\mathscr{M}_j^{(\alpha)}(x)dx = \sum_{j=1}^5 \int_{I_j} \mathscr{M}_k^{(\alpha)}(x)\mathscr{M}_j^{(\alpha)}(x)dx := \sum_{j=1}^5 Q_j, \tag{15}$$

where

$$I_1 = I\bigcap\{x:\, x \le k^{-1}\}; \qquad I_2 = I\bigcap\{x:\, k^{-1} \le x \le j^{-1}\};$$
$$I_3 = I\bigcap\{x:\, j^{-1} \le x \le 2j\}; \qquad I_4 = I\bigcap\{x:\, 2j \le x \le 2k\};$$
$$I_5 = I\bigcap\{x:\, x \ge 2k\}.$$

Using Lemma 2.2, we deal with each Q_j as follows:

$$|Q_1| \lesssim \int_{I_1} x^{1/4}\mathscr{M}_k^{(\alpha)}(x)x^{1/4}\mathscr{M}_j^{(\alpha)}(x)x^{-1/2}dx \lesssim k^{-1/4}j^{-1/4}\int_{I_1} x^{-1/2}dx \lesssim k^{-3/4}j^{-1/4};$$
$$|Q_2| \lesssim k^{-1/4}\int_{I_2} x^{-1/4}dx \lesssim j^{-3/4}k^{-1/4};$$
$$|Q_3| \lesssim k^{-1/4}j^{-1/4}\int_{I_3} x^{-1/2}dx \lesssim k^{-1/4}j^{1/4}|I|;$$
$$|Q_4| \lesssim k^{-1/4}j^{-1/12}\int_{I_4} x^{-1/2}dx \lesssim k^{-1/4}j^{-7/12}|I|;$$
$$|Q_5| \lesssim k^{-1/12}j^{-1/12}\int_{I_5} x^{-1/2}dx \lesssim k^{-7/12}j^{-1/12}|I|.$$

Substituting these estimates into (15) proves that $\left|\int_I \mathscr{M}_k^{(\alpha)}(x)\mathscr{M}_j^{(\alpha)}(x)dx\right| \lesssim j^{-3/4}k^{-1/4}+k^{-1/4}j^{1/4}|I|$ with $j \le k/2$. Furthermore, inserting this into (14), we get the desired inequality (13). □

3. Proof of Theorem 1.1.

Now we prove Theorem 1.1. Our approach is based on the duality of $H^1(\mathbb{R})$ and $BMO(\mathbb{R})$.

Proof. We first note that the conclusion for $2 < q < \infty$ follows from that for $q = 2$. Indeed, let $\nu_n = |\lambda_n|^{q/2}$, then (3) implies

$$\sum_{k=n}^{2n} |\nu_k|^2 = \sum_{k=n}^{2n} |\lambda_k|^q = O(1),$$

and, since $|c_n^{(\alpha)}(f)| \lesssim \|f\|_{H^1([0,\infty))}$ by Lemma 2.1 with $m = 0$, we obtain

$$\sum_{n=0}^{\infty} \left|\lambda_n c_n^{(\alpha)}(f)\right|^q \lesssim \|f\|_{H^1([0,\infty))}^{q-2} \sum_{n=0}^{\infty} \left|\nu_n c_n^{(\alpha)}(f)\right|^2 \lesssim \|f\|_{H^1([0,\infty))}^q.$$

Now we turn to the proof of the theorem for $q = 2$. We fix a sequence $\{b_n\}_{n=0}^{\infty} \in \ell^2$ and for $n = 0, 1, 2, \ldots,$ put

$$g_n(x) = \sum_{k=0}^{n} \lambda_k b_k \tilde{\mathcal{L}}_k^{(\alpha)}(x). \tag{16}$$

By the duality between H^1 and BMO, we have $\left|\int_{-\infty}^{\infty} f(x) g_n(x) dx\right| \lesssim \|g_n\|_{\mathrm{BMO}} \|f\|_{H^1([0,\infty))}$, that is,

$$\left|\sum_{k=0}^{n} \lambda_k b_k c_k^{(\alpha)}(f)\right| \lesssim \|g_n\|_{\mathrm{BMO}} \|f\|_{H^1([0,\infty))}, \tag{17}$$

where $\|g\|_{\mathrm{BMO}} = \sup_I (1/|I|) \int_I |g(t) - g_I| dt$ with supremum taken over all intervals I of the real line $\mathbb{R}$, and $g_I = (1/|I|) \int_I g(t) dt$ with $|I|$ being the length of I. We shall show that $g_n(x)$ is a BMO function and

$$\|g_n\|_{\mathrm{BMO}} \lesssim \left(\sum_{k=0}^{n} |b_k|^2\right)^{1/2} \tag{18}$$

for all $\{b_k\}_{k=0}^{\infty} \in \ell^2$. Once (18) is established, then from (17) we deduce that $\left(\sum_{k=0}^{n} |\lambda_k c_k^{(\alpha)}(f)|^2\right)^{1/2} \lesssim \|f\|_{H^1([0,\infty))}$, which proves the theorem by letting $n \to \infty$.

To prove (18), we have only to find a constant η_I, for any interval I, such that

$$\frac{1}{|I|} \int_I |g_n(x) - \eta_I| dx \lesssim \left(\sum_{k=0}^{n} |b_k|^2\right)^{1/2}. \tag{19}$$

For an interval I, let $m = [|I|^{-1}]$, the integer part of the number $|I|^{-1}$, and choose x_I to be one of the end points of I. If $n \le m$, then applying Lemma 2.1,

$$\begin{aligned} |g_n(x) - g_n(x_I)|^2 &\le \left(\sum_{k=0}^{n} |b_k|^2\right)\left(\sum_{k=0}^{n} |\lambda_k|^2 \left|\tilde{\mathcal{L}}_k^{(\alpha)}(x) - \tilde{\mathcal{L}}_k^{(\alpha)}(x_I)\right|^2\right) \\ &\lesssim \left(\sum_{k=0}^{n} |b_k|^2\right)\left(\sum_{k=0}^{n} |\lambda_k|^2 k^{2\sigma} |x - x_I|^{2\sigma}\right), \end{aligned}$$

where $\sigma = \alpha/2$ for $0 < \alpha/2 < 1$, and $\sigma = 1$ otherwise. By the condition (3) with $q = 2$, summing by parts gives $\sum_{k=0}^{n} |\lambda_k|^2 k^{2\sigma} \lesssim n^{2\sigma}$, then

$$|g_n(x) - g_n(x_I)|^2 \lesssim \sum_{k=0}^{n} |b_k|^2 (n|x - x_I|)^{\sigma} \lesssim \sum_{k=0}^{n} |b_k|^2.$$

Hence (19) holds with $\eta_I = g_n(x_I)$.

If $n > m$, we again choose x_I to be one of the end points of I to obtain

$$|g_n(x) - g_m(x_I)| \le |g_m(x) - g_m(x_I)| + \left|\sum_{m<k\le n} \lambda_k b_k \tilde{\mathcal{L}}_k^{(\alpha)}(x)\right|.$$

Hence by what has been verified,

$$\frac{1}{|I|}\int_I |g_n(x)-g_m(x_I)|dx \lesssim \Big(\sum_{k=0}^{m}|b_k|^2\Big)^{1/2} + F_{m,n}. \tag{20}$$

where $F_{m,n}=|I|^{-1}\int_I\big|\sum_{m<k\le n}\lambda_k b_k\tilde{\mathcal{L}}_k^{(\alpha)}(x)\big|dx$. But for $F_{m,n}$, we have

$$\begin{aligned} F_{m,n}^2 &\le \frac{1}{|I|}\int_I\Big|\sum_{m<k\le n}\lambda_k b_k\tilde{\mathcal{L}}_k^{(\alpha)}(x)\Big|^2 dx \\ &\le \sum_{m<k\le n}\sum_{m<j\le n}|\lambda_k b_k\overline{\lambda_j b_j}|\frac{1}{|I|}\Big|\int_I \tilde{\mathcal{L}}_k^{(\alpha)}(x)\tilde{\mathcal{L}}_j^{(\alpha)}(x)dx\Big|. \end{aligned}$$

By symmetry, it suffices to treat the part $\sum_{m<k\le n}\sum_{m<j\le k}$. For these j,k, $|I|^{-1}\le m+1\le j$, and by Lemma 2.3,

$$\frac{1}{|I|}\Big|\int_I \tilde{\mathcal{L}}_k^{(\alpha)}(x)\tilde{\mathcal{L}}_j^{(\alpha)}(x)dx\Big| \lesssim \frac{j^{1/4}}{k^{1/4}}+\frac{|I|^{-1}}{k^{1/4}j^{3/4}}\lesssim\frac{j^{1/4}}{k^{1/4}}.$$

Thus the evaluation of $F_{m,n}^2$ is reduced to showing the following inequality

$$S_{m,n}:=\sum_{m<k\le n}\sum_{m<j\le k}|\lambda_k b_k\overline{\lambda_j b_j}|\frac{j^{1/4}}{k^{1/4}}\lesssim\sum_{m<k\le n}|b_k|^2.$$

For the purpose we rewrite $S_{m,n}$ as

$$\begin{aligned} S_{m,n} &\le \frac{1}{2}\sum_{m<k\le n}\sum_{m<j\le k}\big(|\lambda_j b_k|^2+|\lambda_k b_j|^2\big)\frac{j^{1/4}}{k^{1/4}} \\ &= \frac{1}{2}\sum_{m<k\le n}\frac{|b_k|^2}{k^{1/4}}\sum_{m<j\le k}|\lambda_j|^2 j^{1/4}+\frac{1}{2}\sum_{m<j\le n}|b_j|^2 j^{1/4}\sum_{j\le k\le n}\frac{|\lambda_k|^2}{k^{1/4}}. \end{aligned} \tag{21}$$

Under the condition (3) with $q=2$, summing by parts again implies

$$\sum_{j\le k}|\lambda_j|^2 j^{1/4}\lesssim k^{1/4},\qquad \sum_{k\ge j}\frac{|\lambda_k|^2}{k^{1/4}}\lesssim j^{-1/4},$$

incorporating these into (21) proves that $S_{m,n}\lesssim\sum_{m<k\le n}|b_k|^2$, moreover, $F_{m,n}\lesssim\big(\sum_{m<k\le n}|b_k|^2\big)^{1/2}$. Inserting this into (20) proves (19) with $\eta_I=g_m(x_I)$.

The proof of Theorem 1.1 is completed. □

Acknowledgments. The authors would like to thank the anonymous referees for thier helpful comments and suggestions which have improved the original manuscript.

References

[1] R. Askey and S. Wainger, Mean convergence of expansions in Laguerre and Hermite series, Amer. J. Math., **87** (1965), 695–708.

[2] R. Balasubramanian and R. Radha, Hardy-type inequalities for Hermite expansions, J. Inequal. in Pure and Appl. Math., **6** (2005), 1–4.

[3] L. Colzani and G. Travaglini, Hardy–Lorentz spaces and expansions in eigenfunctions of the Laplace–Beltrami operator on compact manifolds, Colloq. Math., **58** (1990), 305–316.

[4] P. L. Duren and A. L. Shields, Coefficient multipliers of H^p and B^p spaces, Pacific J. Math., **32** (1970), 69–78.

[5] Y. Kanjin, Hardy's inequalities for Hermite and Laguerre expansions, Bull. London Math. Soc., **29** (1997), 331–337.

[6] Y. Kanjin and K. Sato, Paley's inequality for the Jacobi expansions, Bull. London Math. Soc., **33** (2001), 483–491.

[7] Y. Kanjin and K. Sato, Hardy's inequality for Jacobi expansions, Math. Inequal. Appl., **7** (2004), 551–555.

[8] Zh.-K. Li and Y.-H. Shi, Multipliers of Hardy spaces associated with generalized Hermite expansions, Constr. Approx., **39** (2014), 517–540.

[9] Zh.-K. Li, Y.-H. Shi and X.-L. Zhang, Coefficient multipliers on Hardy spaces for Jacobi expansions of exponential type, to appear in J. Approx. Theory.

[10] B. Muckenhoupt, Asymptotic forms for Laguerre polynomials, Proc. Amer. Math. Soc., **24** (1970), 288–292.

[11] B. Muckenhoupt and D. W. Webb, Two-weight norm inequalities for Cesàro means of Laguerre expansions, Trans. Amer. Math. Soc., **353** (2000), 1119–1149.

[12] B. Osikiewicz, Multipliers of Hardy spaces, Quaestiones Math., **27** (2004), 57–73.

[13] R. Radha and S. Thangavelu, Hardy's inequalities for Hermite and Laguerre expansions, Proc. Amer. Math. Soc., **132** (2004), 3525–3536.

[14] M. Satake, Hardy's inequalities for Laguerre expansions, J. Math. Soc. Japan, **52** (2000), 17–24.

[15] E. M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Univ. Press, Princeton, NJ, 1993.

[16] G. Szegö, Orthogonal Polynomials, 4th ed., Amer. Math. Soc. Colloq. Publ., **23**, Providence, RI, 1975.

[17] Y.-H. Shi and Zh.-K. Li, Multipliers of Hardy spaces associated with Laguerre expansions, J. Math. Soc. Japan, **68** (2016), 91–99.

[18] S. Thangavelu, On regularity of twisted spherical means and special Hermite expansion, Proc. Indian Acad. Sci. Math. Sci., **103** (1993), 303–320.

Yehao Shi
Elementary Education College
Capital Normal University
Beijing 100048, China
E-mail: cnusyh@163.com

Zhongkai Li
Department of Mathematics
Shanghai Normal University
Shanghai 200234, China
E-mail: lizk@shnu.edu.cn