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Spaces of algebraic maps from real projective spaces to toric varieties

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Abstract. The problem of approximating the infinite dimensional space of all continuous maps from an algebraic variety X to an algebraic variety Y by finite dimensional spaces of algebraic maps arises in several areas of geometry and mathematical physics. An often considered formulation of the problem (sometimes called the Atiyah–Jones problem after [**1**]) is to determine a (preferably optimal) integer n_D such that the inclusion from this finite dimensional algebraic space into the corresponding infinite dimensional one induces isomorphisms of homology (or homotopy) groups through dimension n_D, where D denotes a tuple of integers called the "degree" of the algebraic maps and $n_D \to \infty$ as $D \to \infty$. In this paper we investigate this problem in the case when X is a real projective space and Y is a smooth compact toric variety.

0. Introduction.

Let X and Y be manifolds with some additional structure $\mathcal{S}$, e.g. holomorphic, symplectic, real algebraic etc. Let $\mathcal{S}(X,Y)$ denote the space of base-point preserving continuous maps $f : X \to Y$ preserving the structure $\mathcal{S}$ and let $\mathrm{Map}^*(X,Y)$ be the space of corresponding continuous maps. The relation between the topology of the spaces $\mathcal{S}(X,Y)$ and $\mathrm{Map}^*(X,Y)$ has long been an object of study in several areas of topology and geometry (e.g. [**3**], [**5**], [**9**], [**10**], [**12**], [**13**], [**14**], [**15**], [**17**], [**18**], [**21**]). In particular, in [**17**] and [**18**] J. Mostovoy considered the case where the structure $\mathcal{S}$ is that of a complex manifold, and determined an integer n_D such that the inclusion map $j_D : \mathrm{Hol}^*_D(X,Y) \to \mathrm{Map}^*_D(X,Y)$ induces isomorphisms of homology groups through dimension n_D for complex projective spaces X and Y, where $\mathrm{Hol}^*_D(X,Y)$ (resp. $\mathrm{Map}^*_D(X,Y)$) denotes the space of base-point preserving holomorphic (resp. continuous) maps from X to Y of degree D.

In [**2**] and [**14**] the case where the structure $\mathcal{S}$ is that of a real algebraic variety was considered, and integers n_D were found, such that the natural projection map $i_D : A_D(X,Y) \to \mathrm{Map}^*_D(X,Y)$ induces isomorphisms of homology groups through dimension n_D for real projective spaces X and Y, where $A_D(X,Y)$ is a space of tuples of polynomials representing the elements of $\mathrm{Alg}^*_D(X,Y)$—the space of base-point preserving algebraic

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(regular) maps from X to Y of degree D (we will refer to $A_D(X,Y)$ as an "algebraic approximation" to the mapping space $\mathrm{Map}^*_D(X,Y)$).

Recently Mostovoy and Munguia-Villanueva generalized Mostovoy's earlier result [**18**] to the case of holomorphic maps from a complex projective space $\mathbb{C}\mathrm{P}^m$ to a compact smooth toric variety X_Σ in [**19**], where X_Σ denotes the toric variety associated to a fan Σ.

In this paper, we study a real analogue of this result [**19**] (a different kind of analogue will be studied in the subsequent paper [**16**]). More precisely, our original aim was to generalize the results of [**15**] to the spaces of algebraic maps from a real projective space $\mathbb{R}\mathrm{P}^m$ to a compact smooth toric variety X_Σ.[1] Although our approach is based on the original ideas of Mostovoy ([**17**], [**18**]), the real case presents special difficulties. For example, what we call here an algebraic map from $\mathbb{R}\mathrm{P}^m$ to X_Σ is defined on some open dense subset of $\mathbb{C}\mathrm{P}^m$, and it cannot be extended to the entire complexification $\mathbb{C}\mathrm{P}^m$ in general. Moreover, there is another difference between the complex case and the real one, which is the source of a greater difficulty. Namely, in the complex case, base-point preserving algebraic (equivalently holomorphic) maps from $\mathbb{C}\mathrm{P}^m$ to X_Σ are determined by r-tuples of homogenous polynomials taking values outside a certain subvariety of $\mathbb{C}^r$, where r is the number of the Cox's homogenous coordinates of X_Σ.

On the other hand, in the real case, such r-tuples of polynomials determine algebraic maps only up to multiplication by certain positive valued homogenous polynomial functions. This means that base-point preserving algebraic maps from $\mathbb{R}\mathrm{P}^m$ to X_Σ cannot be uniquely represented (in homogeneous coordinates) by r-tuples of polynomials with a fixed "degree" since we can multiply any r-tuple by a positive valued homogeneous polynomial to obtain another tuple representing the same algebraic map. While an algebraic map from $\mathbb{R}\mathrm{P}^m$ to X_Σ has a uniquely defined "minimal degree" and the space of algebraic maps with a fixed minimal degree can be described in terms of r-tuples of homogenous polynomials of the same degree, the topology of the space of algebraic maps with a fixed minimal degree is very complicated to analyze and we shall not make here any use of this concept (besides defining it).[2]

Now, let $i_D : A_D(m, X_\Sigma) \to \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ denote the natural map given by sending r-tuple of polynomials to its representing algebraic map, and consider the natural surjective projection $\Psi_D : A_D(m, X_\Sigma) \to \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) = i_D(A_D(m, X_\Sigma))$ on its image. Because any fibre of Ψ_D is contractible, if we could prove that this map is a quasi-fibration, it would imply that it is a homotopy equivalence and we could then just imitate the method of [**19**]. Unfortunately, it seems difficult to prove that this map is a quasi-fibration or satisfies some other condition that leads to the conclusion that it is a homotopy equivalence.

To get around this problem we adopt the approach used in [**2**]. Namely, we restrict ourselves to considering only the spaces of r-tuples of polynomials which represent algebraic maps rather than the spaces of algebraic maps themselves. These will be our finite dimensional approximations to the space of all continuous maps. (We conjecture

[1] Note that an algebraic map from a real variety V to a complex variety W is a morphism defined on an open dense subset of the complexification $V_\mathbb{C}$ of V which contains all the $\mathbb{R}$-valued points of V.

[2] See Proposition 1.3 and Definition 8.2.

that both kinds of algebraic approximations, by tuples of polynomials and by algebraic maps which they determine, are homotopy equivalent). Because we cannot work with spaces of algebraic maps we cannot make use of the real analogue of [**19**, Proposition 3] (although such an analogue can be proved by a similar method).

Instead, we prove a similar theorem (Theorem 4.6), where in place of the stabilized space of algebraic maps we use the stabilized space of tuples of polynomials representing them and obtain a homology equivalence instead of a homotopy one with the space of all continuous maps. Our method of proof is more complicated since we have to rely on the spectral sequence constructed by Vassiliev [**22**] for computing the homology of the space of continuous maps from a q-dimensional CW complex to a q-connected one (see Section 7 for more details).

More precisely, our proof makes use of three spectral sequences corresponding to three kinds of simplicial resolutions of discriminants; the first one defined by Vassiliev, the other two, its variants due to Mostovoy.

The first resolution, called by Vassiliev "the simplicial resolution" (and in this paper the "non-degenerate simplicial resolution" to distinguish it from the other two), has a $(k-1)$-simplex as its fibre whenever the inverse image consists of k points. Vassiliev used this resolution to construct a spectral sequence for computing cohomology of spaces of continuous mappings. The same kind of resolution and a corresponding spectral sequence can be defined for the space of algebraic maps and clearly there is a natural map between the resolutions which induces a homomorphism between the spectral sequences. However, this is not enough to prove our result as the spectral sequence has too many non-vanishing terms, due to the fact that the values of an algebraic map of fixed degree at different points are not necessarily independent. Mostovoy's idea in [**17**] was to use a degenerate simplicial resolution (which we have decided to name *the Veronese resolution* because it based on the use of a Veronese-like embedding). This resolution collapses some of the fibres of the non-degenerate resolution and makes the corresponding terms 0. The problem with the Veronese resolution is that it does not map into the Vassiliev's non-degenerate simplicial resolution of the space of continuous maps. However, by combining the two resolutions we can prove Theorem 4.6.

It may be worth noting that by using only these two resolutions we can prove Theorem 6.2, which differs from our main Theorem 1.5 only in giving a worse stabilization dimension. To obtain a stronger result we turn to another idea of Mostovoy: that of the (non-degenerate) simplicial resolution truncated after some term.

The basic idea behind the proofs is as follows. We want to compare the topology of two successive algebraic approximations. We make use of the fact that these spaces are complements of discriminants in affine spaces and their topology can be related to that of discriminants by Alexander duality. To compare the homology we resolve the singularities of both spaces using the non-degenerate simplicial resolution. The non-degenerate resolution has a natural filtration, the k-th term of which is the union of the $(k-1)$-th skeleta of the simplices in the fibres of the non-degenerate resolution. The first few terms of the filtration are easy to describe but after a certain dimension they become intractable. To deal with this problem we truncate the resolution after this dimension, by taking only skeleta of lower dimensions and collapsing non-contractible fibres (thus obtaining again a space homotopy equivalent to the discriminant). This truncated resolution inherits a filtration, which turns out to be much more manageable. A comparison of these truncated resolutions and their spectral sequences gives our main result.

This paper is organized as follows. In Section 1, we define the basic notions of toric varieties and state our main results. In Section 2, we recall the definitions of the various simplicial resolutions used in this paper. In Section 3 we study the spectral sequence induced from the non-degenerate simplicial resolution of discriminants, and in Section 4 we prove our key stability result (Theorem 4.8). In Section 5 we recall the elementary results of polyhedral products and investigate the connectivity of the complement $\mathbb{C}^r\setminus Z_\Sigma$. In Section 6, we give the proofs of Theorem 4.6 and the main result (Theorem 1.5) by using Theorem 6.2. In Section 7 we prove Theorem 6.2 by using the Vassiliev spectral sequence [**22**] and that induced from the Veronese (degenerate) resolution. In Section 8 we describe some non-trivial facts, define the minimal degree of algebraic maps, and consider the problem concerning the relationship between spaces of polynomial tuples and the spaces of algebraic maps induced by them.

1. Toric varieties and the main results.

1.1. Toric varieties and polynomials representing algebraic maps.

Toric varieties.

An irreducible normal algebraic variety X (over $\mathbb{C}$) is called *a toric variety* if it has an algebraic action of algebraic torus $\mathbb{T}^r=(\mathbb{C}^*)^r$, such that the orbit $\mathbb{T}^r\cdot *$ of some point $*\in X$ is dense in X and isomorphic to $\mathbb{T}^r$. *A strong convex rational polyhedral cone* σ in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form $\sigma=\mathrm{Cone}(\{\boldsymbol{n}_k\}_{k=1}^s)=\{\sum_{k=1}^s a_k\boldsymbol{n}_k|a_k\geq 0\}$, where $\{\boldsymbol{n}_k\}_{k=1}^s\subset\mathbb{Z}^n$, which does not contain any line. A finite collection Σ of strongly convex rational polyhedral cones in $\mathbb{R}^n$ is called *a fan* if every face of an element of Σ belongs to Σ and the intersection of any two elements of Σ is a face of each.

It is known that a toric variety X is completely characterized up to isomorphism by its fan Σ. We denote by X_Σ the toric variety associated to Σ.

The dimension of a cone σ is the minimal dimension of subspaces $W\subset\mathbb{R}^n$ such that $\sigma\subset W$. A cone σ in $\mathbb{R}^n$ is called *smooth* (resp. *simplicial*) if it is generated by a subset of a basis of $\mathbb{Z}^n$ (resp. a subset of a basis of $\mathbb{R}^n$).

A fan Σ is called *smooth* (resp. *simplicial*) if every cone in Σ is smooth (resp. simplicial). A fan is called *complete* if $\bigcup_{\sigma\in\Sigma}\sigma=\mathbb{R}^n$. Note that X_Σ is compact if and only if Σ is complete [**7**, Theorem 3.4.1], and that X_Σ is a smooth toric variety if and only if Σ is smooth [**7**, Theorem 1.3.12]. It is also known that $\pi_1(X_\Sigma)$ is isomorphic to the quotient of $\mathbb{Z}^n$ by the subgroup generated by $\bigcup_{\sigma\in\Sigma}\sigma\cap\mathbb{Z}^n$ [**7**, Theorem 12.1.10]. In particular, if X_Σ is compact, X_Σ is simply connected.

Homogenous coordinates on toric varieties.

We shall use the symbols $\{z_k\}_{k=1}^r$ to denote variables of polynomials. For $f_1,\dots,f_s\in\mathbb{C}[z_1,\dots,z_r]$, let $V(f_1,\dots,f_s)$ denote the affine variety $V(f_1,\dots,f_s)=\{\boldsymbol{x}\in\mathbb{C}^r\mid f_k(\boldsymbol{x})=0\text{ for each }1\leq k\leq s\}$ given by the polynomial equations $f_1=\cdots=f_s=0$.

Let $\Sigma(1)=\{\rho_1,\dots,\rho_r\}$ denote the set of all one-dimensional cones (*rays*) in a fan Σ, and for each $1\leq k\leq r$ let $\boldsymbol{n}_k\in\mathbb{Z}^n$ denote the generator of $\rho_k\cap\mathbb{Z}^n$ (called *the primitive* *element* of ρ_k) such that $\rho_k \cap \mathbb{Z}^n = \mathbb{Z}_{\geq 0} \cdot \boldsymbol{n}_k$. Let $Z_\Sigma \subset \mathbb{C}^r$ denote the affine variety defined by

$$Z_\Sigma = V(z^{\hat{\sigma}} | \sigma \in \Sigma), \tag{1.1}$$

where $z^{\hat{\sigma}}$ denotes the monomial

$$z^{\hat{\sigma}} = \prod_{1 \leq k \leq r, \boldsymbol{n}_k \notin \sigma} z_k \in \mathbb{Z}[z_1, \ldots, z_r] \qquad (\sigma \in \Sigma). \tag{1.2}$$

Let $G_\Sigma \subset \mathbb{T}^r$ denote the subgroup defined by

$$G_\Sigma = \left\{ (\mu_1, \ldots, \mu_r) \in \mathbb{T}^r \;\middle|\; \prod_{k=1}^{r} \mu_k^{\langle \boldsymbol{m}, \boldsymbol{n}_k \rangle} = 1 \text{ for all } \boldsymbol{m} \in \mathbb{Z}^n \right\}, \tag{1.3}$$

where we set $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{k=1}^n x_k y_k$ for $\boldsymbol{x} = (x_1, \ldots, x_n),\ \boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

We say that a set of primitive elements $\{\boldsymbol{n}_{i_1}, \ldots, \boldsymbol{n}_{i_s}\}$ is *primitive* if they do not lie in any cone in Σ but every proper subset has this property. It is known ([**7**, Proposition 5.1.6]) that

$$Z_\Sigma = \bigcup_{\{\boldsymbol{n}_{i_1}, \ldots, \boldsymbol{n}_{i_s}\}:\ \text{primitive}} V(z_{i_1}, \ldots, z_{i_s}). \tag{1.4}$$

Note that Z_Σ is a closed variety of real dimension $2(r - r_{\min})$, where we set

$$r_{\min} = \min \bigl\{ s \in \mathbb{Z}_{\geq 1} \mid \{\boldsymbol{n}_{i_1}, \ldots, \boldsymbol{n}_{i_s}\} \text{ is primitive} \bigr\}. \tag{1.5}$$

It is known that if the set $\{\boldsymbol{n}_1, \ldots, \boldsymbol{n}_r\}$ spans $\mathbb{R}^n$ and X_Σ is smooth, there is an isomorphism ([**7**, Theorem 5.1.11], [**20**, Proposition 6.7])

$$X_\Sigma \cong (\mathbb{C}^r \setminus Z_\Sigma)/G_\Sigma, \tag{1.6}$$

where the group G_Σ acts freely on the complement $\mathbb{C}^r \setminus Z_\Sigma$ by coordinate-wise multiplication.

EXAMPLE 1.1. For each $k \in \mathbb{Z}$, let $H(k)$ denote *the Hirzebruch surface* given by $H(k) = \{([x_0 : x_1 : x_2], [y_1 : y_2]) \in \mathbb{CP}^2 \times \mathbb{CP}^1 \mid x_1 y_1^k = x_2 y_2^k\} \subset \mathbb{CP}^2 \times \mathbb{CP}^1$. Note that there are isomorphisms $H(0) \cong \mathbb{CP}^1 \times \mathbb{CP}^1$ and $H(k) \cong H(-k)$ for $k \geq 1$. Let Σ denote the fan in $\mathbb{R}^2$ given by

$$\bigl\{ \mathrm{Cone}(\boldsymbol{n}_1, \boldsymbol{n}_2), \mathrm{Cone}(\boldsymbol{n}_2, \boldsymbol{n}_3), \mathrm{Cone}(\boldsymbol{n}_3, \boldsymbol{n}_4), \mathrm{Cone}(\boldsymbol{n}_4, \boldsymbol{n}_1), \mathbb{R}_{\geq 0} \cdot \boldsymbol{n}_j \ (1 \leq j \leq 4),\ \boldsymbol{0} \bigr\},$$

where $\boldsymbol{n}_1 = (1, 0)$, $\boldsymbol{n}_2 = (0, 1)$, $\boldsymbol{n}_3 = (-1, k)$ and $\boldsymbol{n}_4 = (0, -1)$. Then we can easily see that $H(k) = X_\Sigma$ [**7**, Example 3.1.16]. Since $\{\boldsymbol{n}_1, \boldsymbol{n}_3\}$ and $\{\boldsymbol{n}_2, \boldsymbol{n}_4\}$ are primitive, $r_{\min} = 2$. Moreover, by using (1.3), (1.4) and (1.6) we also obtain the isomorphism

$$H(k) \cong \{(y_1, y_2, y_3, y_4) \in \mathbb{C}^4 \mid (y_1, y_3) \neq (0,0), (y_2, y_4) \neq (0,0)\}/G_\Sigma, \tag{1.7}$$

where $G_\Sigma = \{(\mu_1, \mu_2, \mu_1, \mu_1^k \mu_2) \mid \mu_1, \mu_2 \in \mathbb{C}^*\} \cong \mathbb{T}^2$.

Spaces of mappings.

For connected spaces X and Y, let $\mathrm{Map}(X, Y)$ be the space of all continuous maps $f : X \to Y$ and $\mathrm{Map}^*(X, Y)$ the corresponding subspace of all based continuous maps. If $m \geq 2$ and $g \in \mathrm{Map}^*(\mathbb{R}\mathrm{P}^{m-1}, X)$, let $F(\mathbb{R}\mathrm{P}^m, X; g)$ denote the subspace of $\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, X)$ given by

$$F(\mathbb{R}\mathrm{P}^m, X; g) = \{f \in \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, X) \mid f|\mathbb{R}\mathrm{P}^{m-1} = g\},$$

where we identify $\mathbb{R}\mathrm{P}^{m-1} \subset \mathbb{R}\mathrm{P}^m$ by putting $x_m = 0$. It is known that there is a homotopy equivalence $F(\mathbb{R}\mathrm{P}^m, X; g) \simeq \Omega^m X$ if it is not an empty set (see Lemma 8.1).

Assumptions.

From now on, we adopt the following notational conventions and two assumptions:

(1.7.1) Let Σ be a complete smooth fan in $\mathbb{R}^n$, $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ the set of all one-dimensional cones in Σ, and suppose that the primitive elements $\{\boldsymbol{n}_1, \dots, \boldsymbol{n}_r\}$ span $\mathbb{R}^n$, where $\boldsymbol{n}_k \in \mathbb{Z}^n$ denotes the primitive element of ρ_k for $1 \leq k \leq r$.

(1.7.2) Let $D = (d_1, \dots, d_r)$ be an r-tuple of integers such that $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}$.

Then, by (1.7.1) we can make the identification $X_\Sigma = (\mathbb{C}^r \setminus Z_\Sigma)/G_\Sigma$ as in (1.6). For each $(a_1, \dots, a_r) \in \mathbb{C}^r \setminus Z_\Sigma$, we denote by $[a_1, \dots, a_r]$ the corresponding element of X_Σ.

Spaces of polynomials.

Let $\mathcal{H}_{d,m} \subset \mathbb{C}[z_0, \dots, z_m]$ denote the space of global sections $H^0(\mathbb{C}\mathrm{P}^m, \mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d))$ of the line bundle $\mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d)$ of degree d. Note that the space $\mathcal{H}_{d,m} \subset \mathbb{C}[z_0, \dots, z_m]$ coincides with the subspace consisting of all homogeneous polynomials of degree d if $d \geq 0$ and that $\mathcal{H}_{d,m} = 0$ if $d < 0$.[3] For each r-tuple $D = (d_1, \dots, d_r) \in \mathbb{Z}^r$, let $A_D(m)$ denote the space

$$A_D(m) = \mathcal{H}_{d_1,m} \times \mathcal{H}_{d_2,m} \times \cdots \times \mathcal{H}_{d_r,m} \tag{1.8}$$

and let $A_{D,\Sigma}(m) \subset A_D(m)$ denote the subspace defined by

$$A_{D,\Sigma}(m) = \big\{(f_1, \dots, f_r) \in A_D(m) \mid F(\boldsymbol{x}) \notin Z_\Sigma \text{ for any } \boldsymbol{x} \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}\}\big\}, \tag{1.9}$$

where we write $F(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \dots, f_r(\boldsymbol{x}))$ for $\boldsymbol{x} \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}\}$.

Next, we define a map $j'_D : A_{D,\Sigma}(m) \to \mathrm{Map}(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ by the formula

$$j'_D(f_1, \dots, f_r)([\boldsymbol{x}]) = [f_1(\boldsymbol{x}), \dots, f_r(\boldsymbol{x})] \quad \text{for } \boldsymbol{x} \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}\}. \tag{1.10}$$

Remark 1.2. Note that the map j'_D is well defined because $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}$. In

[3] This is because $H^0(\mathbb{C}\mathrm{P}^m, \mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d)) = 0$ if $d < 0$.

fact, if $\lambda \in \mathbb{R}^*$, then since $\prod_{k=1}^r (\lambda^{d_k})^{\langle \boldsymbol{m}, \boldsymbol{n}_k\rangle} = \lambda^{\langle \boldsymbol{m}, \sum_{k=1}^r d_k \boldsymbol{n}_k\rangle} = 1$ for any $\boldsymbol{m} \in \mathbb{R}^n$, $(\lambda^{d_1}, \dots, \lambda^{d_r}) \in G_\Sigma$. So for $(f_1, \dots, f_r) \in A_{D,\Sigma}(m)$, $(f_1(\lambda \boldsymbol{x}), \dots, f_r(\lambda \boldsymbol{x})) = (\lambda^{d_1} f_1(\boldsymbol{x}), \dots, \lambda^{d_r} f_r(\boldsymbol{x}))$. Hence $[f_1(\lambda \boldsymbol{x}), \dots, f_r(\lambda \boldsymbol{x})] = [f_1(\boldsymbol{x}), \dots, f_r(\boldsymbol{x})]$ in X_Σ for any $(\lambda, \boldsymbol{x}) \in \mathbb{R}^* \times (\mathbb{R}^{m+1} \setminus \{\boldsymbol{0}\})$ and the map j'_D is well-defined.

Since the space $A_{D,\Sigma}(m)$ is connected, the image of j'_D lies in a connected component of $\mathrm{Map}(\mathbb{R}\mathrm{P}^m, X_\Sigma)$, which will be denoted by $\mathrm{Map}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma)$. This gives a natural map

$$j'_D : A_{D,\Sigma}(m) \to \mathrm{Map}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma). \tag{1.11}$$

Algebraic maps from a real algebraic variety to a complex algebraic variety.

Let X be an algebraic variety over $\mathbb{R}$, and Y an algebraic variety over $\mathbb{C}$. We set $X_\mathbb{C} = X \times_\mathbb{R} \mathbb{C}$, and denote by $X(\mathbb{R})$ the set of $\mathbb{R}$-valued points of X. Let $\Phi : X_\mathbb{C} - \to Y$ be a rational map[4] defined over $\mathbb{C}$ and let U be the largest Zariski dense open subset of $X_\mathbb{C}$ such that (U, φ) represents Φ, where $\varphi : U \to Y$ is a regular map defined over $\mathbb{C}$. If $X(\mathbb{R}) \cap (X \setminus U) = \emptyset$, then we have a map $\varphi|_{X(\mathbb{R})} : X(\mathbb{R}) \to Y$.

An algebraic map $f : X \to Y$ is defined to be a map $X(\mathbb{R}) \to Y$, which is also denoted by f by abuse of notation, such that $f = \varphi|_{X(\mathbb{R})}$, where $\varphi : U \to Y$ is a regular map defined over $\mathbb{C}$ on a Zariski open subset U of $X_\mathbb{C}$ and U contains $X(\mathbb{R})$.

Note that an algebraic map $f : X \to Y$ is not a morphism of schemes, although the algebraic varieties X and Y are (or can be regarded as) schemes and we use the adjective "algebraic": in fact, there does not exist a morphism from a variety defined over $\mathbb{R}$ with some $\mathbb{R}$-valued points to a variety defined over $\mathbb{C}$. Note also that $\Phi : X_\mathbb{C} - \to Y$ is not only a meromorphic map but also a rational map. In particular, the map φ is not only a holomorphic map but also a regular map. In this sense, an algebraic map $f : X \to Y$ defined above is indeed "algebraic" .

Now let us consider the case $(X, Y) = (\mathbb{R}\mathrm{P}^m, X_\Sigma)$. Let $C_D(m)$ denote the *incidence correspondence*

$$C_D(m) = \{(F, \boldsymbol{x}) \in A_{D,\Sigma}(m) \times \mathbb{C}^{m+1} \mid F(\boldsymbol{x}) \in Z_\Sigma\} \tag{1.12}$$

and $pr_1 : C_D(m) \to A_{D,\Sigma}(m)$ the first projection. Because the map $C_D(m) \to \mathbb{Z}$ given by $(F, \boldsymbol{x}) \mapsto \dim_\mathbb{C} pr_1^{-1}(F)$ is upper semicontinuous, the subspace $A_{D,\Sigma}(m)^\circ$ is a Zariski open subspace of $A_{D,\Sigma}(m)$, where

$$A_{D,\Sigma}(m)^\circ = \{F \in A_{D,\Sigma}(m) \mid \dim_\mathbb{C} pr_1^{-1}(F) < m\}. \tag{1.13}$$

Note that we can see that every algebraic map $f : \mathbb{R}\mathrm{P}^m \to X_\Sigma$ can be represented as $f = j'_D(f_1, \dots, f_r)$ for some $D = (d_1, \dots, d_r)$ and $(f_1, \dots, f_r) \in A_{D,\Sigma}(m)$ such that $\sum_{k=1}^r d_k \boldsymbol{n}_k = \boldsymbol{0}$. If $(f_1, \dots, f_r) \notin A_{D,\Sigma}(m)^\circ$, the representation of f the degree D are not unique.[5] However, the following holds.[6]

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[4]The notation $\Phi : X_\mathbb{C} - \to Y$ is used to stress the fact that Φ is not a map, namely is not necessarily defined at every point of $X_\mathbb{C}$ but is defined only on some Zariski dense open subset of $X_\mathbb{C}$.

[5]See Remark 8.3 (ii) (cf. Example 8.4) for the details.

[6]The proof of Proposition 1.3 is given in Section 8.

Proposition 1.3. *Let X_Σ be a smooth compact complex toric variety and let $f : \mathbb{RP}^m \to X_\Sigma$ be an algebraic map. Then there exists a unique r-tuple $D = (d_1, \dots, d_r) \in \mathbb{Z}^r$ such that $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}$ and that $f = j'_D(f_1, \dots, f_r) = [f_1, \dots, f_r]$ for some $(f_1, \dots, f_r) \in A_{D,\Sigma}(m)^\circ$, where $f_k \in H^0(\mathbb{CP}^m, \mathcal{O}(d_k)) = \mathcal{H}_{d_k,m}$ for each $1 \leq k \leq r$. Moreover, the element $(f_1, \dots, f_r) \in A_{D,\Sigma}(m)^\circ$ is uniquely determined by the map f up to G_Σ-action; i.e. if $f = j'_D(h_1, \dots, h_r)$ for another $(h_1, \dots, h_r) \in A_{D,\Sigma}(m)^\circ$, then there exists an element $(\mu_1, \dots, \mu_r) \in G_\Sigma$ such that $(h_1, \dots, h_r) = (\mu_1 f_1, \dots, \mu_r f_r)$.*

Remark 1.4. The above result is the analogue of [**6**, Theorem 3.1] in the case of algebraic maps, and by using it we can define a unique "minimal degree" of an algebraic map (as in Definition 8.2). However, because the topology of the space of algebraic maps of fixed minimal degree is difficult to analyze, we do not use the concept of minimal degree beyond this point in this paper.

Because $\mathbf{e} = (1, 1, \dots, 1) \in \mathbb{C}^r \setminus Z_\Sigma$, we can choose $x_0 = [1, 1, \dots, 1] \in X_\Sigma$ as the base-point of X_Σ. Let $A_D(m, X_\Sigma) \subset A_{D,\Sigma}(m)$ be defined by

$$A_D(m, X_\Sigma) = \{(f_1, \dots, f_r) \in A_{D,\Sigma}(m) \mid (f_1(\boldsymbol{e}_1), \dots, f_r(\boldsymbol{e}_1)) = \mathbf{e}\}, \tag{1.14}$$

where $\boldsymbol{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^{m+1}$. We choose $[\boldsymbol{e}_1] = [1 : 0 : \cdots : 0]$ as the base-point of $\mathbb{RP}^m$. Note that $j'_D(f_1, \dots, f_r) \in \mathrm{Map}^*(\mathbb{RP}^m, X_\Sigma)$ if $(f_1, \dots, f_r) \in A_D(m, X_\Sigma)$. Hence, setting $\mathrm{Map}^*_D(\mathbb{RP}^m, X_\Sigma) = \mathrm{Map}^*(\mathbb{RP}^m, X_\Sigma) \cap \mathrm{Map}_D(\mathbb{RP}^m, X_\Sigma)$, we have a map

$$i_D = j'_D | A_D(m, X_\Sigma) : A_D(m, X_\Sigma) \to \mathrm{Map}^*_D(\mathbb{RP}^m, X_\Sigma). \tag{1.15}$$

Suppose that $m \geq 2$ and let us choose a fixed element $(g_1, \dots, g_r) \in A_D(m-1, X_\Sigma)$. For each $1 \leq k \leq r$, let $B_k = \{g_k + z_m h : h \in \mathcal{H}_{d_k - 1, m}\}$. Let $A_D(m, X_\Sigma; g) \subset A_D(m, X_\Sigma)$ be the subspace

$$A_D(m, X_\Sigma; g) = A_D(m, X_\Sigma) \cap (B_1 \times B_2 \times \cdots \times B_r). \tag{1.16}$$

It is easy to see that $i_D(f_1, \dots, f_r) | \mathbb{RP}^{m-1} = g$ if $(f_1, \dots, f_r) \in A_D(m, X_\Sigma; g)$, where g denotes the element of $\mathrm{Map}^*_D(\mathbb{RP}^{m-1}, X_\Sigma)$ given by $g([\boldsymbol{x}]) = [g_1(\boldsymbol{x}), \dots, g_r(\boldsymbol{x})]$ for $\boldsymbol{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$. Let $i'_D : A_D(m, X_\Sigma; g) \to F(\mathbb{RP}^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma$ be the map defined by

$$i'_D = i_D | A_D(m, X_\Sigma; g) : A_D(m, X_\Sigma; g) \to F(\mathbb{RP}^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma. \tag{1.17}$$

The action of G_Σ on the space $A_{D,\Sigma}(m)$ and its orbit space.

The group G_Σ acts on the space $A_{D,\Sigma}(m)$ by coordinate-wise multiplication. Let $\widetilde{A_D}(m, X_\Sigma)$ denote the orbit space

$$\widetilde{A_D}(m, X_\Sigma) = A_{D,\Sigma}(m) / G_\Sigma. \tag{1.18}$$

Clearly, j'_D also induces the map $j_D : \widetilde{A_D}(m, X_\Sigma) \to \mathrm{Map}_D(\mathbb{RP}^m, X_\Sigma)$ given by

$$j_D([f_1, \dots, f_r])([\boldsymbol{x}]) = [f_1(\boldsymbol{x}), \dots, f_r(\boldsymbol{x})] \tag{1.19}$$

for $\boldsymbol{x} = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}\}$.

Spaces of algebraic maps.

Let $\mathrm{Alg}(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ denote the space of all algebraic maps from $\mathbb{R}\mathrm{P}^m$ to X_Σ and let $\Gamma'_D : A_{D,\Sigma}(m) \to \mathrm{Alg}(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ denote the natural projection given by

$$\Gamma'_D(f_1, \ldots, f_r) = j'_D(f_1, \ldots, f_r) = [f_1, \ldots, f_r] \qquad \text{for } (f_1, \ldots, f_r) \in A_{D,\Sigma}(m).$$

Let $\mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ denote its image

$$\mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) = \Gamma'_D(A_{D,\Sigma}(m)) \subset \mathrm{Map}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma). \tag{1.20}$$

We can identify Γ'_D with the projection map

$$\Gamma'_D : A_{D,\Sigma}(m) \to \mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma). \tag{1.21}$$

Since the action of G_Σ is compatible with Γ'_D, it induces the natural projection map

$$\Gamma_D : \widetilde{A_D}(m, X_\Sigma) \to \mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma). \tag{1.22}$$

It is easy to see that

$$j_D = j^{\mathbb{C}}_D \circ \Gamma_D : \widetilde{A_D}(m, X_\Sigma) \to \mathrm{Map}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma), \tag{1.23}$$

where $j^{\mathbb{C}}_D : \mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \xrightarrow{\subset} \mathrm{Map}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ denotes the inclusion map.

Let $\mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ denote the subspace of $\mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ given by $\mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) = \mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \cap \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, X_\Sigma)$.

If $m \geq 2$ and $g \in \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^{m-1}, X_\Sigma)$, we denote by $\mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma; g)$ the subspace of $\mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ defined by $\mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) = \mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \cap F(\mathbb{R}\mathrm{P}^m, X_\Sigma; g)$.

By the restriction, the map Γ'_D induces maps

$$\begin{cases} \Psi_D : A_D(m, X_\Sigma) \to \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma), \\ \Psi'_D : A_D(m, X_\Sigma; g) \to \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma; g). \end{cases} \tag{1.24}$$

Let

$$\begin{cases} i^{\mathbb{C}}_D : \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \xrightarrow{\subset} \mathrm{Map}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma), \\ \hat{i}^{\mathbb{C}}_D : \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) \xrightarrow{\subset} F(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma \end{cases} \tag{1.25}$$

denote the inclusions. It is easy to see that

$$\begin{cases} i_D = i^{\mathbb{C}}_D \circ \Psi_D : A_D(m, X_\Sigma) \to \mathrm{Map}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma), \\ i'_D = \hat{i}^{\mathbb{C}}_D \circ \Psi'_D : A_D(m, X_\Sigma; g) \to F(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma. \end{cases} \tag{1.26}$$

The contents of the above definitions are be summarized in the following diagram.

$$\begin{tikzcd}
\mathrm{Map}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) & \mathrm{Map}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \arrow[l, hook'] & F(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) \arrow[l, hook'] \\
\mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \arrow[u, hook, "j_D^{\mathbb{C}}"] & \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \arrow[l, hook'] \arrow[u, hook, "i_D^{\mathbb{C}}"] & \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) \arrow[l, hook'] \arrow[u, hook, "\hat{i}_D^{\mathbb{C}}"] \\
\widetilde{A_D}(m, X_\Sigma) \arrow[u, two heads, "\Gamma_D"] \arrow[uu, bend left, "j_D"] & A_D(m, X_\Sigma) \arrow[u, two heads, "\Psi_D"] \arrow[uu, bend left, "i_D"] & A_D(m, X_\Sigma; g) \arrow[u, two heads, "\Psi'_D"] \arrow[uu, bend left, "i'_D"]
\end{tikzcd}$$

1.2. The main results.

For $d_k \in \mathbb{Z}_{\geq 1}$ $(1 \leq k \leq r)$, let $d_{\min}$ and $D(d_1, \dots, d_r; m)$ be the positive integers defined by

$$d_{\min} = \min\{d_1, d_2, \dots, d_r\}, \qquad D(d_1, \dots, d_r; m) = (2r_{\min} - m - 1)d_{\min} - 2. \tag{1.27}$$

Let $g \in \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^{m-1}, X_\Sigma)$ be any fixed based algebraic map and we choose an element $(g_1, \dots, g_r) \in A_D(m-1, X_\Sigma)$ such that $g = i_D(g_1, \dots, g_r) = [g_1, \dots, g_r]$.

The main result of this paper is the following theorem.

Theorem 1.5. *Let Σ be a complete fan in $\mathbb{R}^n$ satisfying the conditions* (1.7.1), (1.7.2)*, and X_Σ be a smooth compact toric variety associated to the fan Σ. If $2 \leq m \leq 2(r_{\min} - 1)$ and $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 1})^r$, the map $i'_D : A_D(m, X_\Sigma; g) \to F(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma$ is a homology equivalence through dimension $D(d_1, \dots, d_r; m)$.*

Remark 1.6. A map $f : X \to Y$ is called *a homology equivalence through dimension N* if $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$ is an isomorphism for any $k \leq N$.

By the same method as in [**2**, Theorem 3.5] (cf. [**8**]), we also obtain the following:

Corollary 1.7. *Under the same assumptions as in Theorem* 1.5*, if $2 \leq m \leq 2(r_{\min} - 1)$ and $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 1})^r$, the maps*

$$\begin{cases} j_D : \widetilde{A_D}(m, X_\Sigma) \to \mathrm{Map}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma), \\ i_D : A_D(m, X_\Sigma) \to \mathrm{Map}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \end{cases}$$

are homology equivalences through dimension $D(d_1, \dots, d_r; m)$.

Remark 1.8. It is known that the assertion of Corollary 1.7 does not hold for $m = 1$. For example, we can easily see this for $X_\Sigma = \mathbb{C}\mathrm{P}^n$ (cf. [**10**, Theorem 3.5]).

Finally we consider an example illustrating these results. If $X_\Sigma = \mathbb{C}\mathrm{P}^n$, it is easy to see that $(r, r_{\min}) = (n+1, n+1)$ (this case was already treated in [**15**]). Now consider the case $X_\Sigma = H(k)$ (the Hirzebruch surface). Since there is an isomorphism $H(k) \cong H(-k)$, we may assume that $k \geq 0$. Let $\{\boldsymbol{n}_j \in \mathbb{Z}^2 \mid 1 \leq j \leq 4\}$ be the set of the primitive elements of the fan Σ given in Example 1.1. It is easily see that

$(r, r_{\min}) = (4, 2)$. Since $\sum_{j=1}^{4} d_j \boldsymbol{n}_j = \mathbf{0}$ if and only if $(d_3, d_4) = (d_1, d_2 + kd_1)$, we obtain the following result.

EXAMPLE 1.9 (The case $(X_\Sigma, m) = (H(k), 2)$). If $k \geq 0$, $d_j \geq 1$ $(j = 1, 2)$ are integers and $D = (d_1, d_2, d_1, d_2 + kd_1)$, the maps

$$\begin{cases} j_D : \widetilde{A_D}(2, H(k)) \to \mathrm{Map}_D(\mathbb{RP}^2, H(k)), \\ i_D : A_D(2, H(k)) \to \mathrm{Map}^*_D(\mathbb{RP}^2, H(k)), \\ i'_D : A_D(2, H(k); g) \to \Omega^2_D H(k) \end{cases}$$

are homology equivalences through dimension $\min\{d_1, d_2\} - 2$.

2. Simplicial resolutions.

In this section, we summarize the definitions of the non-degenerate simplicial resolution and the associated truncated resolutions ([**2**], [**14**], [**17**], [**18**], [**22**]).

DEFINITION 2.1. (i) For a finite set $\boldsymbol{v} \subset \mathbb{R}^N$, let $\sigma(\boldsymbol{v})$ denote the convex hull spanned by $\boldsymbol{v}$. Let $h : X \to Y$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in Y$, and let $i : X \to \mathbb{R}^N$ be an embedding. Let $\mathcal{X}^\Delta$ and $h^\Delta : \mathcal{X}^\Delta \to Y$ denote the space and the map defined by

$$\mathcal{X}^\Delta = \big\{(y, u) \in Y \times \mathbb{R}^N : u \in \sigma(i(h^{-1}(y)))\big\} \subset Y \times \mathbb{R}^N,\ h^\Delta(y, u) = y.$$

The pair $(\mathcal{X}^\Delta, h^\Delta)$ is called *a simplicial resolution of* (h, i). In particular, $(\mathcal{X}^\Delta, h^\Delta)$ is called *a non-degenerate simplicial resolution* if for each $y \in Y$ any k points of $i(h^{-1}(y))$ span $(k-1)$-dimensional simplex of $\mathbb{R}^N$.

(ii) For each $k \geq 0$, let $\mathcal{X}^\Delta_k \subset \mathcal{X}^\Delta$ be the subspace given by

$$\mathcal{X}^\Delta_k = \big\{(y, u) \in \mathcal{X}^\Delta : u \in \sigma(\boldsymbol{v}), \boldsymbol{v} = \{v_1, \dots, v_l\} \subset i(h^{-1}(y)), l \leq k\big\}.$$

We make identification $X = \mathcal{X}^\Delta_1$ by identifying $x \in X$ with the pair $(h(x), i(x)) \in \mathcal{X}^\Delta_1$, and we note that there is an increasing filtration

$$\emptyset = \mathcal{X}^\Delta_0 \subset X = \mathcal{X}^\Delta_1 \subset \mathcal{X}^\Delta_2 \subset \cdots \subset \mathcal{X}^\Delta_k \subset \mathcal{X}^\Delta_{k+1} \subset \cdots \subset \bigcup_{k=0}^{\infty} \mathcal{X}^\Delta_k = \mathcal{X}^\Delta.$$

REMARK 2.2. Even for a surjective map $h : X \to Y$ which is not finite to one, it is still possible to construct an associated non-degenerate simplicial resolution. In fact, a non-degenerate simplicial resolution may be constructed by choosing a sequence of embeddings $\{\tilde{i}_k : X \to \mathbb{R}^{N_k}\}_{k \geq 1}$ satisfying the following two conditions for each $k \geq 1$ (cf. [**22**]).

$(2.1)_k$ (i) For any $y \in Y$, any t points of the set $\tilde{i}_k(h^{-1}(y))$ span $(t-1)$-dimensional affine subspace of $\mathbb{R}^{N_k}$ if $t \leq 2k$.

(ii) $N_k \leq N_{k+1}$ and if we identify $\mathbb{R}^{N_k}$ with a subspace of $\mathbb{R}^{N_{k+1}}$, then $\tilde{i}_{k+1} = \hat{i} \circ \tilde{i}_k$, where $\hat{i} : \mathbb{R}^{N_k} \overset{\subseteq}{\to} \mathbb{R}^{N_{k+1}}$ denotes the inclusion.

Let $\mathcal{X}^{\Delta}_k = \{(y,u) \in Y \times \mathbb{R}^{N_k} : u \in \sigma(\boldsymbol{v}),\ \boldsymbol{v} = \{v_1, \dots, v_l\} \subset \tilde{i}_k(h^{-1}(y)), l \leq k\}$. Then by identifying $\mathcal{X}^{\Delta}_k$ with a subspace of $\mathcal{X}^{\Delta}_{k+1}$, we define the non-degenerate simplicial resolution $\mathcal{X}^{\Delta}$ of h as the union $\mathcal{X}^{\Delta} = \bigcup_{k \geq 1} \mathcal{X}^{\Delta}_k$.

Definition 2.3. Let $h : X \to Y$ be a surjective semi-algebraic map between semi-algebraic spaces, $j : X \to \mathbb{R}^N$ be a semi-algebraic embedding, and let $(\mathcal{X}^{\Delta}, h^{\Delta} : \mathcal{X}^{\Delta} \to Y)$ denote the associated non-degenerate simplicial resolution of h.

Let k be a fixed positive integer and let $h_k : \mathcal{X}^{\Delta} \to Y$ be the map defined by the restriction $h_k := h^{\Delta} | \mathcal{X}^{\Delta}$. The fibres of the map h_k are $(k-1)$-skeleta of the fibres of h^{Δ} and, in general, fail to be simplices over the subspace

$$Y_k = \{y \in Y : h^{-1}(y) \text{ consists of more than } k \text{ points}\}.$$

Let $Y(k)$ denote the closure of the subspace Y_k. We modify the subspace $\mathcal{X}^{\Delta}_k$ so as to make the all the fibres of h_k contractible by adding to each fibre of $Y(k)$ a cone whose base is this fibre. We denote by $X^{\Delta}(k)$ this resulting space and by $h^{\Delta}_k : X^{\Delta}(k) \to Y$ the natural extension of h_k. Following [**18**], we call the map $h^{\Delta}_k : X^{\Delta}(k) \to Y$ *the truncated* (*after the k-th term*) *simplicial resolution* of Y. Note that there is a natural filtration

$$\emptyset = X^{\Delta}_0 \subset X^{\Delta}_1 \subset \cdots \subset X^{\Delta}_l \subset X^{\Delta}_{l+1} \subset \cdots \subset X^{\Delta}_k \subset X^{\Delta}_{k+1} = X^{\Delta}_{k+2} = \cdots = X^{\Delta}(k),$$

where $X^{\Delta}_l = \mathcal{X}^{\Delta}_l$ if $l \leq k$ and $X^{\Delta}_l = X^{\Delta}(k)$ if $l > k$.

3. Non-degenerate simplicial resolutions.

Definition 3.1. Fix a based algebraic map $g \in \mathrm{Alg}^*_D(\mathbb{RP}^{m-1}, X_{\Sigma})$ of degree D together with a representation $(g_1, \dots, g_r) \in A_D(m-1, X_{\Sigma})$ such that $g = [g_1, \dots, g_r]$. Note that $A_D(m, X_{\Sigma}; g)$ is an open subspace of the affine space $B_D = B_1 \times \cdots \times B_r$.

(i) Let $N_D = \dim_{\mathbb{C}} B_D = \sum_{k=1}^{r} \binom{m+d_k-1}{m}$, and let $\Sigma_D \subset B_D$ denote the *discriminant* of $A_D(m, X_{\Sigma}; g)$ in B_D, that is, the complement

$$\begin{aligned}\Sigma_D &= B_D \setminus A_D(m, X_{\Sigma}; g)\\ &= \{(f_1, \dots, f_r) \in B_D \mid (f_1(\boldsymbol{x}), \dots, f_r(\boldsymbol{x})) \in Z_{\Sigma} \text{ for some } \boldsymbol{x} \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}\}\}.\end{aligned}$$

(ii) Let $Z_D \subset \Sigma_D \times \mathbb{R}^m$ denote *the tautological normalization* of Σ_D consisting of all pairs $(F, x) = ((f_1, \dots, f_r), (x_0, \dots, x_{m-1})) \in \Sigma_D \times \mathbb{R}^m$ such that $(f_1(\boldsymbol{x}), \dots, f_r(\boldsymbol{x})) \in Z_{\Sigma}$, where we set $\boldsymbol{x} = (x_0, \dots, x_{m-1}, 1)$. Projection on the first factor gives a surjective map $\pi_D : Z_D \to \Sigma_D$.

Our goal in this section is to construct, by means of the *non-degenerate* simplicial resolution of the discriminant, a spectral sequence converging to the homology of the space $A_D(m, X_{\Sigma}; g)$.

DEFINITION 3.2. Let $(\mathcal{X}^D, \pi_D^{\Delta} : \mathcal{X}^D \to \Sigma_D)$ be the non-degenerate simplicial resolution of the surjective map $\pi_D : Z_D \to \Sigma_D$ with the following natural increasing filtration as in Definition 2.1, $\mathcal{X}_0^D = \emptyset \subset \mathcal{X}_1^D \subset \mathcal{X}_2^D \subset \cdots \subset \mathcal{X}^D = \bigcup_{k=0}^{\infty} \mathcal{X}_k^D$.

By [**14**, Lemma 2.2], the map $\pi_D^{\Delta} : \mathcal{X}^D \xrightarrow{\simeq} \Sigma_D$ is a homotopy equivalence, which extends to a homotopy equivalence $\pi_{d+}^{\Delta} : \mathcal{X}_+^D \xrightarrow{\simeq} \Sigma_{D+}$, where X_+ denotes the one-point compactification of a locally compact space X. Since $\mathcal{X}_k^D{}_+/\mathcal{X}_{k-1}^D{}_+ \cong (\mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D)_+$, we have a spectral sequence

$$\left\{E_{t;D}^{k,s}, d_t : E_{t;D}^{k,s} \to E_{t;D}^{k+t,s+1-t}\right\} \Rightarrow H_c^{k+s}(\Sigma_D, \mathbb{Z}), \tag{3.1}$$

where $E_{1;D}^{k,s} = H_c^{k+s}(\mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D, \mathbb{Z})$ and $H_c^k(X,\mathbb{Z})$ denotes the cohomology group with compact supports given by $H_c^k(X,\mathbb{Z}) = H^k(X_+,\mathbb{Z})$.

By Alexander duality there is a natural isomorphism

$$H_k(A_D(m, X_\Sigma; g), \mathbb{Z}) \cong H_c^{2N_D-k-1}(\Sigma_D, \mathbb{Z}) \quad \text{for } 1 \leq k \leq 2N_D - 2. \tag{3.2}$$

By reindexing we obtain a spectral sequence

$$\left\{\tilde{E}_{k,s}^{t;D}, \tilde{d}^t : \tilde{E}_{k,s}^{t;D} \to \tilde{E}_{k+t,s+t-1}^{t;D}\right\} \Rightarrow H_{s-k}(A_D(m, X_\Sigma; g), \mathbb{Z}) \tag{3.3}$$

if $s - k \leq 2N_D - 2$, where $\tilde{E}_{k,s}^{1;D} = \tilde{H}_c^{2N_D+k-s-1}(\mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D, \mathbb{Z})$.

For a connected space X, let $F(X,k)$ denote the configuration space of distinct k points in X. The symmetric group S_k of k letters acts on $F(X,k)$ freely by permuting coordinates. Let $C_k(X)$ be the configuration space of unordered k-distinct points in X given by the orbit space $C_k(X) = F(X,k)/S_k$. Similarly, let $L_k \subset (\mathbb{R}^m \times Z_\Sigma)^k$ denote the subspace defined by $L_k = \{((x_1,s_1),\ldots,(x_k,s_k)) \mid x_j \in \mathbb{R}^m, s_j \in Z_\Sigma, x_l \neq x_j \text{ if } l \neq j\}$. The group S_k also acts on L_k by permuting coordinates, and let C_k denote the orbit space

$$C_k = L_k/S_k. \tag{3.4}$$

Note that C_k is a cell complex of dimension $(m + 2r - 2r_{\min})k$ (cf. (1.4)).

LEMMA 3.3. *If $1 \leq k \leq d_{\min}$, $\mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D$ is homeomorphic to the total space of a real affine bundle $\xi_{D,k}$ over C_k with rank $l_{D,k} = 2N_D - 2kr + k - 1$.*

PROOF. The argument is exactly analogous to the one in the proof of [**2**, Lemma 4.4]. Namely, an element of $\mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D$ is represented by $(F,u) = ((f_1,\ldots,f_r),u)$, where $F = (f_1,\ldots,f_r)$ is a r-tuple of polynomials in Σ_D and u is an element of the interior of the span of the images of k distinct points $x_1,\ldots,x_k \in \mathbb{R}^m$ such that $F(\boldsymbol{x}_j) = (f_1(\boldsymbol{x}_j),\ldots,f_r(\boldsymbol{x}_j)) \in Z_\Sigma$ for each $1 \leq j \leq k$, under a suitable embedding, where we set $\boldsymbol{x}_j = (x_j,1) \in \mathbb{R}^{m+1}$. Let $\pi_k : \mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D \to C_k$ be the projection map $((f_1,\ldots,f_r),u) \mapsto \{(x_1,F(\boldsymbol{x}_1)),\ldots,(x_k,F(\boldsymbol{x}_k))\}$. (Note that the points x_i are uniquely determined by u by the construction of the non-degenerate simplicial resolution.)

Next, let $c = \{(x_j, s_j)\}_{j=1}^k \in C_k$ ($x_j \in \mathbb{R}^m$, $s_j \in Z_\Sigma$) be any fixed element and consider the fibre $\pi_k^{-1}(c)$. For this purpose, for each $1 \leq j \leq k$ let us consider the condition

$$F(\boldsymbol{x}_j) = (f_1(\boldsymbol{x}_j), \dots, f_r(\boldsymbol{x}_j)) = s_j \quad \Leftrightarrow \quad f_t(\boldsymbol{x}_j) = s_{t,j} \quad \text{for } 1 \leq t \leq r, \tag{3.5}$$

where we set $s_j = (s_{1,j}, \dots, s_{r,j})$. In general, the condition $f_t(\boldsymbol{x}_j) = s_{t,j}$ gives one linear condition on the coefficients of f_t, which determines an affine hyperplane in B_t. Since $\{\boldsymbol{x}_j\}_{j=1}^k$ is mutually distinct, if $1 \leq k \leq d_{\min}$, by [**2**, Lemma 4.3] the condition (3.5) produces exactly k independent conditions on the coefficients of f_t. Thus the space of polynomials f_t in B_t which satisfies (3.5) is the intersection of k affine hyperplanes in general position and it has codimension k in B_t. Hence, if $1 \leq k \leq d_{\min}$ the fibre $\pi_k^{-1}(c)$ is homeomorphic to the product of open $(k-1)$-simplex with the real affine space of dimension $2\sum_{t=1}^r \left(\binom{d_t+m-1}{m} - k\right) = 2N_D - 2kr$. Thus π_k is a real affine bundle over C_k of rank $l_{D,k} = 2N_D - 2kr + k - 1$. $\square$

Lemma 3.4. *If $1 \leq k \leq d_{\min}$, there is a natural isomorphism*

$$\tilde{E}_{k,s}^{1;D} \cong \tilde{H}_c^{2rk-s}(C_k, \pm\mathbb{Z}).$$

Proof. Suppose that $1 \leq k \leq d_{\min}$. By Lemma 3.3, there is a homeomorphism $(\mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D)_+ \cong T(\xi_{D,k})$, where $T(\xi_{D,k})$ denotes the Thom space of $\xi_{D,k}$. Since

$$(2N_D + k - s - 1) - l_{D,k} = (2N_D + k - s - 1) - (2N_D - 2kr + k - 1) = 2rk - s,$$

by using the Thom isomorphism theorem we obtain a natural isomorphism

$$\tilde{E}_{k,s}^{1;D} \cong \tilde{H}^{2N_D+k-s-1}(T(\xi_{D,k}), \mathbb{Z}) \cong \tilde{H}_c^{2rk-s}(C_k, \pm\mathbb{Z}),$$

where the twisted coefficient system $\pm\mathbb{Z}$ appears in the computation of the cohomology of the Thom space induced by the sign representation of the symmetric group as in [**22**, pp. 37–38, p. 114 and p. 254]). This completes the proof. $\square$

4. Truncated spectral sequences.

In this section, we prove a key result (Theorem 4.8) about the homology stability of "stabilization maps" $s_D : A_D(m, X_\Sigma; g) \to A_{D+\boldsymbol{a}}(m, X_\Sigma; g)$.

Definition 4.1. Let X^Δ denote the truncated (after $d_{\min}$-th term) simplicial resolution of Σ_D with its natural filtration as in Definition 2.3,

$$\emptyset = X_0^\Delta \subset X_1^\Delta \subset \cdots \subset X_{d_{\min}}^\Delta \subset X_{d_{\min}+1}^\Delta = X_{d_{\min}+2}^\Delta = \cdots = X^\Delta,$$

where $X_k^\Delta = \mathcal{X}_k^D$ if $k \leq d_{\min}$ and $X_k^\Delta = X^\Delta$ if $k \geq d_{\min} + 1$.

Remark 4.2. Note that our notation X^Δ conflicts with that of [**18**] and Definition

2.3, where $\mathcal{X}^{\Delta}$ denotes the non-degenerate simplicial resolution.

By [**14**, Lemma 2.5], there is a homotopy equivalence $\pi^{\Delta}: X^{\Delta} \xrightarrow{\simeq} \Sigma_D$. Hence, by using the filtration on X^{Δ} given in Definition 4.1, we have a spectral sequence

$$\{\hat{E}^{k,s}_{t;D}, d_t : \hat{E}^{k,s}_{t;D} \to \hat{E}^{k+t,s+1-t}_{t;D}\} \Rightarrow H^{k+s}_c(\Sigma_D, \mathbb{Z}), \tag{4.1}$$

where $\hat{E}^{k,s}_{1;D} = \tilde{H}^{k+s}_c(X^{\Delta}_k \setminus X^{\Delta}_{k-1}, \mathbb{Z})$. Then by reindexing and using Alexander duality (3.2), we obtain the truncated spectral sequence (cf. [**14**, (4.1)], [**18**])

$$\{E^t_{k,s}, d^t : E^t_{k,s} \to E^t_{k+t,s+t-1}\} \Rightarrow H_{s-k}(A_D(m, X_{\Sigma}; g), \mathbb{Z}) \tag{4.2}$$

if $s-k \leq 2N_D - 2$, where $E^1_{k,s} = \tilde{H}^{2N_D+k-s-1}_c(X^{\Delta}_k \setminus X^{\Delta}_{k-1}, \mathbb{Z})$.

LEMMA 4.3. (i) *If $1 \leq k \leq d_{\min}$, there is a natural isomorphism*

$$E^1_{k,s} \cong \tilde{H}^{2rk-s}_c(C_k, \pm\mathbb{Z}).$$

(ii) $E^1_{k,s} = 0$ *if $k < 0$, or if $k \geq d_{\min} + 2$, or if $k = 0$ and $s \neq 2N_D - 1$.*
(iii) *If $1 \leq k \leq d_{\min}$, $E^1_{k,s} = 0$ for any $s \leq (2r_{\min} - m)k - 1$.*
(iv) *If $k = d_{\min} + 1$, $E^1_{d_{\min}+2,s} = 0$ for any $s \leq (2r_{\min} - m)d_{\min} - 1$.*

PROOF. Since $X^{\Delta}_k \setminus X^{\Delta}_{k-1} = \mathcal{X}^D_k \setminus \mathcal{X}^D_{k-1}$ for $1 \leq k \leq d_{\min}$, the assertion (i) follows from Lemma 3.4. Next, because $X^{\Delta}_0 = \emptyset$ and $X^{\Delta} = X^{\Delta}_k$ for $k \geq d_{\min} + 2$, the assertion (ii) easily follows. Now we assume that $1 \leq k \leq d_{\min}$ and try to prove (iii). Since $\dim C_k = (m + 2r - 2r_{\min})k$, $2rk - s > \dim C_k$ if and only if $s \leq (2r_{\min} - m)k - 1$ and (iii) follows from (i). It remains to show (iv). An easy computation shows that

$$\begin{aligned}
\dim(\mathcal{X}^D_k \setminus \mathcal{X}^D_{k-1}) &= (2N_D - 2kr + k - 1) + \dim C_k \\
&= (2N_D - 2kr + k - 1) + (m + 2r - 2r_{\min})k \\
&= 2N_D - (2r_{\min} - m - 1)k - 1.
\end{aligned}$$

Since $\dim(X^{\Delta}_{d_{\min}+1} \setminus X^{\Delta}_{d_{\min}}) = \dim(\mathcal{X}^D_{d_{\min}} \setminus \mathcal{X}^D_{d_{\min}-1}) + 1$ by [**14**, Lemma 2.6],

$$\dim(X^{\Delta}_{d_{\min}+1} \setminus X^{\Delta}_{d_{\min}}) = 2N_D - (2r_{\min} - m - 1)d_{\min}.$$

Since $E^1_{d_{\min}+1,s} = \tilde{H}^{2N_D+d_{\min}-s}_c(X^{\Delta}_{d_{\min}+1} \setminus X^{\Delta}_{d_{\min}}, \mathbb{Z})$ and

$$2N_D + d_{\min} - s > \dim(X^{\Delta}_{d_{\min}+2} \setminus X^{\Delta}_{d_{\min}+1}) \Leftrightarrow s \leq (2r_{\min} - m)d_{\min} - 1,$$

we see that $E^1_{d_{\min}+1,s} = 0$ for $s \leq (2r_{\min} - m)d_{\min} - 1$. □

Let $\boldsymbol{a} = (a_1, \ldots, a_r) \in (\mathbb{Z}_{\geq 1})^r$ be fixed r-tuple of positive integers such that

$$\sum_{k=1}^{r} a_k \boldsymbol{n}_k = \mathbf{0}. \tag{4.3}$$

We set $D+\boldsymbol{a}=(d_1+a_1,\dots,d_r+a_r)$. Similarly, let Y^{Δ} denote the truncated (after $d_{\min}$-th term) simplicial resolution of $\Sigma_{D+\boldsymbol{a}}$ with its natural filtration

$$\emptyset = Y^{\Delta}_0 \subset Y^{\Delta}_1 \subset \cdots \subset Y^{\Delta}_{d_{\min}} \subset Y^{\Delta}_{d_{\min}+1} = Y^{\Delta}_{d_{\min}+2} = \cdots = Y^{\Delta},$$

where $Y^{\Delta}_k = \mathcal{X}^{D+\boldsymbol{a}}_k$ if $k \leq d_{\min}$ and $Y^{\Delta}_k = Y^{\Delta}$ if $k \geq d_{\min}+1$.

By [**14**, Lemma 2.5], there is a homotopy equivalence ${}'\pi^{\Delta} : Y^{\Delta} \xrightarrow{\simeq} \Sigma_{D+\boldsymbol{a}}$. Hence, by using the same method as above, we obtain a spectral sequence

$$\left\{{}'E^t_{k,s}, {}'d^t : E^t_{k,s} \to {}'E^t_{k+t,s+t-1}\right\} \Rightarrow H_{s-k}(A_{D+\boldsymbol{a}}(m, X_\Sigma; g), \mathbb{Z}) \tag{4.4}$$

if $s-k \leq 2N_{D+\boldsymbol{a}}-2$, where ${}'E^1_{k,s} = \tilde{H}^{2N_{D+\boldsymbol{a}}+k-s-1}_c(Y^{\Delta}_k \setminus Y^{\Delta}_{k-1}, \mathbb{Z})$.

Applying again the same argument we obtain the following result.

Lemma 4.4. (i) *If $1 \leq k \leq d_{\min}$, there is a natural isomorphism*

$${}'E^1_{k,s} \cong \tilde{H}^{2rk-s}_c(C_k, \pm\mathbb{Z}).$$

(ii) ${}'E^1_{k,s}=0$ *if $k<0$, or if $k \geq d_{\min}+2$, or if $k=0$ and $s \neq 2N_{D+\boldsymbol{a}}-1$.*
(iii) *If $1 \leq k \leq d_{\min}$, ${}'E^1_{k,s}=0$ for any $s \leq (2r_{\min}-m)k-1$.*
(iv) *If $k=d_{\min}+1$, ${}'E^1_{d_{\min}+2,s}=0$ for any $s \leq (2r_{\min}-m)d_{\min}-1$.*

Definition 4.5. Let $g \in \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^{m-1}, X_\Sigma)$ be a fixed algebraic map, and let $(g_1,\dots,g_r) \in A_D(m-1, X_\Sigma)$ be its fixed representative. Let $\boldsymbol{a}=(a_1,\dots,a_r) \in (\mathbb{Z}_{\geq 1})^r$ be a fixed r-tuple of positive integers satisfying the condition (4.3). If we set $\tilde{g} = \sum_{k=0}^{m} z_k^2$, we see that the tuple $((\tilde{g}|_{z_m=0})^{a_1}g_1,\dots,(\tilde{g}|_{z_m=0})^{a_r}g_r)$ can also be chosen as a representative of the map $g \in \mathrm{Alg}^*_{D+\boldsymbol{a}}(\mathbb{R}\mathrm{P}^{m-1}, X_\Sigma)$. So one can define a stabilization map $s_D : A_D(m, X_\Sigma; g) \to A_{D+\boldsymbol{a}}(m, X_\Sigma; g)$ by

$$s_D(f_1,\dots,f_r) = (\tilde{g}^{a_1}f_1,\dots,\tilde{g}^{a_r}f_r) \quad \text{for} \quad (f_1,\dots,f_r) \in A_D(m, X_\Sigma; g). \tag{4.5}$$

Because there is a commutative diagram

$$\begin{CD} A_D(m, X_\Sigma; g) @>{s_D}>> A_{D+\boldsymbol{a}}(m, X_\Sigma; g) \\ @V{i'_D}VV @VV{i'_{D+\boldsymbol{a}}}V \\ F(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) @>{=}>> F(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) \end{CD} \tag{4.6}$$

it induces a map

$$s_{D,\infty} = \lim_{k\to\infty} s_{D+k\boldsymbol{a}} : A_{D,\infty}(m, X_\Sigma; g) \to F(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) \simeq \Omega^m_0 X_\Sigma, \tag{4.7}$$

where $A_{D,\infty}(m, X_\Sigma; g)$ denotes the colimit $\lim_{k\to\infty} A_{D+k\boldsymbol{a}}(m, X_\Sigma; g)$ induced from the stabilization maps $s_{D+k\boldsymbol{a}}$'s ($k \geq 0$).

THEOREM 4.6. *If $2 \leq m \leq 2(r_{\min} - 1)$, the map $s_{D,\infty} : A_{D,\infty}(m, X_\Sigma; g) \xrightarrow{\simeq} \Omega_0^m X_\Sigma$ is a homology equivalence.*

We postpone the proof of Theorem 4.6 to Section 6. We first prove another key result (Theorem 4.8). Recall the definition of the map s_D, and consider the map $\tilde{s}_D : \Sigma_D \to \Sigma_{D+\boldsymbol{a}}$ given by the multiplication, $\tilde{s}_D(f_1, \dots, f_r) = (\tilde{g}^{a_1} f_1, \dots, \tilde{g}^{a_r} f_r)$. One can show that it extends to the embedding $\tilde{s}_d : \mathbb{R}^{2(N_{D+\boldsymbol{a}} - N_D)} \times \Sigma_D \to \Sigma_{D+\boldsymbol{a}}$ and that it induces the filtration preserving open embedding $\hat{s}_D : \mathbb{R}^{2(N_{D+\boldsymbol{a}} - N_D)} \times \mathcal{X}^D \to \mathcal{X}^{D+\boldsymbol{a}}$ by using [**19**, Proposition 7] or [**22**, pp. 103–106]. Hence, it also induces the filtration preserving open embedding $\hat{s}_D : \mathbb{R}^{2(N_{D+\boldsymbol{a}} - N_D)} \times X^\Delta \to Y^\Delta$. Since one-point compactification is contravariant for open embeddings, it induces a homomorphism of spectral sequences

$$\{\theta_{k,s}^t : E_{k,s}^t \to {}'E_{k,s}^t\}. \tag{4.8}$$

LEMMA 4.7. *If $1 \leq k \leq d_{\min}$, $\theta_{k,s}^1 : E_{k,s}^1 \to {}'E_{k,s}^1$ is an isomorphism for any s.*

PROOF. Suppose that $1 \leq r \leq d_{\min}$. Then it follows from the proof of Lemma 3.3 that there is a homotopy commutative diagram of open affine bundles

$$\begin{tikzcd}
\mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D \arrow[r, "\pi_k"] \arrow[d, "\hat{s}_D"'] & C_k \arrow[d, equal] \\
\mathcal{X}_k^{D+\boldsymbol{a}} \setminus \mathcal{X}_{k-1}^{D+\boldsymbol{a}} \arrow[r, "\pi_k"] & C_k
\end{tikzcd}$$

Since $X_k^\Delta \setminus X_{k-1}^\Delta = \mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D$ and $Y_k^\Delta \setminus Y_{k-1}^\Delta = \mathcal{X}_k^{D+\boldsymbol{a}} \setminus \mathcal{X}_{k-1}^{D+\boldsymbol{a}}$, by Lemma 4.3 and Lemma 4.4, we have a commutative diagram

$$\begin{tikzcd}
E_{k,s}^1 \arrow[r, "T", "\cong"'] \arrow[d, "\theta_{k,s}^1"'] & \tilde{H}_c^{2rk-s}(C_k, \pm\mathbb{Z}) \arrow[d, equal] \\
{}'E_{k,s}^1 \arrow[r, "T", "\cong"'] & \tilde{H}_c^{2rk-s}(C_k, \pm\mathbb{Z})
\end{tikzcd}$$

where T denotes the Thom isomorphism. Hence, $\theta_{k,s}^1$ is an isomorphism. □

THEOREM 4.8. *$s_D : A_D(m, X_\Sigma; g) \to A_{D+\boldsymbol{a}}(m, X_\Sigma; g)$ is a homology equivalence through dimension $D(d_1, \dots, d_r; m)$.*

PROOF. We set $D_0 = D(d_1, \dots, d_r; m) = (2r_{\min} - m - 1)d_{\min} - 2$, and consider two spectral sequences

$$\begin{cases} \{E^t_{k,s}, d^t : E^t_{k,s} \to \ E^t_{k+t,s+t-1}\} & \Rightarrow \quad H_{s-k}(A_D(m, X_\Sigma; g), \mathbb{Z}), \\ \{'E^t_{k,s}, 'd^t : 'E^t_{k,s} \to 'E^t_{k+t,s+t-1}\} & \Rightarrow \quad H_{s-k}(A_{D+\boldsymbol{a}}(m, X_\Sigma; g), \mathbb{Z}), \end{cases}$$

with a homomorphism $\{\theta^t_{k,s} : E^t_{k,s} \to 'E^t_{k,s}\}$ of spectral sequences.

Next, we shall try to estimate the maximal positive integer $D_{\max}$ such that

$$D_{\max} = \max\{N \in \mathbb{Z} : \theta^\infty_{k,s} \text{ is an isomorphism for all } (k,s) \text{ if } s-k \le N\}.$$

By Lemma 4.3 and 4.4, we see that $E^1_{k,s} = 'E^1_{k,s} = 0$ if $k < 0$, or if $k > d_{\min}+1$, or if $k = d_{\min}+1$ with $s \le (2r_{\min}-m)d_{\min}-1$. Since $(2r_{\min}-m)d_{\min}-(d_{\min}+1) = D_0+1$, we deduce that:

$(*)_1$ if $k < 0$ or $k \ge d_{\min}+1$, $\theta^\infty_{k,s}$ is an isomorphism for all (k,s) if $s-k \le D_0$.

Next, we assume that $0 \le k \le d_{\min}$, and again investigate the condition for $\theta^\infty_{k,s}$ to be an isomorphism. Note that the group $E^1_{k_1,s_1}$ is not known for $(k_1,s_1) \in \mathcal{S}_1 = \{(d_{\min}+1, s) \in \mathbb{Z}^2 : s \ge (2r_{\min}-m)d_{\min}\}$. By considering the differentials $d^1 : E^1_{k,s} \to E^1_{k+1,s}$ and $'d^1 : 'E^1_{k,s} \to 'E^1_{k+1,s}$ and applying Lemma 4.7, we see that $\theta^2_{k,s}$ is an isomorphism if $(k,s) \notin \mathcal{S}_1 \cup \mathcal{S}_2$, where

$$\mathcal{S}_2 =: \{(k_1,s_1) \in \mathbb{Z}^2 : (k_1+1, s_1) \in \mathcal{S}_1\} = \{(d_{\min}, s_1) \in \mathbb{Z}^2 : s_1 \ge (2r_{\min}-m)d_{\min}\}.$$

A similar argument for the differentials d^2 and $'d^2$ shows that $\theta^3_{k,s}$ is an isomorphism if $(k,s) \notin \bigcup_{u=1}^3 \mathcal{S}_u$, where $\mathcal{S}_3 = \{(k_1,s_1) \in \mathbb{Z}^2 : (k_1+2, s_1+1) \in \mathcal{S}_1 \cup \mathcal{S}_2\}$. Continuing in the same fashion, considering the differentials $d^t : E^t_{k,s} \to E^t_{k+t,s+t-1}$ and $'d^t : 'E^t_{k,s} \to 'E^t_{k+t,s+t-1}$, and applying Lemma 4.7, we easily see that $\theta^\infty_{k,s}$ is an isomorphism if $(k,s) \notin \mathcal{S} := \bigcup_{t\ge 1} \mathcal{S}_t = \bigcup_{t \ge 1} A_t$, where A_t denotes the set given by

$$A_t := \left\{ (k_1,s_1) \in \mathbb{Z}^2 \;\middle|\; \begin{array}{l} \text{There are positive integers } l_1, l_2, \dots, l_t \text{ such that,} \\ 1 \le l_1 < l_2 < \cdots < l_t,\ k_1 + \sum_{j=1}^t l_j = d_{\min}+1, \\ s_1 + \sum_{j=1}^t (l_j - 1) \ge (2r_{\min}-m)d_{\min} \end{array} \right\}.$$

If $A_t \ne \emptyset$, it is easy to see that

$$a(t) = \min\{s-k : (k,s) \in A_t\} = (2r_{\min}-m)d_{\min} - (d_{\min}+1) + t = D_0 + t + 1.$$

Hence, $\min\{a(t) : t \ge 1, A_t \ne \emptyset\} = D_0 + 2$, and we have the following:

$(*)_2$ If $0 \le k \le d_{\min}$, $\theta^\infty_{k,s}$ is an isomorphism for any (k,s) if $s-k \le D_0+1$.

Then, by $(*)_1$ and $(*)_2$, we see that $\theta^\infty_{k,s} : E^\infty_{k,s} \xrightarrow{\cong} 'E^\infty_{k,s}$ is an isomorphism for any (k,s) if $s-k \le D_0$. Hence, we can now see that s_D is a homology equivalence through dimension D_0. $\square$

5. The complement $\mathbb{C}^r \setminus Z_\Sigma$.

In this section, we recall the basic results on polyhedral products and investigate the connectivity of the complement $\mathbb{C}^r \setminus Z_\Sigma$.

Definition 5.1. Let $[r] = \{1, 2, \dots, r\}$ be a set of indices and let K be a simplicial complex on the vertex set $[r]$.

(i) For each $\sigma = \{i_1, \dots, i_k\} \subset [r]$, let $L_\sigma \subset \mathbb{C}^r$ denote *the coordinate subspace* in $\mathbb{C}^r$ given by $L_\sigma = \{\boldsymbol{x} = (x_1, \dots, x_r) \in \mathbb{C}^r : x_{i_1} = \cdots = x_{i_k} = 0\}$, and $U(K)$ the complement of the *arrangement of coordinate subspaces* in $\mathbb{C}^r$

$$U(K) = \mathbb{C}^r \Big\backslash \bigcup_{\sigma \notin K} L_\sigma. \tag{5.1}$$

(ii) Let $(\underline{X}, \underline{A})$ be a collection of spaces of pairs $\{(X_k, A_k)\}_{k=1}^r$. The *polyhedral product* $\mathcal{Z}_K(\underline{X}, \underline{A})$ of the collection $(\underline{X}, \underline{A})$ with respect to K by

$$\mathcal{Z}_K(\underline{X}, \underline{A}) = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma, \tag{5.2}$$

where we set $(\underline{X}, \underline{A})^\sigma = \{(x_1, \dots, x_r) \in X_1 \times \cdots \times X_r \mid x_j \in A_j \text{ if } j \notin \sigma\}$ for $\sigma \in K$.

In particular, when $(X_j, A_j) = (X, A)$ for each $1 \leq j \leq r$, we set $\mathcal{Z}_K(\underline{X}, \underline{A}) = \mathcal{Z}_K(X, A)$. For $(X, A) = (D^2, S^1)$, $\mathcal{Z}_K = \mathcal{Z}_K(D^2, S^1)$ is called the *moment-angle complex* of type K, while $DJ(K) = \mathcal{Z}_K(\mathbb{C}\mathrm{P}^\infty, *)$ is called the *Davis–Januszkiwicz space* of type K.

(iii) Let $\mathcal{K}_\Sigma$ denote the simplicial complex on the vertex set $[r]$ defined by

$$\mathcal{K}_\Sigma = \big\{\{i_1, \dots, i_k\} \subset [r] \mid \boldsymbol{n}_{i_1}, \dots, \boldsymbol{n}_{i_k} \text{ span a cone} \in \Sigma\big\}, \tag{5.3}$$

and let q_Σ denote the positive integer given by

$$q_\Sigma = \max\{s \in \mathbb{Z}_{\geq 1} \mid \text{Any } s \text{ vectors } \boldsymbol{n}_{i_1}, \boldsymbol{n}_{i_2}, \dots, \boldsymbol{n}_{i_s} \text{ span a cone in } \Sigma\}. \tag{5.4}$$

Lemma 5.2. *$U(\mathcal{K}_\Sigma) = \mathbb{C}^r \setminus Z_\Sigma$, and $r_{\min} = q_\Sigma + 1$.*

Proof. From (1.1), (1.2), (5.1) and (5.3), it is clear that $U(\mathcal{K}_\Sigma) = \mathbb{C}^r \setminus Z_\Sigma$. Recalling the definitions (1.5) and (5.4), we easy see that $r_{\min} = q_\Sigma + 1$. □

Lemma 5.3 ([**4**]). *If K is a simplicial complex on the vertex set $[r]$, there is a homotopy equivalence $\mathcal{Z}_K \simeq U(K)$ and the space $\mathcal{Z}_K$ is a homotopy fibre of the inclusion map $DJ(K) \xrightarrow{\subsetneq} (\mathbb{C}\mathrm{P}^\infty)^r$.*

Proof. This follows from [**4**, Corollary 6.30, Theorem 8.9]. □

Lemma 5.4. *The space $U(\mathcal{K}_\Sigma) = \mathbb{C}^r \setminus Z_\Sigma$ is $2(r_{\min} - 1)$-connected.*

Proof. Because $q_\Sigma = r_{\min} - 1$, it suffices to show that $U(\mathcal{K}_\Sigma)$ is $2q_\Sigma$-connected.

Note that the $2s$-skeleton of $(\mathbb{C}\mathrm{P}^\infty)^r$ is $\bigcup_{i_1+i_2+\cdots+i_r=s}\mathbb{C}\mathrm{P}^{i_1}\times\cdots\times\mathbb{C}\mathrm{P}^{i_r}$. Since $1\leq q_\Sigma<r$, by (5.3) and (5.4) we see that $\bigcup_{(i_1,\dots,i_r)\in\mathcal{I}}\mathbb{C}\mathrm{P}^{i_1}\times\cdots\times\mathbb{C}\mathrm{P}^{i_r}\subset DJ(\mathcal{K}_\Sigma)$, where $\mathcal{I}=\{(i_1,\dots,i_r)\mid i_j\in\{0,\infty\},\ \mathrm{card}(\{i_j\mid 1\leq j\leq r, i_j=\infty\})=q_\Sigma\}$. Hence, $DJ(\mathcal{K}_\Sigma)$ contains the $2q_\Sigma$-skeleton of $(\mathbb{C}\mathrm{P}^\infty)^r$. Since $(\mathbb{C}\mathrm{P}^\infty)^r$ has no odd dimensional cells, $DJ(\mathcal{K}_\Sigma)$ contains the $(2q_\Sigma+1)$-skeleton of $(\mathbb{C}\mathrm{P}^\infty)^r$. Thus the pair $((\mathbb{C}\mathrm{P}^\infty)^r, DJ(\mathcal{K}_\Sigma))$ is $(2q_\Sigma+1)$-connected. Hence, by Lemma 5.3, $\mathcal{Z}_{\mathcal{K}_\Sigma}$ is $2q_\Sigma$-connected and so is $U(\mathcal{K}_\Sigma)$. □

Remark 5.5. The assertion of Lemma 5.4 holds even if Σ in not complete.

6. The proof of the main result.

In this section, we give the proofs of Theorem 4.6 and the main result (Theorem 1.5) by using Lemma 5.4.

Definition 6.1. Define a map $\iota_D : A_D(m,X_\Sigma;g)\to\Omega^m(\mathbb{C}^r\setminus Z_\Sigma)$ by

$$\iota_D(f_1,\dots,f_r)(\boldsymbol{x})=(f_1(\boldsymbol{x}),\dots,f_r(\boldsymbol{x}))\quad\text{for } \boldsymbol{x}\in S^m. \tag{6.1}$$

Consider the natural toric morphism $p_\Sigma:\mathbb{C}^r\setminus Z_\Sigma\to X_\Sigma$. By [**4**, (8.6)] and [**20**, Proposition 6.7], we see that there is an isomorphism $G_\Sigma\cong\mathbb{T}^{r-n}$ and that the group G_Σ acts on $U(\mathcal{K}_\Sigma)$ freely. Hence, we have a fibration sequence

$$\mathbb{T}^{r-n}\longrightarrow U(\mathcal{K}_\Sigma)=\mathbb{C}^r\setminus Z_\Sigma\xrightarrow{p_\Sigma}X_\Sigma. \tag{6.2}$$

Let $\gamma_m:S^m\to\mathbb{R}\mathrm{P}^m$ be the double covering and $\gamma_m^\#:\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m,X_\Sigma)\to\Omega^mX_\Sigma$ the map $\gamma_m^\#(f)=f\circ\gamma_m$. Assume that $m\geq 2$, and consider the commutative diagram:

$$\begin{CD}
A_D(m,X_\Sigma;g) @>{i'_D}>> F(\mathbb{R}\mathrm{P}^m,X_\Sigma;g) @>{i'}>{\subset}> \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m,X_\Sigma)\\
@V{\iota_D}VV @. @|\\
\Omega^mU(\mathcal{K}_\Sigma) @>{\Omega^mp_\Sigma}>{\simeq}> \Omega^mX_\Sigma @<{\gamma_m^\#}<< \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m,X_\Sigma)
\end{CD} \tag{6.3}$$

Let $D_*(d_1,\dots,d_r;m)$ denote the positive integer defined by

$$D_*(d_1,\dots,d_r;m)=(2r_{\min}-m-1)\Bigl(\Bigl\lfloor\frac{d_{\min}+1}{2}\Bigr\rfloor+1\Bigr)-1 \tag{6.4}$$

where $\lfloor x\rfloor$ denotes the integer part of a real number x.

Theorem 6.2. *If $1\leq m\leq 2(r_{\min}-1)$, the map $\iota_D:A_D(m,X_\Sigma;g)\to\Omega^mU(\mathcal{K}_\Sigma)$ is a homology equivalence through dimension $D_*(d_1,\dots,d_r;m)$.*

Remark 6.3. The assertion of Theorem 6.2 holds even if the condition (1.7.2) is not satisfied.

We postpone the proof of Theorem 6.2 to Section 7, and complete the proofs of

Theorem 4.6 and Theorem 1.5 by assuming it.

PROOF OF THEOREM 4.6. We set $D_* = D_*(d_1, \dots, d_r; m)$ and let $\mathbb{F}$ denote the field $\mathbb{Z}/p$ (p: prime) or $\mathbb{Q}$. Since $m \geq 2$, by (6.2) the map $\Omega^m p_\Sigma$ is a homotopy equivalence. Then, from the diagram (6.3) and Theorem 6.2, we see that the map $\gamma_m^{\#} \circ i'$ induces an epimorphism on homology groups $H_k(\ ,\mathbb{F})$ for any $k \leq D_*$. However, since there is a homotopy equivalence $F(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma$, $\dim_{\mathbb{F}} H_k(F(\mathbb{R}\mathrm{P}^m, X_\Sigma; g), \mathbb{F}) = \dim_{\mathbb{F}} H_k(\Omega^m X_\Sigma, \mathbb{F}) < \infty$ for any k. Therefore, the map $\gamma_m^{\#} \circ i'$ induces an isomorphism on $H_k(\ ,\mathbb{F})$ for any $k \leq D_*$. Thus, from the diagram (6.3), we see that so does the map i'_D. It then follows from the universal coefficient theorem that the map i'_D is a homology equivalence through dimension $D_* = D_*(d_1, \dots, d_r; m)$. Because $\lim_{k\to\infty} D_*(d_1 + ka_1, \dots, d_r + ka_r; m) = \infty$, the diagram (4.6), implies that the map $s_{D,\infty}$ is a homology equivalence. □

PROOF OF THEOREM 1.5. The assertion easily follows from (4.6), Theorem 4.6 and Theorem 4.8. □

7. The Vassiliev spectral sequence.

Spectral sequences induced from the Veronese simplicial resolution.
Let Z_D be the tautological normalization of Σ_D, and let $\pi_D : Z_D \to \Sigma_D$ denote the first projection as in (ii) of Definition 3.1. Let $(\mathcal{Z}^D, \pi_D : \mathcal{Z}^D \to \Sigma_D)$ denote the (degenerate) simplicial resolution of the surjective map $\pi_D : Z_D \to \Sigma_D$ defined from the (generalized) Veronese embedding as in [**2**, p. 782]. We have the following natural filtration

$$\phi = \mathcal{Z}_0^D \subset \mathcal{Z}_1^D = \Sigma_D \subset \mathcal{Z}_2^D \subset \mathcal{Z}_3^D \subset \cdots \subset \bigcup_{k\geq 1} \mathcal{Z}_k^D = \mathcal{Z}^D.$$

By the same method as in (3.2) and (3.3), we obtain a spectral sequence

$$\left\{\hat{E}^t_{k,s},\ \hat{d}^t : \hat{E}^t_{k,s} \to \hat{E}^t_{k+t,s+t-1}\right\} \quad \Rightarrow \quad H_{s-k}(A_D(m, X_\Sigma; g), \mathbb{Z}) \tag{7.1}$$

such that $\hat{E}^1_{k,s} = \tilde{H}_c^{2N_D+k-s-1}(\mathcal{Z}_k^D \setminus \mathcal{Z}_{k-1}^D, \mathbb{Z})$.

LEMMA 7.1. (i) *If $1 \leq k \leq \lfloor (d_{\min}+1)/2 \rfloor$, there is a natural isomorphism*

$$\hat{E}^1_{k,s} \cong \tilde{H}_c^{2rk-s}(C_k, \pm\mathbb{Z}).$$

(ii) *If $r < 0$ or $s < 0$ or $s \leq (2r_{\min} - m)k - 1$, then $\hat{E}^1_{k,s} = 0$.*

PROOF. (i) By using the same argument as in the proof of Lemma 3.4 and [**2**, Lemma 4.4 and Lemma 4.6], we can show that $\mathcal{Z}_k^D \setminus \mathcal{Z}_{k-1}^D$ is an open disk bundle over C_k with rank $l_{D,k}$ if $1 \leq k \leq \lfloor (d_{\min}+1)/2 \rfloor$. (Note that the projection $\pi_k : \mathcal{Z}_k^D \setminus \mathcal{Z}_{k-1}^D \to C_k$ is well-defined only if $k \leq \lfloor (d_{\min}+1)/2 \rfloor$, because the condition (i) of $(2.1)_k$ is not satisfied for $k > \lfloor (d_{\min}+1)/2 \rfloor$.) We can now prove the assertion (i) in exactly the same way as

Lemma 3.3.

(ii) It suffices to show that $\hat{E}^1_{k,s} = 0$ if $s \leq (2r_{\min} - m)k - 1$ when $k, s \geq 0$. In general,

$$\begin{aligned}\dim(\mathcal{Z}^D_k \setminus \mathcal{Z}^D_{k-1}) &\leq (2N_D - 2rk) + \dim C_k + (k-1)\\ &= (2N_D - 2rk) + (\dim Z_\Sigma + m)k + (k-1)\\ &= 2N_D - (2r_{\min} - m)k + k - 1.\end{aligned}$$

Since $2N_D + k - 1 - s > 2N_D - (2r_{\min} - m)k + k - 1 \Leftrightarrow s \leq (2r_{\min} - m)k - 1$ and $\hat{E}^1_{k,s} = \tilde{H}^{2N_D+k-s-1}_c(\mathcal{Z}^D_k \setminus \mathcal{Z}^D_{k-1}, \mathbb{Z})$, $\hat{E}^1_{k,s} = 0$ if $s \leq (2r_{\min} - m)k - 1$ and (ii) follows. □

The Vassiliev spectral sequence.

We now recall the spectral sequence constructed by V. Vassiliev [**22**, pp. 109–115].

From now on, we will assume that $m \leq N$ and that X is a finite dimensional N-connected simplicial complex C^∞-imbedded in $\mathbb{R}^L$. We regard S^m and X as subspaces $S^n \subset \mathbb{R}^{m+1}$, $X \subset \mathbb{R}^L$, respectively. We also choose and fix a map $\varphi : S^m \to X$. Observe that $\mathrm{Map}(S^m, \mathbb{R}^L)$ is a linear space and consider the complements $\mathfrak{A}_m(X) = \mathrm{Map}(S^m, \mathbb{R}^L) \setminus \mathrm{Map}(S^m, X)$ and $\tilde{\mathfrak{A}}_m(X) = \mathrm{Map}^*(S^m, \mathbb{R}^L) \setminus \mathrm{Map}^*(S^m, X)$. Note that $\mathfrak{A}_m(X)$ consists of all continuous maps $f : S^m \to \mathbb{R}^L$ intersecting $\mathbb{R}^L \setminus X$. We will denote by $\Theta^d_\varphi(X) \subset \mathrm{Map}(S^m, \mathbb{R}^L)$ the subspace consisting of all maps f of the forms $f = \varphi + p$, where p is the restriction to X of a polynomial map $S^m \to \mathbb{R}^L$ of total degree $\leq d$. Let $\Theta^d_X \subset \Theta^d_\varphi(X)$ denote the subspace consisting of all $f \in \Theta^d_\varphi(X)$ intersecting $\mathbb{R}^L \setminus X$. In [**22**, pp. 111–112] Vassiliev uses the space $\Theta^d(X)$ as a finite dimensional approximation of $\mathfrak{A}_m(X)$.[7]

Let $\tilde{\Theta}^d_X$ denote the subspace of Θ^d_X consisting of all maps $f \in \Theta^d_X$ which preserve the base points. By a variation of the preceding argument, Vassiliev also shows that $\tilde{\Theta}^d_X$ can be used as a finite dimensional approximation of $\tilde{\mathfrak{A}}_m(X)$ [**22**, p. 112].

Let $\mathcal{X}_d \subset \tilde{\Theta}^d_X \times \mathbb{R}^L$ denote the subspace consisting of all pairs $(f, \alpha) \in \tilde{\Theta}^d_X \times \mathbb{R}^L$ such that $f(\alpha) \in \mathbb{R}^L \setminus X$, and let $p_d : \mathcal{X}_d \to \tilde{\Theta}^d_X$ be the projection onto the first factor. Then, by making use of (non-degenerate) simplicial resolutions of the surjective maps $\{p_d : d \geq 1\}$, one can construct a simplicial resolution $\{\tilde{\mathfrak{A}}_m(X)\}$ of $\tilde{\mathfrak{A}}_m$, whose cohomology is naturally isomorphic to the homology of $\mathrm{Map}^*(S^m, X) = \Omega^m X$. From the natural filtration $F_1 \subset F_2 \subset F_3 \subset \cdots \subset \bigcup_{d=1}^\infty F_d = \{\tilde{\mathfrak{A}}_m(X)\}$, we obtain the associated spectral sequence (for $m \leq N$)

$$\left\{E^t_{k,s}, d^t : E^t_{k,s} \to E^t_{k+t,s+t-1}\right\} \Rightarrow H_{s-k}(\Omega^m X, \mathbb{Z}). \tag{7.2}$$

Since $U(\mathcal{K}_\Sigma)$ is $2(r_{\min}-1)$-connected (by Lemma 5.4), we may assume that $m \leq N = 2(r_{\min} - 1)$ and consider the spectral sequence (7.2) for $X = U(\mathcal{K}_\Sigma) = \mathbb{C}^r \setminus Z_\Sigma$. Note that the construction of this simplicial resolution is almost identical to that of the resolution $\mathcal{Z}^D$. The only difference between the two concerns the following two points. First,

[7] Note that the proof of this fact given by Vassiliev makes use of the Stone–Weierstrass theorem, so, although we are now not using the stable result of [**2**, Theorem 2.1], something like it is also implicitly involved here.

we use all polynomials passing through Z_Σ of total degree $\leq d$ instead of homogenous polynomials passing through Z_Σ satisfying the condition (1.7.2), for d some fixed large integer. Secondly we use a family of embeddings satisfying the condition $(2.1)_k$ instead of a fixed embedding. However, since $\hat{E}^1_{k,s}$ is determined independently of the choice of embeddings if d is sufficiently large, one can prove the following result by using the same method as in the case of the Veronese resolution.

LEMMA 7.2 ([**22**]). *If* $1 \leq m \leq 2(r_{\min} - 1)$*, there is a spectral sequence*

$$\left\{E^t_{k,s}, d^t : E^t_{k,s} \to E^t_{k+t,s+t-1}\right\} \Rightarrow H_{s-k}(\Omega^m U(\mathcal{K}_\Sigma), \mathbb{Z}) \tag{7.3}$$

satisfying the following two conditions:

(i) *If* $k \geq 1$, $E^1_{k,s} = \tilde{H}^{2rk-s}_c(C_k, \pm\mathbb{Z})$.
(ii) *If* $k < 0$ *or* $s < 0$ *or* $s \leq (2r_{\min} - m)k - 1$, *then* $E^1_{k,s} = 0$.

Now we can give the proof of Theorem 6.2.

PROOF OF THEOREM 6.2. Consider the spectral sequences (7.1) and (7.3). Note that the image of the map ι_D lies in a space of mappings that arise from restrictions of polynomial mappings $\mathbb{R}^{m+1} \to \mathbb{C}^r = \mathbb{R}^{2r}$. Since $\mathcal{X}^D$ is a non-degenerate simplicial resolution, the map ι_D naturally extends to a filtration preserving map $\tilde{\pi} : \mathcal{X}^D \to \{\tilde{\mathfrak{A}}_m(U(\mathcal{K}_\Sigma))\}$ between resolutions. By [**14**, Lemma 2.2] there is a filtration preserving homotopy equivalence $q^\Delta : \mathcal{X}^D \xrightarrow{\simeq} \mathcal{Z}^D$. The filtration preserving maps $\mathcal{Z}^D \xleftarrow[\simeq]{q^\Delta} \mathcal{X}^D \xrightarrow{\tilde{\pi}} \{\tilde{\mathfrak{A}}_m(U(\mathcal{K}_\Sigma))\}$ induce a homomorphism of spectral sequences $\{\theta^t_{k,s} : \hat{E}^t_{k,s} \to E^t_{k,s}\}$, where

$$\{\hat{E}^t_{k,s}, \hat{d}^t\} \Rightarrow H_{s-k}(A_D(m, X_\Sigma; g), \mathbb{Z}) \quad \text{and} \quad \{E^t_{k,s}, d^t\} \Rightarrow H_{s-k}(\Omega^m U(\mathcal{K}_\Sigma), \mathbb{Z}).$$

Then by the naturality of the Thom isomorphism and the argument used in the proof of Lemma 4.7, we can show that $\theta^1_{k,s} : \hat{E}^1_{k,s} \xrightarrow{\cong} E^1_{k,s}$ is an isomorphism for any s as long as $k \leq \lfloor (d_{\min} + 1)/2 \rfloor$. By Lemmas 7.1 and 7.2, we see that $\theta^\infty_{k,s} : \hat{E}^\infty_{k,s} \xrightarrow{\cong} E^\infty_{k,s}$ is always an isomorphism for any s if $k \leq \lfloor (d_{\min} + 1)/2 \rfloor$. Now, consider the positive integer $D_{\min}$:

$$D_{\min} = \min\left\{N \in \mathbb{Z}_{\geq 1} \mid N \geq s - k,\ s \geq (2r_{\min} - m)k,\ 1 \leq k < \left\lfloor \frac{d_{\min} + 1}{2} \right\rfloor + 1\right\}.$$

Clearly $D_{\min}$ is the largest integer N which satisfies the inequality $(2r_{\min} - m)k > k + N$ for $k = \lfloor (d_{\min} + 1)/2 \rfloor + 1$. Hence, $D_{\min} = (2r_{\min} - m - 1)(\lfloor (d_{\min} + 1)/2 \rfloor + 1) - 1 = D_*(d_1, \ldots, d_r; m)$. We note that, for dimensional reasons, $\theta^\infty_{k,s} : \hat{E}^\infty_{k,s} \xrightarrow{\cong} E^\infty_{k,s}$ is always an isomorphism when $k \leq \lfloor (d_{\min} + 1)/2 \rfloor$ and $s - k \leq D_*(d_1, \ldots, d_r; m)$. Note also that by Lemma 7.1 and Lemma 7.2, $\hat{E}^1_{k,s} = E^1_{k,s} = 0$ when $s - k \leq D_*(d_1, \ldots, d_r; m)$ and $k > \lfloor (d_{\min} + 1)/2 \rfloor$. Hence, we see that $\theta^\infty_{k,s} : \hat{E}^\infty_{k,s} \xrightarrow{\cong} E^\infty_{k,s}$ is always an isomorphism if $s \leq k + D_*(d_1, \ldots, d_r; m)$. Thus, it follows from the Comparison Theorem of spectral sequences that the map ι_D is a homology equivalence through dimension $D_*(d_1, \ldots, d_r; m)$. $\square$

8. Some facts and conjectures.

A basic lemma.

The aim of the first part of this section is to provide a simple basic lemma used in this paper for which we do not know any convenient reference.

Lemma 8.1. *Let K be a CW complex and $X = K \cup_f e^m$ with* $\dim K < m$. *For* $g \in \mathrm{Map}^*(K, Y)$, *let $F(X, Y; g)$ denote the space given by* $F(X, Y; g) = \{h \in \mathrm{Map}^*(X, Y) : h|K = g\}$. *If* $F(X, Y; g) \neq \emptyset$, *there is a homotopy equivalence* $F(X, Y; g) \simeq \Omega^m Y$.

Proof. By using the characteristic map of the top cell in X, $F(X, Y; g)$ can be identified with the space of all based maps $h : D^m \to Y$ which restrict to the same fixed map on the boundary S^{m-1}, and it can be regarded as the fiber of the map $r : \mathrm{Map}^*(D^m, Y) \to \mathrm{Map}^*(S^{m-1}, Y) = \Omega^{m-1}Y$ given by $r(h) = h|S^{m-1}$. Since r is a fibration with fiber $\Omega^m Y$ and $\mathrm{Map}^*(D^m, Y)$ is contractible, there is a homotopy equivalence $F(X, Y; g) \simeq \Omega^m Y$. $\square$

Minimal degree of algebraic maps.

In the second part of this section we define the minimal degree of an algebraic map $\mathbb{R}\mathrm{P}^m$ to X_Σ. It plays no role in the current paper but we think it is of sufficient independent interest and hope to make use of it in the future. First, we give the proof of Proposition 1.3 stated in Section 1.

Proof of Proposition 1.3. Since $f : \mathbb{R}\mathrm{P}^m \to X_\Sigma$ is an algebraic map, there is a Zariski open subset U of $\mathbb{C}\mathrm{P}^m$ such that U contains the set of $\mathbb{R}$-valued points of $\mathbb{R}\mathrm{P}^m$, which we also denote by $\mathbb{R}\mathrm{P}^m$ by abuse of notation, and there is a regular map $\varphi : U \to X_\Sigma$ such that $f = \varphi|_{\mathbb{R}P^m}$. Without loss of generality we may assume that U is the largest open subset of $\mathbb{C}\mathrm{P}^m$ where f is defined. Then $\mathbb{C}\mathrm{P}^m \setminus U$ has codimension at least two in $\mathbb{C}\mathrm{P}^m$, since X_Σ is proper and $\mathbb{C}\mathrm{P}^m$ is normal. The following proof is almost the same as that of [**6**, Theorem 3.1], but in our case, the map φ is defined only on the Zariski open subset U of $\mathbb{C}\mathrm{P}^m$, which requires some modifications. Let $\pi : \mathbb{C}^{m+1} \setminus \{\mathbf{0}\} \to \mathbb{C}\mathrm{P}^m$ be the Hopf fibering, and set $\tilde{U} = \pi^{-1}(U)$. We also denote by π its restriction $\tilde{U} \to U$, and let $p_\Sigma : \mathbb{C}^r \setminus Z_\Sigma \to X_\Sigma$ be the natural toric morphism as in (6.2). We shall show the existence of D and $(f_1, \dots, f_r)$.

Let $(\mathcal{O}_{X_\Sigma}(D_\rho), \iota_\rho, c_{\chi^{\boldsymbol{m}}})_{\rho \in \Sigma(1), \boldsymbol{m} \in \mathbb{Z}^n}$ denote the universal Σ-collection defined in [**6**, p. 252] and let $(L_\rho, u_\rho, c_{\boldsymbol{m}})_{\rho \in \Sigma(1), \boldsymbol{m} \in \mathbb{Z}^n}$ be its pull back by φ. Since φ is a regular map, L_ρ is an algebraic line bundle on U. Moreover, since $\mathbb{C}\mathrm{P}^m \setminus U$ has codimension at least two in $\mathbb{C}\mathrm{P}^m$, L_ρ can be extended to a line bundle on $\mathbb{C}\mathrm{P}^m$ which is isomorphic to $\mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d_\rho)$ for some integer d_ρ. This isomorphism induces an isomorphism $H^0(U, L_\rho) \cong H^0(\mathbb{C}\mathrm{P}^m, \mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d_\rho))$. Let $f'_\rho \in H^0(\mathbb{C}\mathrm{P}^m, \mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d_\rho))$ denote the element corresponding to $u_\rho \in H^0(U, L_\rho)$ by the above isomorphism. The isomorphisms $c_{\boldsymbol{m}}$ on U can be extended to those on $\mathbb{C}\mathrm{P}^m$ which induce isomorphisms $c'_{\boldsymbol{m}} : \bigotimes_\rho \mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d_\rho)^{\langle \boldsymbol{m}, \boldsymbol{n}_\rho \rangle} \cong \mathcal{O}_{\mathbb{C}\mathrm{P}^m}$ on $\mathbb{C}\mathrm{P}^m$. In terms of (1.7.1), this implies that $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}$. A collection $(\mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d_\rho), f'_\rho, c'_{\boldsymbol{m}})$ is not necessarily a Σ-collection on $\mathbb{C}\mathrm{P}^m$ because sections f'_ρ do not satisfy the non-degeneracy condition outside U. However its restriction $(\mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d_\rho), f'_\rho, c'_{\boldsymbol{m}})|_U$ to U is a Σ-collection on U. For each

$\boldsymbol{m} \in \mathbb{Z}^n$, we have a canonical isomorphism $c^{\mathrm{can}}_{\boldsymbol{m}} : \bigotimes_\rho \mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d_\rho)^{\langle \boldsymbol{m}, \boldsymbol{n}_\rho \rangle} \xrightarrow{\cong} \mathcal{O}_{\mathbb{C}\mathrm{P}^m}$ as in the proof of [**6**, Theorem 3.1]. Then, as in the final paragraph of the proof of [**6**, Theorem 3.1], by taking $\lambda_p \in \mathbb{C}$ suitably and setting $f_\rho = \lambda_\rho f'_\rho$, we have an equivalence $(\mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d_\rho), f'_\rho, c'_{\boldsymbol{m}})|_U \sim (\mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d_\rho), f_\rho, c^{\mathrm{can}}_{\boldsymbol{m}})|_U$ of Σ-collection on U. Now define a morphism $F : \tilde{U} \to \mathbb{C}^r \setminus Z_\Sigma$ by $F(\boldsymbol{x}) = (f_\rho(\boldsymbol{x}))_{\rho \in \Sigma(1)}$ for $\boldsymbol{x} \in \tilde{U}$. Note that the non-degeneracy of f_ρ on U ensures that $(f_\rho(\boldsymbol{x}))_{\rho \in \Sigma(1)} \notin Z_\Sigma$ for all $\boldsymbol{x} \in \tilde{U}$. Since the pull back of $(\mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d_\rho), f_\rho, c^{\mathrm{can}}_{\boldsymbol{m}})|_U$ to $\tilde{U}$ is $(\mathcal{O}_{\tilde{U}}, f_\rho, 1)$, we see that $\varphi \circ \pi = p_\Sigma \circ F$. By reindexing $(f_\rho)_{\rho \in \Sigma(1)}$ as $(f_1, \dots, f_r)$ in accordance with (1.7.1) and setting $D = (d_1, \dots, d_r)$, we obtain an element $(f_1, \dots, f_r) \in A_{D,\Sigma}(m)$. Since $\mathbb{C}\mathrm{P}^m \setminus U$ has codimension at least two in $\mathbb{C}\mathrm{P}^m$, we see that $\dim_{\mathbb{C}} pr_1^{-1}(F) < m$, where $F = (f_1, \dots, f_r)$ and pr_1 is the projection as in Section 1. Therefore, we have $(f_1, \dots, f_r) \in A_{D,\Sigma}(m)^\circ$.

Suppose that $(h_1, \dots, h_r) \in A_{D',\Sigma}(m)^\circ$ also represents the same algebraic map f for some r-tuple $D' = (d'_1, \dots, d'_r) \in \mathbb{Z}^r$ such that $\sum_{k=1}^r d'_k \boldsymbol{n}_k = \boldsymbol{0}$. Set $H = (h_1, \dots, h_r)$ and $V = \mathbb{C}\mathrm{P}^m \setminus \pi(pr_1^{-1}(H))$. Then $V \subset U$ and $(\mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d_\rho), f_\rho, c^{\mathrm{can}}_{\boldsymbol{m}})|_V \sim (\mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d'_\rho), h_\rho, c^{\mathrm{can}}_{\boldsymbol{m}})|_V$. Since $\mathbb{C}\mathrm{P}^m \setminus V$ has codimension at least two in $\mathbb{C}\mathrm{P}^m$, the isomorphism $\mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d_\rho)|_V \cong \mathcal{O}_{\mathbb{C}\mathrm{P}^m}(d'_\rho)|_V$ is given by a nonzero constant $\mu_\rho \in \mathbb{C}$. Therefore $d_\rho = d'_\rho$ for all $\rho \in \Sigma(1)$ and we have $D = D'$. Moreover, as in the proof of [**6**, Theorem 3.1], this implies that $(\mu_1, \dots, \mu_r) \in G_\Sigma$ and $(h_1, \dots, h_r) = (\mu_1 f_1, \dots, \mu_r f_r)$. □

By Proposition 1.3, we can define the minimal degree of an algebraic map as follows.

Definition 8.2. Let $D = (d_1, \dots, d_r) \in \mathbb{Z}^r$ and let $f : \mathbb{R}\mathrm{P}^m \to X_\Sigma$ be an algebraic map. Then the map f is called an algebraic map *of minimal degree* D if it can be represented as $f = j'_D(f_1, \dots, f_r) = [f_1, \dots, f_r]$ for some $(f_1, \dots, f_r) \in A_{D,\Sigma}(m)^\circ$.

Note that the minimal degree depends only on the map itself and not on its representative.

Remark 8.3. (i) We denote by $\mathrm{Alg}_{D,\min}(\mathbb{R}\mathrm{P}^m, X_\Sigma) \subset \mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ the subspace consisting of all algebraic maps $f : \mathbb{R}\mathrm{P}^m \to X_\Sigma$ of minimal degree D. Since $A_{D,\Sigma}(m)^\circ$ is a G_Σ-invariant subspace of $A_{D,\Sigma}(m)$, let $\widetilde{A_D}(m, X_\Sigma)^\circ$ denote the orbit space $\widetilde{A_D}(m, X_\Sigma)^\circ = A_{D,\Sigma}(m)^\circ / G_\Sigma$. Then we can easily see that there is a homeomorphism

$$\widetilde{A_D}(m, X_\Sigma)^\circ \xrightarrow{\cong} \mathrm{Alg}_{D,\min}(\mathbb{R}\mathrm{P}^m, X_\Sigma). \tag{8.1}$$

(ii) Let $\boldsymbol{a} = (a_1, \dots, a_r) \in (\mathbb{Z}_{\geq 1})^r$ and suppose that $\sum_{k=1}^r a_k \boldsymbol{n}_k = \boldsymbol{0}$. Then if we set $\tilde{g} = \sum_{k=1}^r z_k^2$ and $f = [f_1, \dots, f_r] \in \mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma)$, then we can easily see that $f = [\tilde{g}^{a_1} f_1, \dots, \tilde{g}^{a_r} f_r] \in \mathrm{Alg}_{D+\boldsymbol{a}}(\mathbb{R}\mathrm{P}^m, X_\Sigma)$. Hence, the space $\mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ can be identified with the subspace of $\mathrm{Alg}_{D+\boldsymbol{a}}(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ by

$$\mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \xrightarrow{\subseteq} \mathrm{Alg}_{D+\boldsymbol{a}}(\mathbb{R}\mathrm{P}^m, X_\Sigma); \qquad [f_1, \dots, f_r] \mapsto [f_1 \tilde{g}^{a_1}, \dots, f_r \tilde{g}^{a_r}].$$

Note that $\mathrm{Map}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) = \mathrm{Map}_{D'}(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ may happen even if $D \neq D'$. However, $\mathrm{Alg}_{D,\min}(\mathbb{R}\mathrm{P}^m, X_\Sigma) \cap \mathrm{Alg}_{D',\min}(\mathbb{R}\mathrm{P}^m, X_\Sigma) = \emptyset$ if $D \neq D'$.

(iii) It may happen that $d_k < 0$ for some k; the section $f_k \in H^0(\mathbb{C}\mathrm{P}^m, \mathcal{O}(d_k))$ will be zero if $d_k < 0$ (see Example 8.4).

EXAMPLE 8.4. Let $H(k)$ and $\{\boldsymbol{n}_k\}_{k=1}^4$ be the Hirzebruch surface and its primitive generators as in Example 1.1. Suppose that $(d_1, d_2, d_3, d_4) = (1, -k, 1, 0)$. Then we have $\sum_{j=1}^4 d_j \boldsymbol{n}_j = \boldsymbol{n}_1 - k\boldsymbol{n}_2 + \boldsymbol{n}_3 = \mathbf{0}$. Let $k \geq 0$ and we may regard $H(k)$ as the orbit space

$$H(k) = \{(y_1, y_2, y_3, y_4) \in \mathbb{C}^4 \mid (y_1, y_3) \neq (0,0),\ (y_2, y_4) \neq (0,0)\}/G_\Sigma,$$

where $G_\Sigma = \{(\mu_1, \mu_2, \mu_1, \mu_1^k \mu_2) \mid \mu_1, \mu_2 \in \mathbb{C}^*\}$ as in (1.7). Now consider the algebraic map $f : \mathbb{RP}^1 \to H(k)$ defined by $f([x_0 : x_1]) = [(x_0^2 + x_1^2)x_0, 0, (x_0^2 + x_1^2)x_1, 1]$. Then clearly $f \in \mathrm{Alg}_{(3,-3k,3,0)}(\mathbb{RP}^1, H(k))$. However, by the G_Σ-action, we have $f([x_0 : x_1]) = [x_0, 0, x_1, 1]$. Hence $f \in \mathrm{Alg}_{(1,-k,1,0),\min}(\mathbb{RP}^1, H(k))$. The map f is also the limit of the family of algebraic maps of minimal degree $(3, -3k, 3, 0)$. Indeed it is the limit of a family $f_t([x_0 : x_1]) = [(x_0^2 + x_1^2)x_0, 0, (x_0^2 + tx_1^2)x_1, 1]$ for $t > 1$.

Conjectures concerning spaces of algebraic maps.

The purpose of the third part of this section is to formally state the analogues of Theorem 1.5 and Corollary 1.7 concerning approximation of spaces of continuous maps by algebraic maps. As explained in the introduction, the analogous results are true in the complex case.

CONJECTURE 8.5 (cf. [**2**, Conjecture 3.8]). *Under the same assumptions as Theorem* 1.5, *the natural projection maps*

$$\begin{cases} \Psi'_D : A_D(m, X_\Sigma; g) \to \mathrm{Alg}^*_D(\mathbb{RP}^m, X_\Sigma; g), \\ \Psi_D : A_D(m, X_\Sigma) \to \mathrm{Alg}^*_D(\mathbb{RP}^m, X_\Sigma), \\ \Gamma_D : \widetilde{A_D}(m, X_\Sigma) \to \mathrm{Alg}_D(\mathbb{RP}^m, X_\Sigma) \end{cases}$$

are homotopy equivalences.

We strongly believe that Conjecture 8.5 is true. As we have mentioned before, the natural projection Ψ_D has contractible fibers. If Ψ_D is a quasi-fibration or satisfies the condition of the Vietoris-Begle theorem or has some other property of this kind, then it must be a homotopy (or at least homology) equivalence. We proved this for the simplest case in [**15**], but the general case seems difficult as the topology of the quotient space $\mathrm{Alg}^*_D(\mathbb{RP}^m, X_\Sigma)$ is complicated.

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