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Classes of weights and second order Riesz transforms associated to Schrödinger operators

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Abstract. We consider the Schrödinger operator $-\Delta + V$ on $\mathbb{R}^n$ with $n \geq 3$ and V a member of the reverse Hölder class $\mathcal{B}_s$ for some $s > n/2$. We obtain the boundedness of the second order Riesz transform $\nabla^2(-\Delta + V)^{-1}$ on the weighted spaces $L^p(w)$ where w belongs to a class of weights related to V. To prove this, we develop a good-λ inequality adapted to this setting along with some new heat kernel estimates.

1. Introduction and statement of main result.

We consider the Schrödinger operator on $\mathbb{R}^n$ with $n \geq 3$ given by

$$L = -\Delta + V$$

where V is a non-negative and locally integrable function on $\mathbb{R}^n$ satisfying, for some $s > n/2$, the reverse Hölder inequality

$$\left(\frac{1}{|B|} \int_B V^s \right)^{1/s} \leq \frac{C}{|B|} \int_B V \tag{1.1}$$

for all balls $B \subset \mathbb{R}^n$. We shall denote the set of all locally integrable functions that satisfy (1.1) by $\mathcal{B}_s$, the class of reverse Hölder weights of order s.

The seminal paper by Z. Shen [**26**] investigated the operator L in a systematic way, obtaining amongst other things, the $L^p(\mathbb{R}^n)$ estimates for Riesz transforms associated to L. Shen's article has since inspired a host of work on the harmonic analysis related to L. We give a partial list [**1**], [**4**], [**8**], [**10**], [**11**], [**14**], [**16**], [**19**] and refer the reader to their references for further information.

In this article we are interested in a new class of weights introduced by Bongioanni, Harboure, and Salinas in [**10**]. Our main goal is to derive a set of good-λ inequalities (modelled on the work in [**6**]) for these classes of weights, and then show how these inequalities may be applied to obtain the weighted estimates for the second order Riesz transform $\nabla^2 L^{-1}$.

These weight classes $\mathcal{A}_p^L$ are defined in the following manner. Given $p > 1$ we say that $w \in \mathcal{A}_p^L$ if there exists $\theta \geq 0$ and $C > 0$ such that

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$$\left(\frac{1}{|B|}\int_B w\right)^{1/p}\left(\frac{1}{|B|}\int_B w^{1-p'}\right)^{1/p'} \le C\left(1+\frac{r}{\rho(x)}\right)^{\theta} \tag{1.2}$$

for every ball $B = B(x, r)$. The function ρ is the 'critical radius' function introduced by Shen [**26**] and for a definition see (2.1) below. Observe that when $\theta = 0$ these weights coincide with the Muckenhoupt classes $\mathcal{A}_p$, but in general they form a larger class of weights. One can also define reverse Hölder type weights $\mathcal{B}_q^L$ in an analogous fashion. See Definitions 3.1 and 3.2.

Subsequently these weight classes were further investigated in [**9**], [**11**], [**32**], [**33**], [**34**]. In our article we are focused on the weighted estimates of the following Riesz transforms

$$\text{First order:}\quad \nabla L^{-1/2},\quad V^{1/2}L^{-1/2}$$
$$\text{Second order:}\quad \nabla^2 L^{-1},\quad VL^{-1}$$

Before stating our main result we summarize what is known for both the weighted and unweighted estimates of these transforms. In the following and throughout the rest of this article, the notation s^* will denote $ns/(n-s)$ if $s < n$, and will denote ∞ if $s \ge n$.

(i) We first describe the unweighted estimates.

Table 1. Intervals of boundedness on $L^p(\mathbb{R}^n)$.

Operator:	$V^{1/2}L^{-1/2}$	$\nabla L^{-1/2}$	VL^{-1}	$\nabla^2 L^{-1}$
$V \ge 0$	$(1, 2]$	$(1, 2]$	$p = 1$	Weak $(1, 1)$
$V \in \mathcal{B}_s$	$(1, 2s]$	$(1, s^*]$	$(1, s]$	$(1, s]$

The first row describes the situation when V is a non-negative and locally integrable function. In this situation it is known that the operators $\nabla L^{-1/2}$ and $V^{1/2}L^{-1/2}$ are bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, 2]$ (see [**27**], [**13**]). The operators ΔL^{-1} and VL^{-1} are bounded on $L^1(\mathbb{R}^n)$ (see [**4**]), and hence $\nabla^2 L^{-1}$ is of weak-type $(1, 1)$.
The second row of Table 1 specializes to the situation where V belongs to a reverse Hölder $\mathcal{B}_s$ class for $s > n/2$ and $n \ge 3$ and the boundedness of the above Riesz transforms were obtained by Shen [**26**]. See also [**4**] where this is proved for all $n \ge 1$ and $s > 1$.

(ii) We turn to the weighted estimates.

Table 2. Weighted estimates for $V \in \mathcal{B}_s$ with $s \ge n/2$ and $n \ge 3$.

Operator:	$V^{1/2}L^{-1/2}$	$\nabla L^{-1/2}$	VL^{-1}	$\nabla^2 L^{-1}$
$w \in \mathcal{A}_\infty$	$\mathcal{W}_w(1, 2s)$	$\mathcal{W}_w(1, s^*)$	$\mathcal{W}_w(1, s)$	$\mathcal{W}_w(1, s)$
$w \in \mathcal{A}_\infty^L$	$\mathcal{W}_w^L(1, 2s)$	$\mathcal{W}_w^L(1, s^*)$	$\mathcal{W}_w^L(1, s)$	?

Here we use the notation introduced in [**6**]:

$$\mathcal{W}_w(p_0, q_0) := \{p \in (p_0, q_0) : w \in \mathcal{A}_{(p/p_0)} \cap \mathcal{B}_{(q_0/p)'}\}.$$

One can define the sets of exponents $\mathcal{W}_w^L(p_0, q_0)$ analogously for the weights in (1.2).

Table 2 describes the weighted estimates for the case when $V \in \mathcal{B}_s$ with $s \geq n/2$ and $n \geq 3$. For the more general setting of non-negative potentials see [**2**], [**29**]. The first row in Table 2 gives the $L^p(w)$ estimates with w a Muckenhoupt weight and the earliest result in this direction [**22**] was for the operator VL^{-1}. The second row depicts estimates for the $\mathcal{A}_\infty^L$ weights and was obtained for $\nabla L^{-1/2}$ in [**10**], and for $V^{1/2}L^{-1/2}$, VL^{-1} in [**34**]. Since the $\mathcal{A}_\infty^L$ classes contain the $\mathcal{A}_\infty$ classes, these estimates imply the corresponding estimates for the latter weight classes in the first row.

We remark that by standard Calderón–Zygmund theory, the estimates on $L^p(\mathbb{R}^n)$ and $L^p(w)$ with $\mathcal{A}_\infty$ weights for the operator $\nabla^2 L^{-1}$ follow from that of VL^{-1}. However this is not automatically true for the $\mathcal{A}_\infty^L$ weights. The following theorem completes Table 2 above, and is the main result of our article.

Theorem 1.1. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 3$ and $V \in \mathcal{B}_s$ for some $s \geq n/2$. If $w \in \mathcal{A}_\infty^L$ and $p \in \mathcal{W}_w^L(1, s)$ then $\nabla^2 L^{-1}$ is bounded on $L^p(w)$.*

Our techniques are based on good-λ inequalities. These inequalities form an important part of classical harmonic analysis (see [**31**] or [**17**]). Unfortunately, the operators considered in our context are beyond the Calderón–Zygmund class of operators that are treated in say [**31**] and hence there is a need to adapt these methods to the other situations. This adaptation is done in [**3**], [**5**], [**6**], [**23**] and we refer the reader to these works for a more complete description and historical references. Our work here is also largely inspired by [**6**] and part of the motivation for this work is extend the machinery developed there to classes of weights beyond the Muckenhoupt setting. We do this for weights in $\mathcal{A}_p^L$ in Theorem 3.9 below, and this allows us to handle operators such as $\nabla^2 L^{-1}$ that can not be treated by techniques in [**10**], [**32**], [**33**], [**34**].

The other tool we need are estimates on the heat kernel of L. Let e^{-tL} and $p_t(x, y)$ be the heat semigroup and heat kernel associated to L respectively. It is now well known that $p_t(x, y)$ admits stronger decay than the Gaussian for large times (see [**15**], [**21**] and Proposition 2.3 below). Our approach in this article is to study the Riesz transform $\nabla^2 L^{-1}$ through the formula

$$\nabla^2 L^{-1} = \int_0^\infty \nabla^2 e^{-tL}\, dt, \tag{1.3}$$

and as such we require suitable estimates on second derivatives $\nabla_x^2 p_t(x, y)$. We obtain these estimates in Proposition 2.4 and furthermore show that the extra decay on the heat kernel mentioned above can be carried through to its derivatives. Loosely speaking this extra decay allows us to overcome the global growth in the weights defined by (1.2) when proving Theorem 1.1.

This paper is organized as follows. In Section 2 we prove the required estimates for the heat kernel $p_t(x,y)$ and the heat semigroup e^{-tL}. The main result of this section is Proposition 2.3. We turn to the $\mathcal{A}_\infty^L$ weights in Section 3 and we first gather the required definitions and properties of these weights before developing the good-λ inequalities. The key result here is Theorem 3.9. Finally in Section 3.2 we combine the heat kernel estimates with the good-λ inequalities to prove Theorem 1.1.

We gather here some notation that we will use throughout this article. We write $\int_E f$ to mean the Lebesgue integral of f over the measureable set E, and the quantity $⨍_E f$ will be used to denote $(1/|E|)\int_E f$, the average of f over E. By a 'ball B' we mean the set $B = B(x_B, r_B)$, the ball centred at x_B and radius r_B. Given a ball B and $\alpha \geq 1$ we define $\alpha B = B(x_B, \alpha r_B)$ to be the dilation of B by a factor of α. We also define the annuli $U_j(B) = 2^j B \backslash 2^{j-1}B$ for $j \geq 1$, and $U_0(B) = B$. Finally we follow the convention that the symbols C, c in a string of inequalities will mean a constant that may change over the course of the inequalities, but does not depend on the essential variables.

2. Kernel estimates.

In this section we obtain the kernel estimates needed to prove Theorem 1.1.

Let $n \geq 1$ and V be a non-negative locally integrable function on $\mathbb{R}^n$. We define the form $\mathcal{Q}_V$ by

$$\mathcal{Q}_V(u,v) := \int_{\mathbb{R}^n} \nabla u \cdot \nabla v + \int_{\mathbb{R}^n} Vuv$$

with domain $\mathcal{D}(\mathcal{Q}_V) = \big\{u \in W^{1,2}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V|u|^2 < \infty\big\}$. It is known that this form is closed and that $C_0^\infty(\mathbb{R}^n)$ is a core for the form. See [**28**]. We denote by L the self-adjoint operator associated with $\mathcal{Q}_V$, with domain

$$\mathcal{D}(L) = \left\{u \in \mathcal{D}(\mathcal{Q}_V) : \exists v \in L^2(\mathbb{R}^n),\ \text{with } \mathcal{Q}_V(u,\varphi) = \int v\varphi,\ \forall \varphi \in \mathcal{D}(\mathcal{Q}_V)\right\}.$$

We write formally $L = -\Delta + V$.

We need the following auxiliary function, first introduced by Z. Shen [**26**]: for each $x \in \mathbb{R}^n$ we define the *critical radius* associated to V at x by

$$\rho(x) := \sup\left\{r > 0 : \frac{r^2}{B(x,r)} \int_{B(x,r)} V(y)\,dy \leq 1\right\}. \tag{2.1}$$

As an example when $V(x) = |x|^2$ then $\rho(x) \sim 1/(1+|x|)$. We remark also that when $V \in \mathcal{B}_s$ with $s > n/2$ and V is not identically zero, then $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

Next we collect some properties of ρ and $V \in \mathcal{B}_s$ that were proved in [**26**].

Lemma 2.1 ([**26**, Lemma 1.4]). *Let $V \in \mathcal{B}_s$ with $s \geq n/2$. Then there exists $C_0 > 0$ and $\kappa_0 \geq 1$ with*

$$C_0^{-1}\,\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-\kappa_0}\le\rho(y)\le C_0\,\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{\kappa_0}{\kappa_0+1}}. \tag{2.2}$$

In particular if $x,y\in B(x_B,\lambda\rho(x_B))$ for some $\lambda>0$, then

$$\rho(x)\le C_\lambda\rho(y) \tag{2.3}$$

where $C_\lambda=C_0^2(1+\lambda)^{(2\kappa_0+1)/(\kappa_0+1)}$.

LEMMA 2.2 ([**26**, Lemmas 1.2 and 1.8]). *If $n\ge 1$ and $V\in\mathcal{B}_s$ for some $s>1$ then there exists $C>0$ and $\sigma>0$ such that the following holds*:

(a) *for each $\lambda>1$ and all balls B,*

$$r_B^2\fint_B V\le C\lambda^{n/s-2}(\lambda r_B)^2\fint_{\lambda B}V,$$

(b) *for all balls B satisfying $r_B\ge\rho(x_B)$,*

$$r_B^2\fint_B V\le C\bigg(\frac{r_B}{\rho(x_B)}\bigg)^{\sigma}.$$

The operator L generates a semigroup e^{-tL} on $L^2(\mathbb{R}^n)$ with integral kernel $p_t(x,y)$, which we shall refer to as the heat kernel of L. Recall that if V is non-negative and locally-integrable then the heat kernel of L admits the so-called *Gaussian upper bounds* (see p. 195 of [**24**]). However if V is a reverse Hölder potential then the heat kernel satisfies stronger bounds.

PROPOSITION 2.3 ([**21**, Theorem 1]). *Assume that $V\in\mathcal{B}_s$ with $s\ge n/2$ for $n\ge 3$, or $s>1$ for $n=2$. Then there exists $C_0,c_0,c>0$ and $\delta\in(0,1)$ such that for all $x,y\in\mathbb{R}^n$ and $t>0$,*

$$p_t(x,y)\le\frac{C_0}{t^{n/2}}e^{-c_0\frac{|x-y|^2}{t}}\,e^{-c\left(1+\frac{t}{\rho(x)^2}\right)^{\delta}}. \tag{2.4}$$

We remark that δ depends on the constant κ_0 in Lemma 2.1. Similar estimates can be found in [**15**].

The following is the main result of this section. We show that the extra decay term in (2.4) can be carried over to estimates on the derivatives. The estimates on the second derivatives in part (c) of the next result will be needed in the proof of Theorem 1.1. The estimates in parts (a) and (b) are needed to prove part (c).

PROPOSITION 2.4. *Assume $V\in\mathcal{B}_s$ with $s\ge n/2$ for $n\ge 3$ and $s>1$ for $n=2$. Let δ be the constant from (2.4). Then the following holds.*

(a) *There exists $c=c(\delta)>0$ and $c_1>0$ such that for each $k\in\mathbb{N}$ there exists $C_k>0$ satisfying*

$$\left|\frac{\partial^k}{\partial t^k}p_t(x,y)\right| \le \frac{C_k}{t^{n/2+k}}e^{-c_1\frac{|x-y|^2}{t}}e^{-c\left(1+\frac{t}{\rho(x)^2}\right)^\delta} \tag{2.5}$$

for every $x, y \in \mathbb{R}^n$, and $t > 0$.

(b) *For each $p \in [1, s^*)$ there exists $\alpha_p, C_p, c > 0$ such that for all $y \in \mathbb{R}^n$, and $t > 0$,*

$$\left(\int |\nabla_x p_t(x,y)|^p \, e^{\alpha_p \frac{|x-y|^2}{t}} \, dx\right)^{1/p} \le \frac{C_p}{t^{1/2+n/2p'}}e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^\delta}. \tag{2.6}$$

Also for each $p \in [1, 2s)$ there exists $\alpha_p, C_p, c > 0$ such that for all $y \in \mathbb{R}^n$, and $t > 0$,

$$\left(\int \left|V^{1/2}(x)p_t(x,y)\right|^p e^{\alpha_p \frac{|x-y|^2}{t}} \, dx\right)^{1/p} \le \frac{C_p}{t^{1/2+n/2p'}}e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^\delta}. \tag{2.7}$$

Note that α_p also depends on s.

(c) *For each $p \in [1, s)$ there exists $\beta_p, C_p, c > 0$ such that for all $y \in \mathbb{R}^n$, and $t > 0$,*

$$\left(\int \left|\nabla_x^2 p_t(x,y)\right|^p e^{\beta_p \frac{|x-y|^2}{t}} \, dx\right)^{1/p} \le \frac{C_p}{t^{1+n/2p'}}e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^\delta}, \tag{2.8}$$

$$\left(\int |V(x)p_t(x,y)|^p \, e^{\beta_p \frac{|x-y|^2}{t}} \, dx\right)^{1/p} \le \frac{C_p}{t^{1+n/2p'}}e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^\delta}. \tag{2.9}$$

PROOF OF PROPOSITION 2.4 (a). Our approach is to work with a holomorphic extension of the heat semigroup to an appropriate sector in the complex plane, and then evoke Cauchy's integral formula. This holomorphic extension is contained in

LEMMA 2.5 ([**16**, Corollary 6.2]). *The semigroup $\left\{e^{-tL}\right\}$ has a unique holomorphic extension on $L^2(e^{\eta|x-y|}dx)$ for every $\eta > 0$ and $y \in \mathbb{R}^n$ in the sector $\Sigma_{\pi/4} := \{\xi \in \mathbb{C} : |\arg \xi| < \pi/4\}$. Moreover there exists constants $C, c > 0$ such that*

$$\|e^{-zL}\|_{L^2(e^{\eta|x-y|}dx) \to L^2(e^{\eta|x-y|}dx)} \le Ce^{c\eta^2 \Re z}$$

for every $y \in \mathbb{R}^n$, $z \in \Sigma_{\pi/4}$, and $\eta > 0$.

In the following we shall write $p_z(x,y)$ to mean the integral kernel of the operator e^{-zL}. Our aim is to obtain the following pointwise bounds on this integral kernel, which is an extension of (2.4) to complex times.

LEMMA 2.6. *Assume that the conditions in Proposition 2.4 hold. Then there exists $C, c > 0$ such that for all $x, y \in \mathbb{R}^n$ and $z \in \Sigma_{\pi/5}$, one has*

$$|p_z(x,y)| \le \frac{C}{(\Re z)^{n/2}}\, e^{-c\left(1+\frac{\Re z}{\rho(x)^2}\right)^\delta} e^{-c\frac{|x-y|^2}{\Re z}}. \tag{2.10}$$

Let us demonstrate how (2.10) readily leads to (2.5). Fix $x, y \in \mathbb{R}^n$ and $t > 0$. We shall apply Cauchy's integral formula to $p_z(x,y)$ in the disk

$$\Gamma(t) := \{\xi \in \mathbb{C} : |\xi - t| \leq t/2\}.$$

Observe that $\Gamma(t) \subset \Sigma_{\pi/5}$. Hence $p_z(x,y)$ is holomorphic over $\Gamma(t)$, and so for each $k \in \mathbb{N}$, Cauchy's integral formula gives

$$\frac{\partial^k}{\partial t^k} p_t(x,y) = \frac{k!}{2\pi i} \int_{\partial\Gamma(t)} \frac{p_z(x,y)}{(z-t)^{k+1}}\, dz.$$

Using (2.10) and noting that when $z \in \partial\Gamma(t)$ one has $t/2 \leq \Re z \leq 3t/2$ and $|z - t| = t/2$, we get

$$\begin{aligned}
\left| \frac{\partial^k}{\partial t^k} p_t(x,y) \right| &\leq C_k \int_{\partial\Gamma(t)} e^{-c\frac{|x-y|^2}{\Re z}} e^{-c\left(1+\frac{\Re z}{\rho(x)^2}\right)^\delta} \frac{|dz|}{(\Re z)^{n/2}(t/2)^{k+1}} \\
&\leq \frac{C_k}{t^{n/2+k+1}} e^{-c_1\frac{|x-y|^2}{t}} e^{-c\left(1+\frac{t}{2\rho(x)^2}\right)^\delta} \int_{\partial\Gamma(t)} |dz| \\
&\leq \frac{C_k}{t^{n/2+k}} e^{-c_1\frac{|x-y|^2}{t}} e^{-c2^{-\delta}\left(1+\frac{t}{\rho(x)^2}\right)^\delta}
\end{aligned}$$

which is (2.5). □

To complete the proof of Proposition 2.4 (a) we give the

Proof of Lemma 2.6. We claim that (2.10) follows from the following weighted estimate: there exists C, c, and $\epsilon > 0$ such that for every $y \in \mathbb{R}^n$, $\eta > 0$, and $z \in \Sigma_{\pi/5}$,

$$\int_{\mathbb{R}^n} |p_z(x,y)|^2 e^{\eta|x-y|}\, dx \leq \frac{Ce^{\epsilon\eta^2\Re z}}{(\Re z)^{n/2}} e^{-c\left(1+\frac{\Re z}{\rho(y)^2}\right)^\delta}. \tag{2.11}$$

Assume this estimate for the moment. Then the semigroup property, the Cauchy–Schwarz inequality, and estimate (2.11) gives

$$\begin{aligned}
|p_z(x,y)|\, e^{\eta|x-y|} &\leq \left\|p_{z/2}(x,\cdot)\, e^{\eta|x-\cdot|}\right\|_{L^2} \left\|p_{z/2}(\cdot,y)\, e^{\eta|\cdot-y|}\right\|_{L^2} \\
&\leq \frac{Ce^{4\epsilon\eta^2\Re z}}{(\Re z)^{n/2}} e^{-c\left(1+\frac{\Re z}{\rho(x)^2}\right)^\delta}.
\end{aligned}$$

Now fix $\epsilon_0 \in (0, 1/4\epsilon)$ and choose $\eta = \epsilon_0\, |x-y|\, /\Re z$. Then our estimate becomes

$$|p_z(x,y)| \leq \frac{C}{(\Re z)^{n/2}} e^{(4\epsilon\epsilon_0^2-\epsilon_0)\frac{|x-y|^2}{\Re z}} e^{-c\left(1+\frac{\Re z}{\rho(x)^2}\right)^\delta}.$$

Since $4\epsilon\epsilon_0^2 - \epsilon_0 < 0$, this establishes (2.10).

Hence our proof of Lemma 2.6 will be complete provided we show (2.11). Accordingly fix $x, y \in \mathbb{R}^n$, $\eta > 0$, $z \in \Sigma_{\pi/5}$ and set $t := \Re z$. Then the semigroup property implies that

$$p_z(x,y) = \big(e^{-(z-(t/10))L} p_{t/10}(\cdot, y)\big)(x).$$

Since $z \in \Sigma_{\pi/5}$ then $z - (t/10) \in \Sigma_{\pi/4}$, and hence by Lemma 2.5

$$\begin{aligned}\big\|p_z(\cdot,y)\, e^{\eta|\cdot - y|}\big\|_{L^2} &= \left(\int_{\mathbb{R}^n} \big|e^{-(z-(t/10))L} p_{t/10}(\cdot,y)(x)\big|^2 e^{\eta|x-y|}\,dx\right)^{1/2} \\ &\le Ce^{c\eta^2 t}\big\|p_{t/10}(\cdot,y)\, e^{\eta|\cdot-y|}\big\|_{L^2}.\end{aligned}$$

The bounds for the heat kernel from (2.4) give

$$\big\|p_{t/10}(\cdot,y)\, e^{\eta|\cdot-y|}\big\|_{L^2} \le \frac{C}{t^{n/2}} e^{-c10^{-\delta}\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}} \left(\int_{\mathbb{R}^n} e^{-20c_0\frac{|x-y|^2}{t}} e^{\eta|x-y|}\,dx\right)^{1/2}.$$

We shall prove that for any $\theta > 0$ there exists $C_\theta > 0$ and $c_\theta > 0$ such that for all $\eta > 0$ and $t > 0$,

$$\int_{\mathbb{R}^n} e^{-\theta\frac{|x-y|^2}{t}} e^{\eta|x-y|}\,dx \le C_\theta\, t^{n/2}\, e^{c_\theta \eta^2 t}. \tag{2.12}$$

Combining (2.12) with the previous two estimates will give (2.11).

We shall obtain (2.12) by considering two cases: (i) $\eta\sqrt{t} \ge 1$, and (ii) $\eta\sqrt{t} < 1$. Fix a constant $c \ge 8/\theta$. In the first case we write

$$\begin{aligned}&\int_{\mathbb{R}^n} e^{-\theta\frac{|x-y|^2}{t}} e^{\eta|x-y|}\,dx \\ &\quad\le 2\int_{B(y,2c\eta t)} e^{\eta|x-y|}\,dx + \sum_{j=2}^{\infty}\int_{U_j(B(y,c\eta t))} e^{-\theta\frac{|x-y|^2}{t}} e^{\eta|x-y|}\,dx \\ &\quad\le 2e^{2c\eta^2 t}\,|B(y,2c\eta t)| + \sum_{j=2}^{\infty} e^{-\theta\frac{c^2}{4}4^j\eta^2 t} e^{2^j c\eta^2 t}\,\big|B(y,2^j c\eta t)\big|.\end{aligned}$$

Now using that $\theta c \ge 8$ we have that $e^{\eta^2 t(c2^j - \theta 4^j c^2/8)} \le 1$, and hence

$$\int_{\mathbb{R}^n} e^{-\theta\frac{|x-y|^2}{t}} e^{\eta|x-y|}\,dx \le Ct^{n/2}e^{3c\eta^2 t} + C\sum_{j=2}^{\infty} e^{-\theta\frac{c^2}{8}4^j\eta^2 t}(2^j c\eta t)^n$$

$$\le Ct^{n/2}e^{3c\eta^2 t} + C\frac{t^{n/2}}{(\eta^2 t)^n}\sum_{j=2}^{\infty} 2^{-nj}$$
$$\le Ct^{n/2}e^{c_\theta \eta^2 t}$$

where in the next to last line we have used the fact that $\eta^2 t \ge 1$.

For the second case, with the same $c \ge 8\theta$, we write

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-\theta\frac{|x-y|^2}{t}} e^{\eta|x-y|}\,dx\\
&\le 2e^{2c\eta\sqrt{t}}\left|B(y,2c\sqrt{t})\right| + \sum_{j=2}^{\infty}\int_{U_j(B(y,c\sqrt{t}))} e^{-\theta\frac{|x-y|^2}{t}} e^{\eta|x-y|}\,dx\\
&\le Ct^{n/2} + \sum_{j=2}^{\infty} e^{-\theta\frac{c^2}{4}4^j} e^{2^j c\eta\sqrt{t}}\left|B(y,2^j c\sqrt{t})\right|\\
&\le Ct^{n/2} + C\sum_{j=2}^{\infty} e^{-\theta\frac{c^2}{8}4^j}(2^j\sqrt{t})^n\\
&\le Ct^{n/2} \le Ct^{n/2}e^{c_\theta\eta^2 t}.
\end{aligned}$$

In the second line we have used that $\eta\sqrt{t} < 1$.

This completes the proof of (2.12), and hence also of Lemma 2.6. □

Proof of Proposition 2.4 (b). We will consider three separate cases: $p = 2$, $p < 2$, and $p > 2$.

We first obtain the case $p = 2$. Let c_0 be the constant in (2.4), and choose $\alpha_2 \in (0,(2/3)c_0)$. We shall proceed as in [**13**] with some slight modifications. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $0 \le \varphi \le 1$, support in $B(0,2)$, $|\nabla\varphi| \le 1$, and $\varphi \equiv 1$ on $B(0,1)$. Define for each $R \ge 1$,

$$\varphi_R(\cdot) := \varphi\left(\frac{\cdot}{R}\right).$$

Then it follows that $|\nabla\varphi_R| \lesssim 1/R$.

Fix $y \in \mathbb{R}^n$, $t > 0$, $R \ge 1$, and set

$$I_R(t,y) := \sum_{k=1}^{n}\int_{\mathbb{R}^n} |\partial_k p_t(x,y)|^2\, e^{\alpha_2\frac{|x-y|^2}{t}}\varphi_R(x)\,dx.$$

Then one has

$$I_R(t,y) = I_R^1(t,y) - I_R^2(t,y)$$

where

$$I_R^1(t,y) := \sum_{k=1}^n \int_{\mathbb{R}^n} \partial_k p_t(x,y)\, \partial_k \left[p_t(x,y)\, e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x) \right] dx$$

$$I_R^2(t,y) := \sum_{k=1}^n \int_{\mathbb{R}^n} \partial_k p_t(x,y)\, p_t(x,y)\, \partial_k \left[e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x) \right] dx.$$

Let us study the first term. Since φ_R has compact support then

$$p_t(\cdot,y)\, e^{\alpha_2 \frac{|\cdot-y|^2}{t}} \varphi_R(\cdot) \in \mathcal{D}(\mathcal{Q}_V).$$

Therefore since both V and φ_R are non-negative,

$$\begin{aligned} I_R^1(t,y) &\le I_R^1(t,y) + \sum_{k=1}^n \int_{\mathbb{R}^n} V(x)\, p_t(x,y)^2 e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x)\, dx \\ &= \mathcal{Q}_V\left(p_t(\cdot,y)\, ,\, p_t(\cdot,y)\, e^{\alpha_2 \frac{|\cdot-y|^2}{t}} \varphi_R(\cdot) \right) \\ &= \int_{\mathbb{R}^n} L p_t(x,y)\, p_t(x,y)\, e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x)\, dx \\ &= \int_{\mathbb{R}^n} \frac{\partial}{\partial t} p_t(x,y)\, p_t(x,y)\, e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x)\, dx. \end{aligned}$$

Now using the bounds on the heat kernel (2.4) and on its time derivative (2.5) we have

$$I_R^1(t,y) \le \frac{C}{t^{n+1}} e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^\delta} \int_{\mathbb{R}^n} e^{-(c_0-\alpha_2)\frac{|x-y|^2}{t}} \varphi_R(x)\, dx.$$

Since $\alpha_2 < c_0$ and $\varphi_R \le 1$ we can control the integral by a multiple of $t^{n/2}$ and obtain

$$I_R^1(t,y) \le \frac{C}{t^{n/2+1}} e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^\delta}. \tag{2.13}$$

For the second term we have

$$\begin{aligned} I_R^2(t,y) &= \sum_{k=1}^n \int_{\mathbb{R}^n} \partial_k p_t(x,y)\, p_t(x,y)\, e^{\alpha_2 \frac{|x-y|^2}{t}} \left[\partial_k \varphi_R(x) + \frac{2\alpha_2}{t}(x_k - y_k)\, \varphi_R(x) \right] dx \\ &\le \sum_{k=1}^n \frac{C}{\sqrt{t}} \int_{\mathbb{R}^n} |\partial_k p_t(x,y)| |p_t(x,y)|\, e^{2\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x)\, dx \\ &\quad + \sum_{k=1}^n \int_{\mathbb{R}^n} |\partial_k p_t(x,y)| |p_t(x,y)|\, e^{\alpha_2 \frac{|x-y|^2}{t}} |\partial_k \varphi_R(x)|\, dx \\ &=: I_R^{2.1}(t,y) + I_R^{2.2}(t,y). \end{aligned} \tag{2.14}$$

To estimate the first term we use the Cauchy–Schwarz inequality, the heat kernel bounds (2.4), and that $2c_0 > 3\alpha_2$ to obtain

$$
\begin{aligned}
I_R^{2.1}(t,y) &\le \sum_{k=1}^{n} \frac{C}{\sqrt{t}} \Big\| p_t(\cdot,y) e^{\frac{3\alpha_2}{2}\frac{|\cdot-y|^2}{t}} \varphi_R \Big\|_{L^2} \Big\| |\partial_k p_t(\cdot,y)|\, e^{\frac{\alpha_2}{2}\frac{|\cdot-y|^2}{t}} \varphi_R \Big\|_{L^2} \\
&\le \frac{C e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}}}{t^{n/2+1/2}} \sum_{k=1}^{n} \Big\| e^{-\frac{(2c_0-3\alpha_2)}{2}\frac{|\cdot-y|^2}{t}} \Big\|_{L^2} \Big\| |\partial_k p_t(\cdot,y)|\, e^{\frac{\alpha_2}{2}\frac{|\cdot-y|^2}{t}} \varphi_R \Big\|_{L^2} \\
&\le \frac{C}{\sqrt{t^{n/2+1}}} \sqrt{I_R(t,y)}\, e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}}. \qquad (2.15)
\end{aligned}
$$

Combining (2.13), (2.14), and (2.15), with the inequality $\sqrt{AB} \le (\varepsilon/2)A + (1/2\varepsilon)B$, valid for all $\varepsilon, A, B > 0$, we obtain

$$
\begin{aligned}
I_R(t,y) &\le C e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}} \left(\frac{1}{t^{n/2+1}} + \frac{1}{\sqrt{t^{n/2+1}}} \sqrt{I_R(t,y)} \right) + I_R^{2.2}(t,y) \\
&\le C e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}} \left(\frac{1+2\varepsilon}{t^{n/2+1}} + \frac{1}{2\varepsilon} I_R(t,y) \right) + I_R^{2.2}(t,y).
\end{aligned}
$$

Choosing ε large enough therefore gives

$$
I_R(t,y) \le \frac{C}{t^{n/2+1}} e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}} + C\, I_R^{2.2}(t,y).
$$

Now using that $|\nabla\varphi_R| \lesssim 1/R$ we see that

$$
I_R^{2.2}(t,y) \le \frac{C}{R} \left\{ \sum_{k=1}^{n} \int_{\mathbb{R}^n} |\partial_k p_t(x,y)|\, |p_t(x,y)|\, e^{\alpha_2 \frac{|x-y|^2}{t}}\, dx \right\} \longrightarrow 0
$$

as $R \to \infty$. Hence by Fatou's Lemma,

$$
\begin{aligned}
\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^2 e^{\alpha_2 \frac{|x-y|^2}{t}}\, dx &\le \int_{\mathbb{R}^n} \liminf_{R\to\infty} \left\{ |\nabla_x p_t(x,y)|^2 e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x) \right\} dx \\
&\le \liminf_{R\to\infty} \int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^2 e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x)\, dx \\
&= \liminf_{R\to\infty} I_R(t,y) \\
&\le \liminf_{R\to\infty} \left\{ \frac{C}{t^{n/2+1}} e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}} + I_R^{2.2}(t,y) \right\} \\
&\le \frac{C}{t^{n/2+1}} e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}}.
\end{aligned}
$$

This proves (2.6) for $p = 2$.

To obtain (2.7) for $p = 2$, we observe that

$$\int_{\mathbb{R}^n} V(x)\, p_t(x,y)^2 e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x)\, dx = \mathcal{Q}_V\Big(p_t(\cdot,y),\, p_t(\cdot,y)\, e^{\alpha_2 \frac{|\cdot-y|^2}{t}} \varphi_R\Big) - I_R^1(t,y).$$

Since both terms have been estimated we can apply the same computations as in (2.6) and yield (2.7). This completes the proof of Proposition 2.4 part (b) for the case $p = 2$.

Next we turn to the case $p < 2$. Let $p \in [1,2)$ and fix $\alpha_p \in (0, \alpha_2/4)$. Applying Hölder's inequality with exponents $2/p$ and $(2/p)' = 2/(2-p)$ gives

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^p\, e^{\alpha_p \frac{|x-y|^2}{t}}\, dx \\
&= \int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^p\, e^{2p\alpha_p \frac{|x-y|^2}{t}}\, e^{-(2p-1)\alpha_p \frac{|x-y|^2}{t}}\, dx \\
&\le \left(\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^2\, e^{4\alpha_p \frac{|x-y|^2}{t}}\, dx\right)^{p/2} \left(\int_{\mathbb{R}^n} e^{-\frac{2(2p-1)}{2-p}\alpha_p \frac{|x-y|^2}{t}}\, dx\right)^{1-p/2}.
\end{aligned}$$

Since $4\alpha_p < \alpha_2$ we can control the first term by a constant multiple of

$$\left[\frac{1}{t^{n/2+1}}\, e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}}\right]^{p/2},$$

and since $(2p-1)/(2-p) > 0$ we can bound the second integral by a multiple of $(t^{n/2})^{1-p/2}$. Therefore

$$\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^p\, e^{\alpha_p \frac{|x-y|^2}{t}}\, dx \le \frac{C}{t^{p/2+(p-1)n/2}}\, e^{-\frac{cp}{2}\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}}$$

which gives (2.6) for $p \in [1,2)$. Similar calculations gives (2.7) for the same range of p.

We now consider the case $2 < p < s^*$. We shall make use of the following estimate, valid for each $q \in (2, s^*)$,

$$\|\nabla p_t(\cdot,y)\|_q \le \frac{C_q}{t^{1/2+n/2q'}} e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}} \qquad \forall\, y \in \mathbb{R}^n,\ t > 0. \tag{2.16}$$

Assume this estimate for the moment. We shall show how an interpolation between (2.16) and the estimate (2.6) for $p = 2$ yields (2.6) for all $p \in (2, s^*)$. Indeed for each $p \in (2, s^*)$ set (recall that $s^* = \infty$ if and only if $s \ge n$)

$$q := \begin{cases} \dfrac{p+s^*}{2} & \text{if } s^* < \infty, \\ 2p & \text{if } s^* = \infty \end{cases}$$

and $\alpha_p := \alpha_2(q-p)/(q-2)$. Note that p and q satisfy

$$p = 2\left(\frac{q-p}{q-2}\right) + q\left(\frac{p-2}{q-2}\right), \qquad 0 < \frac{q-p}{q-2} < 1, \qquad 1 < \frac{q-2}{q-p} < \infty.$$

Applying Hölder's inequality with exponents

$$\frac{q-2}{q-p} \qquad \text{and} \qquad \left(\frac{q-2}{q-p}\right)' = \frac{q-2}{p-2}$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^p \, e^{\alpha_p \frac{|x-y|^2}{t}} \, dx \\
&\quad = \int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^{2\frac{q-p}{q-2}} \, e^{\alpha_p \frac{|x-y|^2}{t}} \, |\nabla_x p_t(x,y)|^{q\frac{p-2}{q-2}} \, dx \\
&\quad \le \left(\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^2 \, e^{\alpha_2 \frac{|x-y|^2}{t}} \, dx\right)^{\frac{q-p}{q-2}} \left(\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^q \, dx\right)^{\frac{p-2}{q-2}}.
\end{aligned}$$

Estimate (2.6) for the case $p = 2$ allows us to control the first term by a multiple of

$$\left[t^{\frac{n}{2}+1}\right]^{-\frac{q-p}{q-2}} e^{-2c\frac{q-p}{q-2}\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}},$$

while estimate (2.16) allows us to control the second by a multiple of

$$\left[t^{\frac{q}{2}+\frac{n(q-1)}{2}}\right]^{-\frac{p-2}{q-2}} e^{-cq\frac{p-2}{q-2}\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}}.$$

Combining these estimates we obtain

$$\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^p \, e^{\alpha_p \frac{|x-y|^2}{t}} \, dx \le \frac{C}{t^{p/2+(p-1)n/2}} e^{-pc\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}}$$

which is (2.6).

It remains to obtain (2.16). Firstly observe that the semigroup property implies

$$\nabla_x p_{2t}(x,y) = \nabla_x e^{-tL} p_t(\cdot,y)(x). \tag{2.17}$$

Now recall from Table 1 in Section 1 that under our assumptions on $L = -\Delta + V$ the Riesz transform $\nabla L^{-1/2}$ is bounded on $L^q(\mathbb{R}^n)$ for every $q \in (1, s^*)$. This implies that for each $q \in (1, s^*)$

$$\left\|\nabla e^{-tL}\right\|_{q\to q} \le \frac{C_q}{\sqrt{t}}.$$

Indeed by the analyticity of the semigroup $\left\{e^{-tL}\right\}_{t>0}$ (see [**25**, Theorem 6.13, p. 74])

$$\left\|\sqrt{t}\nabla e^{-tL}f\right\|_q = \left\|\sqrt{t}\nabla L^{-1/2}L^{1/2}e^{-tL}f\right\|_q \lesssim \left\|\sqrt{t}L^{1/2}e^{-tL}\right\|_q \lesssim \|f\|_q\,.$$

Hence from (2.17)

$$\|\nabla p_2 t(\cdot,y)\|_q = \left\|\nabla e^{-tL}p_t(\cdot,y)\right\|_q \lesssim \frac{1}{\sqrt{t}}\|p_t(\cdot,y)\|_q\,. \tag{2.18}$$

Now using the bounds (2.4), we have

$$\|p_t(\cdot,y)\|_q^q \le \frac{C}{t^{qn/2}}e^{-qc\left(1+\frac{t}{\rho(y)^2}\right)^\delta}\int_{\mathbb{R}^n} e^{-qc\frac{|x-y|^2}{t}}\,dx \le \frac{C}{t^{(q-1)n/2}}e^{-qc\left(1+\frac{t}{\rho(y)^2}\right)^\delta}.$$

Combining this with (2.18) gives (2.16).

Finally to obtain (2.7) for $p\in(2,2s)$ we may argue in a similar fashion as above, except in place of (2.16) we use

$$\left\|V(\cdot)^{1/2}p_t(\cdot,y)\right\|_q \le \frac{C_q}{t^{1/2+n/2q'}}e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^\delta} \qquad \forall\, y\in\mathbb{R}^n, t>0$$

which follows similarly from the heat kernel bounds (2.4), and the boundedness of $V^{1/2}L^{-1/2}$ on $L^q(\mathbb{R}^n)$ for all $q\in(1,2s)$ (see Table 1 from Section 1).

This concludes the proof of Proposition 2.4 (b). □

Proof of Proposition 2.4 (c). We shall first obtain the Proposition for $p\in(1,s)$. The case $p=1$ can then be obtained by Hölder's inequality (we omit the details for this case). Fix $p\in(1,s)$. Let α_p be the constant in Proposition 2.4 (b), c_1 be the constant in Proposition 2.4 (a), and c_0 the constant in (2.4). Pick $\beta\in(0,\min\{\alpha_p,pc_1,pc_0\})$ and set $\beta_p=\beta/2$.

We shall require the following inequality that is in a sense based on the Calderón–Zygmund inequality. It is inspired by a similar inequality in [**12**] but valid only on certain domains of $\mathbb{R}^n$. The following applies to $\mathbb{R}^n$ and we defer its proof to the end.

Lemma 2.7. *Let $p\in(1,\infty)$ and $f\in W^{2,p}(\mathbb{R}^n)$. Then there exists $C=C(p,n)$ such that for each $1\le j,k\le n$ one has*

$$\|\phi\,\partial_j\partial_k f\|_{L^p} \le C\left(\left\|f|\nabla^2\phi|\right\|_{L^p} + \left\||\nabla f||\nabla\phi|\right\|_{L^p} + \left\|\phi\Delta f\right\|_{L^p}\right)$$

for every $\phi\in C_0^\infty(\mathbb{R}^n)$.

We will prove (2.8) by using a family of weight functions $\{w_{t,R}(\cdot,y)\}_R\subset C_0^\infty(\mathbb{R}^n)$ that forms a smooth cutoff of $e^{\beta|x-y|^2/t}$, and then applying an approximation argument. Accordingly fix $t>0$ and let $\varphi\in C_0^\infty(\mathbb{R}^n)$ be a function satisfying the following (for some fixed constant C):

$$\operatorname{supp}\varphi\subset B(0,2\sqrt{t}),\quad \varphi\equiv 1 \text{ on } B(0,\sqrt{t}),\quad |\varphi|\le 1,\quad |\nabla\varphi|\le\frac{C}{\sqrt{t}},\quad \left|\nabla^2\varphi\right|\le\frac{C}{t}.$$

Now for each $R\ge 1$ set $\varphi_R:=\varphi(\cdot/R)$. Then φ_R satisfies:

$$\varphi_R\equiv 1 \text{ on } B(0,R\sqrt{t}),\quad |\varphi_R|\le 1,\quad |\nabla\varphi_R|\le\frac{C}{\sqrt{t}},\quad \left|\nabla^2\varphi_R\right|\le\frac{C}{t}.$$

Now define

$$w_{t,R}(x,y):=\varphi_R\big(|x-y|\big)e^{\beta_p\frac{|x-y|^2}{pt}}.$$

Then $\operatorname{supp} w_{t,R}(x,y)\subset B(y,2R\sqrt{t})$ and one can show easily that

$$|\nabla_x w_{t,R}(x,y)|\le\frac{C}{\sqrt{t}}e^{\beta\frac{|x-y|^2}{pt}}\quad\text{and}\quad \left|\nabla_x^2 w_{t,R}(x,y)\right|\le\frac{C}{t}e^{\beta\frac{|x-y|^2}{pt}}. \tag{2.19}$$

Next define for each $t>0$, $y\in\mathbb{R}^n$ and $R\ge 1$,

$$J_R(t,y):=\left\|w_{t,R}(\cdot,y)\left|\nabla^2 p_t(\cdot,y)\right|\right\|_p.$$

We apply Lemma 2.7 with $f:=p_t(\cdot,y)$ and $\phi:=w_{t,R}(\cdot,y)$. Note that $p_t(\cdot,y)\in W^{2,p}(\mathbb{R}^n)$. To see this recall firstly that ∇^2L^{-1} is bounded on $L^p(\mathbb{R}^n)$ for $p\in(1,s)$ (from Table 1 of Section 1), and secondly that $(\partial/\partial t)p_t(\cdot,y)\in L^p(\mathbb{R}^n)$ (due to the pointwise bounds (2.5) on the time derivative of the heat kernel of L). Therefore one has

$$\left\|\nabla^2 p_t(\cdot,y)\right\|_p=\left\|-\nabla^2L^{-1}\frac{\partial}{\partial t}p_t(\cdot,y)\right\|_p\lesssim\left\|\frac{\partial}{\partial t}p_t(\cdot,y)\right\|_p<\infty$$

so that $\nabla^2 p_t(\cdot,y)\in L^p(\mathbb{R}^n)$. Hence by Lemma 2.7, for each $t>0$, $y\in\mathbb{R}^n$, and $R\ge 1$, we obtain

$$\begin{aligned}J_R(t,y)&\lesssim\left\||\nabla^2 w_{t,R}(\cdot,y)|p_t(\cdot,y)\right\|_p+\left\||\nabla w_{t,R}(\cdot,y)||\nabla p_t(\cdot,y)|\right\|_p+\left\|w_{t,R}(\cdot,y)\Delta p_t(\cdot,y)\right\|_p\\&=:J_R^1(t,y)+J_R^2(t,y)+J_R^3(t,y).\end{aligned}$$

To estimate the first term we use the bounds of our constructed weight functions (2.19), the bounds on the heat kernel in (2.4), and that $\beta-pc_0<0$:

$$\begin{aligned}J_R^1(t,y)^p&=\int_{\mathbb{R}^n}\left|\nabla_x^2 w_{t,R}(x,y)\right|^p p_t(x,y)^p\,dx\\&\le\frac{C}{t^{p+pn/2}}e^{-pc\left(1+\frac{t}{\rho(y)^2}\right)^\delta}\int_{\mathbb{R}^n}e^{(\beta-pc_0)\frac{|x-y|^2}{t}}\,dx\\&\le\frac{C}{t^{p+(p-1)n/2}}e^{-pc\left(1+\frac{t}{\rho(y)^2}\right)^\delta}.\end{aligned}$$

For the second term J_R^2 we observe that since $s^* \geq s$ then Proposition 2.4 (b) applies. Therefore because $\beta \leq \alpha_p$ we may combine (2.6) with (2.19) to obtain

$$\begin{aligned} J_R^2(t,y)^p &= \int_{\mathbb{R}^n} |\nabla_x w_{t,R}(x,y)|^p \, |\nabla_x p_t(x,y)|^p \, dx \\ &\leq \frac{C}{t^{p/2}} \int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^p \, e^{\beta \frac{|x-y|^2}{t}} \, dx \\ &\leq \frac{C}{t^{p+(p-1)n/2}} e^{-pc\left(1+\frac{t}{\rho(y)^2}\right)^\delta}. \end{aligned}$$

Now for the third term

$$\begin{aligned} J_R^3(t,y) &= \|w_{t,R}(\cdot,y)(L-V)p_t(\cdot,y)\|_p \\ &\leq \|w_{t,R}(\cdot,y)Lp_t(\cdot,y)\|_p + \|w_{t,R}(\cdot,y)Vp_t(\cdot,y)\|_p \\ &=: J_R^{3.1}(t,y) + J_R^{3.2}(t,y). \end{aligned}$$

Using the pointwise bounds on the time derivative of the heat kernel (2.5) and that $|w_{t,R}(x,y)| \leq e^{\beta_p |x-y|^2/t}$ we have

$$\begin{aligned} J_R^{3.1}(t,y)^p &= \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial t} p_t(x,y) \right|^p w_{t,R}(x,y)^p \, dx \\ &\leq \frac{C}{t^{p+pn/2}} e^{-pc\left(1+\frac{t}{\rho(y)^2}\right)^\delta} \int_{\mathbb{R}^n} e^{(\beta_p - pc_1)\frac{|x-y|^2}{t}} \, dx \\ &\leq \frac{C}{t^{p+(p-1)n/2}} e^{-pc\left(1+\frac{t}{\rho(y)^2}\right)^\delta}, \end{aligned}$$

where in the last line we have used that $\beta_p - pc_1 < 0$. For the final term $J_R^{3.2}(t,y)$ we employ the reverse Hölder properties of V, and the improved decay inherent in the heat kernel of L, namely (2.4). Indeed one has

$$\begin{aligned} J_R^{3.2}(t,y)^p &= \int_{\mathbb{R}^n} V(x)^p p_t(x,y)^p w_{t,R}(x,y)^p \, dx \\ &\leq \frac{C}{t^{pn/2}} e^{-pc\left(1+\frac{t}{\rho(y)^2}\right)^\delta} \int_{\mathbb{R}^n} V(x)^p e^{(\beta_p - pc_0)\frac{|x-y|^2}{t}} \, dx \\ &= \frac{C}{t^{pn/2}} e^{-pc\left(1+\frac{t}{\rho(y)^2}\right)^\delta} \sum_{j=0}^{\infty} \int_{U_j(B(y,\sqrt{t}))} V(x)^p e^{-\beta_0 \frac{|x-y|^2}{t}} \, dx \end{aligned}$$

where $\beta_0 := pc_0 - \beta_p > 0$. Now for each $j \geq 1$,

$$\begin{aligned}
\int_{U_j(B(y,\sqrt{t}))} V(x)^p e^{-\beta_0 \frac{|x-y|^2}{t}}\,dx &\le e^{-\beta_0 2^{2j}} \int_{B(y,2^j\sqrt{t})} V(x)^p\,dx \\
&\le Ce^{-\beta_0 4^j} \big|B(y,2^j\sqrt{t})\big| \Big(\fint_{B(y,2^j\sqrt{t})} V(x)\,dx \Big)^p \\
&\le Ce^{-\beta_0 4^j} 2^{jn} t^{n/2} \Big(2^{j(n_0-n)} \fint_{B(y,\sqrt{t})} V(x)\,dx \Big)^p \\
&= \frac{Ce^{-\beta_0 4^j} 2^{j(n+n_0p-np)}}{t^{p-n/2}} \Big(t \fint_{B(y,\sqrt{t})} V(x)\,dx \Big)^p.
\end{aligned}$$

In the second inequality we have used that $V \in \mathcal{B}_p$ because $p < s$ and hence $\mathcal{B}_p \supset \mathcal{B}_s$. In the next to last line we have used that $V\,dx$ is a doubling measure. Next we remark that if $\sqrt{t} \le \rho(y)$, then by Lemma 2.2 (a) and the definition of ρ in (2.1), one has

$$t \fint_{B(y,\sqrt{t})} V(x)\,dx \le C \Big(\frac{\sqrt{t}}{\rho(y)} \Big)^{2-n/s} \le C$$

since $s > n/2$. On the other hand if $\sqrt{t} > \rho(y)$, then Lemma 2.2 (b) implies that

$$t \fint_{B(y,\sqrt{t})} V(x)\,dx \le C \Big(\frac{\sqrt{t}}{\rho(y)} \Big)^{\sigma} \le C \Big(\frac{\sqrt{t}}{\rho(y)} \Big)^{|\sigma|}.$$

In either case we can bound

$$e^{-\frac{pc}{2}\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}} \Big(t \fint_{B(y,\sqrt{t})} V(x)\,dx \Big)^p$$

by a fixed constant independent of t and y. Therefore it follows that

$$\begin{aligned}
J_R^{3.2}(t,y)^p &\le C\, \frac{e^{-pc\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}}}{t^{p+(p-1)n/2}} \Big(t \fint_{B(y,\sqrt{t})} V(x)\,dx \Big)^p \Big\{ 1 + \sum_{j=1}^{\infty} e^{-\beta_0 4^j} 2^{j(n+n_0p-np)} \Big\} \\
&\le C\, \frac{e^{-\frac{pc}{2}\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}}}{t^{p+(p-1)n/2}}.
\end{aligned}$$

Collecting the estimates for J_R^1, J_R^2 and J_R^3 we obtain

$$J_R(t,y) \le \frac{C}{t^{1+n/2p'}} e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^{\delta}}$$

with C, c independent of R. Therefore

$$\left(\int_{\mathbb{R}^n} \left|\nabla_x^2 p_t(x,y)\right|^p e^{\beta\frac{|x-y|^2}{t}}\,dx\right)^{1/p} = \sup_{R\geq 1} J_R(t,y) \leq \frac{C}{t^{1+n/2p'}} e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^\delta}.$$

This establishes (2.8).

To prove (2.9) we simply note that

$$\left(\int |V(x)p_t(x,y)|^p e^{\beta_p\frac{|x-y|^2}{t}}\,dx\right)^{1/p} = \sup_{R\geq 1} J_R^{3,2}(t,y) \leq \frac{C}{t^{1+n/2p'}} e^{-c\left(1+\frac{t}{\rho(y)^2}\right)^\delta}$$

which follows from our previous estimates. □

This concludes the proof of Proposition 2.4 part (c), save for the proof of Lemma 2.7 which was deferred. We turn to this now.

Proof of Lemma 2.7. Fix $p \in (1,\infty)$, $f \in W^{2,p}(\mathbb{R}^n)$ and $j,k \in \{1,2,\dots,n\}$. Let $\phi \in C_0^\infty(\mathbb{R}^n)$. Then the product rule gives the following

$$\begin{aligned}
\phi\,\partial_j\partial_k f &= \partial_j(\phi\,\partial_k f) - \partial_j\phi\,\partial_k f\\
&= \partial_j(\partial_k(\phi f) - f\partial_k\phi) - \partial_j\phi\,\partial_k f\\
&= \partial_j\partial_k(\phi f) - \partial_j(f\partial_k\phi) - \partial_j\phi\,\partial_k f\\
&= \partial_j\partial_k(\phi f) - f\partial_j\partial_k\phi - \partial_j f\partial_k\phi - \partial_j\phi\,\partial_k f.
\end{aligned}$$

Taking L^p norms gives

$$\|\phi\,\partial_j\partial_k f\|_p \leq \|\partial_j\partial_k(\phi f)\|_p + \|f\partial_j\partial_k\phi\|_p + \|\partial_j f\partial_k\phi\|_p + \|\partial_j\phi\,\partial_k f\|_p. \tag{2.20}$$

Note that the left hand side is finite because $f \in W^{2,p}(\mathbb{R}^n)$ and $\phi \in C_0^\infty(\mathbb{R}^n)$. Let us consider each term on the right hand side in turn.

Firstly by noting that $|\partial_j\phi| \leq \left(\sum_k |\partial_k\phi|^2\right)^{1/2} \leq |\nabla\phi|$ for every $j \in \{1,\dots,n\}$, we have

$$\|\partial_j f\partial_k\phi\|_p + \|\partial_j\phi\partial_k f\|_p \leq 2\big\||\nabla f|\,|\nabla\phi|\big\|_p. \tag{2.21}$$

Similarly $|\partial_j\partial_k\phi| \leq \left(\sum_j\sum_k |\partial_j\partial_k\phi|^2\right)^{1/2} = |\nabla^2\phi|$ for every $j,k \in \{1,\dots,n\}$, so that

$$\|f\partial_j\partial_k\phi\|_p \leq \big\|f|\nabla^2\phi|\big\|_p. \tag{2.22}$$

Next since $\phi f \in W^{2,p}(\mathbb{R}^n)$ then by the Calderón–Zygmund inequality (see [**30**, Chapter 3, Proposition 3]) on $\mathbb{R}^n$,

$$\|\partial_j\partial_k(\phi\,f)\|_p \leq \big\||\nabla^2(\phi\,f)|\big\|_p \leq C_p\,\|\Delta(\phi\,f)\|_p.$$

Now direct computations give

$$\begin{aligned}
\Delta(\phi\, f) &= \sum_{j=1}^{n} \partial_j^2(\phi\, f) = \sum_{j=1}^{n} \partial_j\big(\phi\, \partial_j f + f\partial_j \phi\big)\\
&= \sum_{j=1}^{n} \big(\partial_j \phi\, \partial_j f + \phi\, \partial_j^2 f + \partial_j f \partial_j \phi + f\partial_j^2 \phi\big)\\
&= \phi \sum_{j=1}^{n} \partial_j^2 f + f \sum_{j=1}^{n} \partial_j^2 \phi + 2 \sum_{j=1}^{n} \partial_j \phi\, \partial_j f\\
&= \phi\Delta f + f\Delta\phi + 2\nabla\phi\cdot\nabla f.
\end{aligned}$$

By Cauchy–Schwarz,

$$|\Delta(\phi\, f)| \le |\phi\Delta f| + |f\Delta\phi| + 2\,|\nabla\phi||\nabla f| \le |\phi\Delta f| + |f||\nabla^2\phi| + 2|\nabla\phi||\nabla f|.$$

Hence

$$\|\partial_j\partial_k(\phi\, f)\|_p \le C_p\, \|\phi\Delta f\|_p + C_p\big\|f|\nabla^2\phi|\big\|_p + 2C_p\big\||\nabla\phi|\,|\nabla f|\big\|_p. \tag{2.23}$$

Inserting (2.21), (2.22), and (2.23) into (2.20) we obtain

$$\|\phi\, \partial_j\partial_k f\|_p \le C_p\, \|\phi\Delta f\|_p + C_p\big\|f|\nabla^2\phi|\big\|_p + C_p\big\||\nabla\phi|\,|\nabla f|\big\|_p,$$

and in fact

$$\big\|\phi|\nabla^2 f|\big\|_p \le \sum_{j,k=1}^{n} \|\phi\, \partial_j\partial_k f\|_p \le C\big(\|\phi\Delta f\|_p + \big\|f|\nabla^2\phi|\big\|_p + \big\||\nabla\phi||\nabla f|\big\|_p\big),$$

where C depends on p and the dimension n. This ends the proof of Lemma 2.7. □

The estimates in Proposition 2.4 (c) allow us to obtain the following estimates on the heat semigroup. It will be required in the proof of Theorem 1.1.

LEMMA 2.8. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \ge 2$. Then for each $j \ge 2$, $m \ge 1$, $p \in (1, s)$, ball B, and $f \in L^1(B)$ we have*

$$\bigg(\fint_{U_j(B)} |\nabla^2 L^{-1}(I - e^{-r_B^2 L})^m f|^p\bigg)^{1/p} \le Ce^{-c4^j} \fint_B |f|\,, \tag{2.24}$$

$$\bigg(\fint_{U_j(B)} |V L^{-1}(I - e^{-r_B^2 L})^m f|^p\bigg)^{1/p} \le Ce^{-c4^j} \fint_B |f|\,. \tag{2.25}$$

PROOF. We first prove (2.24). The first step is to write, using the binomial theorem,

$$\begin{aligned}\nabla^2 L^{-1}(I-e^{-r_B^2 L})^m &= \sum_{k=0}^m \tbinom{m}{k}(-1)^k \int_0^\infty \nabla^2 e^{-(kr_B^2+t)L}\,dt\\ &= \sum_{k=0}^m \tbinom{m}{k}(-1)^k \int_0^\infty \nabla^2 e^{-tL}\mathbf{1}_{(kr_B^2,\infty)}(t)\,dt\\ &= \int_0^\infty h_{r_B}(t)\nabla^2 e^{-tL}\,dt\end{aligned}$$

where

$$h_r(t) := \sum_{k=0}^m (-1)^k \tbinom{m}{k}\mathbf{1}_{(kr_B^2,\infty)}(t).$$

Now observing that $\sum_{k=0}^m (-1)^k\binom{m}{k} = 0$ we can write

$$\begin{aligned}h_r(t) &= \sum_{k=0}^m (-1)^k \tbinom{m}{k}\mathbf{1}_{(mr^2,\infty)}(t) + \sum_{k=0}^m (-1)^k \tbinom{m}{k}\mathbf{1}_{(kr^2,mr^2]}(t)\\ &= \sum_{k=0}^m (-1)^k \tbinom{m}{k}\mathbf{1}_{(kr^2,mr^2]}(t).\end{aligned}$$

Therefore

$$|h_r(t)| \le \sum_{k=0}^m \tbinom{m}{k}\mathbf{1}_{(0,mr^2]}(t) \le 2^m \mathbf{1}_{(0,mr^2]}(t).$$

Now by Minkowski's inequality,

$$\begin{aligned}\left\|\nabla^2 L^{-1}(I-e^{-r_B^2 L})^m f\right\|_{L^p(U_j(B))} &= \left\|\int_0^\infty h_{r_B}(t)\nabla^2 e^{-tL} f\,dt\right\|_{L^p(U_j(B))}\\ &\le \int_0^\infty |h_{r_B}(t)|\left\|\nabla^2 e^{-tL} f\right\|_{L^p(U_j(B))}\,dt\\ &\le \int_0^\infty |h_{r_B}(t)|\int_B |f(y)|\left\|\nabla^2 p_t(\cdot,y)\right\|_{L^p(U_j(B))}\,dy\,dt.\end{aligned}$$

Next for each $y\in B$ and $t>0$, by estimate (2.8),

$$\begin{aligned}\left\|\nabla^2 p_t(\cdot,y)\right\|_{L^p(U_j(B))} &\le \left(\int_{U_j(B)} \left|\nabla_x^2 p_t(x,y)\right|^p e^{\beta_p\frac{|x-y|^2}{t}} e^{-\beta_p\frac{|x-y|^2}{t}}\,dx\right)^{1/p}\\ &\le \sup_{x\in U_j(B)} e^{-\beta_p\frac{|x-y|^2}{t}}\left\|\nabla^2 p_t(\cdot,y)e^{\beta_p\frac{|\cdot-y|^2}{t}}\right\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

$$\lesssim \frac{1}{t^{1+n/2-n/2p}} e^{-c4^j r_B^2/t}.$$

Therefore one has

$$\begin{aligned}\left(\fint_{U_j(B)} \left|\nabla^2 L^{-1}(I-e^{-r_B^2 L})^m f\right|^p\right)^{1/p} &= \frac{1}{|2^j B|^{1/p}} \|\nabla^2 L^{-1}(I-e^{-r_B^2 L})^m f\|_{L^p(U_j(B))} \\ &\lesssim \left(\int_0^\infty |h_{r_B}(t)| \frac{|B|}{|2^j B|^{1/p}} \frac{e^{-c4^j r_B^2/t}}{t^{1+n/2-n/2p}}\, dt\right)\left(\fint_B |f|\right) \\ &\lesssim \left(\int_0^{mr_B^2} \frac{|B|}{|2^j B|^{1/p}} \frac{e^{-c4^j r_B^2/t}}{t^{1+n/2-n/2p}}\, dt\right)\left(\fint_B |f|\right).\end{aligned}$$

Since

$$\frac{|B|}{|2^j B|^{1/p}} \approx \frac{r_B^n}{(2^{jn} r_B^n)^{1/p}} \approx \frac{r_B^{n(1-1/p)}}{2^{jn/p}} = 2^{-jn}(2^j r_B)^{n(1-1/p)},$$

then it follows that for some $\epsilon > 0$

$$\frac{|B|}{|2^j B|^{1/p}} \frac{1}{t^{n/2(1-1/p)}} \lesssim 2^{-jn}\left(\frac{2^j r_B}{\sqrt{t}}\right)^{n(1-1/p)} \le \left(\frac{2^j r_B}{\sqrt{t}}\right)^{n(1-1/p)} \lesssim e^{\epsilon 4^j r_B^2/t}.$$

Collecting these estimates we obtain

$$\begin{aligned}\left(\fint_{U_j(B)} \left|\nabla^2 L^{-1}(I-e^{-r_B^2 L})^m f\right|^p\right)^{1/p} &\lesssim \left(\int_0^{mr_B^2} e^{-c' 4^j r_B^2/t} \frac{dt}{t}\right)\left(\fint_B |f|\right) \\ &\lesssim e^{-c4^j/m}\left(\int_0^{mr_B^2} \frac{t}{4^j r_B^2} \frac{dt}{t}\right)\left(\fint_B |f|\right) \\ &\lesssim e^{-c4^j/m} \fint_B |f|\end{aligned}$$

provided $m > 0$.

The proof of (2.25) is similar but uses (2.9) in place of (2.8) and we omit the details. □

3. Weights associated to the Schrödinger operator.

In this section we define weights adapted to Schrödinger operators and give some of their properties that we shall need. For further properties we refer the reader to [**10**], [**32**], [**33**], [**34**].

Throughout the rest of this section we use the following notation. For a given ball B and a number $\theta \ge 0$, we set

$$\psi_\theta(B) := \left(1 + \frac{r_B}{\rho(x_B)}\right)^\theta. \tag{3.1}$$

Here $\rho : \mathbb{R}^n \to (0, \infty)$ is the auxiliary weight function defined in (2.1). Observe that for any $\lambda \geq 1$, we have $\psi_\theta(B) \leq \psi_\theta(\lambda B) \leq \lambda^\theta \psi_\theta(B)$. We will also often interchange balls with cubes in our estimates. In this case if Q is a cube, the expression for $\psi_\theta(Q)$ is the same as above but with r_B replaced by $\ell(Q)$ (the sidelength of Q), and x_B replaced by x_Q (the centre of Q).

The following maximal operator was defined in [**11**], [**34**] and will be an essential tool throughout the rest of this section. For each $\theta \geq 0$, we set

$$\mathcal{M}_\theta^L f(x) := \sup_{B \ni x} \frac{1}{\psi_\theta(B)} \fint_B |f(y)|\, dy. \tag{3.2}$$

We mention here that f is pointwise controlled by $\mathcal{M}_\theta^L f$. Indeed, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\theta \geq 0$, we have for almost every x,

$$|f(x)| \leq 2^\theta \mathcal{M}_\theta^L f(x). \tag{3.3}$$

To see this, we let $r \leq \rho(x)$ and observe that

$$\fint_{B(x,r)} |f| \leq \psi_\theta(B(x,r))\, \mathcal{M}_\theta^L f(x) \leq 2^\theta \mathcal{M}_\theta^L f(x).$$

Now let $r \to 0$ and apply Lebesgue's differentiation theorem (see [**30**]) to obtain (3.3).

Definition 3.1 (Weights adapted to the Schrödinger operator). Let w be a non-negative locally-integrable function. For $p \in (1, \infty)$ and $\theta \geq 0$, we say that $w \in \mathcal{A}_p^{L,\theta}$ if there exists $C = C(w, \theta, p) > 0$ such that for all balls B,

$$\left(\fint_B w\right)^{1/p} \left(\fint_B w^{1-p'}\right)^{1/p'} \leq C\psi_\theta(B).$$

We say that $w \in \mathcal{A}_1^{L,\theta}$ if there exists $C = C(w, \theta) > 0$ such that for all balls B

$$\fint_B w \leq C\, \psi_\theta(B)\, w(x) \qquad \text{a.e. } x \in B.$$

For $p \in [1, \infty)$ we set $\mathcal{A}_p^L := \bigcup_{\theta \geq 0} \mathcal{A}_p^{L,\theta}$. We also define $\mathcal{A}_\infty^L := \bigcup_{1 \leq p < \infty} \mathcal{A}_p^L$.

By taking $\theta = 0$ we see that these weights contain the $\mathcal{A}_\infty$ weights. However the inclusion is proper. For example let $V \equiv 1$ and take $w(x) = 1 + |x|^\epsilon$ with $\epsilon > n(p-1)$. Then w is a member of $\mathcal{A}_p^L$ but is does not belong to $\mathcal{A}_p$.

We next define the class of reverse Hölder weights adapted to the Schrödinger operator.

Definition 3.2 (Reverse Hölder weights adapted to the Schrödinger operator). Let w be a non-negative locally-integrable function. For $q \in (1,\infty)$ and $\theta \geq 0$, we say that $w \in \mathcal{B}_q^{L,\theta}$ if there exists $C = C(w,q,\theta) > 0$ such that for all balls B,

$$\left(\fint_B w^q \right)^{1/q} \leq C\, \psi_\theta(B) \fint_B w.$$

We say that $w \in \mathcal{B}_\infty^{L,\theta}$ if there exists $C = C(w,\theta) > 0$ such that for all balls B,

$$w(x) \leq C\, \psi_\theta(B) \fint_B w, \qquad \text{a.e. } x \in B.$$

For $q \in (1,\infty]$ we set $\mathcal{B}_q^L := \bigcup_{\theta \geq 0} \mathcal{B}_q^{L,\theta}$.

We remark that in the definitions one can interchange balls by cubes and obtain the same classes of weights.

Lemma 3.3. *Let $w \in \mathcal{B}_{s'}^{L,\theta}$ for some $\theta \geq 0$ and $1 \leq s \leq \infty$. Then there exists $C_w > 0$ such that for any cube Q and measurable $E \subset Q$,*

$$\frac{w(E)}{w(Q)} \leq C_w\, \psi_\theta(Q) \left(\frac{|E|}{|Q|} \right)^{1/s}.$$

Proof. If $s' < \infty$ then by Hölder's inequality with exponents s' and s,

$$\begin{aligned}
\frac{w(E)}{w(Q)} = \frac{|Q|}{w(Q)} \frac{1}{|Q|} \int_E w &\leq \frac{|Q|}{w(Q)} \left(\fint_Q w^{s'} \right)^{1/s'} \left(\frac{|E|}{|Q|} \right)^{1/s} \\
&\leq C_w \frac{|Q|}{w(Q)} \psi_\theta(Q) \left(\fint_Q w \right) \left(\frac{|E|}{|Q|} \right)^{1/s} \\
&= C_w \psi_\theta(Q) \left(\frac{|E|}{|Q|} \right)^{1/s}.
\end{aligned}$$

If $s' = \infty$ then the same conclusion holds. □

As in the classical situation these two weight classes are intimately connected. It was shown in [**10**] that if $w \in \mathcal{A}_p^L$ for some $p \in [1,\infty)$, then $w \in \mathcal{B}_q^{L,\theta}$ for some $q > 1$ and $\theta \geq 0$ (see [**10**, Lemma 5]). We give a more explicit statement of this connection in the next result, itself modelled on [**4**, Proposition 11.1].

Lemma 3.4. *Let $w \geq 0$ be a measurable function. Then the following are equivalent.*

(a) $w \in \mathcal{A}_\infty^L$.
(b) *For all $\sigma \in (0,1)$, $w^\sigma \in \mathcal{B}_{1/\sigma}^L$.*
(c) *There exists $\sigma \in (0,1)$ such that $w^\sigma \in \mathcal{B}_{1/\sigma}^L$.*

PROOF. If $w^\sigma \in \mathcal{B}^L_{1/\sigma}$ for some $\sigma \in (0,1)$, then the self improvement property of these classes (see Lemma 3.5 (v) below) implies that $w^\sigma \in \mathcal{B}^L_{1/\sigma+\varepsilon}$ for some $\varepsilon > 0$. Therefore $w \in \mathcal{B}^L_{1+\sigma\varepsilon}$, which implies that $w \in \mathcal{A}^L_\infty$. Hence we have (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a).

We now show (a) $\Longrightarrow$ (b). Let $w \in \mathcal{A}^L_\infty$ and $\sigma \in (0,1)$. Then $w \in \mathcal{B}^{L,\theta}_r$ for some $r > 1$ and $\theta \geq 0$ (by [**10**, Lemma 5]). Therefore for any $\alpha > 1$ and cube Q, the set

$$E_Q := \left\{x \in Q : \ w^\sigma(x) > \alpha \fint_Q w^\sigma\right\}$$

satisfies, by Lemma 3.3,

$$\frac{w(E_Q)}{w(Q)} \leq C\psi_\theta(Q)\left(\frac{|E_Q|}{|Q|}\right)^{1/r'}.$$

Then it follows that

$$|E_Q| = \frac{1}{\alpha}\int_{E_Q} \alpha\, dx < \frac{1}{\alpha}\int_{E_Q} \frac{w^\sigma}{\fint_Q w^\sigma} dx \leq \frac{|Q|}{\alpha}.$$

Hence we obtain that

$$w(E_Q) \leq C\,\alpha^{-1/r'}\psi_\theta(Q)\, w(Q).$$

We choose α such that $C\alpha^{-1/r'}\psi_\theta(Q) = 1/2$ (note that $\alpha > 1$). Next, observe that for each $x \in Q\backslash E_Q$ we have $w(x) \leq \left(\alpha \fint_Q w^\sigma\right)^{1/\sigma}$. Therefore

$$\begin{aligned}
\int_Q w\, dx &= \int_{E_Q} w\, dx + \int_{Q\backslash E_Q} w\, dx \\
&\leq \frac{1}{2}\int_Q w\, dx + \left(\alpha \fint_Q w^\sigma\right)^{1/\sigma} \int_{Q\backslash E_Q} dx \\
&\leq \frac{1}{2}\int_Q w\, dx + \alpha^{1/\sigma}\, |Q| \left(\fint_Q w^\sigma\right)^{1/\sigma}.
\end{aligned}$$

Rearranging this statement gives us

$$\fint_Q w\, dx \leq 2\alpha^{1/\sigma}\left(\fint_Q w^\sigma\right)^{1/\sigma} = 2^{r'/\sigma+1}\, C^{r'/\sigma}\, \psi_{\theta r'/\sigma}(Q)\left(\fint_Q w^\sigma\right)^{1/\sigma}.$$

That is, $w^\sigma \in \mathcal{B}^{L,\theta r'}_{1/\sigma} \subset \mathcal{B}^L_{1/\sigma}$. □

We now describe some further properties of these weights.

LEMMA 3.5. *One has*

(i) *For each $\theta \geq 0$, if $1 \leq p_1 \leq p_2 < \infty$ then $\mathcal{A}_1^{L,\theta} \subset \mathcal{A}_{p_1}^{L,\theta} \subset \mathcal{A}_{p_2}^{L,\theta}$.*
(ii) *For each $\theta \geq 0$, if $1 < p_1 \leq p_2 \leq \infty$ then $\mathcal{B}_{p_1}^{L,\theta} \supset \mathcal{B}_{p_2}^{L,\theta} \supset \mathcal{B}_{\infty}^{L,\theta}$.*
(iii) *For each $1 \leq p \leq \infty$ and $\theta \geq 0$, $w \in \mathcal{A}_p^{L,\theta}$ if and only if $w^{1-p'} \in \mathcal{A}_{p'}^{L,\theta}$.*
(iv) *If $w \in \mathcal{A}_p^L$ for some $p \in (1,\infty)$ then there exists $p_0 \in (1,p)$ with $w \in \mathcal{A}_{p_0}^L$.*
(v) *If $w \in \mathcal{B}_q^L$ for some $q \in (1,\infty)$ then there exists $q_0 \in (q,\infty)$ with $w \in \mathcal{B}_{q_0}^L$.*
(vi) *For each $r \in (1,\infty)$, $w^r \in \mathcal{A}_\infty^L \Longleftrightarrow w \in \mathcal{B}_r^L$.*
(vii) *Suppose $w^\sigma \in \mathcal{A}_{\sigma(s-1)+1}^L$ for some $\sigma \in (0,\infty)$ and $s \in [1,\infty)$. Then $w \in \mathcal{A}_s^L$ if and only if $w \in \mathcal{A}_\infty^L$.*
(viii) *For each $1 \leq p \leq \infty$ and $1 \leq q < \infty$, we have*

$$w^q \in \mathcal{A}_{q(p-1)+1}^L \quad \Longleftrightarrow \quad w \in \mathcal{A}_p^L \cap \mathcal{B}_q^L.$$

(ix) *Suppose $p_0 < p < q_0$ and $w \in \mathcal{A}_{p/p_0}^L \cap \mathcal{B}_{(q_0/p)'}^L$. Then there exists p_1 and q_1 such that*

$$p_0 < p_1 < p < q_1 < q_0 \qquad \text{and} \qquad w \in \mathcal{A}_{\frac{p}{p_1}}^L \cap \mathcal{B}_{\left(\frac{q_1}{p}\right)'}^L.$$

(x) *Given $p_0 < p < q_0$, we have*

$$w \in \mathcal{A}_{\frac{p}{p_0}}^L \cap \mathcal{B}_{\left(\frac{q_0}{p}\right)'}^L \quad \Longleftrightarrow \quad w^{1-p'} \in \mathcal{A}_{\frac{p'}{q_0'}}^L \cap \mathcal{B}_{\left(\frac{p_0'}{p'}\right)'}^L.$$

Proof. The proofs of (i), (ii) and (iii) follow easily from the definition of the $\mathcal{A}_p^L$ and $\mathcal{B}_q^L$ classes. For the proof of (iv) see [**10**] and also [**32**, Proposition 2.1 (iii)]. Property (v) is the self-improvement property of the $\mathcal{B}_q^L$ classes mentioned in [**10**]. Property (vi) is a restatement of Lemma 3.4. Indeed by replacing $1/\sigma$ by r and w^σ by w in Lemma 3.4 we obtain (vi).

The proofs of the next two properties are adapted from [**17**] and [**20**].

Proof of (vii). We note that $\mathcal{A}_s^L \subset \mathcal{A}_\infty^L$ for every $s \geq 1$, and so necessity is clear. It suffices to consider the converse. Let $w \in \mathcal{A}_\infty^L$. Suppose firstly that $0 < \sigma < 1$. Since $w \in \mathcal{A}_\infty^L$ then by Lemma 3.4 (or property (vi) above) we have $w^\sigma \in \mathcal{B}_{1/\sigma}^{L,\theta}$ for some $\theta \geq 0$. This means that for any ball B,

$$\left(\fint_B w\right)^\sigma = \left(\fint_B (w^\sigma)^{1/\sigma}\right)^\sigma \leq C\,\psi_\theta(B) \fint_B w^\sigma.$$

Let $r := \sigma(s-1)+1$. Then since $w^\sigma \in \mathcal{A}_r^L$,

$$\begin{aligned}
\left(\fint_B w\right)\left(\fint_B w^{-1/(s-1)}\right)^{s-1} &\leq C\,\psi_{\theta/\sigma}(B)\left(\fint_B w^\sigma\right)^{1/\sigma}\left(\fint_B w^{-1/(s-1)}\right)^{s-1} \\
&= C\,\psi_{\theta/\sigma}(B)\left(\fint_B w^\sigma\right)^{1/\sigma}\left(\fint_B (w^\sigma)^{-1/(r-1)}\right)^{(r-1)/\sigma} \\
&\leq C\,\psi_{(r+1)\theta/\sigma}(B).
\end{aligned}$$

That is, $w \in \mathcal{A}_s^{L,(r+1)\theta/(s\sigma)} \subset \mathcal{A}_s^L$. Suppose now that $1 \le \sigma < \infty$. Let B be a ball and $r := \sigma(s-1)+1$. Note $w^\sigma \in \mathcal{A}_r^L$ implies that $w^\sigma \in \mathcal{A}_r^{L,\theta}$ for some $\theta \ge 0$. Since $\sigma \ge 1$ we may apply Hölder's inequality with exponents σ and σ' to get

$$\begin{aligned}\Big(\fint_B w\Big)\Big(\fint_B w^{-1/(s-1)}\Big)^{s-1} &\le \Big(\fint_B w^\sigma\Big)^{1/\sigma}\Big(\fint_B w^{-1/(s-1)}\Big)^{s-1}\\ &= \Big(\fint_B w^\sigma\Big)^{1/\sigma}\Big(\fint_B (w^\sigma)^{-1/(r-1)}\Big)^{(r-1)/\sigma}\\ &\le C\,\psi_{r\theta/\sigma}(B).\end{aligned}$$

We have shown that $w \in \mathcal{A}_s^{L,\theta r/(\sigma s)} \subset \mathcal{A}_s^L$. This concludes the proof of (vii).

Proof of (viii). We first show the $\Longrightarrow$ direction . Assume that $w^q \in \mathcal{A}^L_{q(p-1)+1}$. Then $w^q \in \mathcal{A}_\infty^L$, and by property (vi) above $w \in \mathcal{B}_q^L$. If in addition $w \in \mathcal{A}_\infty^L$, then applying property (vii) with $\sigma = q$ and $s = p$ we obtain $w \in \mathcal{A}_p^L$. We now prove the converse $\Longleftarrow$ direction. Assume that $w \in \mathcal{A}_p^L \cap \mathcal{B}_q^L$. Then $w \in \mathcal{B}_q^L$ and this implies, by property (vi), that $w^q \in \mathcal{A}_\infty^L$. Hence $(w^q)^{1/q} = w \in \mathcal{A}_p^L$, and property (vii) with $\sigma = 1/q$ and $p = \sigma(s-1)+1$ gives $w^q \in \mathcal{A}_s^L \equiv \mathcal{A}^L_{q(p-1)+1}$.

Proof of (ix). Firstly, property (iv) implies there exists p_1 such that

$$1 < \frac{p}{p_1} < \frac{p}{p_0} \quad \text{and} \quad w \in \mathcal{A}^L_{\frac{p}{p_1}}.$$

This implies $p_0 < p_1 < p$. Secondly, property (v) implies there exists q_1 such that

$$\Big(\frac{q_0}{p}\Big)' < \Big(\frac{q_1}{p}\Big)' < \infty \quad \text{and} \quad w \in \mathcal{B}^L_{\left(\frac{q_1}{p}\right)'}.$$

This implies $p < q_1 < q_0$.

Proof of (x). The proof is almost the same as that of Lemma 4.4 from [**6**]. We give the details here for convenience. Set $q = (q_0/p)'(p/p_0-1)+1$. Using properties (iii) and (viii), we have

$$\begin{aligned} w \in \mathcal{A}^L_{\frac{p}{p_0}} \cap \mathcal{B}^L_{\left(\frac{q_0}{p}\right)'} &\Longleftrightarrow w^{\left(\frac{q_0}{p}\right)'} \in \mathcal{A}^L_{\left(\frac{q_0}{p}\right)'\left(\frac{p}{p_0}-1\right)+1} \equiv \mathcal{A}_q^L\\ &\Longleftrightarrow w^{\left(\frac{q_0}{p}\right)'(1-q')} \in \mathcal{A}^L_{q'}\end{aligned}$$

and

$$w^{1-p'} \in \mathcal{A}^L_{\frac{p'}{q_0'}} \cap \mathcal{B}^L_{\left(\frac{p_0'}{p'}\right)'} \quad \Longleftrightarrow \quad w^{(1-p')\left(\frac{p_0'}{p'}\right)'} \in \mathcal{A}^L_{\left(\frac{p_0'}{p'}\right)'\left(\frac{p'}{q_0'}-1\right)+1}.$$

Direct computations show

$$\left(\frac{q_0}{p}\right)'(1-q') = (1-p')\left(\frac{p_0'}{p'}\right)' \quad \text{and} \quad q' = \left(\frac{p_0'}{p'}\right)'\left(\frac{p'}{q_0'}-1\right)+1. \qquad \square$$

The following weak type property of the operator $\mathcal{M}^L$ is implicit throughout [**34**].

Lemma 3.6. *For each $\eta \geq 0$, the operator $\mathcal{M}_\eta^L$ is weak type (p,p) for every $p \in [1,\infty)$.*

Proof. We simply observe that $\mathcal{M}_\eta^L$ is controlled pointwise by M, the Hardy–Littlewood maximal function, and hence the weak (p,p) bound of $\mathcal{M}_\eta^L$ is controlled by that of M. $\square$

The main mapping property of the operator $\mathcal{M}_\eta^L$ we will require is the following.

Lemma 3.7 ([**32**, Lemma 2.2]). *Let $p \in (1,\infty)$ and $\theta \geq 0$. Then for each $w \in \mathcal{A}_p^{L,\theta}$ there exists $C > 0$ such that*

$$\left\|\mathcal{M}_{p'\theta}^L f\right\|_{L^p(w)} \leq C \left\|f\right\|_{L^p(w)}.$$

Proof. A proof of this result in a more general form can be found in [**34**, Theorem 2.2]. $\square$

Remark 3.8. As a consequence of Lemma 3.7, if $p > s$, $w \in \mathcal{A}_{p/s}^{L,\theta}$ and $\eta = (p/s)'\theta$ then the operator $\mathcal{M}_\eta^L\left(|\cdot|^s\right)^{1/s}$ is bounded on $L^p(w)$. In fact, since $\mathcal{M}_\eta^L$ is bounded on $L^{p/s}(w)$ for each $w \in \mathcal{A}_{p/s}^{L,\theta}$, then we have

$$\|\mathcal{M}_\eta^L\left(|f|^s\right)^{1/s}\|_{L^p(w)}^p = \int \left(\mathcal{M}_\eta^L\,|f|^s\right)^{p/s} w \lesssim \int |f|^p\, w.$$

3.1. A good-λ inequality.

The main result of this section is the following extension of [**6**, Theorem 3.1] from $\mathcal{A}_\infty$ to $\mathcal{A}_\infty^L$ weights.

We remind the reader that the functions ρ, ψ_η, and the operator $\mathcal{M}_\eta^L$ has been defined in (2.1), (3.1), and (3.2) respectively.

Theorem 3.9. *Fix $\eta > 0$, $q \in (1,\infty]$, $\xi \geq 1$, $s \in [1,\infty)$, and $\nu \in \mathcal{B}_{s'}^L$. Assume that F,G, and H are non-negative functions on $\mathbb{R}^n$ such that for each ball B with $r_B \leq 12\sqrt{n}\,\rho(x_B)$, there exist non-negative functions H_B and G_B with*

$$F(x) \leq H_B(x) + G_B(x) \qquad a.e.\ x \in B, \tag{3.4}$$

$$\fint_B G_B \leq G(x), \qquad \forall x \in B, \tag{3.5}$$

$$\left(\fint_B H_B^q\right)^{1/q} \leq \xi\left(\mathcal{M}_\eta^L F(x) + H(y)\right), \qquad \forall x,y \in B \tag{3.6}$$

and for each ball B with $r_B > 12\sqrt{n}\,\rho(x_B)$,

$$\frac{1}{\psi_\eta(B)} \fint_B F \leq G(x), \qquad \forall x \in B. \tag{3.7}$$

Then there exists $C = C(n, q, \nu, \xi, s, \eta, \rho) > 0$ and $K_0 = K_0(m, \eta, \xi, \rho) \geq 1$ with the following property: for each $\lambda > 0$, $K \geq K_0$, and $\gamma \in (0,1)$,

$$\nu\big(\{\mathcal{M}_\eta^L F > K\lambda \text{ and } G \leq \gamma\lambda\}\big) \leq C\bigg(\frac{\xi^q}{K^q} + \frac{\gamma}{K}\bigg)^{1/s} \nu\big(\{\mathcal{M}_\eta^L F > \lambda\}\big). \tag{3.8}$$

As a consequence, for all $r \in \big(0, q/s\big)$, we have

$$\big\|\mathcal{M}_\eta^L F\big\|_{L^r(\nu)} \leq C\big(\|G\|_{L^r(\nu)} + \|H\|_{L^r(\nu)}\big) \tag{3.9}$$

provided $\big\|\mathcal{M}_\eta^L F\big\|_{L^r(\nu)} < \infty$. If $r \geq 1$ then (3.9) holds provided $F \in L^1(\mathbb{R}^n)$.

Remark 3.10. We mention that the term H is an error term, which is useful in applications. For instance it allows us to consider commutators (see Theorem 3.16 in [**6**] for the case of $\mathcal{A}_\infty$ weights). However we do not give any results in this direction here.

Proof of Theorem 3.9. The proof is an adaptation of the proof of Theorem 3.1 from [**6**]. Indeed for balls B satisfying $r_B \leq 12\sqrt{n}\rho(x_B)$ the argument is almost the same but with the operator $\mathcal{M}_\eta^L$ in place of the maximal operator M. The key difference is the scale $r_B > 12\sqrt{n}\rho(x_B)$.

We begin by mentioning that it will suffice to consider the case $G = H$. Indeed if we set $\widetilde{G} := G + H$, then (3.5) holds with $\widetilde{G}$ in place of G and (3.6) holds with $\widetilde{G}$ in place of H. Henceforth we shall assume that $H = G$.

We shall first demonstrate (3.8). Fix $\lambda > 0$ and set

$$\begin{aligned} \Omega_\lambda &:= \big\{x \in \mathbb{R}^n : \mathcal{M}_\eta^L F(x) > \lambda\big\} \\ E_\lambda &:= \big\{x \in \mathbb{R}^n : \mathcal{M}_\eta^L F(x) > K\lambda,\ 2G(x) \leq \gamma\lambda\big\}. \end{aligned}$$

Note that Ω_λ is an open set, and hence the Whitney decomposition lemma (see [**18**]) allows us to decompose it into a family of pairwise disjoint cubes $\mathcal{Q} = \{Q_j\}_j$, with $\Omega_\lambda = \bigcup_j Q_j$, and such that $4Q_j$ meets Ω_λ^c for every j. Our aim is to show the following estimate: there exists $C > 0$ such that for every j for which $E_\lambda \cap Q_j$ is not empty,

$$\nu(E_\lambda \cap Q_j) \leq C\bigg(\frac{\xi^q}{K^q} + \frac{\gamma}{K}\bigg)^{1/s} \nu(Q_j). \tag{3.10}$$

Then since $E_\lambda \subset \bigcup_j E_\lambda \cap Q_j$, we may sum over all the disjoint cubes in $\mathcal{Q}$ to obtain

$$\nu(E_\lambda) \le \sum_j \nu(E_\lambda \cap Q_j) \le C\Big(\frac{\xi^q}{K^q} + \frac{\gamma}{K}\Big)^{1/s} \sum_j \nu(Q_j) = C\Big(\frac{\xi^q}{K^q} + \frac{\gamma}{K}\Big)^{1/s} \nu(\Omega_\lambda)$$

which is (3.8).

We proceed with the proof of (3.10). We shall consider two regimes.

$$\mathcal{J}_0 := \big\{ j : Q_j \in \mathcal{Q} \quad \text{and} \quad \ell(Q_j) \le 2\rho(x_{Q_j}) \big\}$$
$$\mathcal{J}_\infty := \big\{ j : Q_j \in \mathcal{Q} \quad \text{and} \quad \ell(Q_j) > 2\rho(x_{Q_j}) \big\}.$$

We first study the case $j \in \mathcal{J}_0$. For each such j we define B_j to be the ball with the same centre as Q_j but with radius $r_{B_j} = (\sqrt{n}/2)\ell(Q_j)$. (That is, B_j is the 'smallest' ball concentric with and containing Q_j). Our task will be to show that for each $j \in \mathcal{J}_0$ with $E_\lambda \cap Q_j$ non-empty, the following estimate holds:

$$|E_\lambda \cap Q_j| \le C\Big(\frac{\xi^q}{K^q} + \frac{\gamma}{K}\Big) |Q_j| \tag{3.11}$$

with C depending only on q, n, η, ρ, and the weak type bounds of $\mathcal{M}^L_\eta$. (We remark here that if $q = \infty$ then the first term ξ^q/K^q is taken to be zero in (3.11)). Once (3.11) is proven we may obtain (3.10) as follows. Recall that since $\nu \in \mathcal{B}^L_{s'}$, then there exists $\theta \ge 0$ for which $\nu \in \mathcal{B}^{L,\theta}_{s'}$. We then apply Lemma 3.3 to ν, and to the sets $E_\lambda \cap Q_j \subset Q_j$, to obtain

$$\nu(E_\lambda \cap Q_j) \le C_\nu\, \psi_\theta(Q_j) \Big(\frac{|E_\lambda \cap Q_j|}{|Q_j|}\Big)^{1/s} \nu(Q_j) \le C\left(\frac{\xi^q}{K^q} + \frac{\gamma}{K}\right)^{1/s} \nu(Q_j). \tag{3.12}$$

Note we have used that $\psi_\theta(Q_j) \le 3^\theta$ since $j \in \mathcal{J}_0$. This gives estimate (3.10).

We proceed with obtaining (3.11). We shall need a localisation lemma whose proof we postpone to the end of the section.

Lemma 3.11. *There exists $\widetilde{K_0} > 1$ depending only on n, η, and the constant C_0 in Lemma* 2.1 *with the following property*: *for each $K \ge \widetilde{K_0}$, and each ball $\widetilde{B}$ for which there exists $\widetilde{x} \in \widetilde{B}$ with $\mathcal{M}^L_\eta F(\widetilde{x}) \le \lambda$, we have*

$$\big\{ E_\lambda \cap \widetilde{B} \big\} \subset \big\{ x \in \mathbb{R}^n : \mathcal{M}^L_\eta \big(F \mathbf{1}_{3\widetilde{B}}\big)(x) > (K/\widetilde{K_0})\lambda \big\}.$$

Now recall that $4Q_j$ meets Ω^c_λ. This means that there exists $x_j \in 4Q_j \subset 4B_j$ with

$$\mathcal{M}^L_\eta F(x_j) \le \lambda. \tag{3.13}$$

Hence applying Lemma 3.11 to the ball $4B_j$ implies that there for all $K \ge \widetilde{K_0}$,

$$\big\{ E_\lambda \cap 4B_j \big\} \subset \big\{ x \in \mathbb{R}^n : \mathcal{M}^L_\eta \big(F \mathbf{1}_{12B_j}\big)(x) > (K/\widetilde{K_0})\lambda \big\}. \tag{3.14}$$

Now we observe that the hypotheses (3.4), (3.5), and (3.6) may be applied to the ball $12B_j$ (since $j \in \mathcal{J}_0$) and hence $12B_j$ satisfies

$$r_{12B_j} = 12r_{B_j} = 6\sqrt{n}\,\ell(Q_j) \leq 12\sqrt{n}\,\rho(x_{Q_j}) = 12\sqrt{n}\rho\big(x_{12B_j}\big). \tag{3.15}$$

Combining (3.4) with (3.14), and the fact that $\mathcal{M}_\eta^L$ is sublinear,

$$\begin{aligned}|E_\lambda \cap B_j| &\leq \big|\{E_\lambda \cap 4B_j\}\big| \\ &\leq \big|\big\{x \in \mathbb{R}^n : \mathcal{M}_\eta^L\big(F\mathbf{1}_{12B_j}\big) > (K/\widetilde{K_0})\lambda\big\}\big| \\ &\leq \big|\big\{x \in \mathbb{R}^n : \mathcal{M}_\eta^L\big(G_{12B_j}\mathbf{1}_{12B_j}\big)(x) > (K/2\widetilde{K_0})\lambda\big\}\big| \\ &\quad + \big|\big\{x \in \mathbb{R}^n : \mathcal{M}_\eta^L\big(H_{12B_j}\mathbf{1}_{12B_j}\big)(x) > (K/2\widetilde{K_0})\lambda\big\}\big|. \end{aligned}\tag{3.16}$$

Now recall that $E_\lambda \cap Q_j$ is assumed to be not empty. Hence there exists $\widetilde{x_j} \in Q_j \subset B_j$ with

$$G(\widetilde{x_j}) \leq \frac{\gamma}{2}\lambda. \tag{3.17}$$

Let c_p be the weak (p,p) bound of $\mathcal{M}_\eta^L$ (from Lemma 3.6). Applying assumption (3.5), valid because of (3.15), we obtain

$$\begin{aligned}\big|\big\{x \in \mathbb{R}^n : \mathcal{M}_\eta^L\big(G_{12B_j}\mathbf{1}_{12B_j}\big)(x) > (K/2\widetilde{K_0})\lambda\big\}\big| &\leq \frac{c_1 2\widetilde{K_0}}{K\lambda}\int_{12B_j} G_{12B_j} \\ &\leq \frac{c_1 2\widetilde{K_0}}{K\lambda}\,|12B_j|\,G(\widetilde{x_j}) \\ &\leq \frac{12^n c_1\widetilde{K_0}}{K}\,|B_j|\,\gamma. \end{aligned}\tag{3.18}$$

Next suppose that $q < \infty$. We apply (3.6)—again since (3.15) holds—to get

$$\begin{aligned}&\big|\big\{x \in \mathbb{R}^n : \mathcal{M}_\eta^L\big(H_{12B_j}\mathbf{1}_{12B_j}\big)(x) > (K/2\widetilde{K_0})\lambda\big\}\big| \\ &\leq \left(\frac{2\widetilde{K_0}c_q}{K\lambda}\right)^q \int_{12B_j} H_{12B_j}^q \\ &\leq \left(\frac{2\widetilde{K_0}c_q}{K\lambda}\right)^q \xi^q\left(\mathcal{M}_\eta^L F(x_j) + G(\widetilde{x_j})\right)^q\,|12B_j| \\ &\leq (4\widetilde{K_0}c_q)^q\,12^n\frac{\xi^q}{K^q}\,|B_j|, \end{aligned}\tag{3.19}$$

where the points x_j and $\widetilde{x_j}$ satisfy (3.14) and (3.17) respectively. We insert now estimates (3.18) and (3.19) into (3.16) to arrive at

$$|E_\lambda \cap Q_j| \le |E_\lambda \cap B_j| \le C\bigg(\frac{\xi^q}{K^q} + \frac{\gamma}{K}\bigg)\,|Q_j|$$

where C depends on q, n, $\widetilde{K_0}$ and the weak type bounds of $\mathcal{M}^L_\eta$. This gives (3.11) for the case $q < \infty$, and hence from (3.12) we get (3.10) for those cubes Q_j with $j \in \mathcal{J}_0$.

If $q = \infty$, then firstly notice that

$$\big\|\mathcal{M}^L_\eta\big(H_{12B_j}\mathbf{1}_{12B_j}\big)\big\|_{L^\infty} \le \big\|H_{12B_j}\mathbf{1}_{12B_j}\big\|_{L^\infty} \le \xi\left(\mathcal{M}^L_\eta F(x_j) + G(\widetilde{x_j})\right) \le 2\xi\lambda.$$

Therefore it follows that whenever $K \ge 4\xi\widetilde{K_0}$, then

$$\big\{x \in \mathbb{R}^n : \mathcal{M}^L_\eta\big(H_{12B_j}\mathbf{1}_{12B_j}\big)(x) > (K/2\widetilde{K_0})\lambda\big\} = \emptyset.$$

So we set $K_0 = 4\xi\widetilde{K_0} \ge 1$, and for each $K \ge K_0$ we may proceed as before with estimates (3.18) and (3.19) to obtain the following variant of (3.11):

$$|E_\lambda \cap Q_j| \le C\left(\frac{\gamma}{K}\right)^{1/s}|Q_j|\,.$$

Before concluding the proof of the case $j \in \mathcal{J}_0$, we remark that taking the choice $K_0 = 4\xi\widetilde{K_0}$ will allow us to cover both of the situations $q < \infty$, and $q = \infty$.

We turn to the proof of (3.10) for the case $j \in \mathcal{J}_\infty$. We shall require the following decomposition lemma.

Lemma 3.12 ([**34**, Lemma 3.1]). *For any cube Q with $\ell(Q) > 2\rho(x_Q)$ there exists a finite collection of disjoint subcubes $\{Q_k\}_{k=1}^N$ such that $Q = \bigcup_{k=1}^N Q_k$ with the following property: for every $k \in \{1, \dots, N\}$, there exists $x_k \in Q_k$ with*

$$\frac{1}{2}\,\ell(Q_k) \le \rho(x_k) \le 2\sqrt{n}\,C_0\,\ell(Q_k),$$

where C_0 is the constant from Lemma 2.1*.*

Recall that when $j \in \mathcal{J}_\infty$ the cube Q_j satisfies $\ell(Q_j) > 2\rho(x_{Q_j})$. Hence we may apply Lemma 3.12 to Q_j and obtain a finite collection of disjoint subcubes $\{Q_{j,k}\}_{k=1}^{N_j}$, with $Q_j = \bigcup_{k=1}^{N_j} Q_{j,k}$, such that for each $k \in \{1, \dots, N_j\}$ there exists $x_{j,k} \in Q_{j,k}$ with

$$\frac{1}{2}\,\ell(Q_{j,k}) \le \rho(x_{j,k}) \le 2\sqrt{n}\,C_0\,\ell(Q_{j,k}). \tag{3.20}$$

We observe that this implies $\rho(x_{j,k}) \approx \rho(x_{Q_{j,k}})$ with constants depending only on n and C_0, where $x_{Q_{j,k}}$ is the centre of the cube $Q_{j,k}$. Indeed, since $x_{j,k}, x_{Q_{j,k}} \in Q_{j,k}$ then

$$x_{Q_{j,k}} \in B\bigg(x_{j,k}, \frac{\sqrt{n}}{2}\ell(Q_{j,k})\bigg) \subseteq \sqrt{n}\,B\big(x_{j,k}, \rho(x_{j,k})\big)$$

and hence by (2.3) we have $\rho(x_{Q_{j,k}}) \le C_0^2(1+\sqrt{n})^2\rho(x_{j,k})$. The other inequality can be obtained similarly.

Now for each j and k we set $B_{j,k}$ to be the ball concentric with $Q_{j,k}$ but with radius $(\sqrt{n}/2)\ell(Q_{j,k})$. That is, $B_{j,k}$ is the smallest ball concentric with, and containing $Q_{j,k}$. We claim the following property holds, whose proof we defer to the end of this section.

Lemma 3.13. *There exists $\alpha \ge 1$, depending only on n, η and C_0, with the following property: for every cube $Q_{j,k}$ for which $E_\lambda \cap Q_{j,k}$ is non-empty, one has*

$$E_\lambda \cap Q_{j,k} \subset \big\{x \in Q_{j,k} : \mathcal{M}_\eta^L\big(F\mathbf{1}_{\alpha B_{j,k}}\big)(x) > K\lambda\big\} \tag{3.21}$$

$$r_{\alpha B_{j,k}} > 12\sqrt{n}\,\rho\big(x_{\alpha B_{j,k}}\big). \tag{3.22}$$

Let us fix k and assume that $E_\lambda \cap Q_{j,k}$ is not empty, since otherwise there is nothing to prove for the cube $Q_{j,k}$. This implies that there exists a point $\widetilde{x}_{j,k} \in Q_{j,k} \subset \alpha B_{j,k}$ with

$$G\big(\widetilde{x}_{j,k}\big) \le \frac{\gamma}{2}\lambda. \tag{3.23}$$

Let c_1 be the weak $(1,1)$ bound of $\mathcal{M}_\eta^L$. Then (3.21) gives

$$\begin{aligned}|E_\lambda \cap Q_{j,k}| &\le \big|\big\{x \in Q_{j,k} : \mathcal{M}_\eta^L\big(F\mathbf{1}_{\alpha B_{j,k}}\big)(x) > K\lambda\big\}\big| \le \frac{c_1}{K\lambda}\int_{\alpha B_{j,k}} F \\ &\le \frac{c_1}{K\lambda}\,|\alpha B_{j,k}|\;\psi_\eta(\alpha B_{j,k})\,G(\widetilde{x}_{j,k}) \le C\frac{\gamma}{K}\,|Q_{j,k}|\,.\end{aligned} \tag{3.24}$$

In the third inequality we have applied hypothesis (3.7)—since the ball $\alpha B_{j,k}$ satisfies (3.22)—and in the final inequality we used (3.23), the doubling property for the Lebesgue measure, and that

$$\psi_\eta(\alpha B_{j,k}) \le \alpha^\eta\,\psi_\eta(B_{j,k}) \le C,$$

which follows from (3.20). We remark that the constant C in (3.24) depends only on n, η, C_0 and is independent of j and k.

In a similar fashion to estimate (3.12), we apply Lemma 3.3 to $\nu \in \mathcal{B}_{s'}^{L,\theta}$ and the sets $E_\lambda \cap Q_{j,k} \subset Q_{j,k}$ and evoke (3.24) to obtain

$$\nu(E_\lambda \cap Q_{j,k}) \le C_\nu\,\psi_\theta(Q_{j,k})\bigg(\frac{|E_\lambda \cap Q_{j,k}|}{|Q_{j,k}|}\bigg)^{1/s}\nu(Q_{j,k}) \le C\left(\frac{\gamma}{K}\right)^{1/s}\nu(Q_{j,k})$$

where C depends on n, C_0, η and ν. Summing this over k gives

$$\nu(E_\lambda \cap Q_j) \le \sum_{k=1}^{N_j}\nu(E_\lambda \cap Q_{j,k}) \le C\bigg(\frac{\gamma}{K}\bigg)^{1/s}\sum_{k=1}^{N_j}\nu(Q_{j,k}) \le C\bigg(\frac{\xi^q}{K^q}+\frac{\gamma}{K}\bigg)^{1/s}\nu(Q_j)$$

which gives (3.10) for $j \in \mathcal{J}_\infty$. Note that when $q = \infty$ we end the estimate at the second inequality. This concludes the proof of (3.10), and hence of (3.8).

Since (3.8) holds we may prove (3.9) using the same approach as the final part of the proof of Theorem 3.1 from [**6**, pp. 20–21]. In fact the proof is identical but with $\mathcal{M}_\eta^L$ in place of the Hardy–Littlewood maximal operator M, and $\mathcal{B}_{s'}^{L,\theta}$ in place of $\mathcal{B}_{s'}$. We omit the details. □

We end this section with the proofs of the lemmata that were deferred during the proof of Theorem 3.9.

Proof of Lemma 3.11. This proof is an adaptation of the localisation lemma from [**3**]. Let $x \in E_\lambda \cap \widetilde{B}$. Then it follows that

$$G(x) \leq \frac{\gamma}{2}\lambda, \tag{3.25}$$

$$\mathcal{M}_\eta^L F(x) > K\lambda. \tag{3.26}$$

The latter property ensures that there exists a ball B containing x such that

$$\frac{1}{\psi_\eta(B)} \fint_B F > K\lambda. \tag{3.27}$$

Then we necessarily have

$$r_B \leq 12\sqrt{n}\,\rho(x_B). \tag{3.28}$$

Suppose otherwise. Then hypothesis (3.7) applies to B. Combining this with (3.25) and (3.27), we arrive at the statement

$$K\lambda < \frac{1}{\psi_\eta(B)} \fint_B F \leq G(x) \leq \frac{\gamma}{2}\lambda,$$

which is impossible, since $K \geq 1$ and $\gamma \in (0,1)$. Therefore the ball B necessarily satisfies (3.28).

From Lemma 2.1 and (3.28), since $x \in B \subset 12\sqrt{n}\,B\big(x_B, \rho(x_B)\big)$, we have

$$\rho(x_B) \leq C_1\,\rho(x), \tag{3.29}$$

where $C_1 = C_0^2(1+12\sqrt{n})^2 > 1$. This gives

$$\psi_\eta(B) \geq \left(1 + \frac{r_B}{C_1\rho(x)}\right)^\eta \geq (2C_1)^{-\eta}\left(1 + \frac{2r_B}{\rho(x)}\right)^\eta = (2C_1)^{-\eta}\psi_\eta\big(B(x,2r_B)\big).$$

Now note also that $B \subset B(x,2r_B) \subset 3B$ so that $|B| \geq 3^{-n}\,|B(x,2r_B)|$.

Therefore

$$\int_{B(x,2r_B)} F > \int_B F > K\lambda\,|B|\,\psi_\eta(B) \geq \frac{K\lambda\,|B(x,2r_B)|}{3^n}\,\frac{\psi_\eta\big(B(x,2r_B)\big)}{(2C_1)^\eta}.$$

This implies

$$\frac{1}{\psi_\eta(B(x,2r_B))} \fint_{B(x,2r_B)} F > \frac{K}{\widetilde{K_0}}\lambda \tag{3.30}$$

where $\widetilde{K_0} = 3^n(2C_1)^\eta$. Now since $K \geq \widetilde{K_0}$, then in fact

$$\frac{1}{\psi_\eta(B(x,2r_B))} \fint_{B(x,2r_B)} F > \lambda$$

and this combined with the point $\widetilde{x}$ from the hypothesis implies that $\widetilde{x} \notin B(x,2r_B)$, for otherwise this contradicts $\mathcal{M}_\eta^L F(\widetilde{x}) \leq \lambda$. This final fact implies that $B(x,2r_B) \subset 3\widetilde{B}$, and combining this with (3.30) gives

$$\frac{1}{\psi_\eta(B(x,2r_B))} \fint_{B(x,2r_B)} F\mathbf{1}_{3\widetilde{B}} = \frac{1}{\psi_\eta(B(x,2r_B))} \fint_{B(x,2r_B)} F > \frac{K}{\widetilde{K_0}}\lambda.$$

This last step ensures $\mathcal{M}_\eta^L\big(F\mathbf{1}_{3\widetilde{B}}\big)(x) > (K/\widetilde{K_0})\lambda$. $\square$

Proof of Lemma 3.13. Let $x \in E_\lambda \cap Q_{j,k}$. Then arguing as in the proof of Lemma 3.11 we see that there exists a ball B containing x satisfying

$$\frac{1}{\psi_\eta(B)} \fint_B F > K\lambda, \tag{3.31}$$

$$r_B \leq 12\sqrt{n}\,\rho(x_B), \tag{3.32}$$

$$\rho(x_B) \leq C_1\,\rho(x) \tag{3.33}$$

where $C_1 = C_0^2(1+12\sqrt{n})^2$.

We claim now that there exists $\alpha \geq 1$, depending only on C_0, n and η, such that (3.22) holds and

$$B \subset \alpha B_{j,k}. \tag{3.34}$$

Let us demonstrate this claim. This will involve repeated application of (2.3).

Since both $x, x_{j,k} \in Q_{j,k}$, then the distance between x and $x_{j,k}$ is at most the diameter of $Q_{j,k}$. That is,

$$|x - x_{j,k}| \leq \operatorname{diam}(Q_{j,k}) = \sqrt{n}\,\ell(Q_{j,k}).$$

It follows that $x \in B\big(x_{j,k}, \sqrt{n}\,\ell(Q_{j,k})\big) \subset 2\sqrt{n}\,B\big(x_{j,k}, \rho(x_{j,k})\big)$, and hence

$$\rho(x) \leq C_2\,\rho(x_{j,k}) \tag{3.35}$$

where $C_2 = C_0^2(1+2\sqrt{n})^2$. We now combine (3.33) and (3.35) with (3.20) and (3.32) to obtain

$$\begin{aligned} r_B &\leq 12\sqrt{n}\,\rho(x_B) \leq 12\sqrt{n}\,C_1\,\rho(x) \\ &\leq 12\sqrt{n}\,C_1\,C_2\,\rho(x_{j,k}) \leq \alpha_0\frac{\sqrt{n}}{2}\,\ell(Q_{j,k}) = \alpha_0\,r_{B_{j,k}}, \end{aligned}$$

where $\alpha_0 = 48C_0C_1C_2\sqrt{n}$. Therefore it follows that $B \subset (1+2\alpha_0)B_{j,k}$. Next we set $\widetilde{\alpha}$ to be a number such that

$$r_{\widetilde{\alpha}B_{j,k}} > 12\sqrt{n}\,\rho\big(x_{Q_{j,k}}\big).$$

Note that this number exists because we recall that $\rho(x_{Q_{j,k}}) \approx \rho(x_{j,k}) \approx \ell(Q_{j,k}) \approx r_{B_{j,k}}$ with constants depending on C_0 and n. In fact, $\rho(x_{Q_{j,k}}) \geq C_3\rho(x_{j,k})$ where $C_3 = C_0^2(1+\sqrt{n})^2$, so that $\widetilde{\alpha}(\sqrt{n}/2)\ell(Q_{j,k}) > 12\sqrt{n}C_3\,\rho(x_{j,k})$, which holds provided $\widetilde{\alpha} \geq 43\sqrt{n}C_0C_3$ by (3.20). On choosing $\alpha = \max\{1+2\alpha_0, \widetilde{\alpha}\}$, the estimate (3.22) and the claim (3.34) both hold.

Finally to obtain the inclusion (3.21), we see that (3.34) with (3.31) implies

$$\frac{1}{\psi_\eta(B)}\fint_B F\mathbf{1}_{\alpha B_{j,k}} = \frac{1}{\psi_\eta(B)}\fint_B F > K\lambda.$$

It necessarily follows that

$$\mathcal{M}_\eta^L(F\mathbf{1}_{\alpha B_{j,k}})(x) > K\lambda,$$

and as a consequence (3.21) holds. This ends the proof of Lemma 3.13. □

Next we show how Theorem 3.9 may be applied to the study of operators in the following result. It is inspired by and formalizes the method used in [**7**]. We shall apply this tool to prove Theorem 1.1.

Theorem 3.14. *Let $1 \leq p_0 < q_0 \leq \infty$ and T be a linear operator. Assume that for each $\tilde{q} \in (p_0, q_0)$ and $\eta > 0$ there exists a family of operators $\{A_B\}_B$ indexed by balls and a collection of scalars $\{\alpha_j\}_{j=0}^\infty$ such that the following holds.*

(i) *T is bounded on $L^{\tilde{q}}(\mathbb{R}^n)$.*
(ii) *For every ball B with $r_B \leq 12\sqrt{n}\,\rho(x_B)$, and every $f \in L_c^\infty(\mathbb{R}^n)$ supported in B,*

$$\left(\fint_{U_j(B)} |A_B f|^{\tilde{q}}\right)^{1/\tilde{q}} \leq \alpha_j\left(\fint_B |f|^{p_0}\right)^{1/p_0}, \qquad \forall j \geq 0 \tag{3.36}$$

$$\left(\fint_{U_j(B)} |T(I-A_B) f|^{\tilde{q}}\right)^{1/\tilde{q}} \leq \alpha_j\left(\fint_B |f|^{p_0}\right)^{1/p_0}, \qquad \forall j \geq 2. \tag{3.37}$$

(iii) *There exists $\widetilde{C} > 0$ such that for every ball B with $r_B > 12\sqrt{n}\,\rho(x_B)$ and $f \in L_c^\infty(\mathbb{R}^n)$,*

$$\left(\frac{1}{\psi_\eta(B)} \fint_B |T^* f|^{\tilde{q}'}\right)^{1/\tilde{q}'} \leq \widetilde{C}\,\mathcal{M}_\eta^L\big(|f|^{\tilde{q}'}\big)(x)^{1/\tilde{q}'}, \qquad \forall x \in B. \tag{3.38}$$

(iv) *The constants $\{\alpha_j\}$ satisfy $\sum_j \alpha_j 2^{j(n+\eta)} < \infty$.*

Let $p \in (p_0, q_0)$ and $w \in \mathcal{A}^L_{p/p_0} \cap \mathcal{B}^L_{(q_0/p)'}$. Then T extends to a bounded operator on $L^p(w)$.

Proof of Theorem 3.14. The proof is an adaptation of the argument in [**7**], [**2**]. We fix $p \in (p_0, q_0)$ and $w \in \mathcal{A}^L_{p/p_0} \cap \mathcal{B}^L_{(q_0/p)'}$. Denote by T^* the adjoint of T. Then it will suffice to prove that T^* is bounded on $L^{p'}(w^{1-p'})$, because this is equivalent to the $L^p(w)$ boundedness of T (see [**6**, Remark 4.5]). We shall apply Theorem 3.9 to T^*.

Firstly, by Lemma 3.5 property (ix), there exists numbers p_1 and q_1 such that

$$p_0 < p_1 < p < q_1 < q_0 \quad \text{and} \quad w \in \mathcal{A}^L_{p/p_1} \cap \mathcal{B}^L_{(q_1/p)'}.$$

Then it follows from property (x) of Lemma 3.5 that

$$w^{1-p'} \in \mathcal{A}^L_{p'/q_1'} \cap \mathcal{B}^L_{(p_1'/p')'}.$$

Then there also exists $\theta \geq 0$ such that

$$w^{1-p'} \in \mathcal{A}^{L,\theta}_{p'/q_1'}.$$

We now apply Theorem 3.9 to the following datum. For each $f \in L_c^\infty(\mathbb{R}^n)$ we set

$$s := \frac{p_1'}{p'}, \qquad q := \frac{p_1'}{q_1'}, \qquad r := \frac{p'}{q_1'}, \qquad \eta := r'\theta,$$
$$F := |T^* f|^{q_1'}, \qquad H := 0, \qquad \nu := w^{1-p'}.$$

Let $\tilde{q} = q_1$. Take $\{A_B\}_B$ and $\{\alpha_j\}_j$ to be as in the hypotheses. We shall show that conditions (3.4)–(3.7) hold with

$$G_B := 2^{q_1'-1}\,|(I - A_B)^* T^* f|^{q_1'} \quad \text{and} \quad H_B := 2^{q_1'-1}\,|A_B^* T^* f|^{q_1'},$$

and G is a fixed constant multiple of $\mathcal{M}_\eta^L\big(|f|^{q_1'}\big)$ (with the constant to be specified later).

We first check condition (3.4). By noting that $(I - A_B^*) = (I - A_B)^*$, one has

$$\begin{aligned} F(x) = |T^* f(x)|^{q_1'} &= |(I - A_B)^* T^* f(x) + A_B^* T^* f(x)|^{q_1'} \\ &\leq 2^{q_1'-1}\,|(I - A_B^*) T^* f(x)|^{q_1'} + 2^{q_1'-1}\,|A_B^* T^* f(x)|^{q_1'} \end{aligned}$$

$$= G_B(x) + H_B(x).$$

We now check condition (3.6). Let B be a ball with $r_B \le 12\sqrt{n}\,\rho(x_B)$. We first write

$$\left(\fint_B H_B^q\right)^{1/q} = \left(\fint_B 2^{p_1'-p_1'/q_1'}\,|A_B^*T^*f|^{p_1'}\right)^{q_1'/p_1'} \lesssim \left(\fint_B |A_B^*T^*f|^{p_1'}\right)^{q_1'/p_1'}.$$

To estimate the integral we apply duality to T^* and A_B^* with some $g \in L^{p_1}(B, dx/|B|)$ with norm 1, to obtain for each $x \in B$,

$$\begin{aligned}
\left(\fint_B H_B^q\right)^{1/qq_1'} &\lesssim \left(\fint_B |A_B^*T^*f|^{p_1'}\right)^{1/p_1'} \le \fint_B |T^*f|\,|A_Bg| \le \sum_{j=0}^{\infty} 2^{jn} \fint_{U_j(B)} |T^*f|\,|A_Bg| \\
&\le \sum_{j=0}^{\infty} 2^{jn}\psi_\eta(2^jB)\left(\frac{1}{\psi_\eta(2^jB)}\fint_{2^jB} |T^*f|^{q_1'}\right)^{1/q_1'} \\
&\quad\times\left(\frac{1}{\psi_\eta(2^jB)}\fint_{U_j(B)} |A_Bg|^{q_1}\right)^{1/q_1} \\
&\lesssim \mathcal{M}_\eta^L\big(|T^*f|^{q_1'}\big)(x)^{1/q_1'}\sum_{j=0}^{\infty} 2^{j(n+\eta)}\left(\fint_{U_j(B)} |A_Bg|^{q_1}\right)^{1/q_1}.
\end{aligned} \tag{3.39}$$

In the last line we have used that since $r_B \le 12\sqrt{n}\,\rho(x_B)$, then

$$\psi_\eta(2^jB) \le 2^{j\eta}\,\psi_\eta(B) \le 2^{j\eta}\,(1+12\sqrt{n})^\eta \tag{3.40}$$

valid for every $j \ge 0$. Now from (3.36) with exponent $\tilde{q} = q_1$, we have for each $j \ge 0$,

$$\left(\fint_{U_j(B)} |A_Bg|^{q_1}\right)^{1/q_1} \le \alpha_j\left(\fint_B |g|^{p_0}\right)^{1/p_0} \le \alpha_j\left(\fint_B |g|^{p_1}\right)^{1/p_1} = \alpha_j,$$

where we have used Hölder's inequality (with exponents p_1/p_0 and $(p_1/p_0)'$) and the normalisation of g. Inserting this estimate into (3.39) gives, for each $x \in B$,

$$\left(\fint_B H_B^q\right)^{1/qq_1'} \lesssim \mathcal{M}_\eta^L\big(|T^*f|^{q_1'}\big)(x)^{1/q_1'}\sum_{j=0}^{\infty}\alpha_j 2^{j(n+\eta)} \le C_1\mathcal{M}_\eta^L\big(|T^*f|^{q_1'}\big)(x)^{1/q_1'}$$

by hypothesis (iv). Hence (3.6) holds with $H = 0$ and $\xi = C_1$.

Next we check condition (3.5). Let B be a ball with $r_B \le 12\sqrt{n}\,\rho(x_B)$. We first write

$$\left(\fint_B G_B\right)^{1/q_1'} = \left(\fint_B 2^{q_1'-1}\,|(I-A_B)^*T^*f|^{q_1'}\,dx\right)^{1/q_1'} \lesssim \left(\fint_B |(I-A_B)^*T^*f|^{q_1'}\,dx\right)^{1/q_1'}.$$

We apply duality again now with I and $(I-A_B)^*T^*$ with $g \in L^{q_1}(B, dx/|B|)$ of norm 1. Then for each $x \in B$,

$$\begin{aligned}\left(\fint_B G_B\right)^{1/q_1'} &\lesssim \fint_B |f|\,|T(I-A_B)g| \le \sum_{j=0}^{\infty} 2^{jn} \fint_{U_j(B)} |f|\,|T(I-A_B)g| \\ &\le \sum_{j=0}^{\infty} 2^{jn}\psi_\eta(2^jB)\left(\frac{1}{\psi_\eta(2^jB)}\fint_{2^jB}|f|^{q_1'}\right)^{1/q_1'} \\ &\quad\times\left(\frac{1}{\psi_\eta(2^jB)}\fint_{U_j(B)}|T(I-A_B)g|^{q_1}\right)^{1/q_1} \\ &\lesssim \mathcal{M}_\eta^L(|f|^{q_1'})(x)^{1/q_1'}\sum_{j=0}^{\infty}2^{j(n+\eta)}\left(\fint_{U_j(B)}|T(I-A_B)g|^{q_1}\right)^{1/q_1}. \qquad (3.41)\end{aligned}$$

In the last line we applied (3.40) again. Now for each $j \ge 2$, estimate (3.37) with exponent $\tilde{q} = q_1$ gives

$$\left(\fint_{U_j(B)}|T(I-A_B)g|^{q_1}\right)^{1/q_1} \le \alpha_j\left(\fint_B |g|^{p_0}\right)^{1/p_0} \le \alpha_j\left(\fint_B |g|^{q_1}\right)^{1/q_1} = \alpha_j, \qquad (3.42)$$

where we have used Hölder's inequality (with exponents q_1/p_0 and $(q_1/p_0)'$) and the normalisation of g. For $j = 0, 1$ we use hypothesis (i) with $\tilde{q} = q_1$ to give

$$\begin{aligned}\fint_{U_j(B)}|T(I-A_B)g|^{q_1} &\lesssim \frac{1}{|B|}\int_{\mathbb{R}^n}|(I-A_B)g|^{q_1} \\ &\lesssim \frac{1}{|B|}\left\{\int_B |g|^{q_1} + \sum_{k=0}^{\infty}\int_{U_k(B)}|A_Bg|^{q_1}\right\}.\end{aligned}$$

For the summands we use the approach as before, namely applying (3.36) for $k \ge 0$, and Hölder's inequality to get

$$\left(\fint_{U_k(B)}|A_Bg|^{q_1}\right)^{1/q_1} \le \alpha_k\left(\fint_B |g|^{p_0}\right)^{1/p_0} \le \alpha_k\left(\fint_B |g|^{q_1}\right)^{1/q_1} = \alpha_k.$$

Collecting these estimates we have for $j = 0, 1$,

$$\fint_{U_j(B)}|T(I-A_B)g|^{q_1} \lesssim \fint_B |g|^{q_1} + \sum_{k=0}^{\infty}2^{kn}\fint_{U_k(B)}|A_Bg|^{q_1}$$

$$\lesssim \fint_B |g|^{q_1} + \sum_{k=0}^{\infty} \alpha_k^{q_1} 2^{kn}, \tag{3.43}$$

which is finite because the expression $\sum_k \alpha_k 2^{k(n+\eta)}$ is finite. Inserting (3.42) and (3.43) into (3.41) gives

$$\begin{aligned}\left(\fint_B G_B\right)^{1/q_1'} &\lesssim \mathcal{M}_\eta^L\big(|f|^{q_1'}\big)(x)^{1/q_1'} \left\{\sum_{j=2}^{\infty} \alpha_j 2^{j(n+\eta)} + C\right\} \\ &\le C_2 \mathcal{M}_\eta^L\big(|f|^{q_1'}\big)(x)^{1/q_1'} \end{aligned}\tag{3.44}$$

for each $x \in B$.

Now let $G(x) := C_3\, \mathcal{M}_\eta^L\big(|f|^{q_1'}\big)(x)^{1/q_1'}$, where $C_3 = \max\{\widetilde{C}, C_2\}$. Here $\widetilde{C}$ is the constant from hypothesis (iii), and C_2 is the constant from (3.44). With this choice of G, firstly estimate (3.44) implies that (3.5) holds, and secondly estimate (3.38) implies that (3.7) holds.

We have shown that (3.4)–(3.7) holds. Therefore, since $\nu \in \mathcal{B}^L_{(p_1'/p')'} \equiv \mathcal{B}^L_{s'}$, then Theorem 3.9 allows us to conclude that

$$\big\|\mathcal{M}_\eta^L\big(|T^* f|^{q_1'}\big)\big\|_{L^r(\nu)} \le C \big\|\mathcal{M}_\eta^L\big(|f|^{q_1'}\big)\big\|_{L^r(\nu)} \tag{3.45}$$

for some $C > 0$, depending only on ν, q, n, ξ, s, η, γ, C_3, and hence only on w, p, p_1, q_1, C_1, C_2, $\widetilde{C}$. Recalling that $r = p'/q_1'$ and $\nu = w^{1-p'}$, we observe that the $L^{p'}(w^{1-p'})$ boundedness of T^* now follows, because

$$\|T^* f\|^{q_1'}_{L^{p'}(\nu)} \le 2^\eta \big\|\mathcal{M}_\eta^L\big(|T^* f|^{q_1'}\big)\big\|_{L^r(\nu)} \le C \big\|\mathcal{M}_\eta^L\big(|f|^{q_1'}\big)\big\|_{L^r(\nu)} \le C\, \|f\|^{q_1'}_{L^{p'}(\nu)}. \tag{3.46}$$

The first inequality in (3.46) holds by the pointwise control of the operator $\mathcal{M}_\eta^L$ (see (3.3)). The second inequality in (3.46) follows from the conclusion (3.45) above. The final inequality in (3.46) follows from the boundedness of the maximal operator $\mathcal{M}_\eta^L\big(|\cdot|^{q_1'}\big)^{1/q_1'}$ on $L^{p'}(\nu)$. Indeed, Remark 3.8 applies in this situation because firstly $p' > q_1'$, secondly $\nu = w^{1-p'} \in \mathcal{A}^{L,\theta}_{p'/q_1'}$, and lastly $\eta = r'\theta = (p'/q_1')'\theta$.

By duality, (3.46) implies the boundedness of T on $L^p(w)$. □

3.2. Proof of Theorem 1.1.

We first consider the operator $\nabla^2 L^{-1}$. We apply Theorem 3.14 to $T = \nabla^2 L^{-1}$, $p_0 = 1$, $q_0 = s$, and $A_B = e^{-r_B^2 L}$. Fix $\widetilde{q} \in (1, s)$ and $\eta > 0$. We shall show that conditions (i)–(iv) of Theorem 3.14 hold. For simplicity we shall write q to denote $\widetilde{q}$ throughout the rest of this proof.

Firstly the $L^q(\mathbb{R}^n)$ boundedness of T holds from Table 1 of Section 1, and so Theorem 3.14 (i) holds easily. Next we check conditions Theorem 3.14 (ii) and (iv). Fix a ball B and a function $f \in L^\infty_c(\mathbb{R}^n)$ supported in B. We have (via the bounds (2.4) on the heat kernel of L):

$$\left(\fint_{U_j(B)} |A_B f|^q \right)^{1/q} \le C_1 e^{-c_1 4^j} \fint_B |f|, \tag{3.47}$$

if $j \ge 0$. Note that the constants C_1, c_1 depend on q and n only. Indeed, for each $j \ge 2$, $x \in U_j(B)$, and $y \in B$, we observe that $|x-y| \ge 2^j r_B/4$, and hence the bounds (2.4) imply that

$$\sup_{x \in U_j(B)} \left| e^{-r_B^2 L} f(x) \right| \le \sup_{x \in U_j(B)} \int_B \left| p_{r_B^2}(x,y) \right| |f(y)|\, dy \lesssim e^{-c4^j} \fint_B |f|.$$

By Hölder's inequality these bounds give for each $j \ge 2$ and $q \ge 1$

$$\left(\fint_{U_j(B)} |A_B f|^q \, dx \right)^{1/q} \lesssim \left(\fint_{U_j(B)} e^{-c4^j} \left(\fint_B |f| \right)^q dx \right)^{1/q} \le e^{-c4^j} \fint_B |f|.$$

The same approach gives for $j = 0, 1$

$$\left(\fint_{U_j(B)} |A_B f|^q \right)^{1/q} \lesssim \fint_B |f|.$$

These two estimates give (3.47).

Next we recall from Lemma 2.8, and in particular estimate (2.24), that

$$\left(\fint_{U_j(B)} \left| \nabla^2 L^{-1} (I - e^{-r_B^2 L}) f \right|^q \right)^{1/q} \le C_2\, e^{-c_2 4^j} \fint_B |f|, \quad \forall j \ge 2. \tag{3.48}$$

Let us take $\alpha_j = C e^{-c4^j}$ for $j \ge 0$, where $C = \max\{C_1, C_2\}$ and $c = \min\{c_1, c_2\}$. Then Theorem 3.14 (iv) is satisfied, and by (3.47) and (3.48), conditions (3.36) and (3.37) are also satisfied. This proves (ii) and (iv).

Finally we turn to condition (iii) of Theorem 3.14. Let $f \in L_c^\infty(\mathbb{R}^n)$ and fix a ball B with $r_B > 12\sqrt{n}\, \rho(x_B)$. We write

$$f = \sum_{j=0}^{\infty} f \mathbf{1}_{U_j(B)} =: \sum_{j=0}^{\infty} f_j.$$

Then

$$\left(\frac{1}{\psi_\eta(B)} \fint_B |T^* f|^{q'} \right)^{1/q'} \le \sum_{j=0}^{\infty} \left(\frac{1}{\psi_\eta(B)} \fint_B |T^* f_j|^{q'} \right)^{1/q'}. \tag{3.49}$$

To estimate the terms for $j = 0, 1$, we use that T^* is bounded on $L^{q'}(\mathbb{R}^n)$, and that $\psi_\eta(2B) \le 2^\eta\, \psi_\eta(B)$ to obtain, for any $x \in B$,

$$\begin{aligned}\Big(\frac{1}{\psi_\eta(B)}\fint_B |T^*f_j|^{q'}\Big)^{1/q'} &\le C\Big(\frac{1}{\psi_\eta(B)\,|B|}\int_B |f_j|^{q'}\Big)^{1/q'}\\ &\le C\Big(\frac{\psi_\eta(2B)\,|2B|}{\psi_\eta(B)\,|B|}\Big)^{1/q'}\Big(\frac{1}{\psi_\eta(2B)}\fint_{2B}|f|^{q'}\Big)^{1/q'}\\ &\le C\mathcal{M}^L_\eta\big(|f|^{q'}\big)(x)^{1/q'}. \end{aligned}\tag{3.50}$$

Note that C depends on n, q and η. To estimate the terms for $j \ge 2$, we first write

$$\begin{aligned}|T^*f_j(y)| &= \Big|\int_0^\infty\int_{U_j(B)} \nabla_z^2 p_t(z,y)\, f(z)\,dz\,dt\Big|\\ &\le \Big(\int_{U_j(B)}|f|^{q'}\Big)^{1/q'}\int_0^\infty\Big(\int_{U_j(B)}\big|\nabla_z^2 p_t(z,y)\big|^q\,dz\Big)^{1/q}dt \end{aligned}\tag{3.51}$$

by Hölder's inequality. Next, using that $\psi_\eta(2^jB) \le 2^{j\eta}\,\psi_\eta(B)$ we have for any $x \in B$,

$$\begin{aligned}\Big(\int_{U_j(B)}|f|^{q'}\Big)^{1/q'} &= \big(\psi_\eta(2^jB)\,\big|2^jB\big|\big)^{1/q'}\Big(\frac{1}{\psi_\eta(2^jB)}\int_{U_j(B)}|f|^{q'}\Big)^{1/q'}\\ &\le \big(\psi_\eta(B)\,|B|\big)^{1/q'}\,2^{j(n+\eta)/q'}\,\mathcal{M}^L_\eta\big(|f|^{q'}\big)(x)^{1/q'}. \end{aligned}\tag{3.52}$$

Therefore using (3.51) and (3.52) we obtain, for each $j \ge 2$,

$$\begin{aligned}&\Big(\frac{1}{\psi_\eta(B)}\fint_B |T^*f_j|^{q'}\Big)^{1/q'}\\ &\le \frac{1}{\psi_\eta(B)^{1/q'}}\Big(\int_{U_j(B)}|f|^{q'}\Big)^{1/q'}\Big(\fint_B\Big(\int_0^\infty \big\|\nabla^2 p_t(\cdot,y)\big\|_{L^q(U_j(B))}\,dt\Big)^{q'}dy\Big)^{1/q'}\\ &\le 2^{j(n+\eta)/q'}\,|B|^{1/q'}\,\mathcal{M}^L_\eta\big(|f|^{q'}\big)(x)^{1/q'}\,\mathcal{I}(j,q,B) \end{aligned}\tag{3.53}$$

where

$$\mathcal{I}(j,q,B) := \Big(\fint_B\Big(\int_0^\infty \big\|\nabla^2 p_t(\cdot,y)\big\|_{L^q(U_j(B))}\,dt\Big)^{q'}dy\Big)^{1/q'}.$$

Now we estimate the final term in (3.53) by using the heat kernel bounds in Proposition 2.4 (c). For each $j \ge 2$ and $y \in B$, we have $|z-y| \ge 2^{j-2}r_B$. Hence for all $t > 0$ estimate (2.8) gives

$$\begin{aligned}\big\|\nabla^2 p_t(\cdot,y)\big\|_{L^q(U_j(B))} &= \big\|\nabla^2 p_t(\cdot,y)\,e^{\beta_q|\cdot-y|^2/t}\,e^{-\beta_q|\cdot-y|^2/t}\big\|_{L^q(U_j(B))}\\ &\le e^{-c4^jr_B^2/t}\big\|\nabla^2 p_t(\cdot,y)\,e^{\beta_q|\cdot-y|^2/t}\big\|_{L^q(U_j(B))}\end{aligned}$$

$$\leq \frac{C}{t^{1+n/2q'}}\, e^{-c4^j r_B^2/t}\, e^{-c\left(1+t/\rho(y)^2\right)^\delta}$$
$$\leq \frac{C}{t^{1+n/2q'}}\, e^{-c4^j r_B^2/t}\, e^{-c\left(1+t/r_B^2\right)^\delta}. \tag{3.54}$$

In the last step we used that since $r_B > 12\sqrt{n}\,\rho(x_B)$, then for each $y \in B$, by Lemma 2.1,

$$\rho(y) \leq C_0\rho(x_B)\left(1+\frac{r_B}{\rho(x_B)}\right) < C_0\left(\frac{1}{12\sqrt{n}}+1\right) r_B = C' r_B.$$

Estimate (3.54) gives us

$$\mathcal{I}(j,q,B) \leq C\int_0^\infty e^{-c4^j r_B^2/t} e^{-c\left(1+t/r_B^2\right)^\delta} \frac{dt}{t^{1+n/2q'}} = C\left\{\mathcal{I}_j + \mathcal{II}_j\right\} \tag{3.55}$$

where

$$\mathcal{I}_j := \int_0^{2^j r_B^2} e^{-c4^j r_B^2/t} e^{-c\left(1+t/r_B^2\right)^\delta} \frac{dt}{t^{1+n/2q'}},$$
$$\mathcal{II}_j := \int_{2^j r_B^2}^\infty e^{-c4^j r_B^2/t} e^{-c\left(1+t/r_B^2\right)^\delta} \frac{dt}{t^{1+n/2q'}}.$$

To estimate the first term we observe that since $t \leq 2^j r_B^2$ then $e^{-c4^j r_B^2/t} \leq e^{-c2^j}$, so that

$$\mathcal{I}_j \leq Ce^{-c2^j}\int_0^{2^j r_B^2}\left(\frac{t}{4^j r_B^2}\right)^{1+n/2q'} \frac{dt}{t^{1+n/2q'}}$$
$$\leq \frac{Ce^{-c2^j}}{4^{j(1+n/2q')} r_B^{2+n/q'}}\int_0^{2^j r_B^2} dt$$
$$= \frac{Ce^{-c2^j}}{2^{j(1+n/q')} r_B^{n/q'}} \leq \frac{Ce^{-c2^{j\delta}}}{r_B^{n/q'}} \tag{3.56}$$

since $0<\delta<1$. To estimate the second term we observe now that $t \geq 2^j r_B^2$ implies that $e^{-c\left(1+t/r_B^2\right)^\delta} \leq e^{-c2^{j\delta}}$, and hence

$$\mathcal{II}_j \leq \int_{2^j r_B^2}^\infty e^{-c\left(1+t/r_B^2\right)^\delta} \frac{dt}{t^{1+n/2q'}}$$
$$\leq Ce^{-c2^{j\delta}}\int_{2^j r_B^2}^\infty \frac{dt}{t^{1+n/2q'}} \leq \frac{Ce^{-c2^{j\delta}}}{2^{jn/2q'} r_B^{n/q'}} \leq \frac{Ce^{-c2^{j\delta}}}{r_B^{n/q'}}. \tag{3.57}$$

By collecting (3.56) and (3.57) into (3.55), and then inserting the result into (3.53), gives for each $j \geq 2$,

$$\left(\frac{1}{\psi_\eta(B)} \fint_B |T^* f_j|^{q'} \right)^{1/q'} \leq C\, 2^{j(n+\eta)/q'}\, e^{-c2^{j\delta}}\, \mathcal{M}^L_\eta \big(|f|^{q'}\big)(x)^{1/q'} \tag{3.58}$$

for any $x \in B$. Finally on combining (3.58) with (3.50) into (3.49) we have, for every $x \in B$,

$$\begin{aligned} \left(\frac{1}{\psi_\eta(B)} \fint_B |T^* f|^{q'} \right)^{1/q'} &\leq C \mathcal{M}^L_\eta \big(|f|^{q'}\big)(x)^{1/q'} \left\{ 1 + \sum_{j=2}^{\infty} 2^{j(n+\eta)/q'} e^{-c2^{j\delta}} \right\} \\ &\leq C_4 \mathcal{M}^L_\eta \big(|f|^{q'}\big)(x)^{1/q'} \end{aligned}$$

which gives us (3.38) with $\widetilde{C} = C_4$.

Therefore Theorem 3.14 applies and we obtain the required result for $T = \nabla^2 L^{-1}$. The proof of Theorem 1.1 is complete.

Remark 3.15. Finally we mention that the same approach used in the proof of Theorem 1.1 can be applied to give new proofs of the weighted estimates for the operators $\nabla L^{1/2}$, $V^{1/2} L^{-1/2}$, and $V L^{-1}$. These estimates are known—see Table 2 in Section 1 and the works [**10**], [**33**]. However we leave the details to the interested reader.

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After submitting his thesis, the author learned that the good-λ inequality in Theorem 3.9 has also appeared in [**35**]. That article gives applications to spectral multipliers, Littlewood–Paley type operators, and commutators. However our result concerning $\nabla^2 L^{-1}$ is still new.

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