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Counting subgraphs in hyperbolic graphs with symmetry

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Abstract. We confirm a conjecture of Kyoji Saito on the growth functions of graphs, which was originally posed for hyperbolic groups.

1. Introduction.

This note addresses some questions that arise in the series of works by Kyoji Saito on the growth functions of graphs [**Sa**], [**Sa2**]. We study "hyperbolike" graphs, which include Cayley graphs of hyperbolic groups. We generalize some well-known results on hyperbolic groups to the hyperbolike setting (Theorem 2.9, Theorem 4.1), including rationality of generating functions, and sharp estimates on the growth rate of vertices. We then apply these results to confirm a conjecture of Saito on the "opposite series", which was originally posed for hyperbolic groups (Theorem 5.3, Corollary 5.4). We also give a (standard) example of a hyperbolike graph with positive density of dead ends, and point out its implications for the applicability of the main theorems in [**Sa**].

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2. Generating functions of hyperbolike graphs.

Definition 2.1 (hyperbolike graph). A connected graph X of finite valence is δ-*hyperbolike* for some $\delta \geq 0$ if it satisfies the following properties:

(1) X is δ-hyperbolic; and
(2) $\mathrm{Aut}(X)$ is transitive on the vertices.

Condition (2) implies that all vertices have the same valence — i.e. X is regular. Moreover, by hypothesis, this (common) valence is finite. Thus X is proper as a (path) metric space.

Example 2.2 (Hyperbolic group). The main example of a hyperbolike graph is the Cayley graph of a hyperbolic group with respect to a finite generating set. Different choices of generating sets give rise to graphs which are δ-hyperbolike for different δ.

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Moreover, the automorphism group of the graph depends on the choice of generating set. We always have $G \subset \mathrm{Aut}(X)$ where G acts (freely and transitively on the vertices) on its Cayley graph by left multiplication.

EXAMPLE 2.3 (Free group). Even for X the Cayley graph of a hyperbolic group G, The group $\mathrm{Aut}(X)$ may be much bigger than G. For example, we can take G to be a free group, with a free generating set. Then X is a regular tree, and $\mathrm{Aut}(X)$ is uncountable.

EXAMPLE 2.4 (Quasi tree with parabolic symmetry group). The following example was described to us by Pierre-Emmanuel Caprace. Let T be a k-regular tree (with $k \geq 3$ finite) and fix an end e of T. For each vertex v, let γ_v denote the geodesic from v to e, and let v' denote the vertex on γ_v at distance 2 from v. Build a new graph X by attaching an edge from each vertex v to the corresponding v'. Then $\mathrm{Aut}(X)$ is just the subgroup of $\mathrm{Aut}(T)$ fixing e; in particular, it is vertex transitive, but not unimodular, therefore there is no discrete subgroup acting cocompactly on X.

Pick a base point $x \in X$. Since $\mathrm{Aut}(X)$ is transitive, any two choices are isomorphic. For any n let X_n denote the ball of radius n about x in X (i.e. the complete subgraph spanned by the vertices at distance $\leq n$ from x).

We let $\mathrm{Aut}(X, x)$ denote the subgroup of $\mathrm{Aut}(X)$ fixing x. Evidently, $\mathrm{Aut}(X, x)$ fixes each subgraph X_n, so that there are homomorphisms

$$p_n : \mathrm{Aut}(X, x) \to \mathrm{Aut}(X_n)$$

and we can identify $\mathrm{Aut}(X, x)$ with the inverse limit

$$\mathrm{Aut}(X, x) = \varprojlim p_n(\mathrm{Aut}(X, x)).$$

In particular, $\mathrm{Aut}(X, x)$ is compact, and therefore either finite or uncountable. In the first case, $\mathrm{Aut}(X)$ is itself finitely generated and a hyperbolic group, and the orbit map to X is a quasi-isometry. But in general we do not know the answer to the following:

QUESTION 2.5. Let X be hyperbolike. Is there a hyperbolic group G quasi-isometric to X?

REMARK 2.6. If one removes the hypothesis that X be δ-hyperbolic, the analogue of Question 2.5 has a *negative* answer in general.

If T_r and T_s are regular trees of valence r, s respectively (where $r \neq s$ and $\infty > r, s > 2$), and if h_r, h_s are horofunctions on T_r and T_s respectively, the *Diestel Leader graph* $DL(r, s)$ is the subgraph of the product $T_r \times T_s$ where $h_r + h_s = 0$. These graphs were introduced in [**DL**]. Firstly, it was shown in [**BNW**] that they do not admit a group action with finitely many orbits and finite vertex stabilizers, and then it was shown in [**EFW**] that $DL(r, s)$ is not even quasi-isometric to a Cayley graph.

The Diestel Leader graphs are reminiscent of non-unimodular solvable groups, and it is harder to imagine an analogue in the hyperbolic world.

REMARK 2.7. Random walks on hyperbolike graphs (and more general graphs with vertex transitive symmetry groups, which might not be hyperbolic or finite valence) are studied in [**KW**].

Definition 2.8. Let Y, Z be any two *finite* graphs. Let $(Y|Z)$ denote the number of distinct embeddings of Y as a complete subgraph of Z. For any finite graph Y, define the generating function

$$b_Y(t) := \sum (Y|X_n)t^n.$$

In words: the coefficients of $b_Y(t)$ count the number of copies of Y in the balls of each fixed radius in X.

By abuse of notation, we can think of x as a graph with 1 vertex, so that $b_x(t)$ is the generating function for the sizes of the balls $|X_n|$.

With this notation, we have the formula

$$b_{X_n}(t)/|\operatorname{Aut}(X_n)| = b_x(t)/t^n - \text{polar part at } 0.$$

To see this, observe first that every embedding of X_n into X as a complete subgraph (taking x to x without loss of generality) has image equal to exactly X_n. For, every point in X_n is within distance n of x, so the image is contained in X_n. So the claim follows by counting.

Then the formula follows, since embeddings of X_n in X_m up to automorphisms are in bijection with points in X_{m-n}.

The following theorem generalizes a result in [**Ep2**]:

Theorem 2.9 (Rationality). *Let X be δ-hyperbolike. For any connected graph Y, let $b_Y(t)$ be the generating function whose coefficient of t^n is the number of distinct embeddings of Y as a complete subgraph of X_n. Then $b_Y(t)$ is rational.*

The result is known for a Cayley graph of a hyperbolic group, [**Ep2**].

3. Proof of the rationality theorem.

In this section we give the proof of Theorem 2.9. The argument borrows heavily from the well-known proof by Cannon [**Ca**] in the case of a hyperbolic group; but there are some subtleties, which are worth spelling out now in informal language.

The main subtlety is the possibility that there are distinct geodesics γ, γ' between points x and y, and some $\phi \in \operatorname{Aut}(X)$ with $\phi(\gamma) = \gamma'$. This situation can certainly occur: consider a surface group with a presentation like $\langle a, b, c, d \mid [a, b][c, d] \rangle$. The Cayley graph is the 1-skeleton of the tiling of $\mathbb{H}^2$ by regular octagons with angles $\pi/4$ at the vertices. Two antipodal vertices of an octagon may be joined by two distinct paths of length 4 in the Cayley graph, and these paths may be interchanged by an automorphism of the graph.

This ambiguity makes it tricky to define a regular language of geodesics in bijection with the elements of X. Simply put, there is no way to make such a choice without breaking the symmetry — in other words, without finding a subgroup G of $\mathrm{Aut}(X)$ which is still (coarsely) transitive, but acts freely on some rich set of (sufficiently long) geodesics. Such a subgroup does not exist in general (e.g. Example 2.4), and it is not clear what to use as a substitute; morally this is the sort of issue we are raising with Question 2.5.

DEFINITION 3.1 (Synchronous fellow travelers). Let γ and γ' be two geodesics with the same initial vertex. Let their lengths be ℓ and ℓ' respectively. For any $T \geq 0$, the geodesics γ, γ' are said to *T-synchronously fellow travel* if for all i up to $\min(\ell, \ell')$, there is an inequality

$$d(\gamma(\ell - i), \gamma'(\ell' - i)) \leq T.$$

DEFINITION 3.2 (Competitor). Let $B_{2\delta+1}(y)$ be the ball of radius $(2\delta + 1)$ about y in X. For any y, an element $z \in B_{2\delta+1}(y)$ is a *competitor* of y if $d(x, z) \leq d(x, y)$, and if *some* geodesic from x to z $(2\delta + 1)$-synchronously fellow travels *every* geodesic from x to y.

Note that with this definition, y is a competitor of itself, since every geodesic from x to y $(2\delta + 1)$-synchronously fellow travels every (other) geodesic from x to y.

DEFINITION 3.3 (Tournament and tournament type). A function F from the vertices of $B_{2\delta+1}(y)$ to $\mathbb{Z}$ is a *tournament* if it satisfies the following conditions:

(1) for any z there is an inequality $d(x, z) - d(x, y) \leq F(z) \leq d(y, z)$; and
(2) if z is a competitor of y, then $d(x, z) = d(x, y) + F(z)$.

Two tournaments $F : B_{2\delta+1}(y) \to \mathbb{Z}$ and $F' : B_{2\delta+1}(y') \to \mathbb{Z}$ have the same *type* if there is an automorphism $\phi \in \mathrm{Aut}(X)$ with $\phi(y) = \phi(y')$ so that $F = F' \circ \phi$.

Note that if F is a tournament then $|F(z)| \leq 2\delta + 1$ for all z, so there are only finitely many types of tournament.

REMARK 3.4. The meaning of a tournament is roughly as follows. As we march along a path, we would like to know the relative distance from x to the different elements z in $B_{2\delta+1}(y)$ in order to certify that we are really traveling along a geodesic. The problem is that it is hard to keep track of relative distance to points z that are on the periphery. So we keep track of an *upper bound* on their relative distance (i.e. the value $F(z)$), which measures (roughly) the length of the shortest path from x to z which stays (synchronously) close to the geodesic we have traveled along.

DEFINITION 3.5 (cone and cone type). The *cone* associated to a point y, denoted $\mathrm{cone}(y)$, is the full subgraph of X consisting of points z so that $d(x, z) = d(x, y) + d(y, z)$.

We say that y and y' have the same *cone-type* if there is $\phi \in \mathrm{Aut}(X)$ taking y to y' and taking $\mathrm{cone}(y)$ to $\mathrm{cone}(y')$.

The following lemma is the analogue of Cannon's key lemma, that $(2\delta + 1)$-level determines cone type in hyperbolic groups.

LEMMA 3.6 (Tournament determines cone type). *Let X be δ-hyperbolike with base point x. Let y and y' with tournaments F and F' be given.*

Suppose there is $\phi \in \mathrm{Aut}(X)$ with $F = F' \circ \phi$. Then ϕ takes $\mathrm{cone}(y)$ *to* $\mathrm{cone}(y')$.

PROOF. Let $z \in \mathrm{cone}(y)$. We need to show that $\phi(z) \in \mathrm{cone}(y')$. This is proved by induction on $d(y, z)$. If γ is a geodesic from y to z, and w is the penultimate point on the geodesic, then $\phi(w) \in \mathrm{cone}(y')$ by the induction hypothesis. So if $\phi(z)$ is not in $\mathrm{cone}(y')$ we must have $d(x, y') + d(y', \phi(z)) \geq d(x, \phi(z)) + 1$. A geodesic from x to $\phi(z)$ must pass through $B_\delta(y')$ and therefore some point on that geodesic must be a competitor to y'. Applying ϕ^{-1} to the restriction of this geodesic gives a shortcut from a corresponding competitor to z, contrary to the fact that z is in $\mathrm{cone}(y)$. So $\phi(z) \in \mathrm{cone}(y')$ as claimed. □

LEMMA 3.7. *Let $y \in X$, and let $F : B_{2\delta+1}(y) \to \mathbb{Z}$ be a tournament. Then for any $y' \in X$ with $d(y, y') = 1$ and $d(x, y') = d(x, y) + 1$ there is a tournament $F' : B_{2\delta+1}(y') \to \mathbb{Z}$ whose type depends only on the type of F and the choice of y' in the type of* $\mathrm{cone}(y)$.

PROOF. We construct F' as follows. First, note that $d(x, y') = d(x, y) + 1$ means that there is a geodesic γ from x to y' whose penultimate vertex is y.

Now, let $z' \in B_{2\delta+1}(y')$ be a competitor of y'. Thus, by definition, there is some geodesic γ' from x to z' which $(2\delta + 1)$-synchronously fellow travels γ.

Let z be on γ' with $d(z, z') = 1$. Then by the definition of synchronous fellow-traveling, $d(z, y) \leq 2\delta + 1$, and z is a competitor of y.

So, we define

$$F'(z') = \min\{F(z) + d(z, z') - 1 \mid z \in B_{2\delta+1}(y)\}.$$

Then $F'(z') \leq d(y', z')$ since $F(z) \leq d(y, z)$. Evidently, $d(x, z') = d(x, y') + F'(z')$ for every competitor z' of y'. Moreover, for every z' there is some $z \in B_{2\delta+1}(y)$ with

$$d(x, z') - d(x, y') \leq d(x, z) + d(z, z') - d(x, y) - 1 \leq F(z) + d(z, z') - 1 = F'(z')$$

(take z to attain the minimum in the definition of $F'(z')$. If z' is a competitor of y', then each $\leq$ becomes $=$). By definition, the type of F' depends only on the type of F and the choice of y' in the type of $\mathrm{cone}(y)$. □

DEFINITION 3.8 (child and parent). A point y' is a *child* of y, and y is a *parent* of y', if $d(y, y') = 1$ and $d(x, y') = d(x, y) + 1$.

Every parent of y' is a competitor of every other parent. Therefore the tournament type of any parent y determines the number of parents of each child of y.

We now give the proof of Theorem 2.9.

PROOF. Define a finite directed graph as follows. Each vertex corresponds to a possible tournament type. There is a directed edge from the tournament type of y, F to the tournament type of y', F' if y' is a child of y, and F' is the tournament type constructed from the tournament type of F by Lemma 3.7. There is a unique vertex, the base vertex, for the tournament type for the base vertex x. We take the connected component of the base vertex in the following argument.

We put a (rational) weight w on the edge from (y, F) to (y', F') which is equal to the reciprocal of the number of parents of y'. We need to show that this is well-defined; i.e. it can be determined from the tournament type of (y, F) and (y', F'). In the ball $B_2(y)$, count the number of z with $F(z) = 0$ and $d(z, y') = 1$. We claim that these are in bijection with the parents of y'. For, if $F(z) = 0$ then $d(x, z) \leq d(x, y)$ by definition, so if furthermore $d(z, y') = 1$ then $d(x, y') \leq d(x, z) + 1 \leq d(x, y) + 1 = d(x, y')$, so these inequalities are equalities. Conversely, every parent z is a competitor, and thus $F(z) = 0$ since $d(z, y') = 1$. This proves the claim, and shows that the weight is well-defined.

Label the vertices of the graph by distinct integers, and define a non-negative matrix M whose ij entry is equal to $w(e)$ if there is an edge e from vertex i to vertex j, and 0 otherwise. Let ι be the row vector $(1, 0, 0 \cdots 0)$ and let $\mathbf{1}$ be the column vector whose entries are all 1s. Then there is a formula

$$b_x(t) = \sum_n (\iota M^n \mathbf{1}) t^n$$

whose coefficients by their form satisfy a finite linear recurrence (due to the fact that M is a root of its own characteristic polynomial), and therefore $b_x(t)$ is a rational function.

In Section 2 we saw that $b_{X_n}(t)/|\operatorname{Aut}(X_n)|$ is derived from $b_x(t)/t^n$ by throwing away the polar part at 0. Thus $b_{X_n}(t)$ is also a rational function.

Finally, for any connected graph Y we choose a base point $y \in Y$ and an integer n which is at least as big as the diameter of Y, and we count how many copies of Y there are in X_n with y at the center. For each of these copies, we let D be the least number so that Y is in X_D, and call these the *D-copies*. From these finitely many coefficients we can reconstruct $b_Y(t)$ from $b_{X_n}(t)$ in an obvious way and express it as a finite linear combination of series of the form $b_D(t)$ for $D \leq n$. □

Remark 3.9 (explanation of the formula). Let v_0 be the base vertex of the directed graph Γ in the argument. Lemma 3.7 gives a natural map B from the set of all finite geodesics starting at x in X to the set of all directed finite paths starting at v_0 in Γ. Indeed the map B is a bijection. The inverse B^{-1} is given by the induction on the length of a path. We call the inverse image a *lift*. Suppose $v(i_j)$, $j = 0, 1, 2, \ldots, n+1$ is a directed path in Γ with $v(i_0) = v_0$ and it is lifted for $0 \leq j \leq n$ to a geodesic starting at x and ending at z in X. Since there is a directed edge from $v(i_n)$ to $v(i_{n+1})$, there must be a point $y \in X$ and a child of y, y' such that the tournament type of y is $v(i_n)$ and the tournament type of y' is $v(i_{n+1})$, and that there is $\phi \in \operatorname{Aut}(X)$ with $\phi(y) = z$. Now extend the geodesic by adding $\phi(y')$ after z, which is the lift of $v(i_{n+1})$. This is a geodesic by Lemma 3.6.

The map P assigning the end points to those geodesics is a surjection to X. Notice that for each point $y \in X$ with $y \neq x$, by the definition of the weight of each edge in Γ, the total weight of the paths in the set $BP^{-1}(y)$ is always 1 (again, by the induction on $d(x, y)$). Now the formula follows.

4. Patterson–Sullivan measures for hyperbolike graphs.

From Theorem 2.9 and from elementary linear algebra it follows that if X is δ-hyperbolike for some δ, and is not quasi-isometric to a point or a line, then there are constants $\lambda > 1$ and $C > 1$, and an integer $k \geq 0$ so that there is an estimate of the form

$$C^{-1}\lambda^n n^k \leq |X_n| \leq C\lambda^n n^k.$$

In this section we refine this estimate, showing that $k = 0$. Explicitly, we show

Theorem 4.1 (Exponential). *Let X be δ-hyperbolike. Then there are constants $\lambda > 1$ and $C > 1$ so that there is an estimate of the form*

$$C^{-1}\lambda^n \leq |X_n| \leq C\lambda^n.$$

Remark 4.2. If we use the notation $X_{=n}$ for the subset of X at distance *exactly* n from the base point, then a similar estimate

$$C^{-1}\lambda^n \leq |X_{=n}| \leq C\lambda^n$$

holds, with the same constant λ but a possibly different constant C.

If X is the Cayley graph of a hyperbolic group, Theorem 4.1 is due to Coornaert [**Co**], and is proved by generalizing the theory of Patterson–Sullivan measures. As explained in [**C**, Section 2.5], the proof of Coornaert's theorem can be considerably simplified by first showing that the generating function $b_x(t)$ is rational, as a corollary of Cannon's theorem for hyperbolic groups.

Our proof of Theorem 4.1 runs along very similar lines, and amounts to little more than the verification that the steps in the argument given in [**C**] hold in the more general context of hyperbolike graphs. We carry out this verification in the remainder of the section.

4.1. Visual boundary.

The first step is to metrize $\partial_\infty X$ following Gromov. Let d_X denote the ordinary (path) metric in X.

Definition 4.3. Fix some base point $x \in X$ and some constant $a > 1$. The *a-length* of a rectifiable path γ in X, denoted $\text{length}_a(\gamma)$, is the integral along γ of $a^{-d_X(x,\cdot)}$ with respect to its ordinary length, and the *a-distance* from y to z, denoted $d_X^a(y, z)$ is the infimum of the a-lengths of paths between y and z.

The following comparison lemma, due to Gromov, lets us compare a-length to ordinary length.

Lemma 4.4 (Gromov). *There is some $a_0 > 1$ so that for $1 < a < a_0$ the completion $\overline{X}$ of X in the a-length metric is homeomorphic to $X \cup \partial_\infty X$. Moreover, for such an a there is a constant C so that for all $y, z \in \partial_\infty X$ there is an inequality*

$$C^{-1}a^{-(y|z)} \le d_X^a(y,z) \le Ca^{-(y|z)}$$

where $(y|z)$ *denotes the Gromov product.*

The Gromov product $(y|z)$ is usually taken to denote the expression

$$(y|z) := \frac{1}{2}(d_X(x,y) + d_X(x,z) - d_X(y,z))$$

but since we are only ever interested in the value of this expression up to a (uniformly bounded) additive constant, we could just as easily use the normalization $(y|z) = d_X(x, yz)$, i.e. the distance from x to some (equivalently, any) geodesic yz from y to z. We stress that this expression is to be interpreted as denoting "equality up to a uniform additive constant"; this (unspecified but effective) constant will later be absorbed into a multiplicative constant.

4.2. Patterson–Sullivan measure.

The next step is to construct a (so-called) Patterson–Sullivan (probability) measure on $\overline{X}$. Theorem 2.9 has the key corollary that this measure will be supported on the *boundary.*

Define the Poincaré *zeta function* $\zeta_X(s)$, well-defined for s sufficiently large, by the formula

$$\zeta_X(s) := \sum_{y \in X} e^{-s d_X(x,y)}.$$

Recall that we have already shown that there is an estimate of the form

$$C^{-1}\lambda^n n^k \le |X_n| \le C\lambda^n n^k.$$

It follows that ζ_X converges if $s > h := \log(\lambda)$ and *diverges* at h. We may therefore define, for each $s > h$, a probability measure ν_s on $\overline{X}$ (supported in X) by putting an atom of size $e^{-s d_X(x,y)}/\zeta_X(s)$ at each $y \in X$. Take a subsequence of measures that converges as $s \to h$ from above, and define ν to be the limit. By construction, this is a probability measure *supported on* $\partial_\infty X$.

4.3. Quasiconformal measure.

Recall that if $y \in \partial_\infty X$, a *horofunction* b_y centered at y is a limit of a convergent subsequence of functions of the form $d_X(y_i, \cdot) - d_X(y_i, x)$ for $y_i \to y$. Such a horofunction is not unique, but is well-defined up to a uniformly bounded additive constant.

Definition 4.5 (Coornaert). For $\phi \in \mathrm{Aut}(X)$ define $j_\phi : \partial_\infty X \to \mathbb{R}$ by

$$j_\phi(y) = a^{b_y(x) - b_y(\phi(x))}$$

for some horofunction b_y centered at y. A probability measure ν on $\partial_\infty X$ is *quasiconformal of dimension* D if for every $\phi \in \mathrm{Aut}(X)$, the measure $\phi_*\nu$ is absolutely continuous with respect to ν, and there is a constant C (independent of ϕ) so that

$$C^{-1}j_\phi(y)^D \leq d(\phi_*\nu)/d\nu \leq Cj_\phi(y)^D.$$

Note that the uniform additive ambiguity in the definition of b_y is absorbed into a uniform multiplicative ambiguity in the definition of j_ϕ, which is then absorbed into the constant C; so this definition makes sense.

PROPOSITION 4.6. *The measure ν is quasiconformal of dimension D, where $D = h/\log a$.*

PROOF. From the definition of Radon-Nikodym derivative, it suffices to show that there is a constant C, so that for all $y \in \partial_\infty X$ there is a neighborhood V of y in $\overline{X}$ so that for all $A \subset V$,

$$C^{-1}j_\phi(y)^D\nu(A) \leq \nu(\phi^{-1}A) \leq Cj_\phi(y)^D\nu(A).$$

By the definition of a horofunction, and δ-thinness, there is a neighborhood V of y in $\overline{X}$ so that

$$d_X(x,\phi^{-1}z) - d_X(x,z) - C \leq b_y(\phi(x)) - b_y(x) \leq d_X(x,\phi^{-1}z) - d_X(x,z) + C$$

for some C, and for all $z \in V$.

For each $s > h$ we have

$$\phi_*\nu_s(z)/\nu_s(z) = \nu_s(\phi^{-1}z)/\nu_s(z) = e^{-s(d_X(x,\phi^{-1}z)-d_X(x,z))}.$$

Taking $s \to h$ and defining $a^D = e^h$ proves the proposition. □

4.4. Shadows.

We now recall Sullivan's definition of *shadows*:

DEFINITION 4.7. For $y \in X$ and $R > 0$ the *shadow* $S(y,R)$ is the set of $z \in \partial_\infty X$ such that every geodesic ray from x to z comes within distance R of y.

LEMMA 4.8. *Fix $R > 2\delta$. Then there is a constant N so that for any $z \in \partial_\infty X$ and any n there is at least* 1 *and there are at most N elements y with $d_X(x,y) = n$ and $z \in S(y,R)$.*

PROOF. If γ is any geodesic from x to z, and if y is any point on γ, then $z \in S(y,R)$. Conversely, if y and y' are two elements with $d_X(x,y) = d_X(x,y')$ and $z \in S(y,R) \cap S(y',R)$ then $d_X(y,y') \leq 2R$. □

LEMMA 4.9. *Fix R. Then there is a constant C so that for any $y \in X$ there is an inequality*

$$C^{-1}a^{-d_X(x,y)D} \leq \nu(S(y,R)) \leq Ca^{-d_X(x,y)D}.$$

Proof. First observe by δ-thinness and the definition of a shadow, that there is some constant C' so that

$$d_X(x,y) - C' \leq b_z(x) - b_z(y) \leq d_X(x,y) + C'$$

for any $z \in S(y,R)$. Since $j_\phi(z) = a^{b_z(x)-b_z(\phi(x))}$ it follows that there is a constant C so that

$$C^{-1} a^{d_X(x,\phi(x))} \leq j_\phi(z) \leq C a^{d_X(x,\phi(x))}$$

for any $\phi \in \mathrm{Aut}(X)$ and any $z \in S(\phi(x),R)$.

Now, since ν is a quasiconformal measure, ν cannot consist of a single atom. So let $m_0 < 1$ be the measure of the biggest atom of ν, and fix $m_0 < m < 1$. By compactness of $\partial_\infty X$ there is some ϵ so that every ball in $\partial_\infty X$ of diameter $\leq \epsilon$ (in the a-metric) has mass at most m. Now, for any $\phi \in \mathrm{Aut}(X)$, the set $\phi^{-1}S(\phi(x),R)$ consists of exactly the $y \in \partial_\infty X$ for which every geodesic ray from $\phi^{-1}(x)$ to y comes within distance R of x. As $R \to \infty$, the diameter of $\partial_\infty X - \phi^{-1}S(\phi(x),R)$ goes to zero uniformly in ϕ, and so for some R_0, and for all $R \geq R_0$, we have

$$1 - m \leq \nu(\phi^{-1}S(\phi(x),R)) \leq 1$$

independent of ϕ.

But by Proposition 4.6 and the discussion above, there is some constant C_1 so that

$$C_1 a^{d_X(x,\phi(x))D} \leq \nu(\phi^{-1}S(\phi(x),R))/\nu(S(\phi(x),R)) \leq C_1 a^{d_X(x,\phi(x))D}.$$

Taking reciprocals, and using $1 - m \leq \nu(\phi^{-1}S(\phi(x),R)) \leq 1$ completes the proof. □

We now give the proof of Theorem 4.1.

Proof. We already know the lower bound. For each y with $d(x,y) = n$ we have $e^{-hn} = a^{-Dn} \leq C\nu(S(y,R))$. On the other hand, by Lemma 4.8, every point $z \in \partial_\infty X$ is contained in at least 1 and at most N sets $S(y,R)$ with $d(x,y) = n$. So

$$|X_{=n}| e^{-hn} C^{-1} \leq \sum_{d(x,y)=n} \nu(S(y,R)) \leq N\nu\Bigg(\bigcup_{d(x,y)=n} S(y,R)\Bigg) = N. \qquad \square$$

5. The opposite series $\Omega(P)$ for a power series P.

5.1. Definitions.

We recall the definition of *opposite series* from Saito [**Sa**]. Let $P(t) = \sum_{n=0}^\infty a_n t^n$ be a power series in t with a_n real numbers. Assume there exist u, v such that for all n, $u \leq a_{n-1}/a_n \leq v$.

Define a polynomial in s for each $n \geq 0$ as follows:

$$X_n(P) = \sum_{k=0}^{n} \frac{a_{n-k}}{a_n} s^k.$$

Define $\Omega(P)$ as the set of accumulation points of the sequence $\{X_n\}_n$ in the set of formal power series on s (with respect to the product topology on each coefficients). An element in $\Omega(P)$ is called an *opposite series* in [**Sa**, Section 11.2].

Now, let G be a group with a finite generating set S. Let a_n denote the number of elements $g \in G$ whose word length is n with respect to S. Using a_n's, we define $P(t)$, denoted by $P_{G,S}$, and obtain $\Omega(P_{G,S})$.

Saito also defined another set $\Omega(G, S)$, a map $\pi_\Omega : \Omega(G, S) \to \Omega(P_{G,S})$ and proved (Theorem in Section 11.2) that the map is surjective under two assumptions ($\boldsymbol{S}$ and $\boldsymbol{I}$ in his paper; we will discuss $\boldsymbol{S}$). Saito's theory is most interesting when $\Omega(G, S)$ or $\Omega(P_{G,S})$ is finite, but his paper gives only a few examples where finiteness is shown to hold.

Saito proposed the following conjecture in the last section of his paper [**Sa**, Section 12. Conjecture 4]:

CONJECTURE 5.1 (Saito). *$\Omega(G, S)$ is finite if G is a word hyperbolic group.*

In view of Saito's theorem relating $\Omega(G, S)$ to $\Omega(P_{G,S})$, it is natural to ask:

QUESTION 5.2. Is $\Omega(P_{G,S})$ finite if G is hyperbolic?

Saito conjectures that this will be the case [**Sa2**], and we will answer this question in the affirmative (Corollary 5.4). Conjecture 5.1 is still open. Interestingly it turns out that there is an example of a hyperbolic group which does not satisfy the assumption $\boldsymbol{S}$ (see Example 6.2).

THEOREM 5.3 (finiteness). *Let X be a hyperbolike graph, and for any connected graph Y, let $b_Y(t)$ be the generating function whose coefficient of t^n is the number of distinct embeddings of Y as a complete subgraph of X_n. Then $\Omega(b_Y)$ is finite.*

A special case, which answers Saito's question, is:

COROLLARY 5.4. *If G is a word hyperbolic group, then $\Omega(P_{G,S})$ is finite for any finite generating set S.*

5.2. Proof of Theorem 5.3 and Corollary 5.4.

We recall a well-known result from analytic combinatorics.

THEOREM 5.5 ([**FlSe**, Theorem IV.9]). *If $f(z)$ is a rational function that is analytic at zero and has poles at points $\alpha_1, \alpha_2, \ldots \alpha_m$, then its coefficients are a sum of* exponential-polynomials*: there exist m polynomials $\Pi_j(x)$ such that, for n larger than some fixed n_0,*

$$f_n = \sum_j \Pi_j(n) \alpha_j^{-n}$$

where f_n is the coefficient of z^n in $f(z)$. Furthermore, the degree of Π_j is equal to the *order of the pole of f at α_j minus one.*

Let's apply this to prove Theorem 5.3.

Proof. The power series $b_Y(t)$ is a rational function, whose poles are (a subset of) the reciprocals of the roots of the matrix M constructed in the proof of Theorem 2.9. Since M is a non-negative matrix, Perron–Frobenius theory says that there is a root of largest absolute value which is real and positive, and all other roots with this absolute value differ by multiplication by a root of unity. From Theorem 4.1 and Theorem 5.5 we conclude that these roots of maximum modulus are *simple*, or else the dominant term in the growth rate of the coefficients of $b_Y(t)$ would be of the form polynomial times exponential, where the polynomial had positive degree (contrary to Theorem 4.1).

It follows from Theorem 5.5 that for n sufficiently big, there is some $m_0 \le m$ so that after reordering the poles of $b_Y(t)$ in non-decreasing modulus, we have an expression of the form

$$f_n = \sum_{j \le m_0} \pi_j \alpha_j^{-n} + \sum_{j > m_0} \Pi_j(n) \alpha_j^{-n}$$

where α_1 is real and positive, where α_j for $j \le m_0$ is of the form $\alpha_1 \omega_j$ for some root of unity ω_j, and where $|\alpha_j| > \alpha_1$ for $j > m_0$.

Evidently $\alpha_1^{-1} = \lambda$ with notation from Theorem 4.1. Moreover, if N is the least common multiple of the order of the roots of unity ω_j, then we can rewrite this expression as

$$f_n = C_{[n]} \lambda^n + o(\lambda^n)$$

where $C_{[n]}$ depends only on the residue of n mod N. Again, by Theorem 4.1 we can conclude that $C_{[n]}$ is real and *positive* for all n mod N.

If we define the polynomial $X_n(b_Y) = \sum_{k=0}^{n} (f_{n-k}/f_n) s^k$ as in Definition 5.1, then as $n \to \infty$ for every fixed k the coefficient of s^k in $X_n(P)$ approaches a value depending only on n mod N. Hence there are finitely many accumulation points of the X_n, which is exactly the conclusion of Theorem 5.3. □

6. Dead ends.

Definition 6.1. Let X be a graph and x a base point. A vertex y is a *dead end* if there is no $z \ne y$ with $d(x,z) = d(x,y) + d(y,z)$.

It is important for Saito to study graphs with the additional hypothesis that the asymptotic density of dead end elements is zero. This is one of the assumptions he puts in the main theorem in [**Sa**, Section 11.2, Assumption 2. ***S***]. Unfortunately, we show now that this hypothesis is genuinely restrictive, since there are (very simple) hyperbolic groups with finite generating sets whose Cayley graphs have a positive density of dead ends. Actually, these examples are already well-known; we simply bring them up to point out the implications for Saito's theory.

The following example is worked out it detail by Pfeiffer [**Pf**, Appendix C]; we summarize the story.

Example 6.2 (Triangle group). Let G be the $(2,3,7)$ triangle group; i.e. the group with the following presentation

$$G := \langle a, b \mid a^2, b^3, (ab)^7 \rangle.$$

We abbreviate b^{-1} by B. Every geodesic word in G alternates between a and either b or B.

Moreover, *infinite* geodesics are exactly those that don't contain (except possibly at the very start) substrings of the form $ababab$ or $aBaBaB$. For, suppose $ababab$ appears in the middle of the word. It must be followed by an a, and preceded by either b or B. If we have $babababa$ then of course we can replace it by $aBaBaB$ which is shorter. If we have $Babababa$ we can rewrite it as $BBaBaBaB = baBaBaB$ which is shorter.

Now, if W is any word with at most 2 consecutive abs or aBs in a row (and is therefore a geodesic), we can extend it to something like $WXBabaBababab$ which now we claim is a dead-end. For, it can only be extended to

$$WXBabaBabababa = WXBabaBBaBaBaB = WXBababaBaBaB$$

which is definitely shorter. On the other hand, $WXBabaBababab$ is itself a geodesic; trying to rewrite it, one can only replace $ababab$ by $BaBaBaBa$ giving

$$WXBabaBBaBaBaBa = WXBababaBaBaBa$$

which is longer.

Thus this group has dead end elements with positive density (at least 2^{-6}).

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