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# On left-orderability and cyclic branched coverings

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**Abstract.** In a recent paper, Y. Hu has given a sufficient condition for the fundamental group of the $r$-th cyclic branched covering of $S^3$ along a prime knot to be left-orderable in terms of representations of the knot group. Applying her criterion to a large class of two-bridge knots, we determine a range of integers $r > 1$ for which the $r$-th cyclic branched covering of $S^3$ along the knot is left-orderable.

## 1. Introduction.

A non-trivial group $G$ is called left-orderable if there exists a strict total ordering $<$ on its elements such that $g < h$ implies $fg < fh$ for all elements $f, g, h \in G$. Knot groups and more generally the fundamental group of an irreducible 3-manifold with positive first Betti number are examples of left-orderable groups [**HSt**]. Left-orderable groups have recently attracted the attention of many people partly because of their possible connection to L-spaces, a class of rational homology 3-spheres defined by Ozsvath and Szabo [**OS**] using Heegaard Floer homology, via a conjecture of Boyer, Gordon and Watson [**BGW**]. This conjecture predicts that an irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable. The conjecture has been confirmed for Seifert fibered manifolds, Sol manifolds, double branched coverings of non-splitting alternating links [**BGW**], and certain Dehn surgeries on the figure eight knot, on the knot $5_2$ and more generally on genus one two-bridge knots (see [**BGW**], [**CLW**], [**HT1**] and [**HT2**], [**HT3**], [**Tr**] respectively). A technique that has so far worked very well for proving the left-orderability of fundamental groups is lifting a non-abelian $SL_2(\mathbb{R})$ representation of a 3-manifold group to the universal covering group $\widetilde{SL_2(\mathbb{R})}$, and then using the result by Bergman [**Be**] that $\widetilde{SL_2(\mathbb{R})}$ is a left-orderable group. This technique, which is based on an important result of Khoi [**Kh**], was first introduced in [**BGW**] and was applied in [**HT1**], [**HT2**], [**HT3**], [**Tr**] to study the left-orderability of Dehn surgeries on genus one two-bridge knots.

The left-orderability of the fundamental groups of non-hyperbolic geometric rational homology 3-spheres has already been characterized in [**BRW**]. For hyperbolic rational homology 3-spheres, many of them can be constructed from the cyclic branched coverings of $S^3$ along a knot. Based on the Lin's presentation [**Li**] of a knot group and the technique for proving the left-orderability of fundamental groups mentioned above, Y. Hu [**Hu**] has recently given a suffcient condition for the fundamental group of the $r$-th cyclic branched covering of $S^3$ along a prime knot to be left-orderable in terms of representations of the

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knot group. As an application, she proves that for any two-bridge knot $\mathfrak{b}(p,m)$, with $p \equiv 3 \pmod 4$, there are only finitely many cyclic branched coverings whose fundamental groups are not left-orderable. In particular for the two-bridge knots $5_2$ and $7_4$, Y. Hu shows that the fundamental groups of the $r$-th cyclic branched coverings of $S^3$ along them are left-orderable if $r \geq 9$ and $r \geq 13$ respectively.

In this paper by applying Hu's criterion to a large class of two-bridge knots, which includes the knots $5_2$ and $7_4$, we determine a range of integers $r > 1$ for which the $r$-th cyclic branched covering of $S^3$ along the knot is left-orderable.

Let $K = J(k,l)$ be the double twist knot as in Figure 1. Note that $J(k,l)$ is a knot if and only if $kl$ is even, and is the trivial knot if $kl = 0$. Furthermore, $J(k,l) \cong J(l,k)$ and $J(-k,-l)$ is the mirror image of $J(k,l)$. Hence, in the following, we only consider $K = J(k,2n)$ for $k > 0$ and $|n| > 0$.

In the Schubert's normal form $\mathfrak{b}(p,m)$, where $p,m$ are positive integers such that $p$ is odd and $0 < m < p$, of a two-bridge knot one has $J(k,2n) = \mathfrak{b}(2kn-1,2n)$ if $n > 0$ and $J(k,2n) = \mathfrak{b}(1-2kn,-2n)$ if $n < 0$, see e.g. [**BZ**].

For a knot $K$ in $S^3$ and any integer $r > 1$, let $X_K^{(r)}$ be the $r$-th cyclic branched covering of $S^3$ along $K$. The following theorem generalizes Example 4.4 in [**Hu**].

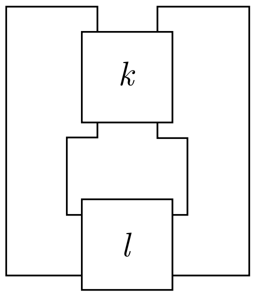


Figure 1. The double twist knot $J(k,l)$. Here $k,l$ denote the numbers of half twists in each box. Positive numbers correspond to right-handed twists and negative numbers correspond to left-handed twists.

THEOREM 1. *Suppose $m$ and $n$ are positive integers. Then the group $\pi_1(X^{(r)}_{J(2m,2n)})$ is left-orderable if $r > \pi/\cos^{-1}\sqrt{1-(4mn)^{-1}}$.*

EXAMPLE 1.1. 1) For the knot $5_2 = J(4,2)$, the manifold $X^{(r)}_{5_2}$ has left-orderable fundamental group if $r > \pi/\cos^{-1}\sqrt{7/8} \approx 8.69$, i.e. $r \geq 9$.

2) For the knot $7_4 = J(4,4)$, the manifold $X^{(r)}_{7_4}$ has left-orderable fundamental group if $r > \pi/\cos^{-1}\sqrt{15/16} \approx 12.43$, i.e. $r \geq 13$.

REMARK 1.2. Dabkowski, Przytycki and Togha [**DPT**] proved that the group $\pi_1(X^{(r)}_{J(2m,-2n)})$, for positive integers $m$ and $n$, is not left-orderable for any integer $r > 1$.

We also prove the following result in this paper.

THEOREM 2. *Suppose $m \geq 0$ and $n > 0$ are integers. Let $q = 2n^2 + 2n\sqrt{4m(m+1)+n^2}$.*

(a) *The group* $\pi_1(X^{(r)}_{J(2m+1,2n)})$ *is left-orderable if one of the following holds*:
   (i) $n$ *is even and* $r > \pi/\cos^{-1}\sqrt{1-q^{-1}}$.
   (ii) $n$ *is odd* $> 1$ *and* $r > \max\{\pi/\cos^{-1}\sqrt{1-q^{-1}}, 4m+2\}$.
(b) *The group* $\pi_1(X^{(r)}_{J(2m+1,-2n)})$ *is left-orderable if one of the following holds*:
   (i) $n$ *is odd and* $r > \pi/\cos^{-1}\sqrt{1-q^{-1}}$.
   (ii) $n$ *is even and* $r > \max\{\pi/\cos^{-1}\sqrt{1-q^{-1}}, 4m+2\}$.

Remark 1.3. We exclude $J(2m+1,2)$, for $m > 0$, from Theorem 2 since it is isomorphic to $J(2m,-2)$, and by Remark 1.2 the group $\pi_1(X^{(r)}_{J(2m,-2)})$, for $m > 0$, is not left-orderable for any integer $r > 1$.

Here is the plan of the paper. We study non-abelian $SL_2(\mathbb{C})$ representations and roots of the Riley polynomial of the knot group of the double twist knots $J(k,l)$ in Section 2. We prove Theorems 1 and 2 in Section 3.

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## 2. Non-abelian representations and roots of the Riley polynomial.

### 2.1. Non-abelian representations.

By [**HSn**], the knot group of $K = J(k,2n)$ is

$$\pi_1(K) = \langle a, b \mid w^n a = b w^n \rangle,$$

where $a, b$ are meridians and

$$w = \begin{cases} (ba^{-1})^m (b^{-1}a)^m, & \text{if } k = 2m, \\ (ba^{-1})^m ba (b^{-1}a)^m, & \text{if } k = 2m+1. \end{cases}$$

A representation $\rho : \pi_1(K) \to SL_2(\mathbb{C})$ is called non-abelian if $\rho(\pi_1(K))$ is a non-abelian subgroup of $SL_2(\mathbb{C})$. Taking conjugation if necessary, we can assume that $\rho$ has the form

$$\rho(a) = A = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = B = \begin{bmatrix} s & 0 \\ 2-y & s^{-1} \end{bmatrix} \tag{2.1}$$

where $(s,y) \in \mathbb{C}^* \times \mathbb{C}$ satisfies the matrix equation $W^n A - B W^n = 0$. Here $W = \rho(w)$.

Let $\{S_j(z)\}_j$ be the sequence of Chebyshev polynomials defined by $S_0(z) = 1$, $S_1(z) = z$, and $S_{j+1}(z) = zS_j(z) - S_{j-1}(z)$ for all integers $j$. Note that if $z = t + t^{-1}$, where $t \neq \pm 1$, then $S_{j-1}(z) = (t^j - t^{-j})/(t - t^{-1})$. Moreover $S_{j-1}(2) = j$ and $S_{j-1}(-2) = (-1)^{j-1} j$ for all integers $j$.

The following lemma is elementary, and hence its proof is omitted.

Lemma 2.1. *For all integers* $j$*, one has*

$$S_j^2(z) - zS_j(z)S_{j-1}(z) + S_{j-1}^2(z) = 1.$$

Let $x = \operatorname{tr} A = s + s^{-1}$ and $\lambda = \operatorname{tr} W$. The following propositions are proved in [**MT**].

Proposition 2.2. *One has*

$$\lambda = \begin{cases} 2 + (y-2)(y+2-x^2)S_{m-1}^2(y), & \text{if } k = 2m, \\ x^2 - y - (y-2)(y+2-x^2)S_m(y)S_{m-1}(y), & \text{if } k = 2m+1. \end{cases}$$

Proposition 2.3. *One has*

$$W^nA - BW^n = \begin{bmatrix} 0 & S_{n-1}(\lambda)\alpha - S_{n-2}(\lambda) \\ (y-2)(S_{n-1}(\lambda)\alpha - S_{n-2}(\lambda)) & 0 \end{bmatrix},$$

*where*

$$\alpha = \begin{cases} 1 - (y+2-x^2)S_{m-1}(y)(S_{m-1}(y) - S_{m-2}(y)), & \text{if } k = 2m, \\ 1 + (y+2-x^2)S_{m-1}(y)(S_m(y) - S_{m-1}(y)), & \text{if } k = 2m+1. \end{cases}$$

Proposition 2.3 implies that the assignment (2.1) gives a non-abelian representation $\rho : \pi_1(K) \to SL_2(\mathbb{C})$ if and only if $(s, y) \in \mathbb{C}^* \times \mathbb{C}$ satisfies the equation

$$\phi_K(x, y) := S_{n-1}(\lambda)\alpha - S_{n-2}(\lambda) = 0.$$

The polynomial $\phi_K(x, y)$ is known as the Riley polynomial [**Ri**] of $K = J(k, 2n)$.

### 2.2. Roots of the Riley polynomial.

In this subsection we prove some properties of the roots of the Riley polynomial of the double twist knots $J(k, l)$.

Lemma 2.4. *One has*

$$\alpha^2 - \alpha\lambda + 1 = \begin{cases} (y+2-x^2)S_{m-1}^2(y)(\lambda+2-x^2), & \text{if } k = 2m, \\ (1 + (y+2-x^2)S_{m-1}(y)S_m(y))(2-\lambda), & \text{if } k = 2m+1. \end{cases}$$

Proof. If $k = 2m$ then $\alpha = 1 - (y+2-x^2)S_{m-1}(y)(S_{m-1}(y) - S_{m-2}(y))$ and $\lambda = 2 + (y-2)(y+2-x^2)S_{m-1}^2(y)$. By direct calculations, we have

$$\begin{aligned} \alpha^2 - \alpha\lambda + 1 &= (y+2-x^2)S_{m-1}^2(y) \\ &\quad \times \big[2 - y + (y+2-x^2)\big((y-1)S_{m-1}^2(y) - yS_{m-1}(y)S_{m-2}(y) + S_{m-2}^2(y)\big)\big]. \end{aligned}$$

Since $S_{m-1}^2(y) - yS_{m-1}(y)S_{m-2}(y) + S_{m-2}^2(y) = 1$ (by Lemma 2.1), we obtain

$$\begin{aligned} \alpha^2 - \alpha\lambda + 1 &= (y+2-x^2)S_{m-1}^2(y)\big(4 - x^2 + (y+2-x^2)(y-2)S_{m-1}^2(y)\big) \\ &= (y+2-x^2)S_{m-1}^2(y)(\lambda+2-x^2). \end{aligned}$$

If $k = 2m+1$ then $\alpha = 1 + (y+2-x^2)S_{m-1}(y)(S_m(y) - S_{m-1}(y))$ and $\lambda = x^2 - y - (y-2)(y+2-x^2)S_m(y)S_{m-1}(y)$. By direct calculations, we have

$$\begin{aligned}\alpha^2 - \alpha\lambda + 1 = (y+2-x^2)\big[1 - (y+2-x^2)S^2_{m-1}(y) + (2y-x^2)S_{m-1}(y)S_m(y) \\ + (y+2-x^2)S^2_{m-1}(y) \\ \times \big(S^2_{m-1}(y) - yS_{m-1}(y)S_m(y) + (y-1)S^2_m(y)\big)\big].\end{aligned}$$

Since $S^2_{m-1}(y) - yS_{m-1}(y)S_{m-2}(y) + S^2_{m-2}(y) = 1$, we obtain

$$\begin{aligned}&\alpha^2 - \alpha\lambda + 1 \\ &\quad = (y+2-x^2)[1 + (2y-x^2)S_{m-1}(y)S_m(y) + (y+2-x^2)(y-2)S^2_{m-1}(y)S^2_m(y)] \\ &\quad = (y+2-x^2)\big(1 + (y+2-x^2)S_{m-1}(y)S_m(y)\big)\big(1 + (y-2)S_{m-1}(y)S_m(y)\big) \\ &\quad = (1 + (y+2-x^2)S_{m-1}(y)S_m(y))(2-\lambda).\end{aligned}$$

This completes the proof of Lemma 2.4. □

The following lemma is well known. We include a proof for the reader's convenience.

LEMMA 2.5. *For any integer $k$ and any real number $t$, one has*

$$|\sin kt| \le |k \sin t|.$$

PROOF. Without loss of generality we assume that $k \ge 2$. If $k = 2m$ ($m \in \mathbb{Z}_+$) then

$$\sin kt = \sum_{j=1}^{m}(\sin 2jt - \sin(2j-2)t) = 2\sin t \sum_{j=1}^{m} \cos(2j-1)t.$$

It follows that $|\sin kt| \le 2m|\sin t|$, since $\left|\sum_{j=1}^m \cos(2j-1)t\right| \le m$.

If $k = 2m+1$ ($m \in \mathbb{Z}_+$) then

$$\sin kt = \sin t + \sum_{j=1}^{m}(\sin(2j+1)t - \sin(2j-1)t) = \sin t + 2\sin t \sum_{j=1}^{m} \cos 2jt.$$

It follows that $|\sin kt| \le (2m+1)|\sin t|$, since $\left|1 + 2\sum_{j=1}^m \cos 2jt\right| \le 2m+1$. □

LEMMA 2.6. *Suppose $z \in \mathbb{R}$ satisfies $|z| \le 2$. Then $|S_{j-1}(z)| \le |j|$ for all integers $j$.*

PROOF. If $z = 2$ then $S_{j-1}(z) = j$. If $z = -2$ then $S_{j-1}(z) = (-1)^{j-1}j$.

If $-2 < z < 2$ we write $z = 2\cos t$, where $0 < t < \pi$. Then $S_{j-1}(z) = \sin jt/\sin t$, and hence $|S_{j-1}(z)| \le |j|$ by Lemma 2.5. □

PROPOSITION 2.7. *Let $K = J(k, 2n)$ where $k > 0$ and $|n| > 0$. Suppose $x, y \in \mathbb{R}$ satisfy $|x| \leq 2$ and $\phi_K(x, y) = 0$. Then $y > 2$ if one of the following holds*:

(a) $k = 2m$ ($m \in \mathbb{Z}_+$) *and* $|x| > 2\sqrt{1 - 1/|4mn|}$.

(b) $k = 2m+1$ ($m \in \mathbb{Z}_+$) *and* $|x| > 2\sqrt{1 - 1/(2n^2 + 2|n|\sqrt{4m(m+1)+n^2})}$.

PROOF. If $|x| = 2$ then by [**MT**, Proposition 3.2], any real root $y$ of $\phi_K(x, y)$ satisfies $y > 2$. We now consider the case $|x| < 2$.

Suppose $x, y \in \mathbb{R}$ satisfy $|x| < 2$ and $\phi_K(x, y) = 0$. Then $S_{n-1}(\lambda)\alpha - S_{n-2}(\lambda) = 0$ and

$$1 = S_{n-1}^2(\lambda) - \lambda S_{n-1}(\lambda)S_{n-2}(\lambda) + S_{n-2}^2(\lambda) = (\alpha^2 - \alpha\lambda + 1)S_{n-1}^2(\lambda). \tag{2.2}$$

(a) Suppose $k = 2m$ ($m \in \mathbb{Z}_+$) and $|x| > \sqrt{4 - 1/|mn|}$. By Lemma 2.4,

$$\alpha^2 - \alpha\lambda + 1 = (y + 2 - x^2)S_{m-1}^2(y)(\lambda + 2 - x^2).$$

Equation (2.2) then implies that

$$1 = (y + 2 - x^2)S_{m-1}^2(y)(\lambda + 2 - x^2)S_{n-1}^2(\lambda). \tag{2.3}$$

Assume $y \leq 2$. Since $\lambda - 2 = (y-2)(y+2-x^2)S_{m-1}^2(y)$, by Equation (2.3) we have

$$(\lambda - 2)(\lambda + 2 - x^2) = (y-2)(y+2-x^2)S_{m-1}^2(y)(\lambda + 2 - x^2) = (y-2)/S_{n-1}^2(\lambda) \leq 0$$

which implies that $x^2 - 2 < \lambda \leq 2$.

Similarly, since $(y-2)(y+2-x^2) = (\lambda - 2)/S_{m-1}^2(y) \leq 0$, we have $x^2 - 2 < y \leq 2$.

Since $y \in \mathbb{R}$ satisfies $|y| \leq 2$, we have $|S_{m-1}(y)| \leq |m|$ by Lemma 2.6. Similarly $|S_{n-1}(\lambda)| \leq |n|$. Hence, it follows from Equation (2.3) that

$$1 = (y + 2 - x^2)(\lambda + 2 - x^2)S_{m-1}^2(y)S_{n-1}^2(\lambda) \leq (4 - x^2)^2 m^2 n^2$$

which implies that $x^2 \leq 4 - 1/|mn|$, a contradiction.

(b) Suppose $k = 2m+1$ ($m \in \mathbb{Z}_+$) and $|x| > \sqrt{4 - 2/(n^2 + |n|\sqrt{4m(m+1)+n^2})}$. By Lemma 2.4,

$$\alpha^2 - \alpha\lambda + 1 = (1 + (y + 2 - x^2)S_{m-1}(y)S_m(y))(2 - \lambda).$$

Equation (2.2) then implies that

$$1 = (1 + (y + 2 - x^2)S_{m-1}(y)S_m(y))(2 - \lambda)S_{n-1}^2(\lambda). \tag{2.4}$$

Assume that $y \leq 2$. Since $\lambda + x^2 - 2 = (2 - y)(1 + (y + 2 - x^2)S_{m-1}(y)S_m(y))$, by Equation (2.4) we have

$$\begin{aligned}(\lambda + 2 - x^2)(\lambda - 2) &= (2-y)(1 + (y+2-x^2)S_{m-1}(y)S_m(y))(\lambda - 2)\\ &= (y-2)/S^2_{n-1}(\lambda) \le 0\end{aligned}$$

which implies that $x^2 - 2 \le \lambda < 2$.

Similarly, since $S^2_{m-1}(y) + S^2_m(y) - yS_{m-1}(y)S_m(y) = 1$ we have

$$\begin{aligned}2 - \lambda &= (y + 2 - x^2)(1 + (y-2)S_{m-1}(y)S_m(y))\\ &= (y+2-x^2)(S_{m-1}(y) - S_m(y))^2 > 0\end{aligned}$$

which implies that $y > x^2 - 2$. Hence, it follows from Equation (2.4) that

$$\begin{aligned}1 &= (2-\lambda)S^2_{n-1}(\lambda)\big(1 + (y+2-x^2)S_{m-1}(y)S_m(y)\big)\\ &\le (4 - x^2)n^2\big(1 + (4-x^2)m(m+1)\big)\end{aligned}$$

which implies that $x^2 \le 4 - 2/(n^2 + |n|\sqrt{4m(m+1)+n^2})$, a contradiction. □

Proposition 2.8. *Let $K = J(2m+1, 2n)$ where $m \ge 0$ and $n \notin \{0,1,2\}$ are integers. Suppose $x \in \mathbb{R}$ satisfies $|x| \ge 2\cos(\pi/(4m+2))$. Then the equation $\phi_K(x,y) = 0$ has at least one real solution $y > x^2 - 2$.*

Proof. Recall that for $K = J(2m+1, 2n)$, $\alpha = 1 + (y+2-x^2)S_{m-1}(y)(S_m(y) - S_{m-1}(y))$ and $\lambda = x^2 - y - (y-2)(y+2-x^2)S_m(y)S_{m-1}(y)$. It is obvious that if $y = x^2 - 2$ then $\alpha = 1$ and $\lambda = 2$. Hence

$$\phi_K(x, x^2-2) = S_{n-1}(\lambda)\alpha - S_{n-2}(\lambda) = S_{n-1}(2) - S_{n-2}(2) = 1.$$

We consider the following two cases.

Case 1: $n \ge 3$. Note that the polynomial $S_{n-1}(t) - S_{n-2}(t)$ has exactly $n-1$ roots given by $t = 2\cos((2j-1)\pi/(2n-1))$, where $1 \le j \le n-1$. Moreover

$$S_{n-1}\left(2\cos\frac{\pi}{2n-1}\right) > 0 > S_{n-1}\left(2\cos\frac{3\pi}{2n-1}\right).$$

Suppose $m = 0$. Then $\alpha = 1$ and $\lambda = x^2 - y$. We have

$$\phi_K\left(x, x^2 - 2\cos\frac{\pi}{2n-1}\right) = S_{n-1}\left(2\cos\frac{\pi}{2n-1}\right) - S_{n-2}\left(2\cos\frac{\pi}{2n-1}\right) = 0.$$

In this case we choose $y = x^2 - 2\cos(\pi/(2n-1))$. Then $\phi_K(x,y) = 0$ and $y > x^2 - 2$.

We now suppose $m > 0$. Note that $2 - \lambda = (y+2-x^2)(S_m(y) - S_{m-1}(y))^2$. Consider the equation $\lambda = 2\cos(3\pi/(2n-1))$, i.e. $(y+2-x^2)(S_m(y) - S_{m-1}(y))^2 = 2 - 2\cos(3\pi/(2n-1))$. It is easy to see that this equation has at least one solution $y_0 > x^2 - 2$. Note that $x^2 - 2 \ge 2\cos(\pi/(2m+1))$. Since $y_0 > 2\cos(\pi/(2m+1))$, we have $S_m(y_0) > S_{m-1}(y_0) > 0$. Hence

$$\begin{aligned}\phi_K(x, y_0) &= S_{n-1}(\lambda)\alpha - S_{n-2}(\lambda) = (\alpha - 1)S_{n-1}(\lambda) \\ &= (y_0 + 2 - x^2)S_{m-1}(y_0)(S_m(y_0) - S_{m-1}(y_0))S_{n-1}\left(2\cos\frac{3\pi}{2n-1}\right) < 0.\end{aligned}$$

Since $\phi_K(x, x^2 - 2) > 0 > \phi_K(x, y_0)$, there exists $y \in (x^2 - 2, y_0)$ such that $\phi_K(x, y) = 0$.

Case 2: $n \leq -1$. Let $l = -n \geq 1$. We have

$$\phi_K(x, y) := S_{n-1}(\lambda)\alpha - S_{n-2}(\lambda) = S_l(\lambda) - S_{l-1}(\lambda)\alpha.$$

Suppose $m = 0$. Then $\alpha = 1$ and $\lambda = x^2 - y$. In this case we choose $y = x^2 - 2\cos(\pi/(2l+1))$. Then $\phi_K(x, y) = 0$ and $y > x^2 - 2$.

We now suppose $m > 0$. Consider the equation $\lambda = 2\cos(\pi/(2l+1))$, i.e. $(y + 2 - x^2)(S_m(y) - S_{m-1}(y))^2 = 2 - 2\cos(\pi/(2l+1))$. This equation has at least one real solution $y_0 > x^2 - 2 \geq 2\cos(\pi/(2m+1))$. We have

$$\begin{aligned}\phi_K(x, y_0) &= S_l(\lambda) - S_{l-1}(\lambda)\alpha \\ &= -(y_0 + 2 - x^2)S_{m-1}(y_0)\left(S_m(y_0) - S_{m-1}(y_0)\right)S_l\left(2\cos\frac{\pi}{2l+1}\right) < 0.\end{aligned}$$

Hence there exists $y \in (x^2 - 2, y_0)$ such that $\phi_K(x, y) = 0$.

This completes the proof of Proposition 2.8. □

## 3. Proof of Theorems 1 and 2.

For a knot $K$ in $S^3$, let $X_K = S^3 \setminus K$ be the knot complement. Let $I$ denote the identity matrix in $SL(2, \mathbb{C})$. The following theorem of Y. Hu is important to us.

THEOREM 3.1 ([**Hu**]). *Given any prime knot $K$ in $S^3$, let $\mu$ be a meridian element of $\pi_1(X_K)$. If there exists a non-abelian representation $\rho : \pi_1(X_K) \to SL_2(\mathbb{R})$ such that $\rho(\mu^r) = \pm I$, then the fundamental group $\pi_1(X_K^{(r)})$ is left-orderable.*

SKETCH OF THE PROOF OF THEOREM 3.1. Let $\widetilde{SL_2(\mathbb{R})}$ be the universal covering group of $SL_2(\mathbb{R})$. There is a lift of $\rho : \pi_1(X_K) \to SL_2(\mathbb{R})$ to a homomorphism $\widetilde{\rho} : \pi_1(X_K) \to \widetilde{SL_2(\mathbb{R})}$ since the obstruction to its existence is the Euler class $e(\rho) \in H^2(X_K; \mathbb{Z}) \cong 0$, see [**Gh**]. Using the Lin's presentation [**Li**] for the knot group $\pi_1(X_K)$ together with the hypotheses that $\rho(\mu^r) = \pm I$ and $\rho$ is non-abelian, Y. Hu [**Hu**] shows that the homomorphism $\widetilde{\rho}$ induces a non-trivial homomorphism $\pi_1(X_K^{(r)}) \to \widetilde{SL_2(\mathbb{R})}$. By [**BRW**], [**HSt**], a compact, orientable, irreducible 3-manifold has a left-orderable fundamental group if and only if there exists a non-trivial homomorphism from its fundamental group to a left-orderable group. We have that $X_K^{(r)}$ is irreducible (since $K$ is prime) and $\widetilde{SL_2(\mathbb{R})}$ is left-orderable. Hence $\pi_1(X_K^{(r)})$ is left-orderable. This proves Theorem 3.1.

We are ready to prove Theorems 1 and 2. For the two-bridge knot $\mathfrak{b}(p,m)$, it is known that the Riley polynomial $\phi_{\mathfrak{b}(p,m)}(x,y)$ is a polynomial in $\mathbb{Z}[x,y]$ with $y$-leading term $\pm y^d$, where $d=(p-1)/2$, see [**Ri**].

### 3.1. Proof of Theorem 1.

Consider $K=J(2m,2n)$ where $m,n$ are positive integers. Note that $K=\mathfrak{b}(4mn-1,2n)$ and hence the Riley polynomial $\phi_K(x,y)$ is a polynomial in $\mathbb{Z}[x,y]$ with $y$-leading term $\pm y^d$, where $d=2mn-1$. Since $d$ is odd, for each $x\in\mathbb{R}$ the equation $\phi_K(x,y)=0$ has at least one real root $y$.

For any integer $r>\pi/\cos^{-1}\sqrt{1-1/4mn}$, there is a non-abelian representation $\rho:\pi_1(X_K)\to SL_2(\mathbb{C})$ of the form

$$\rho(a)=\begin{bmatrix} e^{i(\pi/r)} & 1\\ 0 & e^{-i(\pi/r)}\end{bmatrix} \quad \text{and} \quad \rho(b)=\begin{bmatrix} e^{i(\pi/r)} & 0\\ 2-y & e^{-i(\pi/r)}\end{bmatrix}$$

where $y\in\mathbb{R}$. Note that $x=\operatorname{tr}\rho(a)=2\cos(\pi/r)$ and $\phi_K(x,y)=0$.

Since $x,y\in\mathbb{R}$ satisfy $2\sqrt{1-1/4mn}<|x|\le 2$ and $\phi_K(x,y)=0$, Proposition 2.7 implies that $y>2$. Since $2-y<0$, a result in [**Kh**, p. 786] says that the representation $\rho$ can be conjugated an $SL_2(\mathbb{R})$ representation, denoted by $\rho':\pi_1(X_K)\to SL_2(\mathbb{R})$. Note that $\rho'(a^r)=-I$, since $\rho(a^r)=-I$. Hence Theorem 3.1 implies that $\pi_1(X_K^{(r)})$ is left-orderable. □

### 3.2. Proof of Theorem 2.

Consider $K=J(2m+1,2n)$ where $m\ge 0$ and $|n|>0$. Note that $K=\mathfrak{b}(4mn+2n-1,2n)$ if $n>0$, and $K=\mathfrak{b}(-4mn-2n+1,-2n)$ if $n<0$.

Let $q=2n^2+2|n|\sqrt{4m(m+1)+n^2}$. We consider the following two cases.

Case 1: $n>0$ even or $n<0$ odd. In this case we have $K=\mathfrak{b}(p,m)$ for some integers $p,m$ such that $p\equiv 3 \pmod 4$. Hence the Riley polynomial $\phi_K(x,y)$ is a polynomial in $\mathbb{Z}[x,y]$ with $y$-leading term $\pm y^d$, where $d=(p-1)/2$ is odd.

Suppose $r>\pi/\cos^{-1}\sqrt{1-q^{-1}}$. Then, by similar arguments as in the proof of Theorem 1, one can show that the group $\pi_1(X_K^{(r)})$ is left-orderable.

Case 2: $n>1$ odd or $n<0$ even. In this case we have $K=\mathfrak{b}(p,m)$ for some integers $p,m$ such that $p\equiv 1 \pmod 4$. Suppose $r>\max\{\pi/\cos^{-1}\sqrt{1-q^{-1}},4m+2\}$.

Let $x=2\cos(\pi/r)$. Since $x\in\mathbb{R}$ satisfies $|x|\ge 2\cos(\pi/(4m+2))$, by Proposition 2.8 there exists $y\in\mathbb{R}$ such that $\phi_K(2\cos(\pi/r),y)=0$. Hence there is a non-abelian representation $\rho:\pi_1(X_K)\to SL_2(\mathbb{C})$ of the form

$$\rho(a)=\begin{bmatrix} e^{i(\pi/r)} & 1\\ 0 & e^{-i(\pi/r)}\end{bmatrix} \quad \text{and} \quad \rho(b)=\begin{bmatrix} e^{i(\pi/r)} & 0\\ 2-y & e^{-i(\pi/r)}\end{bmatrix}.$$

Since $x=2\cos(\pi/r)$ also satisfies $|x|>2\sqrt{1-q^{-1}}$, Proposition 2.7 implies that $y>2$. The rest of the proof is similar to that of Theorem 1.

This completes the proof of Theorem 2. □

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