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Time periodic solutions of the Navier–Stokes equations with the time periodic Poiseuille flow under (GOC) for a symmetric perturbed channel in $\mathbb{R}^2$

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Abstract. Beirão da Veiga [**5**] proves that for a straight channel in $\mathbb{R}^n$ $(n \geq 2)$ and for a given time periodic flux there exists a unique time periodic Poiseuille flow. As a by product, existence of the time periodic Poiseuille flow in perturbed channels (Leray's problem) is shown for the Stokes problem $(n \geq 2)$ and for the Navier–Stokes problem $(n \leq 4)$. Concerning the Navier–Stokes case, in [**5**] a quantitative condition required to show the existence of the solutions depends not just on the flux of the time periodic Poiseuille flow but also on the domain itself.

Kobayashi [**16**], [**18**] proves that for a perturbed channel in $\mathbb{R}^n$ $(n = 2,\ 3)$ there exists a time periodic solution of the Navier–Stokes equations with the Poiseuille flow applying the theory of the steady problem to the time periodic problem.

In this paper, applying Fujita [**8**] and Kobayashi [**18**], we succeed in proving the existence of a time periodic solution for a symmetric perturbed channel in $\mathbb{R}^2$.

1. The time periodic Poiseuille flow.

In this section, for a straight channel in $\mathbb{R}^2$, which is parallel to the x_1-axis, let us consider a time periodic flow of an incompressible viscous fluid which is also parallel to the x_1-axis.

For a given $a > 0$, we suppose $\Sigma := (-a, a)$. We write

$$\omega = \mathbb{R} \times \Sigma.$$

The channel ω is parallel to the x_1-axis.

In the straight channel ω, let us consider the nonstationary Navier–Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{0} \quad \text{in} \quad \mathbb{R} \times \omega, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \mathbb{R} \times \omega, \tag{1.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \mathbb{R} \times \partial\omega \tag{1.3}$$

with the time periodic condition

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$$\boldsymbol{u}(t) = \boldsymbol{u}(t+T) \quad \text{in} \quad \omega \tag{1.4}$$

and the flux condition

$$\int_{\Sigma} \boldsymbol{u}(t) \cdot \boldsymbol{n} dS = \alpha(t) \quad (t \in \mathbb{R}), \tag{1.5}$$

where $\boldsymbol{u} = \boldsymbol{u}(t,x)$ and $p = p(t,x)$ are the unknown velocity and the unknown pressure of the fluid motion in ω, respectively, ν is the given viscosity constant, $\boldsymbol{n} = (1,0)$, $T > 0$ is a given constant and the function $\alpha(t)$ is given with T time periodicity.

Since we look for a solution pallalel to the x_1-axis, we assume that

$$\boldsymbol{u}(t,x) = (v(t,x),0).$$

Then it follows that v does not depend on x_1 from (1.2), $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{0}$ and p depends only on t and x_1 from (1.1). Therefore we obtain the following problem.

$$\frac{\partial v}{\partial t} - \nu \Delta v = -\frac{\partial p}{\partial x_1} \quad \text{in} \quad \mathbb{R} \times \Sigma, \tag{1.6}$$

$$v = 0 \qquad \text{on} \quad \mathbb{R} \times \partial\Sigma \tag{1.7}$$

with the time periodic condition

$$v(t) = v(t+T) \quad \text{in} \quad \Sigma \quad (t \in \mathbb{R}) \tag{1.8}$$

and the flux condition

$$\int_{\Sigma} v(t) dS = \alpha(t) \quad (t \in \mathbb{R}), \tag{1.9}$$

where $\Delta = \partial^2/\partial x_2^2$. We recall that, by appealing to the ideas introduced in reference [**5**], Galdi and Robertson [**10**] show that, for $n = 2$, the axial pressure gradient and the flow rate are connected through a simple relation.

Since v does not depend on x_1 and p depends only on t and x_1, it follows from (1.6) that $\partial v/\partial t - \nu \Delta v$ and $\partial p/\partial x_1$ depends only on t. Moreover, we assume that

$$p(t,x_1) = \psi(t) x_1,$$

where $\psi = \psi(t)$ is the unknown function. Integrating (1.6) on Σ, we obtain

$$\psi(t) = -\frac{1}{|\Sigma|}\left(\alpha'(t) - \nu \int_{\Sigma} \Delta v(t) dS\right),$$

where $|\Sigma|$ is the Lebesgue measure of Σ. Therefore there exists a time periodic solution $\boldsymbol{u}$ of the Navier–Stokes equations (1.1)–(1.5) in ω, with the form $\boldsymbol{u} = (v,0)$, if and only if v is a solution of the problem

$$v' + \nu A v - \frac{\nu}{|\Sigma|}(Av, e)e = \frac{\alpha'}{|\Sigma|}e \tag{1.10}$$

with the time periodic condition

$$v(t) = v(t+T) \quad (t \in \mathbb{R}) \tag{1.11}$$

and the flux condition

$$(v(t), e) = \alpha(t) \quad (t \in \mathbb{R}), \tag{1.12}$$

where $e(y) = 1$ $(y \in \Sigma)$, $A = -\Delta$ with the domain $D(A) = H^2(\Sigma) \cap H_0^1(\Sigma)$, $(v, e) = \int_\Sigma vedS$.

Before stating the time periodic result, we introduce the function space. Let X be a Banach space. We set

$$\begin{aligned} H_\pi^1(\mathbb{R}) &= \{\varphi \in H_{\mathrm{loc}}^1(\mathbb{R}); \varphi(t) = \varphi(t+T) \text{ a.e. } t \in \mathbb{R}\}, \\ L_\pi^2(\mathbb{R}; X) &= \{\varphi \in L_{\mathrm{loc}}^2(\mathbb{R}; X); \varphi(t) = \varphi(t+T) \text{ in } X \text{ for a.e. } t \in \mathbb{R}\}, \\ C_\pi(\mathbb{R}; X) &= \{\varphi \in C(\mathbb{R}; X); \varphi(t) = \varphi(t+T) \text{ in } X \text{ for } t \in \mathbb{R}\}. \end{aligned}$$

Beirão da Veiga [**5**] proves that for $n \geq 2$ if the flux $\alpha \in H_\pi^1(\mathbb{R})$ is given, then there exists a unique time periodic solution v^α of this problem (1.10)–(1.12) satisfying

$$\begin{aligned} v^\alpha &\in L_\pi^2(\mathbb{R}; H_0^1(\Sigma) \cap H^2(\Sigma)) \cap C_\pi(\mathbb{R}; H_0^1(\Sigma)), \\ (v^\alpha)' &\in L_\pi^2(\mathbb{R}; L^2(\Sigma)). \end{aligned}$$

In this paper we consider the case $n = 2$.

Applying the regularity result for the heat equations to v^α, we have

$$\begin{aligned} v^\alpha &\in L_\pi^2(\mathbb{R}; H_0^{1,S}(\Sigma) \cap H^3(\Sigma)) \cap C_\pi(\mathbb{R}; H_0^{1,S}(\Sigma)), \\ (v^\alpha)' &\in L_\pi^2(\mathbb{R}; H^{1,S}(\Sigma)), \end{aligned}$$

where $H_0^{1,S}(\Sigma)$ and $H^{1,S}(\Sigma)$ are symmetric function spaces with respect to the x_1-axis.

Set

$$\boldsymbol{V}^\alpha(t, x) = (v^\alpha(t, x), 0).$$

In this paper, let us call $\boldsymbol{V}^\alpha$ "the time periodic Poiseuille flow".

2. Problem in a perturbed channel.

Let $a_i > 0$ $(i = 1, 2)$ and $\Sigma_i := (-a_i, a_i)$. We set

$$\omega_i = \mathbb{R} \times \Sigma_i \quad (i = 1, 2).$$

If the flux $\alpha \in H^1_\pi(\mathbb{R})$ is given, then there exists a unique solution v_i^α of the time periodic problem (1.10)–(1.12) on Σ_i. Set

$$\boldsymbol{V}_i^\alpha(t, x) = (v_i^\alpha(t, x), 0) \quad \text{in} \quad \omega_i.$$

For a certain $L > 0$ we set

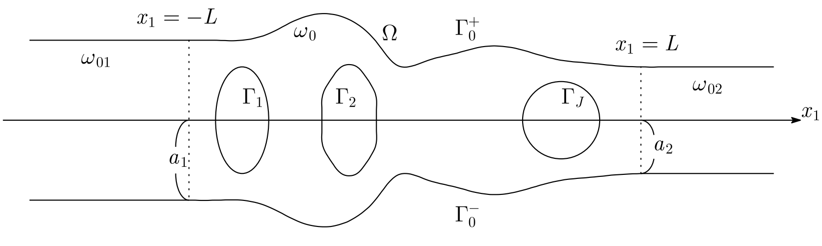
$$\omega_{01} = \{x \in \omega_1; x_1 \leq -L\},$$
$$\omega_{02} = \{x \in \omega_2; x_1 \geq L\}.$$

Let Ω be a smooth and unbounded domain in $\mathbb{R}^2$ and $\partial\Omega$ be the boundary of the domain Ω. A domain Ω is called a perturbed channel if Ω satisfies

$$\Omega \cap \omega_{0i} = \omega_{0i} \quad (i = 1, 2),$$
$$\omega_0 := \Omega \backslash (\omega_{01} \cup \omega_{02}).$$

The boundary $\partial\Omega$ has $J + 2$ disjoint boundary components Γ_0^+, Γ_0^-, $\Gamma_1, \dots, \Gamma_J$. Γ_0^+ is the upper boundary component, Γ_0^- is the lower boundary component and $\Gamma_1, \dots, \Gamma_J$ are the inner boundary components such that $\Gamma_i \cap \Gamma_j = \emptyset$ $(i \neq j)$, $\Gamma_i \cap \Gamma_0^+ = \emptyset$, $\Gamma_i \cap \Gamma_0^- = \emptyset$ $(i = 1, \dots, J)$, and such that $\partial\Omega = \bigcup_{j=1}^J \Gamma_j \cup \Gamma_0^+ \cup \Gamma_0^-$. Furthermore the domain Ω satisfies the following symmetric condition.

Assumption 2.1. *The domain Ω is symmetric with respect to the x_1-axis and the inner boundaries Γ_j $(1 \leq j \leq J)$ intersect the x_1-axis.*



In the domain Ω, we consider the nonstationary Navier–Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad (0, T) \times \Omega, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad (0, T) \times \Omega \tag{2.2}$$

with the boundary condition

$$\boldsymbol{u} = \boldsymbol{\beta} \quad \text{on} \quad (0, T) \times \partial\Omega, \tag{2.3}$$

$$\boldsymbol{u} \to \boldsymbol{V}_i^\alpha \quad \text{as} \quad |x| \to \infty \quad \text{in} \quad \omega_{0i} \quad (i = 1, 2) \tag{2.4}$$

and the time periodic condition

$$\boldsymbol{u}(0) = \boldsymbol{u}(T) \quad \text{in} \quad \Omega, \tag{2.5}$$

where $\boldsymbol{u} = \boldsymbol{u}(t, x)$ and $p = p(t, x)$ are the unknown velocity and the unknown pressure of an incompressible viscous fluid in Ω respectively, while $\nu > 0$ is the kinematic viscosity, $\boldsymbol{f} = \boldsymbol{f}(t, x)$ is the given external force and $\boldsymbol{\beta} = \boldsymbol{\beta}(t, x)$ is the given function on $(0, T) \times \partial\Omega$ with compact support. Since the solution $\boldsymbol{u}(t)$ satisfies $\operatorname{div} \boldsymbol{u}(t) = 0$ in Ω for a fixed $t \in (0, T)$, the given boundary data $\boldsymbol{\beta}(t)$ on $\partial\Omega$ is required to fulfill the compatibility condition which is called "General Outflow Condition" (GOC)

$$\int_{\partial\Omega} \boldsymbol{\beta}(t) \cdot \boldsymbol{n} d\sigma = 0, \tag{2.6}$$

where $\boldsymbol{n}$ is the unit outer normal to $\partial\Omega$. If the given boundary data $\boldsymbol{\beta}$ satisfies the following restricted flux condition

$$\int_{\Gamma_0^+} \boldsymbol{\beta}(t) \cdot \boldsymbol{n} d\sigma = \int_{\Gamma_0^-} \boldsymbol{\beta}(t) \cdot \boldsymbol{n} d\sigma = \int_{\Gamma_j} \boldsymbol{\beta}(t) \cdot \boldsymbol{n} d\sigma = 0 \quad (t \in (0, T),\ j = 1, \dots, J), \tag{2.7}$$

then we call the condition (2.7) "Stringent Outflow Condition" (SOC).

The purpose of this paper is that if the given boundary date $\boldsymbol{\beta}$ satisfies (GOC), we will seek a solution of (2.1)–(2.5) of the form

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{V}^\alpha, \tag{2.8}$$

where the function $\boldsymbol{V}^\alpha$ is to be such that

$$\begin{aligned} \operatorname{div} \boldsymbol{V}^\alpha &= 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{V}^\alpha &= \boldsymbol{0} \quad \text{on} \quad \partial\Omega, \\ \boldsymbol{V}^\alpha &= \boldsymbol{V}_i^\alpha \quad \text{in} \quad \omega_{0i} \quad (i = 1, 2), \\ \boldsymbol{V}^\alpha &\in H^1((0, T); \mathbb{H}^{1,S}(\omega_0)), \\ \boldsymbol{V}^\alpha &\in C_\pi([0, T]; \mathbb{H}^{1,S}(\omega_0)), \\ \boldsymbol{V}^\alpha &\in C([0, T] \times \overline{\Omega}). \end{aligned}$$

Let us call $\boldsymbol{V}^\alpha$ "the extended time periodic Poiseuille flow".

Beirão da Veiga [**5**] proves that for a straight channel in $\mathbb{R}^n$ $(n \geq 2)$ if a flux

$\alpha \in H^1_\pi(\mathbb{R})$ is given there exists a unique time periodic Poiseuille flow by the method of the previous section. Furthermore he proves that, in a perturbed channel of $\mathbb{R}^2$ and $\mathbb{R}^3$, there exists a time periodic solution with the time periodic Poiseuille flow if $\boldsymbol{\beta} = \mathbf{0}$ and a given flux $\alpha \in H^1_\pi(\mathbb{R})$ satisfies

$$c_0\sqrt{\nu + \nu^{-2}}\|\alpha\|_{H^1(0,T)} \leq \frac{1}{2}\nu$$

where the constant c_0 depends on Ω.

Kobayashi [**18**] treats the similar problem to Beirão da Veiga [**5**]. He defines a constant $\hat{\gamma}^\alpha$. See Definition 3.2 and 3.3. He proves that if

$$\hat{\gamma}^\alpha < \nu, \tag{2.9}$$

then there exists a time periodic solution with $\boldsymbol{\beta} = \mathbf{0}$.

In this paper, applying the inequality (2.9) and "the symmetric type of Leray's inequality" (see Section 4.3), we prove that for a perturbed channel satisfying Assumption 2.1 there exists a time periodic solution with the time periodic Poiseuille flow and the symmetric Dirichlet boundary data which fulfills (GOC).

We introduce some function spaces.

Let X be a function space on the symmetric domain Ω. X^S is a set of all symmetric X functions with respect to the x_1-axis. For a vector function $\boldsymbol{v}(x) = (v_1(x), v_2(x))$, $\boldsymbol{v}(x)$ is symmetric with respect to the x_1-axis if and only if v_1 and v_2 satisfy

$$\begin{aligned} v_1(x_1, -x_2) &= v_1(x_1, x_2) \quad ((x_1, x_2) \in \Omega), \\ -v_2(x_1, -x_2) &= v_2(x_1, x_2) \quad ((x_1, x_2) \in \Omega). \end{aligned}$$

$\mathbb{C}^\infty_{0,\sigma}(\Omega)$ is the set of all real smooth vector functions with compact support in Ω and $\operatorname{div}\boldsymbol{\varphi} = 0$. $\mathbb{L}^2_\sigma(\Omega)$ is the closure of $\mathbb{C}^\infty_{0,\sigma}(\Omega)$ for the usual $\mathbb{L}^2(\Omega)$ norm. The $\mathbb{L}^2$ inner product and norm on Ω are denoted as $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_{2,\Omega}$ respectively. $\mathbb{H}^1_0(\Omega)$ and $\mathbb{H}^1_{0,\sigma}(\Omega)$ are the closures of $\mathbb{C}^\infty_0(\Omega)$ and $\mathbb{C}^\infty_{0,\sigma}(\Omega)$ for the usual Dirichlet norm $\|\nabla\cdot\|_{2,\Omega}$, respectively. $\mathbb{H}^1_\sigma(\Omega)$ is the set of all $\mathbb{H}^1(\Omega)$ functions with $\operatorname{div}\boldsymbol{\varphi} = 0$.

Let X be a Banach space. $C([0,T];X)$, $L^2((0,T);X)$, $L^\infty((0,T);X)$ and $H^1((0,T);X)$ are the usual Banach spaces. $C_\pi([0,T];X)$ and $H^1_\pi((0,T);X)$ are the set of all the $C([0,T];X)$ and $H^1((0,T);X)$ functions satisfying the time periodic condition $\boldsymbol{u}(0) = \boldsymbol{u}(T)$ in X.

3. Result.

Our definition of a time periodic weak solution of the Navier–Stokes equations (2.1), (2.2), (2.3), (2.4), (2.5) is as follows.

Definition 3.1. A measurable function $\boldsymbol{u} = \boldsymbol{u}(t,x)$ on $(0,T)\times\Omega$ is called a time periodic weak solution of the Navier–Stokes equations (2.1), (2.2), (2.3), (2.4), (2.5) if $\boldsymbol{u}$ satisfies the following condition.

(1) $\boldsymbol{v} := \boldsymbol{u} - \boldsymbol{V}^\alpha \in L^2((0,T);\mathbb{H}^{1,S}_\sigma(\Omega)) \cap L^\infty((0,T);\mathbb{L}^{2,S}_\sigma(\Omega))$.
(2) $\boldsymbol{u}$ satisfies

$$\begin{aligned}\int_0^T & -(\boldsymbol{u},\boldsymbol{\varphi})\psi' + \{\nu(\nabla\boldsymbol{u},\nabla\boldsymbol{\varphi}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{\varphi})\}\psi dt \\ &= \int_0^T {}_{(\mathbb{H}^{1,S}_{0,\sigma})'}\langle \boldsymbol{f},\boldsymbol{\varphi}\rangle_{\mathbb{H}^{1,S}_{0,\sigma}} \psi dt \quad (\boldsymbol{\varphi} \in \mathbb{H}^{1,S}_{0,\sigma}(\Omega),\ \psi \in C_0^\infty(0,T)).\end{aligned}$$

(3) $\boldsymbol{v} = \boldsymbol{\beta}$ on $(0,T)\times\partial\Omega$ in the trace sense.
(4) $\boldsymbol{v} \in C_\pi([0,T];\mathbb{L}^2(\Omega))$.

Before stating our result, in the channel ω_i (not Ω) we define a constant concerning the time periodic Poiseuille flow.

Definition 3.2. We set

$$\gamma_i^{\alpha,S}(t) = \sup_{\varphi\in\mathbb{H}^{1,S}_{0,\sigma}(\omega_i)} \frac{((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi},\boldsymbol{V}_i^\alpha(t))_{\omega_i}}{\|\nabla\boldsymbol{\varphi}\|^2_{2,\omega_i}} \quad (i=1,2,\ t\in[0,T]). \tag{3.1}$$

Remark 3.1. The $\mathbb{L}^2$ inner product and norm in Definition 3.2 are denoted in the channel ω_i, that is to say, the $\gamma_i^{\alpha,S}$ does not depend on the perturbed part ω_0.

Proposition 3.1 (Kobayashi [**18**]). *Let $\alpha,\beta \in H^1_\pi(\mathbb{R})$ and $\lambda\in\mathbb{R}$. Then*

(1) $\gamma_i^{\alpha,S} \in H^1_\pi(\mathbb{R})$,
(2) $\gamma_i^{\alpha,S}(t) > 0$,
(3) $\gamma_i^{\lambda\alpha,S}(t) = |\lambda|\gamma_i^{\alpha,S}(t)$,
(4) $\gamma_i^{\alpha+\beta,S}(t) = \gamma_i^{\alpha,S}(t) + \gamma_i^{\beta,S}(t)$.

Definition 3.3. We set

$$\begin{aligned}\hat{\gamma}_i^{\alpha,S} &:= \sup_{t\in[0,T]} \gamma_i^{\alpha,S}(t) \quad (i=1,2), \\ \hat{\gamma}^{\alpha,S} &:= \max\{\hat{\gamma}_1^{\alpha,S},\hat{\gamma}_2^{\alpha,S}\}.\end{aligned}$$

Our main theorem on the existence of a time periodic weak solution of the Navier–Stokes equations now reads.

Theorem 3.1. *We assume that the domain Ω satisfies Assumption* 2.1. *We suppose that $\hat{\gamma}^{\alpha,S} < \nu$, $\boldsymbol{f} \in L^2((0,T);(\mathbb{H}^{1,S}_{0,\sigma}(\Omega))')$, $\boldsymbol{\beta} \in H^1_\pi((0,T);\mathbb{H}^{1/2,S}(\partial\Omega))$ with compact support,* (GOC) *and*

$$\int_{\Gamma_0^+} \boldsymbol{\beta}\cdot\boldsymbol{n} d\sigma = \int_{\Gamma_0^-} \boldsymbol{\beta}\cdot\boldsymbol{n} d\sigma = 0 \quad \text{on} \quad [0,T].$$

Then there exists a time periodic weak solution of the Navier–Stokes equations.

Remark 3.2. If Ω has only one inner boundary component, then the boundary data $\boldsymbol{\beta}$ must satisfy (SOC). This is different from the purpose of this paper. Therefore we assume that Ω has at least two inner boundary components.

Remark 3.3. In this paper the domain Ω has two outlets ω_{01} and ω_{02}. We can solve K ($K \geq 3$) outlets problem applying the similar inequality. We consider a straight channel $\omega_i = \mathbb{R} \times \Sigma_i$, where Σ_i is a cross section as Section 1 and the center line of ω_i may not be parallel to the x_1-axis. We assume that a given flux function $\alpha_i \in H^1_\pi(\mathbb{R})$ ($i = 1, \dots, K$) satisfies $\sum_{i=1}^K \alpha_i(t) = 0$ ($t \in \mathbb{R}$). For each α_i we have the time periodic Poiseuille flow $\boldsymbol{V}_i^{\alpha_i}$ in ω_i. We assume that Ω has K outlets ω_{0i} ($i = 1, \dots, K$) where ω_{0i} is a semi-infinite channel with the cross section Σ_i. We suppose that Ω satisfies Assumption 2.1. In the domain Ω, we consider a time periodic problem with the time periodic Poiseuille flow $\boldsymbol{V}_i^{\alpha_i}$. We define constant $\hat{\gamma}^S = \max_{1 \leq i \leq K}\{\hat{\gamma}_i^{\alpha_i,S}\}$ as Definition 3.2 and 3.3. Suppose that $\hat{\gamma}^S < \nu$. Then there exists a time periodic weak solution in Ω with K outlets.

4. Preliminary.

4.1. Lemma.

In this subsection, we show some Lemmas for the proof of the existence of a time periodic weak solution.

Lemma 4.1.

$$
\begin{aligned}
\|\boldsymbol{u}\|^2_{\mathbb{L}^4(\Omega)} &\leq 2^{1/2}\|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2 && (\boldsymbol{u} \in \mathbb{H}^1_0(\Omega)),\\
|((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w})| &\leq C\|\nabla\boldsymbol{u}\|_2\|\nabla\boldsymbol{v}\|_2\|\nabla\boldsymbol{w}\|_2 && (\boldsymbol{u},\,\boldsymbol{v},\,\boldsymbol{w} \in \mathbb{H}^1_0(\Omega)),\\
((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}) &= -((\boldsymbol{u}\cdot\nabla)\boldsymbol{w},\boldsymbol{v}) && (\boldsymbol{u} \in \mathbb{H}^1_{0,\sigma}(\Omega),\,\boldsymbol{v},\,\boldsymbol{w} \in \mathbb{H}^1(\Omega)).
\end{aligned}
$$

Lemma 4.2. *Suppose that a domain Ω satisfies Assumption* 2.1.

Then for any $\varepsilon > 0$ and $\boldsymbol{w} \in C([0,T];\mathbb{L}^{2,S}(\Omega))$, there exist an integer N and functions $\boldsymbol{\psi}_j \in \mathbb{L}^{2,S}(\Omega)$ ($j = 1, \dots, N$) such that the inequality

$$
\int_0^T |((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w})|dt \leq \varepsilon\int_0^T (\|\nabla\boldsymbol{u}\|_2^2 + \|\nabla\boldsymbol{v}\|_2^2 + \|\boldsymbol{u}\|_2\|\nabla\boldsymbol{v}\|_2)dt + \sum_{j=1}^N\int_0^T |(\boldsymbol{u},\boldsymbol{\psi}_j)|^2 dt
$$

holds true, for any $\boldsymbol{u},\boldsymbol{v} \in L^2((0,T);\mathbb{H}^{1,S}_{0,\sigma}(\Omega))$.

This kind of inequality appears in Masuda [**21**, Lemma 2.5, p. 632]. This inequality is its two dimensional and symmetric version.

4.2. Proposition for the extended time periodic Poiseuille flow.

In this subsection, we argue the estimate of the extended time periodic Poiseuille flow.

For a fixed $t \in [0,T]$ we define a functional $\boldsymbol{r}(t)$ as

$$\boldsymbol{\varphi} \in \mathbb{H}^{1,S}_{0,\sigma}(\Omega) \mapsto ((\boldsymbol{V}^{\alpha})'(t), \boldsymbol{\varphi}) + \nu(\nabla \boldsymbol{V}^{\alpha}(t), \nabla \boldsymbol{\varphi}) + ((\boldsymbol{V}^{\alpha}(t) \cdot \nabla)\boldsymbol{V}^{\alpha}(t), \boldsymbol{\varphi}). \tag{4.1}$$

Then we have the following Proposition.

PROPOSITION 4.1. *The map $\boldsymbol{r}$ is a linear and continuous functional on $\mathbb{H}^{1,S}_{0,\sigma}(\Omega)$. Furthermore we have ${}_{(\mathbb{H}^{1,S}_{0,\sigma})'}\langle \boldsymbol{r}, \boldsymbol{\varphi}\rangle_{\mathbb{H}^{1,S}_{0,\sigma}} \in L^2(0,T)$ for any $\boldsymbol{\varphi} \in \mathbb{H}^{1,S}_{0,\sigma}(\Omega)$.*

For the proof, see Amick [**2**, Lemma 3.4].

Suppose that $\theta \in C^{\infty}(\mathbb{R})$ satisfies

$$\begin{aligned} 0 \le \theta(s) \le 1 \quad &(s \in \mathbb{R}), \\ \theta(s) = 1 \quad &(s \ge 1), \\ \theta(s) = 0 \quad &(s \le 0). \end{aligned}$$

For all $\delta > 0$, we set

$$\theta_{\delta}(x) = \begin{cases} \theta(\delta(x_1 - L)) & (x \in \omega_{01}) \\ \theta(-\delta(x_1 + L)) & (x \in \omega_{02}) \\ 0 & \text{otherwise} \end{cases}.$$

Then we have the following Proposition.

PROPOSITION 4.2. *For all $\varepsilon > 0$, there exists an $\boldsymbol{s} \in H^1_{\pi}((0,T); \mathbb{H}^{1,S}_{0,\sigma}(\Omega))$ with compact support such that the following inequality holds true.*

$$((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{V}^{\alpha}) \le ((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{s}) + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{V}^{\alpha}\theta_{\delta}^2)_{\omega_{01}\cup\omega_{02}} + (\varepsilon + c_0\delta)\|\nabla \boldsymbol{v}\|_2^2 \qquad (\boldsymbol{v} \in \mathbb{H}^{1,S}_{0,\sigma}(\Omega)) \quad \text{on} \quad [0,T], \tag{4.2}$$

where the constant c_0 does not depend on ε and δ.

For the proof, see Amick [**2**, p. 495–496].

We set

$$\Gamma^S_{\delta}(t) := \sup_{v \in \mathbb{H}^{1,S}_{0,\sigma}(\Omega)} \frac{((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{V}^{\alpha}(t)\theta_{\delta}^2)_{\omega_{01}\cup\omega_{02}}}{\|\nabla \boldsymbol{v}\|^2_{2,\Omega}}.$$

Then we have the following Proposition.

PROPOSITION 4.3. *We have*

$$\lim_{\delta \to +0} \Gamma^S_{\delta}(t) = \max\{\gamma_1^{\alpha,S}(t), \gamma_2^{\alpha,S}(t)\}.$$

For the proof, see Amick [**2**, Theorem 4.3].

4.3. Leray's inequality.

We need an appropriate extension of the given boundary data $\boldsymbol{\beta}$.

Proposition 4.4. *We assume that a domain $D \subset \mathbb{R}^n$ $(n = 2, 3)$ is bounded and smooth. Suppose that $\boldsymbol{\beta} \in \mathbb{H}^{1/2}(\partial D)$ satisfies* (SOC).

Then for any $\varepsilon > 0$ there exists an extension $\boldsymbol{b}_\varepsilon \in \mathbb{H}^1_\sigma(D)$ of $\boldsymbol{\beta}$ satisfying the inequality

$$|((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{b}_\varepsilon)_D| < \varepsilon \|\nabla \boldsymbol{v}\|^2_{2,D} \quad (\boldsymbol{v} \in \mathbb{H}^1_{0,\sigma}(D)). \tag{4.3}$$

The estimate (4.3) is called "Leray's inequality". For the proof, see Fujita [**7**].

If $\boldsymbol{\beta}$ satisfies (GOC), we have the following Proposition 4.5. For the proof, see Fujita [**8**].

Proposition 4.5. *We assume that a domain $D \subset \mathbb{R}^2$ is bounded, smooth and symmetric with respect to the x_1-axis and all the boundary components $\Gamma_0, \ldots, \Gamma_J$ intersect the x_1-axis. Suppose that $\boldsymbol{\beta} \in \mathbb{H}^{1/2,S}(\partial D)$ satisfies* (GOC).

Then for any $\varepsilon > 0$ there exists an extension $\boldsymbol{b}_\varepsilon \in \mathbb{H}^{1,S}_\sigma(D)$ of $\boldsymbol{\beta}$ satisfying the inequality

$$|((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{b}_\varepsilon)_D| < \varepsilon \|\nabla \boldsymbol{v}\|^2_{2,D} \quad (\boldsymbol{v} \in \mathbb{H}^{1,S}_{0,\sigma}(D)). \tag{4.4}$$

The estimate (4.4) is "the symmetric type of Leray's inequality". The following Proposition 4.6 is its time dependent version in an unbounded perturbed channel.

Proposition 4.6. *We assume that a domain Ω satisfies Assumption* 2.1. *Suppose that $\boldsymbol{\beta} \in H^1_\pi((0,T); \mathbb{H}^{1/2,S}(\partial\Omega))$ satisfies* (GOC)*, the support of $\boldsymbol{\beta}$ is compact and*

$$\int_{\Gamma_0^+} \boldsymbol{\beta} \cdot \boldsymbol{n} d\sigma = \int_{\Gamma_0^-} \boldsymbol{\beta} \cdot \boldsymbol{n} d\sigma = 0 \quad \text{on} \quad [0, T].$$

Then for any $\varepsilon > 0$ there exists an extension $\boldsymbol{b}_\varepsilon \in H^1_\pi((0,T); \mathbb{H}^{1,S}_\sigma(\Omega))$ of $\boldsymbol{\beta}$ such that $\boldsymbol{b}_\varepsilon$ has compact support and the inequality

$$|((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{b}_\varepsilon(t))| < \varepsilon \|\nabla \boldsymbol{v}\|^2_{2,\Omega} \quad (\boldsymbol{v} \in \mathbb{H}^{1,S}_{0,\sigma}(\Omega),\, t \in [0,T]) \tag{4.5}$$

holds true.

The proof of Proposition 4.6 is similar to Proposition 4.5.

Remark 4.1. We assume that Ω is symmetric but does not satisfy Assumption 2.1, that is to say, Ω has pairs of symmetric boundaries Γ_i^+, Γ_i^- $(i = 1, \ldots, N)$ which do not intersect the x_1-axis. If the boundary data $\boldsymbol{\beta}$ satisfies

$$\int_{\Gamma_i^+} \boldsymbol{\beta} \cdot \boldsymbol{n} d\sigma = \int_{\Gamma_i^-} \boldsymbol{\beta} \cdot \boldsymbol{n} d\sigma = 0 \quad (i = 1, \ldots, N),$$

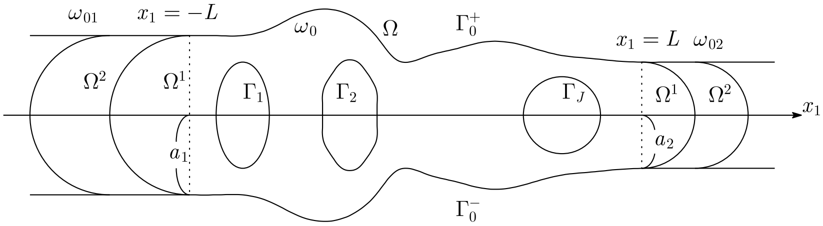
then "the symmetric type of Leray's inequality (4.5)" holds true.

REMARK 4.2. In general domains if we suppose that $\boldsymbol{\beta} \in \mathbb{H}^{1/2}(\partial\Omega)$ satisfies (GOC), then there does not exists an extension $\boldsymbol{b} \in \mathbb{H}^1_\sigma(\Omega)$ satisfying (4.3). See Takeshita [**25**] or Kobayashi [**14**].

5. Proof of Theorem 3.1.

5.1. Time periodic weak solution in a bounded domain.

We suppose that Ω^n is a smooth symmetric and bounded domain within Ω and satisfies $\Omega^n \subset \Omega^{n+1}$ and $\cup_{n\in\mathbb{N}}\Omega^n = \Omega$, where $\partial\Omega^1$ (the boundary of Ω^1) containes the support of $\boldsymbol{\beta}$.



In each bounded domain Ω^n, we consider the following time periodic problem of the Navier–Stokes equations

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in} \quad (0,T)\times\Omega^n,\\
\operatorname{div}\boldsymbol{u} &= 0 && \text{in} \quad (0,T)\times\Omega^n,\\
\boldsymbol{u} &= \boldsymbol{\beta}^n + \boldsymbol{V}^\alpha && \text{on} \quad (0,T)\times\partial\Omega^n,\\
\boldsymbol{u}(0) &= \boldsymbol{u}(T) && \text{in} \quad \Omega^n,
\end{aligned}
$$

where

$$
\boldsymbol{\beta}^n = \begin{cases} \boldsymbol{\beta} & \text{on} \quad (0,T)\times\partial\Omega\cap\partial\Omega^n \\ \boldsymbol{0} & \text{on} \quad (0,T)\times\partial\Omega^n\backslash\partial\Omega \end{cases}.
$$

The domain Ω^n is symmetric with respect to the x_1-axis, the connected components of $\partial\Omega^n$ intersect the x_1-axis and $\boldsymbol{\beta}^n + \boldsymbol{V}^\alpha|_{\partial\Omega^n} \in H^1_\pi((0,T);\mathbb{H}^{1/2,S}(\partial\Omega^n))$ satisfies (GOC) on $\partial\Omega^n$. Therefore there exists a $\boldsymbol{u}_n$ satisfying

$$
\begin{aligned}
&\boldsymbol{u}_n \in L^2((0,T);\mathbb{H}^{1,S}_\sigma(\Omega^n)) \cap L^\infty((0,T);\mathbb{L}^{2,S}_\sigma(\Omega^n)),\\
&\boldsymbol{u}_n' \in L^2((0,T);(\mathbb{H}^{1,S}_{0,\sigma}(\Omega^n))') \quad \text{(weak derivative of } \boldsymbol{u}_n)\\
&\boldsymbol{u}_n \in C_\pi([0,T];\mathbb{L}^{2,S}(\Omega^n))
\end{aligned}
$$

and

$$\frac{d}{dt}(\boldsymbol{u}_n,\boldsymbol{\varphi})_{\Omega^n}+\nu(\nabla\boldsymbol{u}_n,\nabla\boldsymbol{\varphi})_{\Omega^n}+((\boldsymbol{u}_n\cdot\nabla)\boldsymbol{u}_n,\boldsymbol{\varphi})_{\Omega^n}=\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_n \qquad (\boldsymbol{\varphi}\in\mathbb{H}^{1,S}_{0,\sigma}(\Omega^n)),$$
$$\boldsymbol{u}_n=\boldsymbol{\beta}^n+\boldsymbol{V}^\alpha \quad (0,T)\times\text{on}\quad\partial\Omega^n,$$

where $\langle\cdot,\cdot\rangle_n$ denotes the duality pair of $(\mathbb{H}^{1,S}_{0,\sigma}(\Omega^n))'$ and $\mathbb{H}^{1,S}_{0,\sigma}(\Omega^n)$. For the proof, see Kobayashi [**15**, Theorem 1.1].

5.2. The boundness of the initial value.

We set

$$\boldsymbol{w}_n=\begin{cases}\boldsymbol{u}_n-\boldsymbol{b}_\varepsilon-\boldsymbol{V}^\alpha & \text{in}\quad\Omega^n\\ \boldsymbol{0} & \text{in}\quad\Omega\backslash\Omega^n\end{cases},$$

where $\boldsymbol{b}_\varepsilon$ is the extension of $\boldsymbol{\beta}\in H^1_\pi((0,T);\mathbb{H}^{1/2,S}(\partial\Omega))$ satisfying Proposition 4.6. In this subsection, we will prove that $\|\boldsymbol{w}_n(0)\|_{2,\Omega}$ is bounded with respect to n.

It follows that

$$\boldsymbol{w}_n\in L^2((0,T);\mathbb{H}^{1,S}_{0,\sigma}(\Omega))\cap L^\infty((0,T);\mathbb{L}^{2,S}_\sigma(\Omega)),$$
$$\frac{d}{dt}(\boldsymbol{w}_n,\boldsymbol{\varphi})+\nu(\nabla\boldsymbol{w}_n,\nabla\boldsymbol{\varphi})+((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{\varphi})+((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi})+((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{\varphi})$$
$$+((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{V}^\alpha,\boldsymbol{\varphi})+((\boldsymbol{V}^\alpha\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{\varphi})=\langle\boldsymbol{F},\boldsymbol{\varphi}\rangle \quad (\boldsymbol{\varphi}\in\mathbb{H}^{1,S}_{0,\sigma}(\Omega^n)), \tag{5.1}$$

where $\boldsymbol{\varphi}\in\mathbb{H}^{1,S}_{0,\sigma}(\Omega^n)$ is extended as a $\boldsymbol{0}$ function to the outside of Ω^n and

$$\langle\boldsymbol{F},\boldsymbol{\varphi}\rangle=\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle-(\boldsymbol{b}_{\varepsilon,t},\boldsymbol{\varphi})-\nu(\nabla\boldsymbol{b}_\varepsilon,\nabla\boldsymbol{\varphi})-((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi})$$
$$-((\boldsymbol{V}^\alpha\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi})-((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{V}^\alpha,\boldsymbol{\varphi})-\langle\boldsymbol{r},\boldsymbol{\varphi}\rangle \quad (\boldsymbol{\varphi}\in\mathbb{H}^{1,S}_{0,\sigma}(\Omega)).$$

For the definition of $\boldsymbol{r}$, see Proposition 4.1. We have $\boldsymbol{F}\in L^2((0,T);(\mathbb{H}^{1,S}_{0,\sigma}(\Omega))')$.

Now we set $\boldsymbol{\varphi}=\boldsymbol{w}_n$ in the equation (5.1). We have

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}_n\|_2^2+\nu\|\nabla\boldsymbol{w}_n\|_2^2=((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{b}_\varepsilon)+((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{V}^\alpha)+\langle\boldsymbol{F},\boldsymbol{w}_n\rangle. \tag{5.2}$$

It follows from Proposition 4.6 that

$$|((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{b}_\varepsilon)|<\varepsilon\|\nabla\boldsymbol{w}_n\|_2^2. \tag{5.3}$$

Applying Proposition 4.2 to $((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{V}^\alpha)$, then we have

$$((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{V}^\alpha)\le((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{s})+((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{V}^\alpha\theta_\delta^2)_{\omega_{01}\cup\omega_{02}}+(\varepsilon+c_0\delta)\|\nabla\boldsymbol{w}_n\|_2^2$$
$$\le((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{s})+\Gamma^S_\delta\|\nabla\boldsymbol{w}_n\|_2^2+(\varepsilon+c_0\delta)\|\nabla\boldsymbol{w}_n\|_2^2. \tag{5.4}$$

We choose $\varepsilon > 0$ such that

$$\nu - \hat{\gamma}^{\alpha,S} - 5\varepsilon > 0.$$

Furthermore we choose $\delta > 0$ from Proposition 4.3 such that

$$\begin{aligned} \Gamma_\delta^S(t) &\le \hat{\gamma}^{\alpha,S} + \varepsilon, \\ \mu_1 &:= \nu - \hat{\gamma}^{\alpha,S} - 5\varepsilon - c_0\delta > 0. \end{aligned} \tag{5.5}$$

It follows from (5.2), (5.3), (5.4) and (5.5) that

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}_n\|_2^2 + (\nu - \hat{\gamma}^{\alpha,S} - 4\varepsilon - C_0\delta)\|\nabla \boldsymbol{w}_n\|_2^2 \le ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{s}) + C\|\boldsymbol{F}\|^2_{(\mathbb{H}^{1,S}_{0,\sigma})'}, \tag{5.6}$$

where the constant C depends only on ε. There exists an $N \in \mathbb{N}$ such that for any $n \ge N$, $\operatorname{supp}(\boldsymbol{s}) \subset \Omega^n$. Therefore we have by (5.1)

$$\begin{aligned} &((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{s}) \\ &\quad = -\frac{d}{dt}(\boldsymbol{w}_n, \boldsymbol{s}) + (\boldsymbol{w}_n, \boldsymbol{s}') - \nu(\nabla \boldsymbol{w}_n, \nabla \boldsymbol{s}) - ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{b}_\varepsilon, \boldsymbol{s}) \\ &\qquad - ((\boldsymbol{b}_\varepsilon \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{s}) - ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{V}^\alpha, \boldsymbol{s}) - ((\boldsymbol{V}^\alpha \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{s}) + \langle \boldsymbol{F}, \boldsymbol{s}\rangle \\ &\quad \le -\frac{d}{dt}(\boldsymbol{w}_n, \boldsymbol{s}) + \varepsilon\|\nabla \boldsymbol{w}_n\|_2^2 + K_1, \end{aligned} \tag{5.7}$$

where the constant C depnds on $\boldsymbol{V}^\alpha$, ε and the Poincaré inequality,

$$K_1(t) := C\Big\{\Big(\sup_{[0,T]}\|\nabla \boldsymbol{s}\|_2^2\Big)\|\boldsymbol{b}_\varepsilon(t)\|^2_{\mathbb{H}^1} + \|\boldsymbol{s}'(t)\|_2^2 + \|\boldsymbol{F}(t)\|^2_{(\mathbb{H}^{1,S}_{0,\sigma})'} + \|\nabla \boldsymbol{s}(t)\|_2^2\Big\}.$$

It follows from (5.6) and (5.7) that

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}_n\|_2^2 + \mu_1\|\nabla \boldsymbol{w}_n\|_2^2 \le -\frac{d}{dt}(\boldsymbol{w}_n, \boldsymbol{s}) + K_1 + C\|\boldsymbol{F}\|^2_{(\mathbb{H}^{1,S}_{0,\sigma})'}. \tag{5.8}$$

Applying the Poincaré inequality, we have

$$\frac{d}{dt}\|\boldsymbol{w}_n\|_2^2 + \mu_2\|\boldsymbol{w}_n\|_2^2 \le -2\frac{d}{dt}(\boldsymbol{w}_n, \boldsymbol{s}) + 2K_2, \tag{5.9}$$

where

$$K_2(t) := K_1(t) + C\|\boldsymbol{F}(t)\|^2_{(\mathbb{H}^{1,S}_{0,\sigma})'},$$

$$\mu_2 := \frac{2\mu_1}{C(\Omega)^2}.$$

We choose $\xi > 0$ such that $\mu := \mu_2 - \xi > 0$ holds. Multiplying (5.9) by $e^{\mu t}$, then we obtain

$$\begin{aligned}
& e^{\mu t} \frac{d}{dt} \|\boldsymbol{w}_n(t)\|_2^2 + \mu_2 e^{\mu t} \|\boldsymbol{w}_n(t)\|_2^2 \\
& \leq -2e^{\mu t} \frac{d}{dt} (\boldsymbol{w}_n(t), \boldsymbol{s}(t)) + 2K_2(t) e^{\mu t} \\
& = -2 \frac{d}{dt} \{(\boldsymbol{w}_n(t), \boldsymbol{s}(t)) e^{\mu t}\} + 2\mu e^{\mu t} (\boldsymbol{w}_n(t), \boldsymbol{s}(t)) + 2K_2(t) e^{\mu t} \\
& \leq -2 \frac{d}{dt} \{(\boldsymbol{w}_n(t), \boldsymbol{s}(t)) e^{\mu t}\} + \xi e^{\mu t} \|\boldsymbol{w}_n(t)\|_2^2 + (C\|\boldsymbol{s}(t)\|_2^2 + 2K_2(t)) e^{\mu t}, \qquad (5.10)
\end{aligned}$$

where the constant C depends only on μ_2 and ξ. Therefore we have

$$\frac{d}{dt}(e^{\mu t}\|\boldsymbol{w}_n(t)\|_2^2) \leq -2\frac{d}{dt}\{(\boldsymbol{w}_n(t), \boldsymbol{s}(t))e^{\mu t}\} + K_3(t), \qquad (5.11)$$

where

$$K_3(t) = (C\|\boldsymbol{s}(t)\|_2^2 + 2K_2(t))e^{\mu t}.$$

Integrating the inequality (5.11) on $[0, T]$, then we have

$$\|\boldsymbol{w}_n(T)\|_2^2 e^{\mu T} \leq \|\boldsymbol{w}_n(0)\|_2^2 - 2(\boldsymbol{w}_n(T), \boldsymbol{s}(T))e^{\mu T} + 2(\boldsymbol{w}_n(0), \boldsymbol{s}(0)) + K,$$

where

$$K = \int_0^T K_3(t)dt.$$

Since $\boldsymbol{w}_n$ is time periodic in $\mathbb{L}^2(\Omega)$, for any $\lambda > 0$ we have

$$\|\boldsymbol{w}_n(0)\|_2^2 e^{\mu T} \leq \|\boldsymbol{w}_n(0)\|_2^2 + (\lambda\|\boldsymbol{w}_n(0)\|_2^2 + C\|\boldsymbol{s}(T)\|_2^2)e^{\mu T} + \lambda\|\boldsymbol{w}_n(0)\|_2^2 + C\|\boldsymbol{s}(0)\|_2^2 + K,$$

where the constant C depends only on λ. We set

$$\begin{aligned}
H &= Ke^{-\mu T} + C \sup_{[0,T]} \|\boldsymbol{s}\|_2^2 (e^{-\mu T} + 1), \\
h &= 1 - \lambda - (1 + \lambda)e^{-\mu T}.
\end{aligned}$$

We choose $\lambda > 0$ such that $h > 0$. Then we have

$$\|\boldsymbol{w}_n(0)\|_2^2 \leq \frac{H}{h} := M_1.$$

Consequently, $\|\boldsymbol{w}_n(0)\|_2$ is a bounded sequence with respect to n.

5.3. A priori estimate and weak limit.

In this subsection, we will prove that $\{\boldsymbol{w}_n\}_n$ is a bounded sequence in $L^2((0,T);\mathbb{H}^{1,S}_{0,\sigma}(\Omega)) \cap L^\infty((0,T);\mathbb{L}^{2,S}_\sigma(\Omega))$ with respect to n.

Integrating (5.8) from 0 to τ, where $0<\tau<T$, we have in particular

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{w}_n(\tau)\|_2^2 - \frac{1}{2}\|\boldsymbol{w}_n(0)\|_2^2 + \mu_1\int_0^\tau \|\nabla \boldsymbol{w}_n\|_2^2 dt \\
&\quad \le -(\boldsymbol{w}_n(\tau),\boldsymbol{s}(\tau)) + (\boldsymbol{w}_n(0),\boldsymbol{s}(0)) + \int_0^T K_1(t) + \|\boldsymbol{F}(t)\|^2_{(\mathbb{H}^{1,S}_{0,\sigma})'} dt \\
&\quad \le \frac{1}{4}\|\boldsymbol{w}_n(\tau)\|_2^2 + \|\boldsymbol{s}(\tau)\|_2^2 + \frac{1}{2}\|\boldsymbol{w}_n(0)\|_2^2 + \frac{1}{2}\|\boldsymbol{s}(0)\|_2^2 + M_2,
\end{aligned}$$

where

$$M_2 = \int_0^T K_1(t) + \|\boldsymbol{F}(t)\|^2_{(\mathbb{H}^{1,S}_{0,\sigma})'} dt.$$

Therefore it follows that

$$\frac{1}{4}\|\boldsymbol{w}_n(\tau)\|_2^2 + \mu_1\int_0^T \|\nabla \boldsymbol{w}_n\|_2^2 dt \le M_1 + \frac{3}{2}\sup_{t\in[0,T]}\|\boldsymbol{s}(t)\|_2^2 + M_2 := M_3. \tag{5.12}$$

This proves that $\{\boldsymbol{w}_n\}$ is a bounded sequence in $L^2((0,T);\mathbb{H}^{1,S}_{0,\sigma}(\Omega)) \cap L^\infty((0,T);\mathbb{L}^{2,S}_\sigma(\Omega))$.

For any $\boldsymbol{\varphi}\in\mathbb{C}^{\infty,S}_{0,\sigma}(\Omega)$, there exists a $J\in\mathbb{N}$ such that $\mathrm{supp}(\boldsymbol{\varphi})\subset\Omega^n$ for any $n\ge J$. Then we have by (5.1) and (5.12)

$$|(\boldsymbol{w}_n(t),\boldsymbol{\varphi})| \le \|\boldsymbol{w}_n(t)\|_2\|\boldsymbol{\varphi}\|_2 \le 4M_3\|\boldsymbol{\varphi}\|_2$$

and

$$\begin{aligned}
&|(\boldsymbol{w}_n(t),\boldsymbol{\varphi}) - (\boldsymbol{w}_n(s),\boldsymbol{\varphi})| \\
&\quad = \left|\int_s^t \frac{d}{d\tau}(\boldsymbol{w}_n(\tau),\boldsymbol{\varphi})d\tau\right| \\
&\quad \le \int_s^t \nu|(\nabla \boldsymbol{w}_n,\nabla\boldsymbol{\varphi})| + |((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{\varphi})| + |((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi})| + |((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{\varphi})| \\
&\qquad + |((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{V}^\alpha,\boldsymbol{\varphi})| + |((\boldsymbol{V}^\alpha\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{\varphi})| + |\langle\boldsymbol{F},\boldsymbol{\varphi}\rangle| d\tau \\
&\quad \le \int_s^t \{(\nu + 2\|\boldsymbol{w}_n\|_2 + C_3\|\boldsymbol{b}_\varepsilon\|_{\mathbb{H}^1(\Omega)} + C_4)\|\nabla\boldsymbol{w}_n\|_2 + \|\boldsymbol{F}\|_{(\mathbb{H}^{1,S}_{0,\sigma})'})\}\|\nabla\boldsymbol{\varphi}\|_2 d\tau \\
&\quad \le M_4|t-s|^{1/2}\|\nabla\boldsymbol{\varphi}\|_2,
\end{aligned}$$

where the constant M_4 does not depend on n. Therefore $\{(\boldsymbol{w}_n(t),\boldsymbol{\varphi})\}_{n\ge J}$ is uniformly bounded and equicontinuous on $[0,T]$ with respect to $n\ge J$.

Since $\{\boldsymbol{w}_n\}$ is a bounded sequence in $L^2((0,T);\mathbb{H}^{1,S}_{0,\sigma}(\Omega))\cap L^\infty((0,T);\mathbb{L}^{2,S}_{\sigma}(\Omega))$ with respect to n, there exist a subsequence $\{\boldsymbol{w}_{nk}\}_k$ of $\{\boldsymbol{w}_n\}$ and a $\boldsymbol{w}$ of $L^2((0,T);\mathbb{H}^{1,S}_{0,\sigma}(\Omega))\cap L^\infty((0,T);\mathbb{L}^{2,S}_{\sigma}(\Omega))$ such that

$$\boldsymbol{w}_{nk}\to\boldsymbol{w}\quad\text{in}\quad\begin{cases}L^\infty((0,T);\mathbb{L}^{2,S}_{\sigma}(\Omega)) & \text{weak star}\\ L^2((0,T);\mathbb{H}^{1,S}_{0,\sigma}(\Omega)) & \text{weakly}\end{cases}\quad(k\to\infty). \tag{5.13}$$

For any $\boldsymbol{\varphi}\in\mathbb{C}^{\infty,S}_{0,\sigma}(\Omega)$, there exists a subsequence $\{\boldsymbol{w}_{nki}\}$ of $\{\boldsymbol{w}_{nk}\}$ such that

$$\lim_{i\to\infty}(\boldsymbol{w}_{nki},\boldsymbol{\varphi})=(\boldsymbol{w},\boldsymbol{\varphi}) \tag{5.14}$$

by *the Ascoli–Arzelà* Theorem. We will prove the convergence (5.14) for any $\boldsymbol{\varphi}\in\mathbb{L}^{2,S}(\Omega)$. We have the orthogonal decomposition

$$\boldsymbol{\varphi}=\boldsymbol{\varphi}_\sigma+\boldsymbol{\varphi}_p\quad(\boldsymbol{\varphi}_\sigma\in\mathbb{L}^{2,S}_{\sigma}(\Omega),\ \boldsymbol{\varphi}_p\in(\mathbb{L}^{2,S}_{\sigma}(\Omega))^\perp).$$

Since $\mathbb{C}^{\infty,S}_{0,\sigma}(\Omega)$ is dence in $\mathbb{L}^{2,S}_{\sigma}(\Omega)$, for any $\eta>0$ there exists $\boldsymbol{\varphi}^\eta_\sigma\in\mathbb{C}^{\infty,S}_{0,\sigma}(\Omega)$ such that

$$\|\boldsymbol{\varphi}^\eta_\sigma-\boldsymbol{\varphi}_\sigma\|_2<\eta.$$

We have

$$\begin{aligned}|(\boldsymbol{w}-\boldsymbol{w}_n,\boldsymbol{\varphi})|&\le|(\boldsymbol{w}-\boldsymbol{w}_n,\boldsymbol{\varphi}_\sigma-\boldsymbol{\varphi}^\eta_\sigma)|+|(\boldsymbol{w}-\boldsymbol{w}_n,\boldsymbol{\varphi}^\eta_\sigma)|\\&\le 8M_3\eta+|(\boldsymbol{w}-\boldsymbol{w}_n,\boldsymbol{\varphi}^\eta_\sigma)|\end{aligned} \tag{5.15}$$

because $\boldsymbol{w}_n$ is bounded in $L^\infty((0,T);\mathbb{L}^{2,S}_{\sigma}(\Omega))$. We can choose a subsequence $\{\boldsymbol{w}_{nk}\}_k$ of $\{\boldsymbol{w}_n\}_n$ such that the second term of the right hand side of (5.15) goes to 0. Therefore for any $\boldsymbol{\varphi}\in\mathbb{L}^{2,S}(\Omega)$ there exists a subsequence $\{\boldsymbol{w}_{nki}\}$ such that $(\boldsymbol{w}_{nki},\boldsymbol{\varphi})$ converges to $(\boldsymbol{w},\boldsymbol{\varphi})$ uniformly on $[0,T]$.

5.4. Time periodic weak solution.

In this subsection, we prove that $\boldsymbol{w}+\boldsymbol{b}_\varepsilon+\boldsymbol{V}^\alpha$ is a time periodic weak solution.

For any $\boldsymbol{\varphi}\in\mathbb{C}^{\infty,S}_{0,\sigma}(\Omega)$, there exists an $L\in\mathbb{N}$ such that $\operatorname{supp}(\boldsymbol{\varphi})\subset\Omega^n$ for any $n\ge L$. Then we have (5.1) for any $n\ge L$. We multiply it by $\psi\in C_0^\infty(0,T)$ and integrate on $[0,T]$. Then it follows that

$$\begin{aligned}\int_0^T&-(\boldsymbol{w}_n,\boldsymbol{\varphi})\psi'+\{\nu(\nabla\boldsymbol{w}_n,\nabla\boldsymbol{\varphi})+((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{\varphi})+((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi})\\&\qquad+((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{\varphi})+((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{V}^\alpha),\boldsymbol{\varphi})+((\boldsymbol{V}^\alpha\cdot\nabla)\boldsymbol{w}_n,\boldsymbol{\varphi})\}\psi dt\\&=\int_0^T\langle\boldsymbol{F},\boldsymbol{\varphi}\rangle\psi dt.\end{aligned} \tag{5.16}$$

We can choose a subsequence $\{\boldsymbol{w}_{nk}\}_k$ such that the left hand side of (5.16) except the nonlinear term converges to

$$\int_0^T -(\boldsymbol{w},\boldsymbol{\varphi})\psi' + \{\nu(\nabla\boldsymbol{w},\nabla\boldsymbol{\varphi}) + ((\boldsymbol{w}\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi}) + ((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi}) \\ + ((\boldsymbol{w}\cdot\nabla)\boldsymbol{V}^\alpha),\boldsymbol{\varphi}) + ((\boldsymbol{V}^\alpha\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi})\}\psi dt.$$

We prove that there exists a subsequence $\{\boldsymbol{w}_{nki}\}_i$ such that

$$\int_0^T ((\boldsymbol{w}_{nki}\cdot\nabla)\boldsymbol{w}_{nki},\boldsymbol{\varphi})\psi dt \to \int_0^T ((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi})\psi dt \quad (i\to\infty). \tag{5.17}$$

We have

$$\int_0^T ((\boldsymbol{w}_{nk}\cdot\nabla)\boldsymbol{w}_{nk},\boldsymbol{\varphi})\psi dt - \int_0^T ((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi})\psi dt \\ = \int_0^T ((\boldsymbol{w}_{nk}-\boldsymbol{w})\cdot\nabla\boldsymbol{w}_{nk},\boldsymbol{\varphi})\psi dt - \int_0^T (\boldsymbol{w}\cdot\nabla\boldsymbol{\varphi},\boldsymbol{w}_{nk}-\boldsymbol{w})\psi dt \quad (:= I_1 + I_2). \tag{5.18}$$

Firstly, we consider I_1. By Lemma 4.2, for any $\eta > 0$ there exist an integer N_1 and $\boldsymbol{\psi}_l \in \mathbb{L}^{2,S}(\Omega)$ $(l = 1,\dots,N_1)$ such that

$$\left|\int_0^T ((\boldsymbol{w}_{nk}-\boldsymbol{w})\cdot\nabla\boldsymbol{w}_{nk},\psi\boldsymbol{\varphi})dt\right| \\ \le \eta\int_0^T (\|\nabla\boldsymbol{w}_{nk}-\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{w}_{nk}\|_2^2 + \|\boldsymbol{w}_{nk}-\boldsymbol{w}\|_2\|\nabla\boldsymbol{w}_{nk}\|_2)dt \\ + \sum_{l=1}^{N_1}\int_0^T |(\boldsymbol{w}_{nk}-\boldsymbol{w},\boldsymbol{\psi}_l)|^2 dt. \tag{5.19}$$

There exists a constant M_5 independent of n such that

$$\eta\int_0^T (\|\nabla\boldsymbol{w}_{nk}-\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{w}_{nk}\|_2^2 + \|\boldsymbol{w}_{nk}-\boldsymbol{w}\|_2\|\nabla\boldsymbol{w}_{nk}\|_2)dt \le M_5\eta. \tag{5.20}$$

Secondly, we consider I_2. Since we know that $\psi\boldsymbol{w}\cdot\nabla\boldsymbol{\varphi} \in L^2((0,T);\mathbb{L}^{2,S}(\Omega))$, for any $t\in[0,T]$ we obtain the decomposition

$$\psi(t)\boldsymbol{w}(t)\cdot\nabla\boldsymbol{\varphi} = \boldsymbol{\Phi}_\sigma(t) + \boldsymbol{\Phi}_p(t) \quad (\boldsymbol{\Phi}_\sigma(t)\in\mathbb{L}^{2,S}_\sigma(\Omega),\ \boldsymbol{\Phi}_p(t)\in(\mathbb{L}^{2,S}_\sigma(\Omega))^\perp).$$

Since $\boldsymbol{\Phi}_\sigma \in L^2((0,T);\mathbb{L}^{2,S}_\sigma(\Omega))$, we have

$$\int_0^T ((\boldsymbol{w}\cdot\nabla)\boldsymbol{\varphi},\boldsymbol{w}_{nk}-\boldsymbol{w})\psi dt = \int_0^T (\boldsymbol{\Phi}_\sigma,\boldsymbol{w}_{nk}-\boldsymbol{w})dt. \tag{5.21}$$

It follows from (5.19), (5.20) and (5.21) that

$$\begin{aligned}\left|\int_0^T ((\boldsymbol{w}_{nk}\cdot\nabla)\boldsymbol{w}_{nk},\boldsymbol{\varphi})\psi dt-\int_0^T ((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi})\psi dt\right|\\ < M_5\eta+\sum_{l=1}^{N_1}\int_0^T |(\boldsymbol{w}_{nk}-\boldsymbol{w},\boldsymbol{\psi}_l)|^2 dt+\left|\int_0^T (\boldsymbol{\Phi}_\sigma,\boldsymbol{w}_{nk}-\boldsymbol{w})dt\right|. \qquad (5.22)\end{aligned}$$

This proves that there exists a subsequence $\{\boldsymbol{w}_{nki}\}_{i\in\mathbb{N}}$ of $\{\boldsymbol{w}_{nk}\}_{k\in\mathbb{N}}$ such that the limit (5.17) holds. Therefore we obtain

$$\begin{aligned}\int_0^T &-(\boldsymbol{w},\boldsymbol{\varphi})\psi'+\{\nu(\nabla\boldsymbol{w},\nabla\boldsymbol{\varphi})+((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi})+((\boldsymbol{w}\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi})+((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi})\\ &\qquad+((\boldsymbol{w}\cdot\nabla)\boldsymbol{V}^\alpha),\boldsymbol{\varphi})+((\boldsymbol{V}^\alpha\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi})\}\psi dt\\ &=\int_0^T \langle\boldsymbol{F},\boldsymbol{\varphi}\rangle\psi dt\end{aligned}$$

Since the inclusion $\mathbb{C}_{0,\sigma}^{\infty,S}(\Omega)\subset\mathbb{H}_{0,\sigma}^{1,S}(\Omega)$ is dense, we have

$$\begin{aligned}\int_0^T &-(\boldsymbol{w},\boldsymbol{\varphi})\psi'+\{\nu(\nabla\boldsymbol{w},\nabla\boldsymbol{\varphi})+((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi})+((\boldsymbol{w}\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi})+((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi})\\ &\qquad+((\boldsymbol{w}\cdot\nabla)\boldsymbol{V}^\alpha),\boldsymbol{\varphi})+((\boldsymbol{V}^\alpha\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi})\}\psi dt\\ &=\int_0^T \langle\boldsymbol{F},\boldsymbol{\varphi}\rangle\psi dt\quad(\boldsymbol{\varphi}\in\mathbb{H}_{0,\sigma}^{1,S}(\Omega),\ \psi\in C_0^\infty(0,T)). \qquad (5.23)\end{aligned}$$

It is easy to prove $\boldsymbol{w}'\in L^2((0,T);(\mathbb{H}_{0,\sigma}^{1,S}(\Omega))')$, where $\boldsymbol{w}'$ is the weak derivative of $\boldsymbol{w}$. Therefore we have $\boldsymbol{w}\in C([0,T];\mathbb{L}_\sigma^{2,S}(\Omega))$.

Lastly, we prove that $\boldsymbol{w}$ is time periodic in $\mathbb{L}^2(\Omega)$. For any $\boldsymbol{\varphi}\in\mathbb{L}^2(\Omega)$, there exists a subsequence $\{\boldsymbol{w}_{nk}\}$ such that the limit (5.14) holds true. Therefore it follows that

$$(\boldsymbol{w}(0)-\boldsymbol{w}(T),\boldsymbol{\varphi})=(\boldsymbol{w}(0)-\boldsymbol{w}_{nk}(0),\boldsymbol{\varphi})+(\boldsymbol{w}_{nk}(T)-\boldsymbol{w}(T),\boldsymbol{\varphi})\to 0\quad(k\to\infty).$$

We set

$$\boldsymbol{u}=\boldsymbol{w}+\boldsymbol{b}_\varepsilon+\boldsymbol{V}^\alpha.$$

Then $\boldsymbol{u}$ is a time periodic weak solution. □

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