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Instanton-type solutions for the second and the fourth Painlevé hierarchies with a large parameter

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Abstract. A construction of general formal solutions for members $(P_{\mathrm{J}})_m$ ($\mathrm{J} = \mathrm{II}, \mathrm{IV}$) of the second and the fourth Painlevé hierarchies with a large parameter is discussed. We also investigate a relation between formal solutions of $(P_{\mathrm{J}})_m$ ($\mathrm{J} = \mathrm{II}, \mathrm{IV}$).

1. Introduction.

For the traditional Painlevé equations with a large parameter η, in 1990's, T. Aoki, T. Kawai and Y. Takei constructed formal solutions with two free parameters called instanton-type solutions and they succeed in giving the concrete descriptions of the Stokes phenomenon and the connection formula among these solutions (see [**11**], [**6**], [**12**] and [**10**]). Since this success, from a viewpoint of the exact WKB analysis, the higher-order Painlevé equations have been studied in the series of papers [**9**], [**13**] and [**14**]. T. Kawai and Y. Takei [**14**] established structure theorem for instanton-type solutions of Painlevé hierarchies near simple P-turning points of the first kind and instanton-type solutions are expected to be suitable formal solutions for the description of Stokes phenomenon of Painlevé hierarchies with η. The construction of such solutions has been studied in [**1**], [**6**], [**10**], [**19**], [**21**] and [**2**], and so far the existence of the solutions for Painlevé hierarchies $(P_{\mathrm{J}})_m$ ($\mathrm{J} = \mathrm{I}, \mathrm{II}, 34, \mathrm{IV}$) has been proved by the method which is based on their Hamiltonian structures (see Y. Takei [**21**], [**15**] and [**16**]). On the other hand, there is another method to construct instanton-type solutions. The solutions of traditional Painlevé equations with η were constructed by multiple-scale analysis (for example, see [**1**] and [**10**]). Recently, T. Aoki, N. Honda and the author [**5**] showed how to implement the concrete computation of instanton-type solutions for $(P_{\mathrm{I}})_m$ by the multiple-scale analysis and gave the explicit forms of the solutions. The concrete forms of instanton-type solutions of $(P_{\mathrm{II}})_m$ and $(P_{\mathrm{IV}})_m$ have not been obtained yet. We are interested in a question: What kind of classes of differential equations can we apply the method given in [**5**] to? In this paper, following [**5**], we construct instanton-type solutions to $(P_{\mathrm{II}})_m$ and $(P_{\mathrm{IV}})_m$. More specifically, we first introduce a new auxiliary variable θ and rewrite $(P_{\mathrm{II}})_m$ and $(P_{\mathrm{IV}})_m$ in the systems described by generating functions of their unknown functions. Then we consider our problem in the expanded space $\mathcal{A}(\Omega)$ consisting of formal power series of θ with suitable coefficients and construct instanton-type solutions of $(P_{\mathrm{II}})_m$ and

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$(P_{\mathrm{IV}})_m$ so that the solutions are expressed by proper generating functions in $\mathcal{A}(\Omega)$. In the paper, it is interesting that we show not only the concrete forms of solutions but also the shared algebraic structures between $(P_{\mathrm{J}})_m$ (J = I, II, 34, IV) which appear in the procedure of the construction of solutions. As is shown in [**5**], $(P_{\mathrm{I}})_m$ is associated with a system of partial differential equations expressed by the characteristic map Q and, thanks to the map Q, the system has a special multiplicative structure which plays an essential role in the construction of solutions. This fact is a key of success in the construction of solutions for $(P_{\mathrm{I}})_m$. This paper clarifies that $(P_{\mathrm{II}})_m$ and $(P_{\mathrm{IV}})_m$ also have the completely same structure as what we stated for $(P_{\mathrm{I}})_m$. In addition, we see that instanton-type solutions of $(P_{\mathrm{II}})_m$ and $(P_{\mathrm{IV}})_m$ are transformed each other by the replacement of the corresponding variables. (See Theorem 6.1 below for more precise statements.)

The rest of this paper is organized as follows. We first construct instanton-type solutions for $(P_{\mathrm{II}})_m$. In Section 2, we give the explicit form of $(P_{\mathrm{II}})_m$ and some results concerning its 0-parameter solution by using generating functions. In Section 3, we begin with considering a linearized equation of $(P_{\mathrm{II}})_m$ along its 0-parameter solution. We also investigate its algebraic structure (cf. Lemma 3.1) and the system of partial differential equations associated with $(P_{\mathrm{II}})_m$ is given by (45). In Section 3.3, we observe that the system is simplified by results in Section 3.2. To obtain the leading and subleading terms of solutions, we need to see the first member $(\mathcal{E}_1)$ of the non-secularity conditions which naturally appear by the multiple-scale analysis. Although $(\mathcal{E}_1)$ is a system of non-linear equations with $2m$ unknown functions, we can solve globally the system (see Section 4). As a matter of fact, we have the explicit forms of the leading and subleading terms of solutions for $(P_{\mathrm{II}})_m$ in Lemmas 4.1, 4.4 and Proposition 4.8. In Section 5, we prove that the higher-order terms of solutions can be constructed and we obtain our main theorem (Theorem 5.3). In Section 6, instanton-type solutions of $(P_{\mathrm{IV}})_m$ are obtained and the relation of solutions of $(P_{\mathrm{II}})_m$ and $(P_{\mathrm{IV}})_m$ is given by Theorem 6.1. For the sake of completeness, in the appendix, we give some long proofs to be needed in previous sections.

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2. The second Painlevé hierarchy in terms of generating functions.

For $m = 1, 2, \ldots$, the m-th member $(P_{\mathrm{II}})_m$ of the second Painlevé hierarchy with a large parameter η is given by T. Koike [**15**] (See also [**7**]):

$$\begin{cases} \eta^{-1}\dfrac{du_j}{dt} = -2(u_1u_j + v_j + u_{j+1}) + 2c_ju_1, & j = 1, 2, \ldots, m, \\ \eta^{-1}\dfrac{dv_j}{dt} = 2(v_1u_j + v_{j+1} + w_j) - 2c_jv_1, & j = 1, 2, \ldots, m \end{cases} \tag{1}$$

with $u_{m+1} = \gamma t$ and $v_{m+1} = \kappa$. Here u_j, v_j are unknown functions of t and c_j, $\gamma(\neq 0)$, κ are constants, and w_j denotes a polynomial of u_k, v_l $(1 \leq k, l \leq m)$ recursively defined by

$$w_j := \sum_{k=1}^{j-1} u_{j-k} w_k + \sum_{k=1}^{j} u_{j-k+1} v_k + \frac{1}{2} \sum_{k=1}^{j-1} v_{j-k} v_k - \sum_{k=1}^{j-1} c_{j-k} w_k. \tag{2}$$

As is mentioned in [**15**], if we set $m = 1$ and $c_1 = 0$, then we have the traditional second Painlevé equation P_{II} with a large parameter η given in [**10**]:

$$\frac{d^2 u}{dt^2} = \eta^2 (2u^3 + 8\gamma t u + c), \tag{3}$$

where $u = -2u_1$ and $c = 8\kappa + 4\eta^{-1}\gamma$. In [**8**], the parametric Stokes phenomena for 1-parameter solutions of (3) are deeply studied by the exact WKB analysis.

2.1. The form of $(P_{\mathrm{II}})_m$ by generating functions.

Following the idea of [**5**], we represent $(P_{\mathrm{II}})_m$ in terms of generating functions of u_k, v_k, w_k and c_k $(k = 1, 2, \dots)$ defined by

$$U(\theta) := \sum_{k=1}^{\infty} u_k \theta^k, \quad V(\theta) := \sum_{k=1}^{\infty} v_k \theta^k, \quad W(\theta) := \sum_{k=1}^{\infty} w_k \theta^{k+1} \text{ and } C(\theta) := \sum_{k=1}^{\infty} c_k \theta^k, \tag{4}$$

respectively. Here θ denotes an independent variable. Using generating functions, we have the following system:

$$\begin{cases} \eta^{-1} \dfrac{dU}{dt} \theta = 2(u_1(1 - U + C)\theta - U - V\theta), \\ \eta^{-1} \dfrac{dV}{dt} \theta = 2(-v_1(1 - U + C)\theta + W + V) \end{cases} \tag{5}$$

with

$$W = UW + UV + \frac{1}{2} V^2 \theta - CW, \quad \text{that is,} \quad W = \frac{2UV + V^2\theta}{2(1 - U + C)}. \tag{6}$$

Putting (4) into (5) (resp. (6)) and comparing the coefficients of θ^k $(2 \leq k \leq m+1)$ on the both sides, we have (1) (resp. (2)). In what follows, by $A \equiv B$ we mean that $A - B$ is zero modulo θ^{m+2} and we consider

$$\eta^{-1} \frac{d}{dt} \begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv 2 \begin{pmatrix} u_1(1 - U + C)\theta - U - V\theta \\ -v_1(1 - U + C)\theta + \dfrac{2UV + V^2\theta}{2(1 - U + C)} + V \end{pmatrix} \tag{7}$$

with the condition that the coefficients of θ^{m+1} of U and V are equal to γt and κ respectively.

2.2. Generating functions of 0-parameter solutions for $(P_{\mathrm{II}})_m$.

Equation (1) has a 0-*parameter solution* of the form

$$u_k(t) = \sum_{j=0}^{\infty} \eta^{-j} \hat{u}_{k,j}(t), \quad v_k(t) = \sum_{j=0}^{\infty} \eta^{-j} \hat{v}_{k,j}(t), \quad k = 1, \dots, m. \tag{8}$$

The construction and existence of 0-parameter solutions for higher-order Painlevé equations have been deeply investigated by [**9**], [**3**], [**4**], and we know that the higher-order terms $\hat{u}_{k,j}$, $\hat{v}_{k,j}$ $(j \geq 1)$ of (8) can be obtained uniquely once the leading terms $\hat{u}_{k,0}$, $\hat{v}_{k,0}$ $(1 \leq k \leq m)$ are determined by the following system of algebraic equations for $\hat{u}_{k,0}$, $\hat{v}_{k,0}$:

$$\begin{aligned} &-(\hat{u}_{1,0}\hat{u}_{k,0} + \hat{v}_{k,0} + \hat{u}_{k+1,0}) + c_k \hat{u}_{1,0} = 0, \\ &(\hat{v}_{1,0}\hat{u}_{k,0} + \hat{v}_{k+1,0} + \hat{w}_{k,0}) - c_k \hat{v}_{1,0} = 0, \quad k = 1, 2, \dots, m \end{aligned} \tag{9}$$

with $\hat{u}_{m+1,0} = \gamma t$ and $\hat{v}_{m+1,0} = \kappa$. Here $\hat{w}_{k,0}$ is defined in the similarly way as that of w_k by (2).

We define the generating functions of leading terms $\hat{u}_{i,0}$, $\hat{v}_{i,0}$ $(i \geq 1)$ of (8) by

$$\hat{u}_0(\theta) := \sum_{i=1}^{\infty} \hat{u}_{i,0}\theta^i \quad \text{and} \quad \hat{v}_0(\theta) := \sum_{i=1}^{\infty} \hat{v}_{i,0}\theta^i, \tag{10}$$

respectively. By (10), the system (9) is rewritten in the form

$$\frac{\hat{u}_0 + \hat{v}_0\theta}{1 - \hat{u}_0 + C} = \hat{u}_{1,0}\theta, \quad \frac{2\hat{u}_0\hat{v}_0 + (\hat{v}_0)^2\theta}{2(1 - \hat{u}_0 + C)} = \hat{v}_{1,0}(1 - \hat{u}_0 + C)\theta - \hat{v}_0 \tag{11}$$

with $\hat{u}_{m+1,0} = \gamma t$ and $\hat{v}_{m+1,0} = \kappa$. The above equations imply

$$(1 + \hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2 = \frac{(1 + C)^2}{(1 - \hat{u}_0 + C)^2}. \tag{12}$$

Hence, we have

$$\begin{aligned} \hat{u}_0 &= (1 + C)\left(1 - \sqrt{\frac{1}{(1 + \hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2}}\right), \\ \hat{v}_0\theta &= (1 + C)\left(-1 + (1 + \hat{u}_{1,0}\theta)\sqrt{\frac{1}{(1 + \hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2}}\right). \end{aligned} \tag{13}$$

As is shown in (13), $\hat{u}_0$ and $\hat{v}_0$ are determined once $\hat{u}_{1,0}$ and $\hat{v}_{1,0}$ are given. Noticing that $\hat{u}_{m+1,0} = \gamma t$ and $\hat{v}_{m+1,0} = \kappa$, we determine $\hat{u}_{1,0}$ and $\hat{v}_{1,0}$ so that the coefficients of θ^{m+1} in $\hat{u}_0$ and $\hat{v}_0$ of (13) are γt and κ, respectively.

3. Multiple-scale analysis for $(P_{\mathrm{II}})_m$ with generating functions.

We first prepare some definitions. Let α be a negative real number and $\tau := (\tau_1, \ldots, \tau_m)$ be m-independent variables. We denote by Ω an open subset in $\mathbb{C}_t$ satisfying the conditions (S1) and (S2) which will be given later in Section 3.2. Then the solution space to which formal solutions of instanton-type of (7) belong is defined by

$$\begin{aligned}\mathcal{A}_\alpha(\Omega) &:= \mathcal{M}(\Omega)\left[\left[\eta^\alpha e^{\tau_1}, \ldots, \eta^\alpha e^{\tau_m},\ \eta^\alpha e^{-\tau_1}, \ldots, \eta^\alpha e^{-\tau_m},\ \theta\right]\right],\\ \mathcal{A}^{\mathcal{O}}_\alpha(\Omega) &:= \mathcal{O}(\Omega)\left[\left[\eta^\alpha e^{\tau_1}, \ldots, \eta^\alpha e^{\tau_m},\ \eta^\alpha e^{-\tau_1}, \ldots, \eta^\alpha e^{-\tau_m},\ \theta\right]\right],\end{aligned} \tag{14}$$

where $\mathcal{M}(\Omega)$ (resp. $\mathcal{O}(\Omega)$) denotes the set of multi-valued holomorphic functions with a finite number of branching points and poles (resp. holomorphic functions) on Ω. We also denote by $\hat{\mathcal{A}}_\alpha(\Omega)$ (resp. $\hat{\mathcal{A}}^{\mathcal{O}}_\alpha(\Omega)$) the subset in $\mathcal{A}_\alpha(\Omega)$ (resp. $\mathcal{A}^{\mathcal{O}}_\alpha(\Omega)$) consisting of a formal power series of order less than or equal to α with respect to η. The subsequence arguments go well if α satisfies $\alpha = -1/k$ for an integer $k \geq 2$. For details, see Lemma 3.2 in [**5**]. Hence, from now on, we fix $\alpha = -1/2$ and abbreviate $\hat{\mathcal{A}}_\alpha(\Omega)$ to $\hat{\mathcal{A}}(\Omega)$.

In this section, by multiple-scale analysis with generating functions (cf. [**5**], [**10**]), we compute the system of partial differential equations in $\hat{\mathcal{A}}(\Omega)$ associated with (7) and study the algebraic structure of $(P_{\mathrm{II}})_m$.

3.1. A linearized equation of $(P_{\mathrm{II}})_m$ along the leading term of the 0-parameter solution.

We consider a linearized equation of (7) along $(\hat{u}_0, \hat{v}_0)$ defined by (10). Let $u_{i,j\alpha}$ and $v_{i,j\alpha}$ $(i, j \geq 1)$ be unknown functions of the variable t and we define $(u, v) \in \hat{\mathcal{A}}^2(\Omega) := (\hat{\mathcal{A}}(\Omega))^2$ by

$$u := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} u_{i,j\alpha}(t)\,\theta^i\,\eta^{j\alpha}, \qquad v := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} v_{i,j\alpha}(t)\,\theta^i\,\eta^{j\alpha}. \tag{15}$$

We also define $\sigma^\theta_k(u)$ (resp. $\sigma^\theta_k(v)$) by the coefficient of θ^k in u (resp. v). Then (7) is transformed, by a change of

$$U = \hat{u}_0 + (1 - \hat{u}_0 + C)u, \qquad V = \hat{v}_0 + (1 - \hat{u}_0 + C)v, \tag{16}$$

into the system of non-linear equations for (u, v):

$$\begin{aligned}&\eta^{-1}\frac{d}{dt}\begin{pmatrix}\hat{u}_0\\ \hat{v}_0\end{pmatrix}\theta + \eta^{-1}\left(\frac{d}{dt}(1 - \hat{u}_0 + C) + (1 - \hat{u}_0 + C)\frac{d}{dt}\right)\begin{pmatrix}u\\ v\end{pmatrix}\theta\\ &\quad\equiv 2(1 - \hat{u}_0 + C)\left(\begin{pmatrix}(\sigma^\theta_1(u) - \hat{u}_{1,0}u - v)\theta - u\\ (-\sigma^\theta_1(v) + 2\hat{v}_{1,0}u + \hat{u}_{1,0}v)\theta + v\end{pmatrix} + \begin{pmatrix}-\sigma^\theta_1(u)u\\ \sigma^\theta_1(v)u\end{pmatrix}\theta\right)\\ &\qquad + (1 - \hat{u}_0 + C)\frac{1}{1-u}\begin{pmatrix}0\\ 2u(\hat{v}_{1,0}u + \hat{u}_{1,0}v)\theta + (2uv + v^2\theta)\end{pmatrix}.\end{aligned} \tag{17}$$

Here we use (11) in order to obtain (17). Dividing (17) by $1 - \hat{u}_0 + C$, we have

$$
\begin{aligned}
&-\eta^{-1}\begin{pmatrix}\varrho\\ \delta\end{pmatrix}\theta + \eta^{-1}\left(\varrho + \frac{d}{dt}\right)\begin{pmatrix}u\\ v\end{pmatrix}\theta\\
&\equiv 2\Bigg(\begin{pmatrix}(\sigma_1^\theta(u) - \hat{u}_{1,0}u - v)\theta - u\\ (-\sigma_1^\theta(v) + 2\hat{v}_{1,0}u + \hat{u}_{1,0}v)\theta + v\end{pmatrix} + 2\begin{pmatrix}-\sigma_1^\theta(u)u\\ \sigma_1^\theta(v)u\end{pmatrix}\theta\\
&\quad + \frac{1}{1-u}\begin{pmatrix}0\\ 2u(\hat{v}_{1,0}u + \hat{u}_{1,0}v)\theta + (2uv + v^2\theta)\end{pmatrix}\Bigg),
\end{aligned} \tag{18}
$$

where ϱ and δ are defined by

$$
\varrho := \frac{d}{dt}(\log(1 - \hat{u}_0 + C)) \quad \text{and} \quad \delta := \frac{d/dt(-\hat{v}_0)}{1 - \hat{u}_0 + C}, \tag{19}
$$

respectively. A characteristic feature common to $(P_{\mathrm{I}})_m$, $(P_{\mathrm{II}})_m$, $(P_{34})_m$ and $(P_{\mathrm{IV}})_m$ is that the systems obtained as above are described by their special maps Q and each map is defined by the leading term of the right-hand side with respect to η. Now we confirm this fact for $(P_{\mathrm{II}})_m$. Let us define Θ by the set of formal power series of θ without constant terms and the map $Q : (\Theta\theta)^2 \to \Theta^2$ by

$$
Q\begin{pmatrix}x\,\theta\\ y\,\theta\end{pmatrix} := 2\begin{pmatrix}(\sigma_1^\theta(x) - \hat{u}_{1,0}x - y)\theta - x\\ (-\sigma_1^\theta(y) + 2\hat{v}_{1,0}x + \hat{u}_{1,0}y)\theta + y\end{pmatrix} \tag{20}
$$

for any $x = \sum_{j=1}^\infty x_j\theta^j$, $y = \sum_{j=1}^\infty y_j\theta^j \in \Theta$. Then Eq. (18) can be written in the form

$$
\begin{aligned}
&-\eta^{-1}\begin{pmatrix}\varrho\\ \delta\end{pmatrix}\theta + \eta^{-1}\left(\varrho + \frac{d}{dt}\right)\begin{pmatrix}u\\ v\end{pmatrix}\theta\\
&\equiv Q\begin{pmatrix}u\theta\\ v\theta\end{pmatrix} + 2\begin{pmatrix}-\sigma_1^\theta(u)u\\ \sigma_1^\theta(v)u\end{pmatrix}\theta + \frac{1}{1-u}\begin{pmatrix}0\\ \frac{1}{2}(-v, u)Q\begin{pmatrix}u\theta\\ v\theta\end{pmatrix} + (\sigma_1^\theta(u)v + \sigma_1^\theta(v)u)\theta\end{pmatrix}
\end{aligned} \tag{21}
$$

and (21) is equivalent to

$$
\begin{aligned}
\left(\eta^{-1}\frac{d}{dt} - Q\right)\begin{pmatrix}u\theta\\ v\theta\end{pmatrix} \equiv &\left(\begin{pmatrix}-2\sigma_1^\theta(u)u\theta\\ S(u,v)\end{pmatrix} + \eta^{-1}\begin{pmatrix}\varrho\\ \delta\end{pmatrix}\theta - uQ\begin{pmatrix}u\theta\\ v\theta\end{pmatrix}\right)\\
&- \left(2u^2\begin{pmatrix}-\sigma_1^\theta(u)\\ \sigma_1^\theta(v)\end{pmatrix} + \eta^{-1}\begin{pmatrix}2\varrho u\\ \delta u + \varrho v\end{pmatrix}\right)\theta\\
&+ \eta^{-1}u\left(\varrho + \frac{d}{dt}\right)\begin{pmatrix}u\\ v\end{pmatrix}\theta.
\end{aligned} \tag{22}
$$

Here ϱ, δ and $S(u,v)$ are defined by (19) and

$$S(u,v) := \frac{1}{2}(-v,u)Q\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} + (\sigma_1^\theta(u)v + 3\sigma_1^\theta(v)u)\theta, \tag{23}$$

respectively. To obtain a solution of the original system (1) from (22), noticing (16), we naturally impose the condition that the coefficients of θ^{m+1} of $(1-\hat{u}_0+C)u$ and $(1-\hat{u}_0+C)v$ are identically zero.

3.2. The algebraic structure associated with Q.

Let us obtain the eigenvector $(x,y) \in \Theta^2$ corresponding to an eigenvalue λ of Q in the sense of

$$Q\begin{pmatrix} x\theta \\ y\theta \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}\theta \quad \text{with} \quad x = \sum_{j=1}^{\infty} x_j\theta^j, \quad y = \sum_{j=1}^{\infty} y_j\theta^j. \tag{24}$$

By the definition of Q, (24) has the form

$$(E + M(\lambda)\theta)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sigma_1^\theta(x) \\ \sigma_1^\theta(y) \end{pmatrix}\theta. \tag{25}$$

Here E and $M(\lambda)$ are defined by

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M(\lambda) = \begin{pmatrix} \hat{u}_{1,0} + (\lambda/2) & 1 \\ 2\hat{v}_{1,0} & \hat{u}_{1,0} - (\lambda/2) \end{pmatrix}, \tag{26}$$

respectively. By the assumption $\det(E + M(\lambda)\theta) \neq 0$, (25) is solved. Let us try to find $g(\lambda)$ so that the following relation holds.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\chi(\lambda)\theta}{1 - g(\lambda)\theta} \quad \text{with} \quad \chi(\lambda) = \begin{pmatrix} \sigma_1^\theta(x) \\ \sigma_1^\theta(y) \end{pmatrix}, \tag{27}$$

that is,

$$\frac{\chi(\lambda)}{1 - g(\lambda)\theta} = (E + M(\lambda)\theta)^{-1}\chi(\lambda).$$

It is sufficient to find $g(\lambda)$ satisfying $M(\lambda)\chi(\lambda) = -g(\lambda)\chi(\lambda)$ below.

$$\begin{aligned} \left(\hat{u}_{1,0} + \frac{\lambda}{2}\right)\sigma_1^\theta(x) + \sigma_1^\theta(y) &= -g(\lambda)\sigma_1^\theta(x), \\ 2\hat{v}_{1,0}\sigma_1^\theta(x) + \left(\hat{u}_{1,0} - \frac{\lambda}{2}\right)\sigma_1^\theta(y) &= -g(\lambda)\sigma_1^\theta(y). \end{aligned} \tag{28}$$

Eq. (28) is solved and by $\sigma_1^\theta(x) \neq 0$ we have

$$g(\lambda) = -\hat{u}_{1,0} \pm \sqrt{\frac{\lambda^2}{4} + 2\hat{v}_{1,0}}, \quad \sigma_1^\theta(y) = \Big(-\frac{\lambda}{2} \mp \sqrt{\frac{\lambda^2}{4} + 2\hat{v}_{1,0}}\Big)\sigma_1^\theta(x). \tag{29}$$

Note that $1 - \theta g(\lambda) = \det(E + M(\lambda)\theta) \neq 0$. Therefore we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\theta}{1 - \theta g(\lambda)} \begin{pmatrix} 1 \\ -\frac{\lambda}{2} \mp \sqrt{\frac{\lambda^2}{4} + 2\hat{v}_{1,0}} \end{pmatrix} \times \sigma_1^\theta(x). \tag{30}$$

Taking (30) into account, the eigenvector $A(\lambda)$ corresponding to an eigenvalue λ of Q has two forms determined by the choice of the signs $\pm$ of $g(\lambda)$ as follows.

$$A(\lambda) := \begin{pmatrix} a(\lambda) \\ \rho(\lambda)a(\lambda) \end{pmatrix} \quad \text{with} \quad a(\lambda) := \frac{\theta}{1 - \theta g(\lambda)}, \tag{31}$$

where $\rho(\lambda)$ and $g(\lambda)$ are given by (i) or (ii).

$$\text{(i)} \quad \rho(\lambda) := -\frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} + 2\hat{v}_{1,0}}, \quad g(\lambda) := -\hat{u}_{1,0} - \sqrt{\frac{\lambda^2}{4} + 2\hat{v}_{1,0}}. \tag{32}$$

$$\text{(ii)} \quad \rho(\lambda) := -\frac{\lambda}{2} - \sqrt{\frac{\lambda^2}{4} + 2\hat{v}_{1,0}}, \quad g(\lambda) := -\hat{u}_{1,0} + \sqrt{\frac{\lambda^2}{4} + 2\hat{v}_{1,0}}. \tag{33}$$

Remark that the difference of the choice of (i) and (ii) has no effect on the procedure of the construction of solutions in the rest of this paper. However, after we construct a solution, if we substitute the values $\rho(\lambda)$ and $g(\lambda)$ for the solution, the difference might be seen in the coefficients of the solution. For example, as is shown in Remark E.3 (see Appendix E), when we put $\rho(\lambda)$, $g(\lambda)$ into J_k and R_k, then the difference may appear in the coefficients of the non-secularity conditions. Hence there is a possibility that $(P_{\mathrm{II}})_m$ has instanton-type solutions of two forms determined by the choice of (i) and (ii).

The following relations are often used later.

$$\begin{aligned} \begin{pmatrix} a(\lambda) \\ 0 \end{pmatrix} &= \frac{1}{\lambda}(\rho(-\lambda)A(\lambda) - \rho(\lambda)A(-\lambda)), \\ \begin{pmatrix} 0 \\ a(\lambda) \end{pmatrix} &= -\frac{1}{\lambda}(A(\lambda) - A(-\lambda)). \end{aligned} \tag{34}$$

Next we explain how to determine λ. We construct the solution (u, v) for (22) so that it is expressed by $A(\lambda)$. By the condition that the coefficients of θ^{m+1} of $(1 - \hat{u}_0 + C)u$ and $(1 - \hat{u}_0 + C)v$ are identically zero, those in $(1 - \hat{u}_0 + C)A(\lambda)$ are zero. Hence the following equation must hold.

$$g(\lambda)^m - \sum_{k=1}^{m} (\hat{u}_{k,0} - c_k) g(\lambda)^{m-k} = 0, \tag{35}$$

where $\hat{u}_{k,0}$ and c_k have been defined by (8) and (1). On the other hand, by (29), $g(\lambda)$ satisfies

$$g(\lambda)^2 + 2\hat{u}_{1,0}\, g(\lambda) - \frac{\lambda^2}{4} + (\hat{u}_{1,0})^2 - 2\hat{v}_{1,0} = 0. \tag{36}$$

We determine λ so that the resultant of the following two polynomials of X equals zero.

$$\begin{aligned} X^m - \sum_{k=1}^{m} (\hat{u}_{k,0} - c_k) X^{m-k} = 0, \\ X^2 + 2\hat{u}_{1,0} X + (\hat{u}_{1,0})^2 - \frac{\lambda^2}{4} - 2\hat{v}_{1,0} = 0. \end{aligned} \tag{37}$$

We remark that the resultant coincides with the characteristic equation of the Fréchet derivative of $(P_{\mathrm{II}})_m$ at its 0-parameter solution. For the details, see Appendix A. Noticing that the resultant becomes an even polynomial of λ and its degree is $2m$, we define $\nu_{\pm 1}(t), \ldots, \nu_{\pm m}(t)$ by the roots of the resultant with $\nu_{-k} = -\nu_k$. Throughout this paper we always suppose conditions:

(S1) The roots $\nu_i(t)$'s $(1 \leq |i| \leq m)$ are mutually distinct for each t in Ω, i.e. t is neither a turning point of the first kind nor a turning point of the second kind.
(S2) The function $p_1\nu_1(t) + \cdots + p_m\nu_m(t)$ does not vanish identically on Ω for any $(p_1, \ldots, p_m) \in \mathbb{Z}^m \setminus \{0\}$.

It follows from the specific form of $a(\lambda)$ that we obtain the following lemma.

Lemma 3.1. 1. *For any $k \neq j$ $(1 \leq k, j \leq m)$, we have*

$$a(\nu_k)a(\nu_j) = \frac{1}{g(\nu_k) - g(\nu_j)} (a(\nu_k) - a(\nu_j)). \tag{38}$$

Furthermore, for any integers $1 \leq i_1 < i_2 < \cdots < i_k \leq m$, we get

$$a(\nu_{i_1}) \cdots a(\nu_{i_k}) = \sum_{l=1}^{k} \frac{a(\nu_{i_l})}{(g(\nu_{i_l}) - g(\nu_{i_1})) \cdots (g(\nu_{i_l}) - g(\nu_{i_{l-1}}))(g(\nu_{i_l}) - g(\nu_{i_{l+1}})) \cdots (g(\nu_{i_l}) - g(\nu_{i_k}))}. \tag{39}$$

Note that these equations are strict (not $\equiv$).

2. *For any $1 \leq k \leq m$, we have*

$$a(\nu_k)^2 \equiv \sum_{j=1}^{m} h_{k,j} a(\nu_j), \tag{40}$$

where $h_{k,j}$ are defined by

$$h_{k,j} = \frac{\prod_{\substack{1\le l\le m,\\ l\ne k,j}}(g(\nu_k)-g(\nu_l))}{\prod_{\substack{1\le l\le m,\\ l\ne j}}(g(\nu_j)-g(\nu_l))} \quad (j \ne k), \quad h_{k,k} = \sum_{\substack{l=1,\\ l\ne k}}^{m} \frac{1}{g(\nu_k)-g(\nu_l)}. \tag{41}$$

3. *We have*

$$\frac{\partial a(\nu_k)}{\partial t} \equiv g(\nu_k)' \sum_{j=1}^{m} h_{k,j} a(\nu_j), \tag{42}$$

where $g(\nu_k)'$ *denotes the derivative of* $g(\nu_k)$ *with respect to* t.

The proof of Lemma 3.1 has already been given at Appendix A in [**5**]. The multiplicative relations in the lemma are common to $(P_{\mathrm{J}})_m$ ($\mathrm{J} = \mathrm{I}$, II, 34, IV) and the more common structure of $(P_{\mathrm{J}})_m$ ($\mathrm{J} = \mathrm{I}$, II, 34, IV) is well understood when we compare this subsection with Section 3.2 in [**5**].

3.3. The system of partial differential equations associated with $(P_{\mathrm{II}})_m$. We define the morphism ι by

$$\iota(\psi) = \psi\left(\eta\int^t \nu_1(s)ds, \ldots, \eta\int^t \nu_m(s)ds,\, t,\, \theta,\, \eta\right) \tag{43}$$

for $\psi(\tau_1, \ldots, \tau_m, t, \theta, \eta) \in \hat{\mathcal{A}}(\Omega)$. Let us go back to (22) and replace d/dt in (22) by

$$\frac{\partial}{\partial t} + \eta\nu_1\frac{\partial}{\partial\tau_1} + \eta\nu_2\frac{\partial}{\partial\tau_2} + \cdots + \eta\nu_m\frac{\partial}{\partial\tau_m}. \tag{44}$$

Then we have the system of partial differential equations associated with $(P_{\mathrm{II}})_m$.

$$\begin{aligned} P\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} \equiv &\left(\begin{pmatrix} -2\sigma_1^\theta(u)u\theta \\ S(u,v) \end{pmatrix} + \eta^{-1}\begin{pmatrix} \varrho \\ \delta \end{pmatrix}\theta + uP\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} \right) \\ &- \left(2u^2\begin{pmatrix} -\sigma_1^\theta(u) \\ \sigma_1^\theta(v) \end{pmatrix} + \eta^{-1}\left(\begin{pmatrix} 2\varrho u \\ \delta u + \varrho v \end{pmatrix} + \frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} \right) \right)\theta \\ &+ \eta^{-1}u\left(\varrho + \frac{\partial}{\partial t}\right)\begin{pmatrix} u \\ v \end{pmatrix}\theta. \end{aligned} \tag{45}$$

Here the operator P is defined by

$$P := \chi_\tau - Q, \qquad \chi_\tau := \nu_1\frac{\partial}{\partial\tau_1} + \nu_2\frac{\partial}{\partial\tau_2} + \cdots + \nu_m\frac{\partial}{\partial\tau_m}. \tag{46}$$

For a solution $(u,v) \in \hat{\mathcal{A}}^2(\Omega)$ of (45), $(\iota(u), \iota(v))$ becomes a formal solution of (22). We recall the definition (cf. [**14**], [**21**]) of instanton-type solutions for $(P_{\mathrm{II}})_m$ in our formulation.

Definition 3.2. We say that a formal solution (U, V) on Ω of (7) is of instanton-type if (U, V) has the form $(\hat{u}_0, \hat{v}_0) + (1 - \hat{u}_0 + C)(\iota(u), \iota(v))$ for $(u, v) \in \hat{\mathcal{A}}^2(\Omega)$ which is a solution of (45).

In the rest of this subsection, we see that Lemma 3.1 neatly manipulates the algebraic structure of $(P_{\mathrm{II}})_m$ and the computation of non-linear terms of (45) is extremely simplified. Now we assume that an element (u, v) in $\hat{\mathcal{A}}^2(\Omega)$ has the expansion

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{1 \le |k| \le m} f_k(\tau, t; \eta) A(\nu_k) \quad \text{with} \quad f_k(\tau, t; \eta) := \sum_{j=1}^{\infty} f_{k, j\alpha}(\tau, t) \eta^{j\alpha}. \tag{47}$$

We remark that $A(\nu_k)$'s contain θ and f_k's are independent of θ. By putting the above expansion into (45), we obtain some results which imply that the right-hand side of (45) is written by a linear combination of eigenvector $A(\nu_k)$'s. In what follows, for the simplicity, we use the notation below.

$$\rho_{k,j} := \rho(\nu_k) + \rho(\nu_j) \quad ({}^{\forall}k, {}^{\forall}j \in \mathbb{Z}), \tag{48}$$

where $\rho(\nu_k)$'s have been defined by (32) or (33).

Proposition 3.3. *We have*

$$\begin{pmatrix} -2\sigma_1^{\theta}(u) u \theta \\ S(u, v) \end{pmatrix} + \eta^{-1} \begin{pmatrix} \varrho \\ \delta \end{pmatrix} \theta \equiv \sum_{1 \le |k| \le m} \frac{1}{\nu_k} \big(\Lambda_k(t) + \eta^{-1} (\gamma_k \rho(\nu_{-k}) - \delta_k) \big) A(\nu_k) \theta. \tag{49}$$

Here $\Lambda_k(t)$ is expressed by

$$\begin{aligned} \Lambda_k(t) := & -2 \sum_{\substack{1 \le |j| \le m, \\ k+j \ne 0}} \left(\frac{2\nu_k + \nu_j}{\nu_k + \nu_j} (\rho_{k,j} f_k f_j + \rho_{-k,-j} f_{-k} f_{-j}) + \nu_k f_k f_j \right) \\ & + \sum_{j=1}^{m} \nu_j^2 h_{j,k} f_j f_{-j} - 2(3\rho_{k,-k} + \nu_k) f_k f_{-k}, \end{aligned} \tag{50}$$

where $h_{j,k}$ have been given by (41) with convention $h_{j,k} := h_{|j|,|k|}$, and γ_k, δ_k $(k > 0)$ are multi-valued functions of t satisfying

$$\gamma_{-k} = \gamma_k, \quad \delta_{-k} = \delta_k \quad (1 \le k \le m). \tag{51}$$

More precisely, see Appendix B.

Proof. This is proved by Lemmas 3.4 and 3.5 below. □

Lemma 3.4. *We have*

$$\begin{pmatrix} -2\sigma_1^\theta(u)u\,\theta \\ S(u,v) \end{pmatrix} \equiv \sum_{1\le|k|\le m} \frac{1}{\nu_k}\Lambda_k(t)A(\nu_k)\theta. \tag{52}$$

Here $\Lambda_k(t)$ *has been defined by* (50).

PROOF. Note that $Q(A(\nu_k)\theta) = \nu_k A(\nu_k)\theta$ $(1 \le |k| \le m)$ holds. Taking (38) and (40) into account, we have

$$\begin{aligned}
&\frac{1}{2}(-v,u)Q\begin{pmatrix} u\,\theta \\ v\,\theta \end{pmatrix} \\
&\quad = \frac{1}{2}\sum_{1\le|k|\le m} \nu_k f_k(-v,u)A(\nu_k)\theta \\
&\quad = \frac{1}{2}\sum_{1\le|k|\le m} \nu_k f_k(\rho(\nu_k)u - v)a(\nu_k)\theta \\
&\quad = \frac{1}{2}\sum_{1\le|k|\le m}\sum_{1\le|j|\le m} \nu_k(\rho(\nu_k)-\rho(\nu_j))f_k f_j a(\nu_k)a(\nu_j)\theta \\
&\quad \equiv \sum_{\substack{1\le|k|\le m,\\ 1\le|j|\le m,\\ k+j\ne 0}} \frac{2\nu_k(\rho(\nu_k)-\rho(\nu_j))}{\nu_k^2-\nu_j^2}(g(\nu_k)+g(\nu_j)+2\hat{u}_{1,0})f_k f_j(a(\nu_k)-a(\nu_j))\theta \\
&\qquad - \sum_{j=1}^m \nu_j^2 f_j f_{-j}\sum_{k=1}^m h_{j,k}a(\nu_k)\theta \\
&\quad = \sum_{\substack{1\le|k|\le m,\\ 1\le|j|\le m,\\ k+j\ne 0}} \frac{\nu_k}{\nu_k+\nu_j}\rho_{k,j} f_k f_j(a(\nu_k)-a(\nu_j))\theta - \sum_{j=1}^m \nu_j^2 f_j f_{-j}\sum_{k=1}^m h_{j,k}a(\nu_k)\theta \\
&\quad = \sum_{\substack{1\le|k|\le m,\\ 1\le|j|\le m,\\ k+j\ne 0}} \frac{\nu_k-\nu_j}{\nu_k+\nu_j}\rho_{k,j} f_k f_j a(\nu_k)\theta - \sum_{j=1}^m \nu_j^2 f_j f_{-j}\sum_{k=1}^m h_{j,k}a(\nu_k)\theta.
\end{aligned} \tag{53}$$

In the same way, noticing $\sigma_1^\theta(u) = \sum_{1\le|j|\le m} f_j$ and $\sigma_1^\theta(v) = \sum_{1\le|j|\le m}\rho(\nu_j)f_j$, we obtain

$$\sigma_1^\theta(u)v = \sum_{1\le|k|\le m}\sum_{\substack{1\le|j|\le m,\\ k+j\ne 0}} \rho(\nu_k)f_k f_j a(\nu_k) + \sum_{1\le|k|\le m}\rho(\nu_k)f_k f_{-k}a(\nu_k). \tag{54}$$

$$3\sigma_1^\theta(v)u = 3\sum_{1\le|k|\le m}\sum_{\substack{1\le|j|\le m,\\ k+j\ne 0}} \rho(\nu_j)f_k f_j a(\nu_k) + 3\sum_{1\le|k|\le m}\rho(\nu_{-k})f_k f_{-k}a(\nu_k). \tag{55}$$

It follows from (53), (54) and (55) that we have

$$
\begin{aligned}
S(u,v) \equiv & \sum_{1\le|k|\le m} \sum_{\substack{1\le|j|\le m,\\ k+j\neq 0}} \left(\frac{2(2\nu_k+\nu_j)}{\nu_k+\nu_j}\rho_{k,j} - 2\rho(\nu_k)\right) f_k f_j a(\nu_k)\theta \\
& + 4\sum_{1\le|k|\le m} \rho(\nu_k) f_k f_{-k} a(\nu_k)\theta - \sum_{k=1}^{m}\sum_{j=1}^{m} \nu_j^2 h_{j,k} f_j f_{-j} a(\nu_k)\theta. \qquad (56)
\end{aligned}
$$

Hence, using the second equation of (34), we have

$$
\begin{pmatrix} 0 \\ S(u,v) \end{pmatrix} \equiv \sum_{1\le|k|\le m} \frac{1}{\nu_k}\tilde{\Lambda}_{k,1}(t) A(\nu_k)\theta, \qquad (57)
$$

$$
\begin{aligned}
\tilde{\Lambda}_{k,1}(t) := & \sum_{\substack{1\le|j|\le m,\\ k+j\neq 0}} \frac{-2(2\nu_k+\nu_j)}{\nu_k+\nu_j}\big(\rho_{k,j} f_k f_j + \rho_{-k,-j} f_{-k} f_{-j}\big) \\
& + 2\sum_{\substack{1\le|j|\le m,\\ k+j\neq 0}} \big(\rho(\nu_k) f_k f_j + \rho(\nu_{-k}) f_{-k} f_{-j}\big) - 4\rho_{k,-k} f_k f_{-k} + \sum_{j=1}^{m} \nu_j^2 h_{j,k} f_j f_{-j}.
\end{aligned}
$$

Finally, the following fact is shown by the first equation of (34).

$$
\begin{pmatrix} -2u\sigma_1^{\theta}(u)\theta \\ 0 \end{pmatrix} = \sum_{1\le|k|\le m} \frac{1}{\nu_k}\tilde{\Lambda}_{k,2}(t) A(\nu_k)\theta, \qquad (58)
$$

$$
\tilde{\Lambda}_{k,2}(t) := -2\sum_{\substack{1\le|j|\le m,\\ k+j\neq 0}} \rho(\nu_{-k})(f_k f_j + f_{-k} f_{-j}) - 4\rho(\nu_{-k}) f_k f_{-k}.
$$

Therefore we can see

$$
\Lambda_k(t) = \tilde{\Lambda}_{k,1}(t) + \tilde{\Lambda}_{k,2}(t). \qquad (59)
$$

This proves the lemma. □

Let us continue to calculate the second term on the right-hand side of (45).

LEMMA 3.5. *We obtain*

$$
\begin{pmatrix} \varrho \\ \delta \end{pmatrix} \equiv \sum_{1\le|k|\le m} \frac{1}{\nu_k}(\gamma_k \rho(\nu_{-k}) - \delta_k) A(\nu_k), \qquad (60)
$$

where γ_k and δ_k are multi-valued functions of t satisfying (51).

PROOF. By (12), the following equation is shown.

$$
\big((1+\hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2\big)(1-\hat{u}_0+C)^2 = (1+C)^2. \qquad (61)
$$

Taking the derivative of the above equation with respect to t, we get

$$\varrho = -\frac{\big((1+\hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2\big)'}{2\big((1+\hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2\big)}. \tag{62}$$

Using the first equation of (11), we also have

$$\delta\theta = -(1+\hat{u}_{1,0}\theta)\rho - (\hat{u}_{1,0})'\theta. \tag{63}$$

Hence, putting (62) into (63), we see

$$\delta = \frac{-(\hat{v}_{1,0})'(1+\hat{u}_{1,0}\theta) + 2(\hat{u}_{1,0})'\hat{v}_{1,0}\theta}{(1+\hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2}\theta. \tag{64}$$

On the other hand, as ρ and δ can be written in the forms

$$\varrho \equiv \sum_{k=1}^{m} \gamma_k(t) a(\nu_k) \quad \text{and} \quad \delta \equiv \sum_{k=1}^{m} \delta_k(t) a(\nu_k), \tag{65}$$

we have

$$\begin{aligned} \sum_{k=1}^{m} \frac{\gamma_k(t)}{1-\theta g(\nu_k)} &= \frac{-(\hat{u}_{1,0})'(1+\hat{u}_{1,0}\theta) + (\hat{v}_{1,0})'\theta}{(1+\hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2}, \\ \sum_{k=1}^{m} \frac{\delta_k(t)}{1-\theta g(\nu_k)} &= \frac{-(\hat{v}_{1,0})'(1+\hat{u}_{1,0}\theta) + 2(\hat{u}_{1,0})'\hat{v}_{1,0}\theta}{(1+\hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2}. \end{aligned} \tag{66}$$

We solve the systems obtained by comparing the expansions in θ of (66), then we have γ_k and δ_k. For the explicit forms of γ_k and δ_k, see Appendix B. Finally (60) follows from (34) and (65), and the relation (51) can be verified. This completes the proof. □

Similarly, we also get Proposition 3.6 whose proof will be given in Appendix C.

PROPOSITION 3.6. *We have*

$$\begin{aligned} u\begin{pmatrix} \varrho \\ \delta \end{pmatrix} \equiv \sum_{1\le|k|\le m} \frac{1}{\nu_k} \Bigg(& \sum_{j=1}^{m} (\gamma_j \rho(\nu_{-k}) - \delta_j)(f_j + f_{-j}) h_{j,k} \\ & + \sum_{\substack{j=1,\\ j\neq\pm k}}^{m} \frac{(\gamma_j\rho(\nu_{-k}) - \delta_j)(f_k + f_{-k}) + (\gamma_k\rho(\nu_{-k}) - \delta_k)(f_j + f_{-j})}{g(\nu_k) - g(\nu_j)} \Bigg) A(\nu_k) \end{aligned} \tag{67}$$

and obtain

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} \equiv \sum_{1\le|k|\le m}\left(\frac{1}{\nu_k}\widetilde{\Lambda}_k + \frac{\partial f_k}{\partial t}\right)A(\nu_k). \tag{68}$$

Here $\widetilde{\Lambda}_k$ *has the form*

$$\begin{aligned}&\sum_{\substack{1\le|j|\le m,\\ j\neq\pm k}}(\rho(\nu_{-k})-\rho(\nu_j))g(\nu_j)'h_{j,k}f_j\\ &\quad+\left(-\frac{\partial\rho(\nu_k)}{\partial t}+\nu_k g(\nu_k)'h_{k,k}\right)f_k-\frac{\partial\rho(\nu_{-k})}{\partial t}f_{-k}.\end{aligned} \tag{69}$$

4. The first member of the non-secularity conditions.

In this section, we get the leading term $f_{k,\alpha}$ and subleading term $f_{k,2\alpha}$ of (47).

4.1. The leading and subleading terms.

Substituting (47) for (45) and looking at the coefficient of η^α, we have

$$P\biggl(\sum_{1\le|k|\le m} f_{k,\alpha}(\tau,t)A(\nu_k)\theta\biggr)=\begin{pmatrix}0\\0\end{pmatrix}. \tag{70}$$

Since Ker$P$ is equivalent to the subspace generated by the vectors $\eta^\alpha e^{\tau_i}A(\nu_i)$ over $\mathcal{M}(\Omega)[[\eta^{-1}]]$, the following lemma holds.

Lemma 4.1. *We obtain a solution to* (70) *of the form*

$$f_{k,\alpha}=\omega_k^{(1)}e^{\tau_k}\quad(1\le|k|\le m). \tag{71}$$

Here $\omega_k^{(1)}(t)$*'s* $(1\le|k|\le m)$ *are arbitrary functions of* t.

The corollary below follows from the above lemma.

Corollary 4.2. *The leading term* $(\sigma_\alpha^\eta(u),\sigma_\alpha^\eta(v))$ *of* $(u,v)\in\hat{\mathcal{A}}^2(\Omega)$ *is given by*

$$\begin{aligned}\sigma_\alpha^\eta(u)&=\sum_{k=1}^m\bigl(\omega_k^{(1)}e^{\tau_k}+\omega_{-k}^{(1)}e^{-\tau_k}\bigr)a(\nu_k),\\ \sigma_\alpha^\eta(v)&=\sum_{k=1}^m\bigl(\rho(\nu_k)\omega_k^{(1)}e^{\tau_k}+\rho(\nu_{-k})\omega_{-k}^{(1)}e^{-\tau_k}\bigr)a(\nu_k).\end{aligned} \tag{72}$$

In what follows, we abbreviate $\omega_k^{(1)}$ to ω_k $(1\le|k|\le m)$ and we decide ω_k's containing free parameters by the first member $(\mathcal{E}_1)$ of the non-secularity conditions. Specifically, we first solve the equation for $f_{k,2\alpha}$. Next we calculate the equation for $f_{k,3\alpha}$, then the terms containing $e^{\tau_k}A(\nu_k)$ $(1\le|k|\le m)$ appear in the right-hand side. The $(\mathcal{E}_1)$ is the condition that the coefficients of $e^{\tau_k}A(\nu_k)$ $(1\le|k|\le m)$ are zero and $(\mathcal{E}_1)$ becomes the system of non-linear differential equations for ω_k's. By solving $(\mathcal{E}_1)$, we have the explicit forms of ω_k's. The reason why we determine ω_k's by the non-secularity conditions has been explained in [**5**] and [**10**].

Now let us calculate the form of $f_{k,2\alpha}$.

LEMMA 4.3. *The $f_{k,2\alpha}$ satisfies*

$$P\biggl(\sum_{1\le|k|\le m} f_{k,2\alpha}A(\nu_k)\theta\biggr) \equiv \sum_{1\le|k|\le m} \frac{1}{\nu_k}\Lambda_{k,2\alpha}(t)A(\nu_k)\theta. \tag{73}$$

Here $\Lambda_{k,2\alpha}$ is defined by

$$\begin{aligned}\Lambda_{k,2\alpha}(t) :=& -2\sum_{\substack{1\le|j|\le m,\\ k+j\ne0}}\left(\frac{2\nu_k+\nu_j}{\nu_k+\nu_j}\rho_{k,j}+\nu_k\right)\omega_k\omega_j e^{\tau_k+\tau_j}\\ &-2\sum_{\substack{1\le|j|\le m,\\ k+j\ne0}}\frac{2\nu_k+\nu_j}{\nu_k+\nu_j}\rho_{-k,-j}\,\omega_{-k}\omega_{-j}e^{-\tau_k-\tau_j}\\ &+\sum_{j=1}^m \nu_j^2 h_{j,k}\omega_j\omega_{-j} - 2(3\rho_{k,-k}+\nu_k)\omega_k\omega_{-k} + \gamma_k\rho(\nu_{-k}) - \delta_k,\end{aligned} \tag{74}$$

where $h_{j,k}$ have been defined by (41).

PROOF. Proposition 3.3 immediately implies the lemma. □

LEMMA 4.4. *For any k $(1\le|k|\le m)$, we obtain*

$$\begin{aligned}f_{k,2\alpha} =& -2\sum_{\substack{1\le|j|\le m,\\ k+j\ne0}}\left(\frac{2\nu_k+\nu_j}{\nu_k\nu_j(\nu_k+\nu_j)}\rho_{k,j}+\frac{1}{\nu_j}\right)\omega_k\omega_j e^{\tau_k+\tau_j}\\ &+2\sum_{\substack{1\le|j|\le m,\\ k+j\ne0}}\frac{1}{\nu_k(\nu_k+\nu_j)}\rho_{-k,-j}\,\omega_{-k}\omega_{-j}e^{-\tau_k-\tau_j}\\ &-\frac{1}{\nu_k^2}\left(\sum_{j=1}^m \nu_j^2 h_{j,k}\omega_j\omega_{-j} - 2(3\rho_{k,-k}+\nu_k)\omega_k\omega_{-k}\right)\\ &-\frac{1}{\nu_k^2}(\gamma_k\rho(\nu_{-k})-\delta_k).\end{aligned} \tag{75}$$

PROOF. Taking (73) into account, we see that $f_{k,2\alpha}$ $(1\le|k|\le m)$ satisfies

$$(\chi_\tau-\nu_k)f_{k,2\alpha}(t,\tau) = \frac{1}{\nu_k}\Lambda_{k,2\alpha}(t), \tag{76}$$

where χ_τ is the first-order differential operator with respect to τ given by (46). Looking at the form of $\Lambda_{k,2\alpha}$, we seek for $f_{k,2\alpha}$ in the form

$$f_{k,2\alpha} = \sum_{\substack{1\le|j|\le m,\\ j\neq -k}} (l_1(\nu_k,\nu_j)\omega_k\omega_j e^{\tau_k+\tau_j} + l_2(\nu_k,\nu_j)\omega_{-k}\omega_{-j}e^{-\tau_k-\tau_j}) + \sum_{j=1}^{m} l_3(\nu_k,\nu_j). \quad (77)$$

Here the coefficients $l_1(\nu_k,\nu_j)$, $l_2(\nu_k,\nu_j)$ and $l_3(\nu_k,\nu_j)$ are unknown functions of ν_i's. As $Q(A(\nu_i)\theta) = \nu_i A(\nu_i)\theta$ $(1\le|i|\le m)$ holds, we have

$$\begin{aligned}(\chi_\tau - \nu_k)&f_{k,2\alpha}(t,\tau)\\ &= \sum_{\substack{1\le|j|\le m,\\ j\neq -k}} (\nu_j l_1(\nu_k,\nu_j)\omega_k\omega_j e^{\tau_k+\tau_j} - (2\nu_k+\nu_j)l_2(\nu_k,\nu_j)\omega_{-k}\omega_{-j}e^{-\tau_k-\tau_j})\\ &\quad - \nu_k\sum_{j=1}^{m} l_3(\nu_k,\nu_j). \end{aligned} \quad (78)$$

Clearly the coefficients are obtained by (74), (76) and (78). This completes the proof of lemma. □

4.2. The first member of the non-secularity conditions.

As mentioned previously, the first member of the non-secularity conditions is obtained by seeing the right-hand side of the equation for $f_{k,3\alpha}$. We first compare the coefficients of $\eta^{3\alpha}$ on the both sides of (45), then we have

$$\begin{aligned}P&\Bigl(\sum_{1\le|k|\le m} f_{k,3\alpha}A(\nu_k)\theta\Bigr)\\ &\equiv \begin{pmatrix} 0 \\ (-\bar v_\alpha,\bar u_\alpha)Q\begin{pmatrix}\bar u_{2\alpha} & \theta\\ \bar v_{2\alpha} & \theta\end{pmatrix} + \dfrac{\bar u_\alpha}{2}(-\bar v_\alpha,\bar u_\alpha)Q\begin{pmatrix}\bar u_\alpha & \theta\\ \bar v_\alpha & \theta\end{pmatrix}\end{pmatrix}\\ &\quad + \begin{pmatrix} 0\\ \bar u_\alpha(\sigma_1^\theta(\bar u_\alpha)\bar v_\alpha + \sigma_1^\theta(\bar v_\alpha)\bar u_\alpha)\end{pmatrix}\theta\\ &\quad + 2\begin{pmatrix} -\sigma_1^\theta(\bar u_\alpha)\bar u_{2\alpha} - \sigma_1^\theta(\bar u_{2\alpha})\bar u_\alpha\\ \sigma_1^\theta(\bar u_{2\alpha})\bar v_\alpha + \sigma_1^\theta(\bar v_\alpha)\bar u_{2\alpha} + 2\sigma_1^\theta(\bar v_{2\alpha})\bar u_\alpha\end{pmatrix}\theta - \left(\varrho + \frac{\partial}{\partial t}\right)\begin{pmatrix}\bar u_\alpha\\ \bar v_\alpha\end{pmatrix}\theta. \end{aligned} \quad (79)$$

Here we set $\bar u_{j\alpha} := \sigma_{j\alpha}^\eta(u)$ and $\bar v_{j\alpha} := \sigma_{j\alpha}^\eta(v)$ $(j=1,2)$. The following lemma simplifies (79) (See Appendix D for the proof of Lemma 4.5).

Lemma 4.5. *We have*

$$(-\bar v_\alpha,\bar u_\alpha)Q\begin{pmatrix}\bar u_{2\alpha} & \theta\\ \bar v_{2\alpha} & \theta\end{pmatrix} + \frac{\bar u_\alpha}{2}(-\bar v_\alpha,\bar u_\alpha)Q\begin{pmatrix}\bar u_\alpha & \theta\\ \bar v_\alpha & \theta\end{pmatrix} + \bar u_\alpha\bigl(\sigma_1^\theta(\bar u_\alpha)\bar v_\alpha + \sigma_1^\theta(\bar v_\alpha)\bar u_\alpha\bigr)\theta$$

$$= \varrho \bar{v}_\alpha - \delta \bar{u}_\alpha + \sum_{\substack{1\le|k|\le m,\\ 1\le|j|\le m}} \sum_{\substack{1\le|i|\le m,\\ i\ne\pm k}} \frac{4}{\nu_j}(\rho(\nu_k) - \rho(\nu_i))a(\nu_k) f_{i,\alpha} f_{j,\alpha} f_{k,\alpha}\theta, \tag{80}$$

where ϱ *and* δ *have been defined by* (19).

As a consequence of Lemma 4.5, we can reduce (79) to

$$\begin{aligned} &P\biggl(\sum_{1\le|k|\le m} f_{k,3\alpha} A(\nu_k)\theta\biggr) \\ &\equiv 2\begin{pmatrix} -\sigma_1^\theta(\bar{u}_\alpha)\bar{u}_{2\alpha} - \sigma_1^\theta(\bar{u}_{2\alpha})\bar{u}_\alpha \\ \sigma_1^\theta(\bar{u}_{2\alpha})\bar{v}_\alpha + \sigma_1^\theta(\bar{v}_\alpha)\bar{u}_{2\alpha} + 2\sigma_1^\theta(\bar{v}_{2\alpha})\bar{u}_\alpha \end{pmatrix}\theta - \bar{u}_\alpha\begin{pmatrix}\varrho\\ \delta\end{pmatrix}\theta - \frac{\partial}{\partial t}\begin{pmatrix}\bar{u}_\alpha\\ \bar{v}_\alpha\end{pmatrix}\theta \\ &\quad + \sum_{1\le|k|\le m}\sum_{1\le|j|\le m}\sum_{\substack{1\le|i|\le m,\\ i\ne\pm k}} \frac{4}{\nu_k\nu_j}\bigl((\rho(\nu_i) - \rho(\nu_k)) f_{i,\alpha} f_{j,\alpha} f_{k,\alpha} \\ &\qquad\qquad - (\rho(\nu_{-i}) - \rho(\nu_{-k})) f_{-i,\alpha} f_{-j,\alpha} f_{-k,\alpha}\bigr) A(\nu_k)\theta. \end{aligned} \tag{81}$$

We have the following proposition by Lemmas E.1 and E.2 which will be given in Appendix E.

PROPOSITION 4.6. *For any* k $(1 \le |k| \le m)$, *there exist functions* $\varphi(k,j)$ *of the variables* ν_ℓ*'s and multi-valued functions* J_k *and* R_k *of finite determination in* Ω *satisfying the conditions*

$$\varphi(k,j) = \varphi(-k,j) \quad (1\le j\le m), \qquad J_k = J_{-k}, \qquad R_k = R_{-k} \tag{$*$}$$

such that the coefficient of $e^{\tau_k}A(\nu_k)$ *on the right-hand side of the equation for* $f_{k,3\alpha}$ *is given by*

$$\frac{1}{\nu_k}\biggl(\sum_{j=1}^m \varphi(k,j)\omega_j\omega_{-j} + J_k - \nu_k R_k\biggr)\omega_k - \frac{d\omega_k}{dt}. \tag{82}$$

Therefore the first member $(\mathcal{E}_1)$ of the non-secularity conditions is obtained.

THEOREM 4.7. *The first member* $(\mathcal{E}_1)$ *of the non-secularity conditions has the following form*: *For any* $1 \le k \le m$,

$$\frac{d\omega_k}{dt} = \frac{1}{\nu_k}\biggl(\sum_{j=1}^m \varphi(k,j)\omega_j\omega_{-j} + J_k - \nu_k R_k\biggr)\omega_k, \tag{83}$$

$$\frac{d\omega_{-k}}{dt} = -\frac{1}{\nu_k}\biggl(\sum_{j=1}^m \varphi(-k,j)\omega_j\omega_{-j} + J_{-k} + \nu_k R_{-k}\biggr)\omega_{-k}, \tag{84}$$

where $\varphi(k,j)$, J_k *and* R_k *satisfy the condition* $(*)$ *in Proposition* 4.6 *and their concrete forms are given by Remark* E.3 *in Appendix* E.

4.3. Global solvability of the first member of the non-secularity conditions.

Thanks to the condition $(*)$ given in Proposition 4.6, the first member of the non-secularity conditions can be solved globally except for turning points. Similarly to the case of $(P_{\mathrm{I}})_m$, by multiplying (83) (resp. (84)) by ω_{-k} (resp. ω_k) and summing them up, we obtain

$$\frac{d(\omega_k(t)\omega_{-k}(t))}{dt} = -2R_k\omega_k(t)\omega_{-k}(t). \tag{85}$$

Therefore we have, for a constant b_k,

$$\omega_k(t)\omega_{-k}(t) = b_k \exp\Big(-2\int^t R_k(t)dt\Big), \quad 1 \le k \le m. \tag{86}$$

Putting (86) into (83) and (84), we obtain the following system of first-order linear homogeneous differential equations for ω_k $(1 \le |k| \le m)$:

$$\frac{d\omega_k}{dt} = \frac{1}{\nu_k}\Big(\sum_{j=1}^m \varphi(k,j)b_j \exp\Big(-2\int^t R_j(t)dt\Big) + J_k - \nu_k R_k\Big)\omega_k, \tag{87}$$

$$\frac{d\omega_{-k}}{dt} = -\frac{1}{\nu_k}\Big(\sum_{j=1}^m \varphi(-k,j)b_j \exp\Big(-2\int^t R_j(t)dt\Big) + J_{-k} + \nu_k R_{-k}\Big)\omega_{-k}. \tag{88}$$

We can solve the above system and we have

$$\omega_k = \beta_k^{(1)}S_{k,1}(t), \quad \omega_{-k} = \beta_{-k}^{(1)}S_{-k,1}(t), \tag{89}$$

$$\begin{aligned} S_{k,1}(t) &:= \exp\Big(\int^t \frac{1}{\nu_k}\Big(\sum_{j=1}^m \varphi(k,j)b_j \exp\Big(-2\int^t R_j dt\Big) + J_k - \nu_k R_k\Big)dt\Big), \\ S_{-k,1}(t) &:= \exp\Big(\int^t -\frac{1}{\nu_k}\Big(\sum_{j=1}^m \varphi(k,j)b_j \exp\Big(-2\int^t R_j dt\Big) + J_k + \nu_k R_k\Big)dt\Big) \end{aligned} \tag{90}$$

for $1 \le k \le m$. Note that the product of ω_k and ω_{-k} satisfies (86), we see $\beta_k^{(1)}\beta_{-k}^{(1)} = b_k$. Hence we have the following.

Proposition 4.8. *The system* (83) *and* (84) *has the multi-valued holomorphic solution ω_k on Ω in the form*

$$\begin{aligned} \omega_k &= \beta_k^{(1)} \exp\Big(\int^t \frac{1}{\nu_k}\Big(\sum_{j=1}^m \varphi(k,j)\beta_j^{(1)}\beta_{-j}^{(1)} \exp\Big(-2\int^t R_j dt\Big) + J_k - \nu_k R_k\Big)dt\Big), \\ \omega_{-k} &= \beta_{-k}^{(1)} \exp\Big(\int^t -\frac{1}{\nu_k}\Big(\sum_{j=1}^m \varphi(k,j)\beta_j^{(1)}\beta_{-j}^{(1)} \exp\Big(-2\int^t R_j dt\Big) + J_k + \nu_k R_k\Big)dt\Big) \end{aligned} \tag{91}$$

for $1 \le k \le m$.

5. Construction of higher-order terms of instanton-type solutions for $(P_{\mathrm{II}})_m$.

We prove that we can construct the higher-order terms of (47) by induction, and hence the existence of instanton-type solutions for $(P_{\mathrm{II}})_m$ is shown.

Assume that $l(\ge 2)$ is a natural number and $f_{k,j\alpha}$ $(j = 1, \dots, 2(l-1))$ have been constructed in a similar way as that for the proof in Section 4. Let us see that $f_{k,(2l-1)\alpha}$ and $f_{k,2l\alpha}$ are determined by the l-th member of the non-secularity conditions. We first have $f_{k,(2l-1)\alpha}$ of the form

$$f_{k,(2l-1)\alpha} = d_{k,(2l-1)\alpha} + \xi_k^{(l)}(t)e^{\tau_k}, \qquad 1 \le |k| \le m, \tag{92}$$

where $d_{k,(2l-1)\alpha}$ depends only on $f_{k,j\alpha}$'s $(1 \le j \le 2(l-1))$ and $\xi_k^{(l)}(t)$ $(1 \le |k| \le m)$ are new arbitrary functions of t. Actually, the right-hand side of the equation for $f_{k,(2l-1)\alpha}$ is expressed in terms of known quantities by the induction hypothesis and the coefficients of $A(\nu_k)e^{\tau_k}$ $(1 \le |k| \le m)$ in the right-hand side have vanished by the $(l-1)$-th member of the non-secularity conditions. Hence we have $f_{k,(2l-1)\alpha}$ of the form (92).

Next, let us see that we can determine $\xi_k^{(l)}(t)$ $(1 \le |k| \le m)$ by the l-th member of the non-secularity conditions. The following lemma can be proved in the same way as Lemmas 4.3 and 4.4. We abbreviate $\xi_k^{(l)}$ to ξ_k $(1 \le |k| \le m)$.

LEMMA 5.1. *For any k $(1 \le |k| \le m)$, we have $f_{k,2l\alpha}$ of the form*

$$\begin{aligned}
&-2\sum_{\substack{1\le|j|\le m,\\ j\ne -k}}\left(\frac{2\nu_k+\nu_j}{\nu_k\nu_j(\nu_k+\nu_j)}\rho_{k,j}+\frac{1}{\nu_j}\right)(\omega_k\xi_j+\xi_k\omega_j)e^{\tau_k+\tau_j}\\
&+\sum_{\substack{1\le|j|\le m,\\ j\ne -k}}\frac{2}{\nu_k(\nu_k+\nu_j)}\rho_{-k,-j}(\omega_{-k}\xi_{-j}+\xi_{-k}\omega_{-j})e^{-\tau_k-\tau_j}\\
&-\frac{1}{\nu_k^2}\left(\sum_{j=1}^{m}\nu_j^2h_{j,k}(\omega_j\xi_{-j}+\xi_j\omega_{-j})-2(3\rho_{k,-k}+\nu_k)(\omega_k\xi_{-k}+\xi_k\omega_{-k})\right)+d_{k,2l\alpha},
\end{aligned}\tag{93}$$

where ω_k's have been given by (91) *and $d_{k,2l\alpha}$ can be determined in terms of known quantities that do not contain $\xi_k(t)$ $(1 \le |k| \le m)$.*

PROOF. Set $\bar{u}_{j\alpha} := \sigma^{\eta}_{j\alpha}(u)$ and $\bar{v}_{j\alpha} := \sigma^{\eta}_{j\alpha}(v)$. On the right-hand side of the equation for $f_{k,2l\alpha}$, the terms containing $\xi_k(t)$ $(1 \le |k| \le m)$ are given only in

$$\begin{pmatrix} -2\bar{u}_{(2l-1)\alpha}\sigma_1^{\theta}(\bar{u}_\alpha)\theta \\ \dfrac{1}{2}(-\bar{v}_{(2l-1)\alpha}, \bar{u}_{(2l-1)\alpha})Q\begin{pmatrix}\bar{u}_\alpha\,\theta\\ \bar{v}_\alpha\,\theta\end{pmatrix} + (\sigma_1^{\theta}(\bar{u}_{(2l-1)\alpha})\bar{v}_\alpha + 3\sigma_1^{\theta}(\bar{v}_{(2l-1)\alpha})\bar{u}_\alpha)\theta \end{pmatrix}$$

$$+\begin{pmatrix} -2\bar{u}_\alpha\sigma_1^\theta(\bar{u}_{(2l-1)\alpha})\theta \\ \dfrac{1}{2}(-\bar{v}_\alpha,\bar{u}_\alpha)Q\begin{pmatrix}\bar{u}_{(2l-1)\alpha}\,\theta\\ \bar{v}_{(2l-1)\alpha}\,\theta\end{pmatrix}+(\sigma_1^\theta(\bar{u}_\alpha)\bar{v}_{(2l-1)\alpha}+3\sigma_1^\theta(\bar{v}_\alpha)\bar{u}_{(2l-1)\alpha})\theta\end{pmatrix}. \tag{94}$$

It follows from Lemma 3.4 that (94) can be written in the form

$$\sum_{1\le|k|\le m}\frac{1}{\nu_k}\Psi_k(t)A(\nu_k)\theta. \tag{95}$$

Here $\Psi_k(t)$ is defined by

$$\begin{aligned}
&-2\sum_{\substack{1\le|j|\le m,\\ j\ne -k}}\left(\frac{2\nu_k+\nu_j}{\nu_k+\nu_j}\rho_{k,j}+\nu_k\right)(\omega_k\xi_j+\xi_k\omega_j)e^{\tau_k+\tau_j}\\
&-2\sum_{\substack{1\le|j|\le m,\\ j\ne -k}}\frac{2\nu_k+\nu_j}{\nu_k+\nu_j}\rho_{-k,-j}(\omega_{-k}\xi_{-j}+\xi_{-k}\omega_{-j})e^{-\tau_k-\tau_j}\\
&+\sum_{j=1}^{m}\nu_j^2h_{j,k}(\omega_j\xi_{-j}+\xi_j\omega_{-j})-2(3\rho_{k,-k}+\nu_k)(\omega_k\xi_{-k}+\xi_k\omega_{-k}).
\end{aligned}$$

From the above equation, we have (93). This completes the proof of lemma. □

We study the equation for $f_{k,(2l+1)\alpha}$ obtained by comparing the coefficients of $\eta^{(2l+1)\alpha}$ in (45). To get the form of l-th member of the non-secularity conditions, it suffices to observe only the terms containing $f_{i,(2l-1)\alpha}$'s and $f_{i,2l\alpha}$'s in the right-hand side of the equation for $f_{k,(2l+1)\alpha}$, which appear in

$$\begin{aligned}
&\begin{pmatrix} 0\\ (-\bar{v}_\alpha,\bar{u}_\alpha)Q\begin{pmatrix}\bar{u}_{2l\alpha}\,\theta\\ \bar{v}_{2l\alpha}\,\theta\end{pmatrix}+(-\bar{v}_{(2l-1)\alpha},\bar{u}_{(2l-1)\alpha})Q\begin{pmatrix}\bar{u}_{2\alpha}\,\theta\\ \bar{v}_{2\alpha}\,\theta\end{pmatrix}\end{pmatrix}\\
&+\begin{pmatrix}0\\ \dfrac{\bar{u}_{(2l-1)\alpha}}{2}(-\bar{v}_\alpha,\bar{u}_\alpha)Q\begin{pmatrix}\bar{u}_\alpha\,\theta\\ \bar{v}_\alpha\,\theta\end{pmatrix}\end{pmatrix}\\
&+\begin{pmatrix}0\\ \dfrac{\bar{u}_\alpha}{2}(-\bar{v}_{(2l-1)\alpha},\bar{u}_{(2l-1)\alpha})Q\begin{pmatrix}\bar{u}_\alpha\,\theta\\ \bar{v}_\alpha\,\theta\end{pmatrix}+\dfrac{\bar{u}_\alpha}{2}(-\bar{v}_\alpha,\bar{u}_\alpha)Q\begin{pmatrix}\bar{u}_{(2l-1)\alpha}\,\theta\\ \bar{v}_{(2l-1)\alpha}\,\theta\end{pmatrix}\end{pmatrix}\\
&+\begin{pmatrix}0\\ \bar{u}_{(2l-1)\alpha}\big(\sigma_1^\theta(\bar{u}_\alpha)\bar{v}_\alpha+\sigma_1^\theta(\bar{v}_\alpha)\bar{u}_\alpha\big)\end{pmatrix}\theta\\
&+\begin{pmatrix}0\\ \bar{u}_\alpha\big(\sigma_1^\theta(\bar{u}_{(2l-1)\alpha})\bar{v}_\alpha+\sigma_1^\theta(\bar{u}_\alpha)\bar{v}_{(2l-1)\alpha}+\sigma_1^\theta(\bar{v}_{(2l-1)\alpha})\bar{u}_\alpha+\sigma_1^\theta(\bar{v}_\alpha)\bar{u}_{(2l-1)\alpha}\big)\end{pmatrix}\theta
\end{aligned}$$

$$+ 2 \begin{pmatrix} -\sigma_1^\theta(\bar{u}_\alpha)\bar{u}_{2l\alpha} - \sigma_1^\theta(\bar{u}_{(2l-1)\alpha})\bar{u}_{2\alpha} - \sigma_1^\theta(\bar{u}_{2l\alpha})\bar{u}_\alpha - \sigma_1^\theta(\bar{u}_{2\alpha})\bar{u}_{(2l-1)\alpha} \\ \sigma_1^\theta(\bar{u}_{2\alpha})\bar{v}_{(2l-1)\alpha} + \sigma_1^\theta(\bar{u}_{2l\alpha})\bar{v}_\alpha + \sigma_1^\theta(\bar{v}_\alpha)\bar{u}_{2l\alpha} + \sigma_1^\theta(\bar{v}_{(2l-1)\alpha})\bar{u}_{2\alpha} \end{pmatrix} \theta$$

$$+ 4 \begin{pmatrix} 0 \\ \sigma_1^\theta(\bar{v}_{2\alpha})\bar{u}_{(2l-1)\alpha} + \sigma_1^\theta(\bar{v}_{2l\alpha})\bar{u}_\alpha \end{pmatrix} \theta - \left(\varrho + \frac{\partial}{\partial t}\right) \begin{pmatrix} \bar{u}_{(2l-1)\alpha} \\ \bar{v}_{(2l-1)\alpha} \end{pmatrix} \theta. \tag{96}$$

Comparing (79) and (96), we observe that the coefficient of $A(\nu_k)e^{\tau_k}$ containing ξ_j's in the above equation is expressed as

$$\frac{1}{\nu_k}\left(\sum_{j=1}^{m} \varphi(k,j)(\xi_j\omega_{-j}\omega_k + \omega_j\xi_{-j}\omega_k + \omega_j\omega_{-j}\xi_k) + J_k\xi_k\right) - R_k\xi_k - \frac{d\xi_k}{dt} \tag{97}$$

for k ($1 \leq |k| \leq m$). Therefore the l-th member of the non-secularity conditions for the higher-order terms is obtained.

THEOREM 5.2. *The l-th ($l \geq 2$) member of the non-secularity conditions has the following form*: *For $1 \leq k \leq m$,*

$$\begin{aligned} \frac{d\xi_k}{dt} &= \frac{1}{\nu_k}\left(\sum_{j=1}^{m} \varphi(k,j)(\omega_{-j}\omega_k\xi_j + \omega_j\omega_k\xi_{-j} + \omega_j\omega_{-j}\xi_k) + J_k\xi_k\right) - R_k\xi_k + q_k, \\ \frac{d\xi_{-k}}{dt} &= -\frac{1}{\nu_k}\left(\sum_{j=1}^{m} \varphi(-k,j)(\omega_{-j}\omega_{-k}\xi_j + \omega_j\omega_{-k}\xi_{-j} + \omega_j\omega_{-j}\xi_{-k}) + J_{-k}\xi_{-k}\right) \\ &\quad - R_{-k}\xi_{-k} + q_{-k}. \end{aligned} \tag{98}$$

Here $\varphi(k,j)$, J_k, R_k and ω_j have been defined by (82), (91), *and $q_{\pm k}$ are the inhomogeneous terms containing only the known quantities. For the more explicit forms of $\varphi(k,j)$, J_k and R_k, see Remark* E.3 *in Appendix* E.

Thus, as l-th ($l \geq 2$) member is a system, which can be solved globally, of first-order linear differential equations, we have $f_{k,(2l-1)\alpha}$ and $f_{k,2l\alpha}$. Hence we can successively construct the higher-order terms. Summing up, we have the main theorem.

THEOREM 5.3. *Let Ω be an open subset in $\mathbb{C}_t$ and we assume the conditions* (S1) *and* (S2). *Then we have instanton-type solutions for $(P_{\mathrm{II}})_m$ with $2m$ free parameters $(\beta_{-m}, \ldots, \beta_m) \in \mathbb{C}^{2m}[[\eta^{-1}]]$. Especially, we can construct the solution (u,v) in $\hat{\mathcal{A}}^2(\Omega)$ for* (45) *of the form*

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{1 \leq |k| \leq m} f_k(\tau, t; \eta) A(\nu_k) \tag{99}$$

with

$$f_k(\tau, t; \eta) = \sum_{j=1}^{\infty}\left(\sum_{\ell \geq 0,\, p \in \mathbb{Z}^m,\, 2\ell + |p| = j} f_{k,p,\ell}(t)e^{p\cdot\tau}\right)\eta^{-j/2}.$$

The more precise forms of the leading term $f_{k,\alpha}$ and the subleading term $f_{k,2\alpha}$ of f_k are given in Lemmas 4.1, 4.4 *and Proposition* 4.8.

As an application of the results in this paper and [**22**], taking parameters suitably, we can prove that the solution (u, v) with $(m+1)$ free parameters of (45) is also constructed in $(\hat{\mathcal{A}}_\alpha^{\mathcal{O}}(D))^2$ where D is a specific region described in [**22**].

6. Instanton-type solutions for $(P_{\rm IV})_m$ with a large parameter.

In the last section, we explain that we can construct instanton-type solutions for general members of the fourth Painlevé hierarchy with η by using previous sections.

6.1. The fourth Painlevé hierarchy with η.

For $m = 1, 2, \ldots$, the m-th member $(P_{\rm IV})_m$ of the fourth Painlevé hierarchy with η (cf.[**15**], [**7**]) is a system of non-linear differential equations with $2m$ unknown functions $u_1, \ldots, u_m$, $v_1, \ldots, v_m$ of t:

$$(P_{\rm IV})_m \begin{cases} \eta^{-1}\dfrac{du_j}{dt} = -2(u_1u_j + v_j + u_{j+1}) + 2c_ju_1, & j = 1, 2, \ldots, m, \\ \eta^{-1}\dfrac{dv_j}{dt} = 2(v_1u_j + v_{j+1} + w_j) - 2c_jv_1, & j = 1, 2, \ldots, m \end{cases} \tag{100}$$

with

$$u_{m+1} = -\left(\gamma tu_1 + \alpha_1 + \frac{1}{2}\eta^{-1}\gamma\right), \quad v_{m+1} = -w_m - \gamma tv_1 - \frac{(v_m - \alpha_1)^2 - \alpha_2^2}{2(u_m - \gamma t - c_m)}.$$

Here $\gamma(\neq 0)$, α_1, α_2 and c_j are constants, and w_j are polynomials of u_k, v_l $(1 \le k,\ l \le m)$ recursively defined by

$$w_j := \sum_{k=1}^{j-1} u_{j-k}w_k + \sum_{k=1}^{j} u_{j-k+1}v_k + \frac{1}{2}\sum_{k=1}^{j-1} v_{j-k}v_k - \sum_{k=1}^{j-1} c_{j-k}w_k. \tag{101}$$

Note that the definition of w_j in (101) is same as that in (2).

As u_{m+1} contains a large parameter η, we adopt the formulation below which is obtained by replacing $u_m - \gamma t$ in (100) with u_m:

$$(P_{\rm IV})_m \begin{cases} \eta^{-1}\dfrac{du_j}{dt} = -2(u_1u_j + v_j + u_{j+1}) + 2c_ju_1 - 2\delta_{j,m-1}\gamma t, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \ldots, m, \\ \eta^{-1}\dfrac{dv_j}{dt} = 2(v_1u_j + v_{j+1} + w_j) - 2c_jv_1, \qquad j = 1, 2, \ldots, m \end{cases} \tag{102}$$

with

$$u_{m+1} = -\alpha_1, \quad v_{m+1} = -w_m - \frac{(v_m - \alpha_1)^2 - \alpha_2^2}{2(u_m - c_m)}.$$

Here $\gamma(\neq 0)$, α_1, α_2 and c_j are constants, δ_{jm} stands for Kronecker's delta, and w_j is defined by

$$w_j := \sum_{k=1}^{j-1} u_{j-k}w_k + \sum_{k=1}^{j} u_{j-k+1}v_k + \frac{1}{2}\sum_{k=1}^{j-1} v_{j-k}v_k - \sum_{k=1}^{j-1} c_{j-k}w_k + \delta_{jm}\gamma t v_1. \tag{103}$$

If we can construct a solution $(u_1, \ldots, u_m, v_1, \ldots, v_m)$ of (102), then $(u_1, \ldots, u_{m-1}, u_m + \gamma t, v_1, \ldots, v_m)$ becomes a solution of (100). From now on, we prove that we obtain instanton-type solutions to (102).

6.2. The form of $(P_{\mathrm{IV}})_m$ by generating functions.

Firstly (102) can be written in the form

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv 2\begin{pmatrix} u_1(1-U+C)\theta - U - V\theta - \gamma t\theta^m \\ -v_1(1-U+C)\theta + \dfrac{2UV+V^2\theta}{2(1-U+C)} + V + \gamma t v_1\theta^{m+1} \end{pmatrix}, \tag{104}$$

where U, V and C are defined by the same forms as (4). Let us define the generating functions of $\hat{u}_{i,0}$ and $\hat{v}_{i,0}$ $(i \geq 1)$ by

$$\hat{u}_0(\theta) := \sum_{i=1}^{\infty} \hat{u}_{i,0}\theta^i \quad \text{and} \quad \hat{v}_0(\theta) := \sum_{i=1}^{\infty} \hat{v}_{i,0}\theta^i, \tag{105}$$

respectively. Here $\hat{u}_{i,0}$, $\hat{v}_{i,0}$ $(i \geq 1)$ denote the leading terms of a 0-parameter solution of (102). Then $\hat{u}_0$ and $\hat{v}_0$ satisfy

$$\begin{aligned} \hat{u}_0 + \hat{v}_0\theta &= \hat{u}_{1,0}(1-\hat{u}_0+C)\theta - \gamma t\theta^m, \\ \frac{2\hat{u}_0\hat{v}_0 + (\hat{v}_0)^2\theta}{2(1-\hat{u}_0+C)} &= \hat{v}_{1,0}(1-\hat{u}_0+C)\theta - \hat{v}_0 - \gamma t\hat{v}_{1,0}\theta^{m+1}. \end{aligned} \tag{106}$$

It follows from the above equations that we have

$$(1+\hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2 = \frac{(1+C)^2 - \gamma^2t^2\theta^{2m}}{(1-\hat{u}_0+C)^2} + \frac{2\gamma t\theta^m\big((1+\hat{u}_{1,0}\theta) - \hat{v}_{1,0}\theta^2\big)}{1-\hat{u}_0+C}. \tag{107}$$

Hence we get

$$(1+\hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2 \equiv \frac{(1+C)^2}{(1-\hat{u}_0+C)^2} + 2\gamma t\theta^m(1+2\hat{u}_{1,0}\theta - c_1\theta). \tag{108}$$

Comparing (11) (resp. (12)) with (106) (resp. (108)), we can see

$$\hat{u}_0 \equiv (1+C)\left(1 - \sqrt{\frac{1}{f(t,\theta)}}\right),$$

$$\hat{v}_0\theta \equiv (1+C)\left(-1+(1+\hat{u}_{1,0}\theta)\sqrt{\frac{1}{f(t,\theta)}}\right)-\gamma t\theta^m. \tag{109}$$

Here $f(t,\theta)$ is defined by

$$f(t,\theta) := (1+\hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2 - 2\gamma t\theta^m(1+2\hat{u}_{1,0}\theta - c_1\theta). \tag{110}$$

Note that $\hat{u}_{i,0}$ and $\hat{v}_{i,0}$ $(2 \le i \le m)$ can be described by $\hat{u}_{1,0}$ and $\hat{v}_{1,0}$. By the defining equation of the leading terms of a 0-parameter solution of (102), the following equations should hold.

$$\hat{u}_{m+1,0} = -\alpha_1 \quad \hat{v}_{m+1,0} = -\hat{w}_{m,0} - \frac{(\hat{v}_{m,0}-\alpha_1)^2-\alpha_2^2}{2(\hat{u}_{m,0}-c_m)}, \tag{111}$$

where $\hat{w}_{m,0}$ is defined by (103) with u_k, v_k and w_k being replaced by $\hat{u}_{k,0}, \hat{v}_{k,0}$ and $\hat{w}_{k,0}$, respectively. Therefore we determine $\hat{u}_{1,0}$ and $\hat{v}_{1,0}$ so that the coefficients of θ^{m+1} in $\hat{u}_0$ and $\hat{v}_0$ of (109) are the right-hand sides of (111), respectively.

6.3. The system of partial differential equations associated with $(P_{\mathrm{IV}})_m$. By the same argument as that for $(P_{\mathrm{II}})_m$, we consider a linearized equation of (104) along $(\hat{u}_0, \hat{v}_0)$ given by (109). We take the following change of the unknown functions:

$$U = \hat{u}_0 + (1-\hat{u}_0+C)u, \qquad V = \hat{v}_0 + (1-\hat{u}_0+C)v. \tag{112}$$

Here we use the same notations u and v as that of (15). Putting (112) into (104), as (106) holds, we have

$$\begin{aligned}
&-\eta^{-1}\begin{pmatrix}\varrho\\ \delta\end{pmatrix}\theta + \eta^{-1}\left(\varrho+\frac{d}{dt}\right)\begin{pmatrix}u\\ v\end{pmatrix}\theta \\
&\equiv 2\begin{pmatrix}(\sigma_1^\theta(u)-\hat{u}_{1,0}u-v)\theta - u\\ (-\sigma_1^\theta(v)+2\hat{v}_{1,0}u+\hat{u}_{1,0}v)\theta+v\end{pmatrix} + 2u\begin{pmatrix}-\sigma_1^\theta(u)\\ \sigma_1^\theta(v)\end{pmatrix}\theta \\
&\quad + \frac{1}{1-u}\begin{pmatrix}0\\ 2u(\hat{v}_{1,0}u+\hat{u}_{1,0}v)\theta+(2uv+v^2\theta)\end{pmatrix}.
\end{aligned} \tag{113}$$

Here ϱ and δ are defined by (19) with $\hat{u}_0$ and $\hat{v}_0$ of (109). An important fact is that the above system has the completely same form as (18): The only difference between $(P_{\mathrm{II}})_m$ and $(P_{\mathrm{IV}})_m$ is that the algebraic equation that determines $\hat{u}_0$ and $\hat{v}_0$ differ (cf. (11) and (106)) and consequently they have different values. Now we list up the relevant variables of $(P_{\mathrm{II}})_m$ and how they are determined by the 0-parameter solution of $(P_{\mathrm{II}})_m$:

	$(\hat{u}_0, \hat{v}_0)$ by (13)
	(ϱ, δ) by (19)
$(P_{\mathrm{II}})_m$	$\gamma_k,\ \delta_k\ (1 \le k \le m)$ by (65), (66)
	$\nu_k,\ \nu_{-k}\ (1 \le k \le m)$ by (35)
	$A(\lambda),\ a(\lambda),\ \rho(\lambda)\ g(\lambda)$ by (31), (32), (33)

If we replace the variables of $(P_{\mathrm{II}})_m$ in the above list with the corresponding ones of $(P_{\mathrm{IV}})_m$, then we find that the arguments after (18) for $(P_{\mathrm{II}})_m$ run also for $(P_{\mathrm{IV}})_m$ in the exactly same manner. As a consequence, we have the following.

THEOREM 6.1. *Let Ω be an open subset in $\mathbb{C}_t$ and we assume the conditions* (S1) *and* (S2) *for $(P_{\mathrm{IV}})_m$. Then $(P_{\mathrm{IV}})_m$ has instanton-type solutions with $2m$ free parameters $(\beta_{-m}, \dots, \beta_m) \in \mathbb{C}^{2m}[[\eta^{-1}]]$ of the same form as* (99)*. Moreover, instanton-type solutions of $(P_{\mathrm{II}})_m$ are transformed to those of $(P_{\mathrm{IV}})_m$ by replacing the variables of $(P_{\mathrm{II}})_m$ in the above list with the corresponding ones of $(P_{\mathrm{IV}})_m$.*

Appendix

A. The resultant of (37).

In this appendix, we see that the resultant of (37) coincides with the characteristic equation of the Fréchet derivative of $(P_{\mathrm{II}})_m$ at its 0-parameter solution. We define the polynomials d_j $(1 \le j \le 2m)$ by

$$\begin{aligned} d_j &= -2(u_1 u_j + v_j + u_{j+1}) + 2c_j u_1, \qquad j = 1, 2, \dots, m, \\ d_{j+m} &= 2(v_1 u_j + v_{j+1} + w_j) - 2c_j v_1, \qquad j = 1, 2, \dots, m \end{aligned} \tag{114}$$

with $u_{m+1} = \gamma t$ and $v_{m+1} = \kappa$. We set

$$V := \{(t, u, v) \in \mathbb{C}_t \times \mathbb{C}_u^m \times \mathbb{C}_v^m; d_1 = 0,\, d_2 = 0, \dots, d_{2m} = 0\}.$$

Let $J_{u,v}$ be the Jacobian matrix of functions $(d_1, d_2, \dots, d_{2m})$ with respect to the variables u_i, v_j $(1 \le i,\, j, \le m)$. By (6) and (11), we obtain the explicit form of the restriction $J_{u,v}|_V$ of $J_{u,v}$ to V as follows.

$$J_{u,v}|_V = \begin{pmatrix} -A & -2E_m \\ 4\hat{v}_{1,0}E_m & A \end{pmatrix}, \quad A := \begin{pmatrix} 2a_1 + 2\hat{u}_{1,0} & 2 & 0 & \cdots & \cdots & \cdots & 0 \\ 2a_2 & 2\hat{u}_{1,0} & 2 & 0 & \cdots & \cdots & 0 \\ 2a_3 & 0 & 2\hat{u}_{1,0} & \ddots & \ddots & & \vdots \\ 2a_4 & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & & 2 & 0 \\ 2a_{m-1} & 0 & 0 & \cdots & 0 & 2\hat{u}_{1,0} & 2 \\ 2a_m & 0 & 0 & \cdots & & 0 & 2\hat{u}_{1,0} \end{pmatrix}. \tag{115}$$

Here $\hat{u}_{j,0}, \hat{v}_{j,0}$ denote the leading terms of (8), E_m is the identity matrix of size m and $a_j := \hat{u}_{j,0} - c_j$. By using elementary transformations of matrices, we find that the characteristic polynomial of $J_{u,v}|_V$ has the following form.

$$\Lambda(\lambda, u, v) = \det(\lambda E_{2m} - J_{u,v}|_V) = 2^{2m} \det B, \tag{116}$$

where B is given by

$$\begin{pmatrix} a_2 + (a_1 + 2\hat{u}_{1,0})a_1 - b(\lambda) & a_1 + 2\hat{u}_{1,0} & 1 & 0 & \cdots & \cdots & 0 \\ a_3 + (a_1 + 2\hat{u}_{1,0})a_2 & a_2 - b(\lambda) & 2\hat{u}_{1,0} & 1 & 0 & \cdots & 0 \\ a_4 + (a_1 + 2\hat{u}_{1,0})a_3 & a_3 & -b(\lambda) & 2\hat{u}_{1,0} & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & -b(\lambda) & 2\hat{u}_{1,0} & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & & 2\hat{u}_{1,0} & 1 \\ a_m + (a_1 + 2\hat{u}_{1,0})a_{m-1} & a_{m-1} & 0 & \cdots & 0 & -b(\lambda) & 2\hat{u}_{1,0} \\ (a_1 + 2\hat{u}_{1,0})a_m & a_m & 0 & \cdots & & 0 & -b(\lambda) \end{pmatrix} \tag{117}$$

with $b(\lambda) := \lambda^2/4 - \hat{u}_{1,0}^2 + 2\hat{v}_{1,0}$.

We can easily see that the resultant of (37) coincides with (116).

B. Remark on Lemma 3.5.

We consider the explicit forms of γ_j and δ_j given in Lemma 3.5. Firstly, we define G_k and H_k by the coefficients of θ^k in the right-hand sides of (66) respectively:

$$\begin{aligned} \frac{-(\hat{u}_{1,0})'(1 + \hat{u}_{1,0}\theta) + (\hat{v}_{1,0})'\theta}{(1 + \hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2} &= \sum_{k=0}^{\infty} G_k(t)\theta^k, \\ \frac{-(\hat{v}_{1,0})'(1 + \hat{u}_{1,0}\theta) + 2(\hat{u}_{1,0})'\hat{v}_{1,0}\theta}{(1 + \hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2} &= \sum_{k=0}^{\infty} H_k(t)\theta^k. \end{aligned} \tag{118}$$

Taking (66) into account, we find the following system of unknown functions $\gamma_j = \gamma_j(t)$ and $\delta_j = \delta_j(t)$ $(1 \le j \le m)$:

$$\begin{aligned} g(\nu_1)^k\gamma_1 + g(\nu_2)^k\gamma_2 + \cdots + g(\nu_m)^k\gamma_m &= G_k(t) \quad (k = 0, 1, \ldots, m-1), \\ g(\nu_1)^k\delta_1 + g(\nu_2)^k\delta_2 + \cdots + g(\nu_m)^k\delta_m &= H_k(t) \quad (k = 0, 1, \ldots, m-1). \end{aligned} \tag{119}$$

Therefore, by the Cramer's formula, we have

$$\begin{aligned} \gamma_j = \gamma_{-j} &= \frac{\det M(g(\nu_1), \ldots, g(\nu_{j-1}),\, G_j(t),\, g(\nu_{j+1}), \ldots, g(\nu_m))}{\det M(g(\nu_1), \ldots, g(\nu_{j-1}),\, g(\nu_j),\, g(\nu_{j+1}), \ldots, g(\nu_m))}, \\ \delta_j = \delta_{-j} &= \frac{\det M(g(\nu_1), \ldots, g(\nu_{j-1}),\, H_j(t),\, g(\nu_{j+1}), \ldots, g(\nu_m))}{\det M(g(\nu_1), \ldots, g(\nu_{j-1}),\, g(\nu_j),\, g(\nu_{j+1}), \ldots, g(\nu_m))}, \end{aligned} \tag{120}$$

where the $m \times m$ matrix M is defined by

$$M(\kappa_1, \ldots, \kappa_m) := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \kappa_1 & \kappa_2 & \cdots & \kappa_m \\ \kappa_1^2 & \kappa_2^2 & \cdots & \kappa_m^2 \\ \vdots & \vdots & & \vdots \\ \kappa_1^{m-1} & \kappa_2^{m-1} & \cdots & \kappa_m^{m-1} \end{pmatrix} \tag{121}$$

with m-variables $\kappa_1, \ldots, \kappa_m$. Note that we obtain $\det M(g(\nu_1), \ldots, g(\nu_m)) = \prod_{1 \le i < j \le m}(g(\nu_j) - g(\nu_i)) \ne 0$.

C. Proof of Proposition 3.6.

Let us first prove (67). By using (47), (65), (38) and (40), we have

$$u\varrho \equiv \sum_{\substack{1 \le |k| \le m, \\ 1 \le |j| \le m, \\ j \ne \pm k}} \frac{1}{2(g(\nu_k) - g(\nu_j))}(\gamma_j f_k + \gamma_k f_j)a(\nu_k) + \sum_{1 \le |j| \le m} \gamma_j f_j \sum_{k=1}^{m} h_{j,k} a(\nu_k). \tag{122}$$

It follows from the first equation of (34) and (122) that $\binom{u\varrho}{0}$ is equal to

$$\sum_{1 \le |k| \le m} \frac{1}{\nu_k} \left(\sum_{\substack{j=1, \\ j \ne \pm k}}^{m} \frac{\gamma_j(f_k + f_{-k}) + \gamma_k(f_j + f_{-j})}{g(\nu_k) - g(\nu_j)} + \sum_{j=1}^{m} \gamma_j(f_j + f_{-j})h_{j,k} \right) \rho(\nu_{-k})A(\nu_k) \tag{123}$$

modulo θ^{m+2}. On the other hand, note that $u\delta$ is writtem by (122) with γ_j being replaced by δ_j. By the second equation of (34), we can see that $\binom{0}{u\delta}$ is equal to

$$-\sum_{1 \le |k| \le m} \frac{1}{\nu_k} \left(\sum_{\substack{j=1, \\ j \ne \pm k}}^{m} \frac{\delta_j(f_k + f_{-k}) + \delta_k(f_j + f_{-j})}{g(\nu_k) - g(\nu_j)} + \sum_{j=1}^{m} \delta_j(f_j + f_{-j})h_{j,k} \right) A(\nu_k) \tag{124}$$

modulo θ^{m+2}. Hence, by summing (123) and (124) up, we have (67).

We repeat the same argument as above by using (34) and (42), then we have (68). This completes the proof.

D. Proof of Lemma 4.5.

Taking (53) into account, we have

$$\frac{\bar{u}_\alpha}{2}(-\bar{v}_\alpha, \bar{u}_\alpha)Q \begin{pmatrix} \bar{u}_\alpha\, \theta \\ \bar{v}_\alpha\, \theta \end{pmatrix} = \bar{u}_\alpha \sum_{k=1}^{m} L_k a(\nu_k)\theta \tag{125}$$

with

$$L_k := \sum_{\substack{1\le |i|\le m,\\ i+k\ne 0}} \frac{\nu_k-\nu_i}{\nu_k+\nu_i}(\rho_{k,i}f_{k,\alpha}f_{i,\alpha}+\rho_{-k,-i}f_{-k,\alpha}f_{-i,\alpha}) - \sum_{i=1}^{m}\mu(k,i)f_{i,\alpha}f_{-i,\alpha},$$

where $\mu(k,i) := \nu_i^2 h_{i,k}$ and $\rho_{k,i}$'s have been defined by (48). On the other hand, a direct computation shows

$$\begin{aligned}(-\bar v_\alpha,\bar u_\alpha)Q^{\bar u_{2\alpha}\,\theta}_{\bar v_{2\alpha}\,\theta} &= \sum_{1\le |k|\le m}\nu_k f_{k,2\alpha}(-\bar v_\alpha+\rho(\nu_k)\bar u_\alpha)a(\nu_k)\theta\\ &= -\bar v_\alpha\sum_{k=1}^{m}\nu_k(f_{k,2\alpha}-f_{-k,2\alpha})a(\nu_k)\theta\\ &\quad+\bar u_\alpha\sum_{k=1}^{m}\nu_k\big(\rho(\nu_k)f_{k,2\alpha}-\rho(\nu_{-k})f_{-k,2\alpha}\big)a(\nu_k)\theta. \end{aligned}\tag{126}$$

We also see

$$\begin{aligned}&\nu_k(f_{k,2\alpha}-f_{-k,2\alpha})\\ &\quad=\sum_{1\le|i|\le m}\frac{4}{\nu_i}\bigg(-\Big(\rho_{k,i}+\frac{\nu_k}{2}\Big)f_{k,\alpha}f_{i,\alpha}+\Big(\rho_{-k,-i}-\frac{\nu_k}{2}\Big)f_{-k,\alpha}f_{-i,\alpha}\bigg)-\gamma_k\end{aligned}$$

and

$$\begin{aligned}&\nu_k\big(\rho(\nu_k)f_{k,2\alpha}-\rho(\nu_{-k})f_{-k,2\alpha}\big)\\ &\quad=\sum_{\substack{1\le|i|\le m,\\ i+k\ne0}}\bigg(-\frac{4}{\nu_i}\Big(\rho_{k,i}+\frac{\nu_k}{2}\Big)\rho(\nu_k)-\frac{2\nu_k}{\nu_k+\nu_i}\rho_{k,i}\bigg)f_{k,\alpha}f_{i,\alpha}\\ &\qquad+\sum_{\substack{1\le|i|\le m,\\ i+k\ne0}}\bigg(\frac{4}{\nu_i}\Big(\rho_{-k,-i}-\frac{\nu_k}{2}\Big)\rho(\nu_{-k})-\frac{2\nu_k}{\nu_k+\nu_i}\rho_{-k,-i}\bigg)f_{-k,\alpha}f_{-i,\alpha}\\ &\qquad+\sum_{i=1}^{m}\mu(k,i)f_{i,\alpha}f_{-i,\alpha}-4\rho_{k,-k}f_{k,\alpha}f_{-k,\alpha}-\delta_k.\end{aligned}$$

Finally, it is easy to check that we have

$$\bar u_\alpha\big(\bar v_\alpha\sigma_1^\theta(\bar u_\alpha)+\bar u_\alpha\sigma_1^\theta(\bar v_\alpha)\big)\theta=\bar u_\alpha\sum_{\substack{1\le|k|\le m,\\ 1\le|i|\le m}}\rho_{k,i}f_{k,\alpha}f_{i,\alpha}a(\nu_k)\theta. \tag{127}$$

Summing (125), (126) and (127) up, we can obtain (80). This proves the assertion of lemma.

E. Proof of Proposition 4.6.

Let us prove the proposition which plays an essential role in solving the first member of the non-secularity conditions. It suffices to show the subsequent Lemmas E.1 and E.2.

Lemma E.1. *For any k $(1 \le |k| \le m)$, there exist functions $\varphi(k, j)$ of the variables ν_ℓ's and multi-valued functions $J_{k,1}$ of finite determination in Ω satisfying the conditions*

$$\varphi(k, j) = \varphi(-k, j) \quad (1 \le j \le m), \qquad J_{k,1} = J_{-k,1} \tag{$*$}$$

such that the coefficient of $e^{\tau_k} A(\nu_k)$ in the first and the fourth terms of the right-hand side for (81) *is given by*

$$\frac{1}{\nu_k}\Bigg(\sum_{j=1}^{m} \varphi(k, j)\omega_j\omega_{-j} + J_{k,1}\Bigg)\omega_k. \tag{128}$$

Proof. We compute the first term of right-hand side of (81). Noticing (47) and the definition of $\sigma_1^\theta(\bar{u}_{i\alpha})$ $(i = 1, 2)$, we have

$$-\sigma_1^\theta(\bar{u}_\alpha)\bar{u}_{2\alpha} - \bar{u}_\alpha\sigma_1^\theta(\bar{u}_{2\alpha}) = -\sum_{1\le|k|\le m}\sum_{1\le|j|\le m}\big(f_{j,\alpha}f_{k,2\alpha} + f_{k,\alpha}f_{j,2\alpha}\big)a(\nu_k), \tag{129}$$

$$\begin{aligned}&\sigma_1^\theta(\bar{u}_{2\alpha})\bar{v}_\alpha + \sigma_1^\theta(\bar{v}_\alpha)\bar{u}_{2\alpha} + 2\sigma_1^\theta(\bar{v}_{2\alpha})\bar{u}_\alpha\\ &\quad= \sum_{1\le|k|\le m}\sum_{1\le|j|\le m}\big(\rho(\nu_k)f_{j,2\alpha}f_{k,\alpha} + \rho(\nu_j)f_{j,\alpha}f_{k,2\alpha} + 2\rho(\nu_j)f_{j,2\alpha}f_{k,\alpha}\big)a(\nu_k).\end{aligned} \tag{130}$$

Using (34), we obtain

$$\begin{pmatrix}-\sigma_1^\theta(\bar{u}_\alpha)\bar{u}_{2\alpha} - \bar{u}_\alpha\sigma_1^\theta(\bar{u}_{2\alpha})\\ \sigma_1^\theta(\bar{u}_{2\alpha})\bar{v}_\alpha + \sigma_1^\theta(\bar{v}_\alpha)\bar{u}_{2\alpha} + 2\sigma_1^\theta(\bar{v}_{2\alpha})\bar{u}_\alpha\end{pmatrix} = -\sum_{1\le|k|\le m}\sum_{1\le|j|\le m}\frac{1}{\nu_k}\Gamma_{k,j}A(\nu_k), \tag{131}$$

where $\Gamma_{k,j}$ has the form

$$(\rho_{k,j} + \rho_{-k,j})f_{j,2\alpha}f_{k,\alpha} + \rho_{-k,j}f_{j,\alpha}(f_{k,2\alpha} + f_{-k,2\alpha}) + 2\rho_{-k,j}f_{j,2\alpha}f_{-k,\alpha}. \tag{132}$$

For any fixed k $(1 \le |k| \le m)$, we set

$$\begin{aligned}\mathrm{I}_1 &:= -\sum_{1\le|j|\le m}(\rho_{k,j} + \rho_{-k,j})f_{j,2\alpha}f_{k,\alpha},\\ \mathrm{I}_2 &:= -\sum_{1\le|j|\le m}\rho_{-k,j}\,f_{j,\alpha}(f_{k,2\alpha} + f_{-k,2\alpha}),\\ \mathrm{I}_3 &:= -\sum_{1\le|j|\le m}2\rho_{-k,j}f_{j,2\alpha}f_{-k,\alpha}.\end{aligned} \tag{133}$$

Let us compute I_1 of (133). For the simplicity, we put

$$\begin{aligned} l(j,i) &:= \frac{1}{\nu_j+\nu_i}\rho_{-j,-i}, \qquad \mu(j,i) := \nu_i^2 h_{i,j}, \\ n(j) &:= 6\rho_{j,-j}+2\nu_j, \qquad r(j) := \gamma_j\rho(\nu_{-j})-\delta_j. \end{aligned} \tag{134}$$

Here $h_{i,j}$, γ_j and δ_j have been defined by (41) and Proposition 3.3. Then, by Lemma 4.4, for any j $(1\le |j|\le m)$, $f_{j,2\alpha}$ is written in the form

$$\begin{aligned} f_{j,2\alpha} = \sum_{\substack{1\le|i|\le m,\\ i+j\neq 0}} &\left(\left(\frac{2(2\nu_j+\nu_i)}{\nu_j\nu_i}l(-j,-i)-\frac{2}{\nu_i}\right)f_{i,\alpha}f_{j,\alpha}+\frac{2}{\nu_j}l(j,i)f_{-i,\alpha}f_{-j,\alpha}\right) \\ &-\frac{1}{\nu_j^2}\left(\sum_{i=1}^m \mu(j,i)f_{i,\alpha}f_{-i,\alpha}-n(j)f_{j,\alpha}f_{-j,\alpha}+r(j)\right). \end{aligned} \tag{135}$$

Putting (71) and (135) into the right-hand side of I_1 in (133), we have

$$\begin{aligned} I_1 = &\sum_{1\le|j|\le m}\ \sum_{\substack{1\le|i|\le m,\\ i+j\neq 0}} \frac{4}{\nu_i}\left(-(\rho_{k,-k}+\rho_{j,-j})l(-j,-i)+\frac{1}{2}\rho_{k,-k}-\rho(\nu_i)\right)\omega_k\omega_j\omega_i e^{\tau_k+\tau_j+\tau_i} \\ &+\sum_{1\le|j|\le m}\frac{1}{\nu_j^2}(\rho_{k,j}+\rho_{-k,j})\left(\sum_{i=1}^m \mu(j,i)\omega_i\omega_{-i}-n(j)\omega_j\omega_{-j}+r(j)\right)\omega_k e^{\tau_k}. \end{aligned} \tag{136}$$

For k $(1\le|k|\le m)$, the terms containing e^{τ_k} in I_1 are given by

$$\sum_{1\le|j|\le m}\frac{1}{\nu_j^2}(\rho_{k,j}+\rho_{-k,j})\left(\sum_{i=1}^m \mu(j,i)\omega_i\omega_{-i}-n(j)\omega_j\omega_{-j}+r(j)\right)\omega_k e^{\tau_k}. \tag{137}$$

Next, for I_2 of (133), we repeat the same arguments as those for I_1. Using (135), we have

$$\begin{aligned} I_2 = &\sum_{1\le|j|\le m}\ \sum_{\substack{1\le|i|\le m,\\ i+k\neq 0}} \rho_{-k,j}\left(-\frac{4}{\nu_i}l(-k,-i)+\frac{2}{\nu_i}\right)\omega_k\omega_j\omega_i e^{\tau_k+\tau_j+\tau_i} \\ &+\sum_{1\le|j|\le m}\ \sum_{\substack{1\le|i|\le m,\\ i+k\neq 0}} \rho_{-k,-j}\left(\frac{4}{\nu_i}l(k,i)-\frac{2}{\nu_i}\right)\omega_{-k}\omega_{-j}\omega_{-i}e^{-\tau_k-\tau_j-\tau_i} \\ &+\sum_{1\le|j|\le m}\frac{2}{\nu_k^2}\rho_{-k,j}\left(\sum_{i=1}^m \mu(k,i)\omega_i\omega_{-i}-6\rho_{k,-k}\omega_k\omega_{-k}+\frac{\gamma_k}{2}\rho_{k,-k}-\delta_k\right)\omega_j e^{\tau_j}, \end{aligned} \tag{138}$$

where $l(k,i)$, $\mu(k,j)$ are given by (134). The terms containing e^{τ_k} on the right-hand side of (138) for k $(1 \le |k| \le m)$ are given by

$$\sum_{\substack{1\le|j|\le m,\\ j\ne k}} \rho_{-k,j}\left(\frac{4}{\nu_j}l(-k,j)-\frac{2}{\nu_j}\right)\omega_j\omega_{-j}\omega_k e^{\tau_k}$$
$$+\frac{2}{\nu_k^2}\rho_{-k,k}\left(\sum_{i=1}^{m}\mu(k,i)\omega_i\omega_{-i}-6\rho_{k,-k}\omega_k\omega_{-k}+\frac{\gamma_k}{2}\rho_{k,-k}-\delta_k\right)\omega_k e^{\tau_k}. \tag{139}$$

Finally, let us compute I_3 of (133). The following fact can be easily shown.

$$I_3=\sum_{1\le|j|\le m}\sum_{\substack{1\le|i|\le m,\\ i+j\ne 0}}\left(\frac{4}{\nu_i}(\rho_{k,-k}+\rho_{j,-j})l(j,i)+\frac{4\nu_k}{\nu_i}l(j,i)+\frac{4}{\nu_i}(\rho(\nu_{-i})-\rho(\nu_{-k}))\right)$$
$$\times\,\omega_{-k}\omega_{-i}\omega_{-j}e^{-\tau_k-\tau_j-\tau_i}$$
$$+\sum_{1\le|j|\le m}\frac{2}{\nu_j^2}\rho_{-k,j}\left(\sum_{i=1}^{m}\mu(j,i)\omega_i\omega_{-i}-n(j)\omega_j\omega_{-j}+r(j)\right)\omega_{-k}e^{-\tau_k}. \tag{140}$$

The terms containing e^{τ_k} in the right-hand side of (140) for k $(1 \le |k| \le m)$ are given by

$$\left(-\frac{4}{\nu_k}(\rho_{k,-k}+\rho_{-k,k})l(-k,-k)-4l(-k,-k)-\frac{4}{\nu_k}(\rho(\nu_k)-\rho(\nu_{-k}))\right)\omega_k^2\omega_{-k}e^{\tau_k}. \tag{141}$$

By summing (137), (139) and (141) up, we have

$$\left(\sum_{j=1}^{m}\tilde{\varphi}_1(k,j)\omega_j\omega_{-j}+J_{k,1}\right)\omega_k e^{\tau_k}, \tag{142}$$

where $\tilde{\varphi}_1(k,j)$ and $J_{k,1}$ are given by

$$\tilde{\varphi}_1(k,j):=\sum_{\substack{i=1,\\ i\ne|k|}}^{m}\frac{2}{\nu_i^2}(\rho_{k,-k}+\rho_{i,-i})\mu(i,j)-\frac{12}{\nu_j^2}\rho_{j,-j}(\rho_{k,-k}+\rho_{j,-j})$$
$$+\frac{2}{\nu_j^2-\nu_k^2}(\rho_{k,-k}+\rho_{j,-j})^2+\frac{6}{\nu_k^2}\rho_{k,-k}\mu(k,j)+4 \quad (j\ne|k|),$$
$$\tilde{\varphi}_1(k,k):=\sum_{\substack{i=1,\\ i\ne|k|}}^{m}\frac{2}{\nu_i^2}(\rho_{k,-k}+\rho_{i,-i})\mu(i,k)-\frac{32}{\nu_k^2}(\rho_{k,-k})^2+\frac{8}{\nu_k^2}\rho(\nu_k)\rho(\nu_{-k})$$
$$+\frac{6}{\nu_k^2}\rho_{k,-k}\mu(k,k)+8, \tag{143}$$

$$J_{k,1} := \sum_{\substack{j=1,\\ j\neq|k|}}^{m} \frac{4}{\nu_j^2}\left(\gamma_j\left(\frac{1}{4}\rho_{k,-k}\rho_{j,-j} + \rho(\nu_j)\rho(\nu_{-j})\right) - \frac{\delta_j}{2}(\rho_{k,-k}+\rho_{j,-j})\right)$$
$$+ \frac{4}{\nu_k^2}\left(\gamma_k\left(\frac{1}{2}(\rho_{k,-k})^2 + \rho(\nu_k)\rho(\nu_{-k})\right) - \frac{3}{2}\rho_{k,-k}\delta_k\right). \tag{144}$$

Therefore, noticing $\rho_{k,-k} = \rho_{-k,k}$, $\mu(i,k) = \mu(i,-k)$ and $\mu(k,k) = \mu(-k,-k)$, we can see $\tilde{\varphi}_1(k,j) = \tilde{\varphi}_1(-k,j)$ $(j \neq |k|)$, $\tilde{\varphi}_1(k,k) = \tilde{\varphi}_1(-k,-k)$ and $J_{k,1} = J_{-k,1}$. The terms containing $e^{\tau_k}A(\nu_k)$ in the first term of (81) are written by

$$\frac{2}{\nu_k}\left(\sum_{j=1}^{m}\tilde{\varphi}_1(k,j)\omega_j\omega_{-j} + J_{k,1}\right)\omega_k e^{\tau_k}A(\nu_k)\theta. \tag{145}$$

On the other hand, we take the terms containing $e^{\tau_k}A(\nu_k)$ in the fourth term of (81). Then we have

$$\sum_{\substack{j=1,\\ j\neq|k|}}^{m}\frac{4}{\nu_k}\omega_j\omega_{-j}\omega_k e^{\tau_k}A(\nu_k)\theta. \tag{146}$$

Hence, by (145) and (146), we have the explicit forms of the coefficients of (128) as follows.

$$\varphi(k,j) = 2\tilde{\varphi}_1(k,j) + 4 \quad (j\neq|k|), \qquad \varphi(k,k) = 2\tilde{\varphi}_1(k,k). \tag{147}$$

The assertions of lemma have been obtained. □

Lemma E.2. *For any k $(1 \leq |k| \leq m)$, there exist multi-valued functions $J_{k,2}$ and R_k of finite determination in Ω satisfying the conditions*

$$J_{k,2} = J_{-k,2}, \quad R_k = R_{-k}, \tag{$*$}$$

such that the coefficient of $e^{\tau_k}A(\nu_k)$ in the second and third terms of the right-hand side for (81) is given by

$$\left(\frac{1}{\nu_k}J_{k,2} - R_k\right)\omega_k - \frac{d\omega_k}{dt}. \tag{148}$$

Proof. By Proposition 3.6, we can easily see

$$J_{k,2} := \sum_{\substack{j=1,\\ j\neq|k|}}^{m}\frac{2(\rho_{k,-k}+\rho_{j,-j})}{\nu_j^2-\nu_k^2}\left(\delta_j - \frac{\gamma_j}{2}\rho_{k,-k}\right) + \left(\delta_k - \frac{\gamma_k}{2}\rho_{k,-k}\right)h_{k,k} + \frac{1}{2}(\rho_{k,-k})',$$

$$R_k := \frac{\nu_k'}{2\nu_k} + \sum_{\substack{j=1,\\ j\neq|k|}}^{m} \frac{\rho_{k,-k}+\rho_{j,-j}}{\nu_j^2-\nu_k^2}\gamma_j + \frac{1}{2}\gamma_k h_{k,k} + (g(\nu_k))' h_{k,k}. \tag{149}$$

Hence we have the assertion of lemma. □

By using Lemmas E.1 and E.2, we have Proposition 4.6. Especially, the coefficients are given by (147), (149) and

$$J_k = 2J_{k,1} + J_{k,2}. \tag{150}$$

Finally, we give the remark below.

REMARK E.3. More precise forms of the coefficients appearing in Proposition 4.6 are given by

$$\begin{aligned}
\varphi(k,j) &:= \sum_{\substack{i=1,\\ i\neq|k|}}^{m} \frac{4\nu_j^2}{\nu_i^2}(\rho_{k,-k}+\rho_{i,-i})h_{j,i} - \frac{24}{\nu_j^2}\rho_{j,-j}(\rho_{k,-k}+\rho_{j,-j}) \\
&\quad + \frac{4}{\nu_j^2-\nu_k^2}(\rho_{k,-k}+\rho_{j,-j})^2 + \frac{12\nu_j^2}{\nu_k^2}\rho_{k,-k}h_{j,k} + 12 \quad (j\neq|k|), \\
\varphi(k,k) &:= \sum_{\substack{i=1,\\ i\neq|k|}}^{m} \frac{4\nu_k^2}{\nu_i^2}(\rho_{k,-k}+\rho_{i,-i})h_{k,i} - \frac{64}{\nu_k^2}(\rho_{k,-k})^2 + \frac{16}{\nu_k^2}\rho(\nu_k)\rho(\nu_{-k}) \\
&\quad + 12\rho_{k,-k}h_{k,k} + 16, \\
J_k &:= \sum_{\substack{j=1,\\ j\neq|k|}}^{m} \frac{8}{\nu_j^2}\left(\gamma_j\left(\frac{1}{4}\rho_{k,-k}\rho_{j,-j} + \rho(\nu_j)\rho(\nu_{-j})\right) - \frac{\delta_j}{2}(\rho_{k,-k}+\rho_{j,-j})\right) \\
&\quad + \frac{8}{\nu_k^2}\left(\gamma_k\left(\frac{1}{2}(\rho_{k,-k})^2 + \rho(\nu_k)\rho(\nu_{-k})\right) - \frac{3}{2}\rho_{k,-k}\delta_k\right) \\
&\quad + \sum_{\substack{j=1,\\ j\neq|k|}}^{m} \frac{2(\rho_{k,-k}+\rho_{j,-j})}{\nu_j^2-\nu_k^2}\left(\delta_j - \frac{\gamma_j}{2}\rho_{k,-k}\right) + \left(\delta_k - \frac{1}{2}\rho_{k,-k}\gamma_k\right)h_{k,k} + \frac{1}{2}(\rho_{k,-k})', \\
R_k &:= \frac{\nu_k'}{2\nu_k} + \sum_{\substack{j=1,\\ j\neq|k|}}^{m} \frac{(\rho_{k,-k}+\rho_{j,-j})}{\nu_j^2-\nu_k^2}\gamma_j + \frac{1}{2}\gamma_k h_{k,k} + (g(\nu_k))' h_{k,k}.
\end{aligned} \tag{151}$$

Here $h_{i,j}$, γ_j and δ_j have been defined by (134), (41) and Proposition 3.3.

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