J. Math. Soc. Japan
Vol. 67, No. 2 (2015) pp. 609–636
doi: 10.2969/jmsj/06720609

Double filtration of twisted logarithmic complex and Gauss–Manin connection

By Kazuhiko Aomoto and Yoshinori Machida

(Received Oct. 3, 2012)
(Revised July 10, 2013)

Abstract. The twisted de Rham complex associated with hypergeometric integral of a power product of polynomials is quasi-isomorphic to the corresponding logarithmic complex. We show in this article that the latter has a double filtration with respect to degrees of polynomials and exterior algebras. By a combinatorial method we prove the quasi-isomorphism between the twisted de Rham cohomology and a specially filtered subcomplex in case of polynomials of the same degree. This fact gives a more detailed structure of a basis for the twisted de Rham cohomology.

1. Introduction.

Let $P_k(x)$ $(1 \leq k \leq m)$ be polynomials of $x = (x_1, \ldots, x_n)$ in $\boldsymbol{C}^n$ over the coefficient field $\boldsymbol{C}$. We assume that each P_k is of *the same degree* l $(l \geq 1)$. Let D_k be the divisor in $\boldsymbol{C}^n$ defined by $P_k(x) = 0$ and the union $D = \bigcup_{k=1}^m D_k$. Let $\mathcal{M}$ be the complement $\boldsymbol{C}^n - D$. Denote by $\Omega^{\cdot}(\boldsymbol{C}^n) = \bigoplus_{p=0}^n \Omega^p(\boldsymbol{C}^n)$ the polynomial differential forms on $\boldsymbol{C}^n$, and by $\Omega^{\cdot}(\log D) = \bigoplus_{p=0}^n \Omega^p(\log D)$ the space of logarithmic p-forms $(0 \leq p \leq n)$ φ on $\mathcal{M}$ along D, i.e.,

$$P_1 \cdots P_m \varphi,\ P_1 \cdots P_m d\varphi \in \Omega^{\cdot}(\boldsymbol{C}^n).$$

We define the total degree of φ (denoted by $\mathrm{tdeg}(\varphi)$) to be $\deg(\varphi)+p$. Remark that $\mathrm{tdeg}(dP_k/P_k) = 0$.

Let $F_\mu\Omega^p(\log D)$ $(\mu \in \boldsymbol{Z})$ be the subspace of $\Omega^p(\log D)$ consisting of φ such that $\mathrm{tdeg}(\varphi) \leq \mu$. Note that $F_\mu\Omega^p(\log D) = \{0\}$ for $\mu < -lm+p$. Then we have the increasing filtration

$$\{0\} \subset F_0\Omega^p(\log D) \subset F_1\Omega^p(\log D) \subset \cdots \subset F_p(\Omega^p(\log D)) \subset \cdots \subset \Omega^p(\log D).$$

By definition we have

$$\Omega^p(\log D) = \bigcup_{\mu=0}^{\infty} F_\mu\Omega^p(\log D).$$

2010 *Mathematics Subject Classification.* Primary 33C70; Secondary 14F40.

Key Words and Phrases. hypergeometric integrals, twisted de Rham cohomology, logarithmic forms, de Rham–Saito Lemma, vanishing theorem, double filtration, Gauss–Manin connection.

The second author was partially supported by Grant-in-Aid for Scientific Research (C) (No. 22540109).

For $\lambda_k \in \boldsymbol{C}$ $(1 \leq k \leq m)$, we consider the covariant differentiation ∇ on $\Omega^{\cdot}(\log D)$ and the subcomplex $F_\mu\Omega^{\cdot}(\log D)$ respectively by

$$\nabla\psi = d\psi + \sum_{k=1}^{m} \lambda_k \frac{dP_k}{P_k} \wedge \psi,$$

assuming that $l\sum_{k=1}^{m} \lambda_k \notin \boldsymbol{Z}$. $H^p(\Omega^{\cdot}(\log D), \nabla), H^p(F_\mu\Omega^{\cdot}(\log D), \nabla)$ denote the respective twisted de Rham cohomologies. We also denote by $H^p(\Omega^{\cdot}(*D), \nabla)$ the twisted de Rham cohomology for the complex $\Omega^{\cdot}(*D)$ of rational differential forms on $\mathcal{M}$ with poles only at D.

Similarly denote by $\overline{P}_k$ the homogeneous part of highest degree of P_k and by $\overline{D}_k$ the divisor $\overline{P}_k(x) = 0$ in $\boldsymbol{C}^n$ and $\overline{D} = \bigcup_{k=1}^{m} \overline{D}_k$. By the differentiation $\overline{\nabla}$:

$$\overline{\nabla}\psi = d\psi + \sum_{k=1}^{m} \lambda_k \frac{d\overline{P}_k}{\overline{P}_k} \wedge \psi,$$

we can also define the twisted de Rham cohomologies $H^p(\Omega^{\cdot}(\log \overline{D}), \overline{\nabla})$, $H^p(F_\mu\Omega^{\cdot}(\log \overline{D}), \overline{\nabla})$, $H^p(\Omega^{\cdot}(*\overline{D}), \overline{\nabla})$ respectively.

In the sequel we simply write $\Omega^p(\log D), \Omega^{\cdot}(\log D)$ by $\Omega^p, \Omega^{\cdot}$ respectively. Denote by $[1, m]$ the set of natural numbers ν such that $1 \leq \nu \leq m$. For the set of indices $J = \{j_1, \ldots, j_q\} \subset [1, m]$, $|J|$ denotes q the size of J.

The homogenization of an inhomogeneous polynomial $f(x)$ in $\boldsymbol{C}[x]$ is defined as a homogeneous polynomial in $\boldsymbol{C}[x_0, x_1, \ldots, x_n]$

$$\widetilde{f} = H(f) = x_0^l f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \quad (l = \deg(f)).$$

We also define the homogenization of df by

$$\widetilde{df} = H(df) = \tilde{d}\tilde{f} = \sum_{k=1}^{n} \frac{\partial \widetilde{f}}{\partial x_k} dx_k.$$

For $\varphi \in \Omega^{\cdot}(*D)$, the homogenizations of φ and $d\varphi$ are defined by

$$\widetilde{d\varphi} = \tilde{d}\tilde{\varphi}, \quad \tilde{\varphi} \wedge \tilde{\psi} = \widetilde{\varphi \wedge \psi}.$$

An inhomogeneous ideal $\mathfrak{I}$ in $\boldsymbol{C}[x]$ has the canonical homogenization $H(\mathfrak{I})$ in $\boldsymbol{C}[x_0, x_1, \ldots, x_n]$.

For $r + s$ polynomials $f_1, \ldots, f_r$ and $g_1, \ldots, g_s$, we can consider the ideal generated by the differential forms $df_1 \wedge \cdots \wedge df_r$ $(1 \leq r \leq n)$ and $g_1, \ldots, g_s$:

$$\mathfrak{I} = (df_1 \wedge \cdots \wedge df_r, g_1, \ldots, g_s).$$

We now settle "genericity condition" for the family of the polynomials $P_k(x)$ $(1 \leq k \leq m)$.

The first condition is as follows:

($\mathcal{C}_1$) *Take any integer* r *such that* $1 \le r \le \min\{m, n\}$ *and* $1 \le j_1 < \cdots < j_r \le m$. *Let* $Q_1, Q_2, \ldots, Q_r$ $(1 \le r \le m)$ *be arbitrary different polynomials among* P_k $(1 \le k \le m)$. *Then the homogeneous ideal* $(H(dQ_1 \wedge \cdots \wedge dQ_r, Q_1, \ldots, Q_r))$ *satisfies*

(i) $\mathrm{height}(H(dQ_1 \wedge \cdots \wedge dQ_r, Q_1, \ldots, Q_r)) \ge n+1$ (*in* $\boldsymbol{C}[x_0, x_1, \ldots, x_n]$),
(ii) *For* $1 \le s \le \min\{m, n+1\}$, $H(Q_1), \ldots, H(Q_s)$ *form a regular sequence in* $\boldsymbol{C}[x_0, x_1, \ldots, x_n]$.

The second condition is as follows:

($\mathcal{C}_2$) *Take any integer* r *such that* $1 \le r \le \min\{m, n-1\}$ *and* $1 \le j_1 < \cdots < j_r \le m$. *Let* $\overline{Q}_1, \overline{Q}_2, \ldots, \overline{Q}_r$ $(1 \le r \le m)$ *be arbitrary different polynomials among* $\overline{P}_k$ $(1 \le k \le m)$. *Then the homogeneous ideal* $(d\overline{Q}_1 \wedge \cdots \wedge d\overline{Q}_r, \overline{Q}_1, \ldots, \overline{Q}_r)$ *satisfies*

(i) $\mathrm{height}(d\overline{Q}_1 \wedge \cdots \wedge d\overline{Q}_r, \overline{Q}_1, \ldots, \overline{Q}_r) \ge n$ (*in* $\boldsymbol{C}[x_1, \ldots, x_n]$),
(ii) *For* $1 \le s \le \min\{m, n\}$, $\overline{Q}_1, \ldots, \overline{Q}_s$ *form a regular sequence in* $\boldsymbol{C}[x_1, \ldots, x_n]$.

Throughout our article we set the conditions $(\mathcal{C}_1), (\mathcal{C}_2)$.

Then the following Proposition is valid (see [**1**], [**2**], [**3**], [**7**]):

PROPOSITION 1. (i) *For* $\mu \ge 0$ *we have*

$$H^p(\Omega^{\cdot}, \nabla) \cong H^p(F_\mu \Omega^{\cdot}, \nabla) \cong \{0\} \quad (0 \le p \le n-1),$$

(ii)
$$H^n(\Omega^{\cdot}, \nabla) \cong H^n(\Omega^{\cdot}(*D), \nabla),$$
$$\dim H^n(\Omega^{\cdot}, \nabla) = (-1)^n \mathcal{E}(\mathcal{M}) = \sum_{\nu=0}^{n} \binom{m-1}{\nu} (l-1)^{n-\nu} \binom{m+n-\nu-1}{n-\nu},$$

where $\mathcal{E}(\mathcal{M})$ *denotes the Euler characteristic of* $\mathcal{M}$.

LEMMA 2. *For* $\psi \in \Omega^p$ $(0 \le p \le n-1)$, ψ *can be described as*

$$\psi = \psi_0 + \sum_{q=1}^{\min(p,m)} \sum_{J=\{j_1,\ldots,j_q\} \subset [1,m]} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_q}}{P_{j_q}} \wedge \psi_J \tag{1}$$

for $\psi_0 \in \Omega^p(\boldsymbol{C}^n)$, $\psi_J \in \Omega^{p-q}(\boldsymbol{C}^n)$.

For the proof, see [**1**, Proposition 3.1].

PROPOSITION 3. *For* $\mu \ge 0$ *we have*

$$H^p(\Omega^{\cdot}(*\overline{D}), \overline{\nabla}) \cong H^p(\Omega^{\cdot}(\log \overline{D}), \overline{\nabla}) \cong H^p(F_\mu(\Omega^{\cdot}(\log \overline{D}), \overline{\nabla})) \cong \{0\} \quad (0 \le p \le n).$$

As a Corollary of Lemma 2 and Proposition 3, Lemma 4. *Suppose that* $\psi \in \Omega^p$ $(0 \le p \le n-1)$ *satisfies*

$$\nabla\psi \in F_\mu\Omega^{p+1} \quad (\mu \ge 0),$$

then ψ *can be described as* (1) *such that* $\psi_0 \in F_\mu\Omega^p(\boldsymbol{C}^n)$, $\psi_J \in F_\mu\Omega^{p-q}(\boldsymbol{C}^n)$.

Definition 5. Denote by $F_{\mu,q}\Omega^p$ $(0 \le q \le p)$ the subspace of $F_\mu\Omega^p$ consisting of φ such that φ can be written by

$$\varphi = \varphi_0 + \sum_{\nu=1}^{q} \sum_{J=\{j_1<\cdots<j_\nu\}\subset[1,m]} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_\nu}}{P_{j_\nu}} \wedge \varphi_J \tag{2}$$

for $\varphi_0 \in F_\mu\Omega^p(\boldsymbol{C}^n)$ and $\varphi_J \in F_\mu\Omega^{p-\nu}(\boldsymbol{C}^n)$. We have the double filtration

$$\{0\} \subset F_{\mu,0}\Omega^p \subset F_{\mu,1}\Omega^p \subset \cdots \subset F_{\mu,p}\Omega^p = F_{\mu,p+1}\Omega^p = \cdots = F_{\mu,\infty}\Omega^p \subset F_\mu\Omega^p.$$

In this article we shall give the decomposition formula for the n dimensional de Rham cohomology $H^n(\Omega^\cdot, \nabla)$ associated to the double filtration (Theorem 18 and Theorem 28) and derive the corresponding formula of Gauss–Manin connection for the twisted integrals (Theorem 29).

2. Dimension formula.

From now on, we assume that $\mu \ge (l-1)n$.
The following Lemma is fundamental.

Lemma 6. *Suppose that* $\varphi \in F_{\mu,q}\Omega^p$ *in* (2) *lies in* $F_{\mu,q-1}\Omega^p$ $(q+p \le n)$, *and hence*

$$dP_{j_1} \wedge \cdots \wedge dP_{j_q} \wedge \varphi_J \equiv 0 \mod \sum_{\nu=1}^{q} P_{j_\nu} F_{\mu+l(q-1)}\Omega^p(\boldsymbol{C}^n)$$

for each φ_J $(J = \{j_1, \ldots, j_q\})$, *then* φ_J *can be described as*

$$\varphi_J = \sum_{\nu=1}^{q} \bigl(P_{j_\nu}\theta_\nu + dP_{j_\nu} \wedge \theta'_\nu\bigr), \tag{3}$$

where $\theta_\nu \in F_{\mu-l}\Omega^{p-q}(\boldsymbol{C}^n)$ *and* $\theta'_\nu \in F_{\mu-l}\Omega^{p-q-1}(\boldsymbol{C}^n)$, *in other words,*

$$\varphi_J \equiv 0 \mod \mathcal{F}_\mu^{p-q}(J),$$

where $\mathcal{F}_\mu^{p-q}(J)$ *denotes the subspace of* $\Omega^{p-q}(\boldsymbol{C}^n)$:

$$\mathcal{F}_\mu^{p-q}(J) = \sum_{\nu=1}^{q} \bigl(P_{j_\nu} F_{\mu-l}\Omega^{p-q}(\boldsymbol{C}^n) + dP_{j_\nu} \wedge F_{\mu-l}\Omega^{p-q-1}(\boldsymbol{C}^n)\bigr).$$

The proof can be done based on syzygies of Cohen-Macaulay H-ideals (homogeneous ideals) and on de Rham-Saito lemma (see [**3**, Lemma 2.19], replacing n by $n+1$, and [**1**], [**8**] for related topics).

We also note that

$$F_{\mu,q}\Omega^p/F_{\mu,q-1}\Omega^p \cong \bigoplus_{J\subset[1,m];|J|=q} F_\mu\Omega^{p-q}(\boldsymbol{C}^n)/\mathcal{F}_\mu\Omega^{p-q}(J).$$

We now fix q. We want to give an explicit formula for the dimension of $F_\mu\Omega^{p-q}(\boldsymbol{C}^n)/\mathcal{F}_\mu\Omega^{p-q}(J)$. A numerical computations based on Lemma 6 show the following Propositions 11 and 13.

We fix the set of indices $J=\{j_1,\dots,j_q\}$. For simplicity we rewrite P_{j_ν} by Q_ν. Let σ_0 denote the surjective morphism:

$$\sigma_0 : F_\mu\Omega^{p-q}(\boldsymbol{C}^n) \longrightarrow F_\mu\Omega^{p-q}(\boldsymbol{C}^n)/\mathcal{F}_\mu^{p-q}(J) \subset F_{\mu,q}\Omega^p(\log D)/F_{\mu,q-1}\Omega^p(\log D),$$

which is defined by

$$\sigma_0(\varphi_J) = \frac{dQ_1}{Q_1}\wedge\cdots\wedge\frac{dQ_q}{Q_q}\wedge\varphi_J. \tag{4}$$

First we want to construct a resolution of the morphism σ_0.

Let $S=\bigoplus_{\nu=0}^\infty S_\nu$ be the polynomial ring over $\boldsymbol{C}$ in the indeterminates $y_1,\dots,y_q$, and $\Lambda=\bigoplus_{\nu=0}^q\Lambda^\nu$ be the exterior algebra over $\boldsymbol{C}$ in the indeterminates $dy_1,\dots,dy_q$. Here S_ν and Λ^ν denote the parts of ν-th degree of S and Λ respectively. Let $\mathcal{K}_{r,s}=S_r\otimes\Lambda^s\wedge F_{\mu-(r+s)l}\Omega^{p-q-r}(\boldsymbol{C}^n)$, and put $\mathcal{K}_\nu=\bigoplus_{r+s=\nu;0\le r\le p-q;0\le s\le q}\mathcal{K}_{r,s}$, $\mathcal{K}=\bigoplus_{\nu=0}^\infty\mathcal{K}_\nu$. Remark that $\mathcal{K}_\nu\cong\{0\}$ for $\nu\ge p+1$. We can identify $F_\mu\Omega^{p-q}(\boldsymbol{C}^n)$ with $\mathcal{K}_0=\mathcal{K}_{0,0}$. An arbitrary element ψ of $\mathcal{K}_{r,s}$ can be uniquely written as

$$\psi=\frac{1}{r!s!}\sum_{K,L} y_{k_1}\cdots y_{k_r}dy_{l_1}\wedge\cdots\wedge dy_{l_s}\wedge\psi(K;L),$$

where K moves over the set of sequences consisting of r indices $K=(k_1,\dots,k_r)\in[1,q]^r$ and L moves over the set of sequences consisting of s different indices in $[1,q]$. $|K|$ and $|L|$ denote r the size of K and s the size of L respectively. $\psi(K;L)\in F_{\mu-(r+s)l}\Omega^{p-q-r}(\boldsymbol{C}^n)$ are symmetric with respect to $k_1,\dots,k_r$ and alternating with respect to $l_1,\dots,l_s$. In the sequel we shall call such $\psi(K;L)$ "admissible".

We define the morphism

$$\sigma_\nu : \mathcal{K}_\nu \to \mathcal{K}_{\nu-1} \quad (\nu\ge 1)$$

by the differentiation

$$\sigma_\nu\psi=\sum_{i=1}^q\left(Q_i\frac{\partial}{\partial dy_i}+dQ_i\wedge\frac{\partial}{\partial y_i}\right)\psi$$

for $\psi \in \mathcal{K}_\nu$. In more detail, σ_ν ($\nu = r+s$) is a morphism from $\mathcal{K}_{r,s}$ into $\mathcal{K}_{r,s-1} \oplus \mathcal{K}_{r-1,s}$. Since

$$\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\psi = \frac{\partial}{\partial y_j}\frac{\partial}{\partial y_i}\psi, \quad \frac{\partial}{\partial dy_i}\frac{\partial}{\partial dy_j}\psi = -\frac{\partial}{\partial dy_j}\frac{\partial}{\partial dy_i}\psi,$$

we have $\sigma_{\nu-1} \circ \sigma_\nu = 0$. In this way, we can define the Cartan–Koszul double complex $\{\mathcal{K}, (\sigma_\nu)_\nu\}$:

$$\{0\} \to \cdots \to \mathcal{K}_\nu \to \mathcal{K}_{\nu-1} \to \cdots \to \mathcal{K}_0 \to F_\mu\Omega^{p-q}(\boldsymbol{C}^n)/\mathcal{F}_\mu^{p-q}(J) \to \{0\}. \tag{5}$$

(See [**5**, Chapter 7] for the definitions and basic properties of Cartan and Koszul complexes.)

We will show that the complex is a resolution of σ_0 (Proposition 9).

The morphism σ_ν can be described in terms of indices as follows:

$$(\sigma_1\psi)(\emptyset;\emptyset) = \sum_{i=1}^q Q_i\psi(\emptyset;\{i\}) + dQ_i \wedge \psi(\{i\};\emptyset) \quad (\nu = 1), \tag{6}$$

$$(\sigma_\nu\psi)(K';L) = \sum_{i=1}^q \{Q_i\psi(K';\{i\}\cup L) + (-1)^s dQ_i \wedge \psi(\{i\}\cup K';L)\},$$
$$(|K'| = r-1,\ |L| = s,\ r+s=\nu). \tag{7}$$

The following Lemma corresponds to the acyclicity of the part of Cartan complex.

Lemma 7. *Suppose that admissible $\psi(K;L) \in F_{\mu-\nu l}\Omega^{p-q-r}(\boldsymbol{C}^n)$ ($|K| = r$, $|L| = s$, $r+s=\nu$) satisfy, for $|K'| = r-1$,*

$$\sum_{i=1}^q dQ_i \wedge \psi(\{i\}\cup K';L) \equiv 0$$
$$\mod (Q_1F_{\mu-\nu l}\Omega^{p-q-r+1}(\boldsymbol{C}^n) + \cdots + Q_qF_{\mu-\nu l}\Omega^{p-q-r+1}(\boldsymbol{C}^n)). \tag{8}$$

Then there exist admissible $\theta(\tilde{K};L) \in F_{\mu-(\nu+1)l}\Omega^{p-q-r-1}(\boldsymbol{C}^n)$ ($|\tilde{K}| = r+1$) such that

$$\psi(K;L) \equiv (-1)^s \sum_{i=1}^q dQ_i \wedge \theta(\{i\}\cup K;L)$$
$$\mod (Q_1F_{\mu-(\nu+1)l}\Omega^{p-q-r}(\boldsymbol{C}^n) + \cdots + Q_qF_{\mu-(\nu+1)l}\Omega^{p-q-r}(\boldsymbol{C}^n)). \tag{9}$$

Proof. We put E_ρ to be the set of ρ pieces of the label q contained in K such that $K = \{k_1,\ldots,k_{r-\rho}\}\cup E_\rho$ and $\{k_1,\ldots,k_{r-\rho}\} \subset [1,q-1]$. The proof can be done by double induction on lowering ρ and raising q.

In case where $q=0$ the lemma is trivial.

Suppose first that $q>0$ and $\rho = r$ i.e., $K = E_r$ consists of r pieces of only q. Then by Lemma 6, (8) implies that there exist admissible $\theta_{E_r}(k_1;L) \in F_{\mu-(\nu+1)l}\Omega^{p-q-r-1}(\boldsymbol{C}^n)$

such that

$$\psi(E_r;L) \equiv (-1)^s \sum_{i=1}^{q} dQ_i \wedge \theta_{E_r}(i;L). \tag{10}$$

$\theta_{E_r}(k_1;L)$ may be denoted by $\theta(\{k_1\} \cup E_r;L)$ which can be made admissible too, so that (9) is valid in case of $K = E_r$.

And then (8) for $K' = E_{r-1}$ ($\rho = r-1$) shows

$$\sum_{i=1}^{q-1} dQ_i \wedge \{\psi(\{i\} \cup E_{r-1};L) - (-1)^s dQ_q \wedge \theta(\{i\} \cup E_r;L)\} \equiv 0.$$

By induction hypothesis with respect to q, there exist admissible $\theta_{E_{r-1}}(k_1k_2;L) \in F_{\mu-(\nu+1)l}\Omega^{p-q-r-1}(\boldsymbol{C}^n)$ such that

$$\psi(\{k_1\} \cup E_{r-1};L) \equiv (-1)^s \sum_{i=1}^{q-1} \{dQ_i \wedge \theta_{E_{r-1}}(ik_1;L) + dQ_q \wedge \theta(\{k_1\} \cup E_r;L)\}$$
$$(1 \le k_1 \le q-1). \tag{11}$$

We may put $\theta(\{k_1,k_2\} \cup E_{r-1};L) = \theta_{E_{r-1}}(k_1k_2;L)$, so that we have the identity (9) for $\rho = r-1$. $\theta(\tilde{K};L)$ thus defined for $|\tilde{K}| = r+1$, $|L| = s$ may be made admissible.

Suppose now that the Lemma has been proved in case of $\rho \ge \tau$. We want to prove it in case of $\rho = \tau - 1$. The identity (8) implies that there exist $\theta(\{k_1,\dots,k_{r-\tau+1}\} \cup E_\tau;L) \in F_{\mu-(\nu+1)l}\Omega^{p-q-r-1}(\boldsymbol{C}^n)$ such that

$$\psi(\{k_1,\dots,k_{r-\tau}\} \cup E_\tau;L) \equiv (-1)^s \sum_{i=1}^{q} dQ_i \wedge \theta(\{i,k_1,\dots,k_{r-\tau}\} \cup E_\tau;L). \tag{12}$$

We may assume that $\theta(\{k_1,\dots,k_{r-\tau+1}\} \cup E_\tau;L)$ is admissible. By substitution of (12) into (8) we have the identity

$$\sum_{i=1}^{q-1} \{dQ_i \wedge \psi(\{i,k_1,\dots,k_{r-\tau}\} \cup E_{\tau-1};L) - (-1)^s dQ_q \wedge \theta(\{i,k_1,\dots,k_{r-\tau}\} \cup E_\tau;L)\} \equiv 0.$$

By induction hypothesis, there exist admissible $\theta_{E_{\tau-1}}(k_1 \cdots k_{r-\tau+2};L) \in F_{\mu-(\nu+1)l} \cdot\Omega^{p-q-r-1}(\boldsymbol{C}^n)$ such that

$$\psi(\{k_1,\dots,k_{r-\tau+1}\} \cup E_{\tau-1};L)$$
$$\equiv (-1)^s \left\{ \sum_{i=1}^{q-1} dQ_i \wedge \theta_{E_{\tau-1}}(ik_1 \cdots k_{r-\tau+1};L) + dQ_q \wedge \theta(\{k_1,\dots,k_{r-\tau+1}\} \cup E_\tau;L) \right\}$$
$$(\{k_1,\dots,k_{r-\tau+1}\} \subset [1,q-1]). \tag{13}$$

We may put again

$$\theta(\{k_1,\ldots,k_{r-\tau+2}\}\cup E_{\tau-1};L)=\theta_{E_{\tau-1}}(k_1\cdots k_{r-\tau+2};L).$$

Thus $\theta(\{k_1,\ldots,k_{r-\tau+2}\}\cup E_{\tau-1};L)$ are in $F_{\mu-(\nu+1)l}\Omega^{p-q-r-1}(\boldsymbol{C}^n)$ and made admissible. Hence we have the identity (9) for $E_{\tau-1}$. Lemma 7 has been proved for all K. □

The following Lemma related to the acyclicity of the part of Koszul complex is well-known and can be proved similarly as above (see [**5**], [**6**]).

Lemma 8. *Suppose that admissible* $\psi(K;L)\in F_{\mu-\nu l}\Omega^{p-q-r}(\boldsymbol{C}^n)$ $(|K|=r,\ |L|=s,\ r+s=\nu)$ *satisfy*

$$\sum_{i=1}^{q}Q_i\psi(K;\{i\}\cup L')=0\quad(|L'|=s-1).\tag{14}$$

Then there exist admissible $\theta(K;\tilde L)\in F_{\mu-(\nu+1)l}\Omega^{p-q-r}(\boldsymbol{C}^n)$ $(|\tilde L|=s+1)$ *such that*

$$\psi(K;L)=\sum_{i=1}^{q}Q_i\theta(K;\{i\}\cup L).$$

Under this circumstance the following Proposition holds.

Proposition 9. *The complex* $\{\mathcal{K},(\sigma_\nu)_\nu\}$ *is acyclic.*

Proof. Suppose $\sigma_0(\varphi_J)=0$ for $\varphi_J\in F_\mu\Omega^{p-q}(\boldsymbol{C}^n)$. Then Lemma 6 shows that there exist $\psi(\{i\};\emptyset)\in F_{\mu-l}\Omega^{p-q-1}(\boldsymbol{C}^n)$, $\psi(\emptyset;\{i\})\in F_{\mu-l}\Omega^{p-q}(\boldsymbol{C}^n)$ such that $\varphi_J=(\sigma_1\psi)(\emptyset;\emptyset)$.

Next suppose that $\sigma_\nu\psi=0$ for $\psi\in\mathcal{K}_\nu$ $(\nu\ge 1)$. We must prove that there exists $\theta\in\mathcal{K}_{\nu+1}$ such that $\psi=\sigma_{\nu+1}\theta$. By (7) we have the identity (8). From Lemma 7 there exist admissible $\theta\in\mathcal{K}_{\nu+1}$ such that (9) is valid. We put $\tilde\psi=\psi-\sigma_{\nu+1}\theta$, namely, for $|K|=r$, $|L|=s$, $(r+s=\nu)$

$$\tilde\psi(K;L)=\psi(K;L)-\sum_{i=1}^{q}\{Q_i\theta(K;\{i\})\cup L)+(-1)^s dQ_i\wedge\theta(\{i\}\cup K;L)\}.\tag{15}$$

Then Lemma 7 shows that $\tilde\psi\equiv 0$. In particular, for $|K|=\nu$, $L=\emptyset$ there exist admissible $\tilde\theta(K;\{l_1\})\in F_{\mu-(\nu+1)l}\Omega^{p-q-r}(\boldsymbol{C}^n)$ such that

$$\tilde\psi(K;\emptyset)=\sum_{i=1}^{q}Q_i\tilde\theta(K;\{i\}).$$

We put further $\tilde\theta(\tilde K;\emptyset)$ to be 0 for $|\tilde K|=\nu+1$. By induction on s, we can construct admissible $\tilde\theta(K;\tilde L)$ for $|K|=r$, $|\tilde L|=s+1$ such that

$$\tilde{\psi}(K;L)=\sum_{i=1}^{q}Q_i\tilde{\theta}(K;\{i\}\cup L)+(-1)^s\sum_{i=1}^{q}dQ_i\wedge\tilde{\theta}(\{i\}\cup K;L). \tag{16}$$

In fact (16) is valid for $s=0$. Suppose that $\tilde{\theta}(\tilde{K};L)$ have been constructed for $|L|<s$. The identity $\sigma_\nu\tilde{\psi}=0$ implies the identity for $|K|=r$, $|L'|=s-1$, $r+s=\nu$:

$$\sum_{i=1}^{q}Q_i\tilde{\psi}(K;\{i\}\cup L')+(-1)^{s-1}dQ_i\wedge\tilde{\psi}(\{i\}\cup K;L')=0. \tag{17}$$

By the substitution of (16) for $\tilde{\psi}(\{i\}\cup K;L')$ into (17) we have

$$\sum_{i=1}^{q}Q_i\left\{\tilde{\psi}(K;\{i\}\cup L')+(-1)^{s-1}\sum_{j=1}^{q}dQ_j\wedge\tilde{\theta}(\{j\}\cup K;\{i\}\cup L')\right\}=0.$$

Hence from Lemma 8 there exist admissible $\tilde{\theta}(K;\tilde{L})\in F_{\mu-(\nu+1)l}\Omega^{p-q-r}(\boldsymbol{C}^n)$ for $|\tilde{L}|=s+1$ such that (16) holds for $|L|=s$. In this way we have constructed $\tilde{\theta}(K;L)$ for all K,L with $|K|+|L|=\nu+1$ such that $\tilde{\psi}=\sigma_{\nu+1}\tilde{\theta}$. Therefore the following identity holds:

$$\psi=\sigma_{\nu+1}(\theta+\tilde{\theta}).$$

Proposition 9 has been proved. □

As a result of Proposition 9 we have the equality

Corollary 10. *We have*

$$\dim F_\mu\Omega^{p-q}(\boldsymbol{C}^n)/\mathcal{F}_\mu\Omega^{p-q}(J)=\sum_{\nu=0}^{p}(-1)^\nu\dim\mathcal{K}_\nu. \tag{18}$$

Proposition 11. *Fix the set of indices $J=\{j_1,\ldots,j_q\}\subset[1,m]$. We have the dimension formula*

$$\dim F_\mu\Omega^{p-q}(\boldsymbol{C}^n)/\mathcal{F}_\mu^{p-q}(J)=N_{\mu,q}^{(p)},$$

where

$$N_{\mu,q}^{(p)}=\sum_{\alpha\geq 0,p-q\geq\beta\geq 0}(-1)^{\alpha+\beta}\binom{\mu+n-p+q-l\alpha-(l-1)\beta}{n}\cdot\binom{n}{p-q-\beta}\binom{q}{\alpha}\binom{q+\beta-1}{\beta}.$$

In particular when $p=n$,

$$N^{(n)}_{\mu,q} = \sum_{\alpha\geq 0; n-q\geq\beta\geq 0} (-1)^{\alpha+\beta}\binom{\mu+q-l\alpha-(l-1)\beta}{n}\binom{n}{n-q-\beta}\binom{q}{\alpha}\binom{q+\beta-1}{\beta}.$$

Proof. If $\mu \geq (l-1)n$, then $\mu+n-p+q-l\alpha-(l-1)\beta \geq 0$ for $q\geq\alpha\geq 0$, $p-q\geq\beta\geq 0$. The following formula

$$\dim \mathcal{K}_{\beta,\alpha} = \binom{\mu+n-p+q-l\alpha-(l-1)\beta}{n}\binom{n}{p-q-\beta}\binom{q}{\alpha}\binom{q+\beta-1}{\beta}$$

holds, since

$$\dim S_\beta = \binom{q+\beta-1}{\beta}, \quad \dim \Lambda^\alpha = \binom{q}{\alpha},$$

$$\dim F_{\mu-(\alpha+\beta)l}\Omega^{p-q-\beta}(\boldsymbol{C}^n) = \binom{\mu+n-p+q-l\alpha-(l-1)\beta}{n}\binom{n}{p-q-\beta}.$$

Hence (18) shows Proposition 11. □

Corollary 12. *We have*

$$\dim F_{\mu,q}\Omega^p/F_{\mu,q-1}\Omega^p = \binom{m}{q}N^{(p)}_{\mu,q}.$$

From now on, we simply write $N^{(p)}_q$ instead of $N^{(p)}_{(l-1)n,q}$ for $\mu=(l-1)n$. In particular, in case $p=n$, we have the following Proposition.

Proposition 13. *We have*

$$\dim F_\mu\Omega^n = \sum_{q=0}^{\min(n,m)}\binom{m}{q}N^{(n)}_{\mu,q} = \binom{\mu+lm}{n}. \tag{19}$$

By inversion formula, the identity (19) is equivalent to say

$$N^{(n)}_{\mu,q} = (-1)^q\sum_{\nu=0}^{n}(-1)^\nu\binom{q}{\nu}\binom{\mu+l\nu}{n} \quad (0\leq q\leq \min(n,m)). \tag{20}$$

As a result we have

$$F_\mu\Omega^n = F_{\mu,n}\Omega^n.$$

In particular, for $\mu=(l-1)n$,

$$\dim F_{(l-1)n}\Omega^n = \binom{(l-1)n+lm}{n} = \sum_{q=0}^{\min(n,m)}\binom{m}{q}N^{(n)}_q, \tag{21}$$

$$N_q^{(n)} = (-1)^q \sum_{\nu=0}^{q} (-1)^\nu \binom{(l-1)n + l\nu}{n} \binom{q}{\nu}. \tag{22}$$

REMARK. The identity (20) is still valid for $q \geq n+1$ or $q \geq m+1$ in the sense that both sides of (20) vanish simultaneously.

PROOF OF PROPOSITION 13. It is sufficient to prove the identity (20). We introduce the two generating functions as follows:

$$f(t) = \sum_{0 \leq \alpha \leq q, 0 \leq \beta \leq n-q} \binom{\mu + q - l\alpha - (l-1)\beta}{n} \binom{n}{n-q-\beta} \binom{q}{\alpha} \binom{q+\beta-1}{\beta} t^\beta, \tag{23}$$

$$g(t) = \sum_{0 \leq \alpha \leq q, 0 \leq \beta \leq n-q} \binom{\mu + q - l\alpha - (l-1)\beta}{n} \binom{n}{n-q-\beta} \binom{q}{\alpha} \binom{q+\beta}{\beta} t^\beta. \tag{24}$$

By definition, we have

$$f(1) = N_{\mu,q}^{(n)}, \quad g(t) = \left(1 + \frac{1}{q} t \frac{d}{dt}\right) f(t).$$

Since $f(t), g(t)$ both are polynomials in t, we have

$$f(t) = q t^{-q} \int_0^t t^{q-1} g(t) dt. \tag{25}$$

We first want to find an integral representation of $g(t)$.

LEMMA 14. *$g(t)$ can be represented as the integral*

$$g(t) = \binom{n}{q} \frac{1}{2\pi i} \oint_{s=0} s^{-n-1} (1+s)^{\mu-(l-1)n} \{(1+s)^l - 1\}^q \{(1+s)^{l-1} - t\}^{n-q} ds. \tag{26}$$

PROOF. In fact substituting the equality

$$\binom{\mu + q - l\alpha - (l-1)\beta}{n} = \frac{1}{2\pi i} \oint_{s=0} s^{-n-1} (1+s)^{\mu+q-l\alpha-(l-1)\beta} ds \tag{27}$$

into the RHS of (24),

$$\begin{aligned} g(t) &= \sum_{\alpha,\beta} (-1)^{\alpha+\beta} \binom{q}{\alpha} \frac{n!}{q!\beta!(n-q-\beta)!} t^\beta \cdot \frac{1}{2\pi i} \oint_{s=0} s^{-n-1} (1+s)^{\mu+q-l\alpha-(l-1)\beta} ds \\ &= \binom{n}{q} \sum_\alpha (-1)^\alpha \binom{q}{\alpha} \frac{1}{2\pi i} \oint_{s=0} s^{-n-1} (1+s)^{\mu+q-l\alpha} \left\{1 - \frac{t}{(1+s)^{l-1}}\right\}^{n-q} ds \\ &= \binom{n}{q} \sum_\alpha (-1)^\alpha \binom{q}{\alpha} \frac{1}{2\pi i} \oint_{s=0} s^{-n-1} (1+s)^{\mu-(l-1)n+l(q-\alpha)} \{(1+s)^{l-1} - t\}^{n-q} ds \end{aligned}$$

$$= \binom{n}{q}\frac{1}{2\pi i}\oint_{s=0} s^{-n-1}(1+s)^{\mu-(l-1)n+lq}\left\{1-\frac{1}{(1+s)^l}\right\}^q\{(1+s)^{l-1}-t\}^{n-q}ds$$

$$= \binom{n}{q}\frac{1}{2\pi i}\oint_{s=0} s^{-n-1}(1+s)^{\mu-(l-1)n}\{(1+s)^l-1\}^q\{(1+s)^{l-1}-t\}^{n-q}ds. \qquad \square$$

By substituting (26) into (25), we have the integral formula for $f(t)$:

$$f(t) = \frac{n!}{(q-1)!(n-q)!}t^{-q}\frac{1}{2\pi i}\oint_{s=0} s^{-n-1}(1+s)^{\mu-(l-1)n}\{(1+s)^l-1\}^q ds \cdot \int_0^t t^{q-1}\{(1+s)^{l-1}-t\}^{n-q}dt. \tag{28}$$

Furthermore we have the following.

Lemma 15.

$$\int_0^t t^{q-1}\{(1+s)^{l-1}-t\}^{n-q}dt = -\sum_{\nu=1}^{q}\frac{(q-1)!t^{q-\nu}}{(n-q+1)_\nu(q-\nu)!}\{(1+s)^{l-1}-t\}^{n-q+\nu} + \frac{(q-1)!}{(n-q+1)_q}(1+s)^{n(l-1)}. \tag{29}$$

Proof. (29) can be proved by induction on q, while n being fixed. In fact, for $q=1$ both sides of (29) are equal to

$$-\frac{1}{n}\{(1+s)^{l-1}-t\}^n + \frac{1}{n}(1+s)^{n(l-1)}.$$

Suppose $1 < q \le n$. By integration by parts, the LHS of (29) is equal to

$$-\frac{1}{n-q+1}t^{q-1}\{(1+s)^{l-1}-t\}^{n-q+1} + \frac{q-1}{n-q+1}\int_0^t t^{q-2}\{(1+s)^{l-1}-t\}^{n-q+1}dt.$$

Applying the formula (29) for $q-1$ instead of q, we get the formula (29) for q. $\square$

Hence from (28) and Lemma 15, we have

$$\begin{aligned}\text{LHS of (20)} &= f(1)\\ &= \frac{n!}{(q-1)!(n-q)!}\frac{1}{2\pi i}\oint_{s=0} s^{-n-1}(1+s)^{\mu-(l-1)n}\{(1+s)^l-1\}^q\\ &\quad\cdot\left[-\sum_{\nu=1}^{q}\frac{(q-1)!}{(n-q+1)_\nu(q-\nu)!}\{(1+s)^{l-1}-1\}^{n-q+\nu} + \frac{(q-1)!}{(n-q+1)_q}(1+s)^{n(l-1)}\right]ds\\ &= \frac{n!}{(q-1)!(n-q)!}\frac{1}{2\pi i}\oint_{s=0} s^{-n-1}(1+s)^{\mu}\{(1+s)^l-1\}^q\cdot\frac{(q-1)!}{(n-q+1)_q}ds \qquad (30)\end{aligned}$$

$$= \frac{1}{2\pi i}\oint_{s=0} s^{-n-1}(1+s)^{\mu}\{(1+s)^{l}-1\}^{q}ds. \tag{31}$$

Here the relation (30) follows from the Taylor expansions near $s=0$:

$$\{(1+s)^{l}-1\}^{q}\{(1+s)^{l-1}-1\}^{n-q+\nu} = As^{n+\nu}+\cdots,$$

where A is a constant.

On the other hand, we have

$$\begin{aligned}\text{RHS of (20)} &= (-1)^{q}\sum_{\nu=0}^{q}(-1)^{\nu}\binom{\mu+l\nu}{n}\binom{q}{\nu}\\ &= (-1)^{q}\frac{1}{2\pi i}\oint_{s=0}\sum_{\nu=0}^{q}(-1)^{\nu}s^{-n-1}(1+s)^{\mu+l\nu}\binom{q}{\nu}ds\\ &= (-1)^{q}\frac{1}{2\pi i}\oint_{s=0} s^{-n-1}(1+s)^{\mu}\{1-(1+s)^{l}\}^{q}ds\\ &= \frac{1}{2\pi i}\oint_{s=0} s^{-n-1}(1+s)^{\mu}\{(1+s)^{l}-1\}^{q}ds. \end{aligned}\tag{32}$$

Hence (20) has been proved. □

From now on we only consider the case where $\mu=(l-1)n$.

Definition 16. For each $J=\{j_1,\dots,j_q\}$, there exists an $N_q^{(n)}$ dimensional subspace W_J of $F_{(l-1)n}\Omega^{n-q}(\boldsymbol{C}^n)$ such that

$$F_{(l-1)n}\Omega^{n-q}(\boldsymbol{C}^n) = W_J \oplus \mathcal{F}^{n-q}_{(l-1)n}(J).$$

We also put $W_0 = F_{(l-1)n}\Omega^{n}(\boldsymbol{C}^n)$.

Then it is possible from Lemma 6 and Proposition 13 to make the following identification:

Corollary 17. *We have the isomorphism*

$$\rho:\ F_{(l-1)n}\Omega^{n} \cong W_0 \oplus \sum_{q=1}^{n}\sum_{J\subset[1,m],|J|=q} W_J. \tag{33}$$

Remark. $F_{\mu}\Omega^n$ coincides with the space spanned by

$$\varphi = \frac{f}{P_1P_2\cdots P_m}\varpi \quad (f\in \boldsymbol{C}[x]) \tag{34}$$

such that $\deg f \le \mu - n + lm$, where

$$\varpi = dx_1 \wedge \cdots \wedge dx_n.$$

As regards (34), there exist the unique $\varphi_0 \in W_0, \varphi_J \in W_J$ such that

$$\varphi = \frac{f}{P_1 \cdots P_m}\varpi = \varphi_0 + \sum_{q=1}^{\min(n,m)} \sum_{J,|J|=q} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_q}}{P_{j_q}} \wedge \varphi_J. \tag{35}$$

This is a partial fraction decomposition with the denominators $P_1, \ldots, P_m$.

3. Main Results.

We first prove the following

Theorem 18. *We have the isomorphism*

$$H^n(\Omega^{\cdot}, \nabla) \cong H^n(F_{(l-1)n}\Omega^{\cdot}, \nabla).$$

Proof. It is enough to prove the following two facts:

(i) For an arbitrary $\varphi \in \Omega^n$, there exists $\varphi^* \in F_{(l-1)n}\Omega^n$ such that

$$\varphi \sim \varphi^*.$$

(ii) Two arbitrary $\varphi, \varphi^* \in F_{(l-1)n}\Omega^n$ which are cohomologous to each other in $\Omega^{\cdot}$ are cohomologous in $F_{(l-1)n}\Omega^{\cdot}$.

About (i):
Since $\Omega^n = \bigcup_{\mu=(l-1)n}^{\infty} F_\mu\Omega^n$, there exists μ ($\mu \geq (l-1)n$) such that $\varphi \in F_\mu\Omega^n$.
By the formula (35) φ has the expression

$$\varphi = \varphi_0 + \sum_{q=1}^{\min(n,m)} \sum_{J\subset[1,m],|J|=q} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_q}}{P_{j_q}} \wedge \varphi_J, \tag{36}$$

where $\varphi_0 \in F_\mu\Omega^n(\boldsymbol{C}^n)$ and $\varphi_J \in F_\mu\Omega^{n-q}(\boldsymbol{C}^n)$.

Suppose that $\mu > (l-1)n$. By taking the homogeneous part of highest degree,

$$\overline{\varphi} = \overline{\varphi}_0 + \sum_{q=1}^{\min(n,m)} \sum_{J,|J|=q} \frac{d\overline{P}_{j_1}}{\overline{P}_{j_1}} \wedge \cdots \wedge \frac{d\overline{P}_{j_q}}{\overline{P}_{j_q}} \wedge \overline{\varphi}_J.$$

Owing to Proposition 3 there exists a homogeneous $\overline{\psi} \in F_\mu\Omega^{n-1}(\log \overline{D})$:

$$\overline{\psi} = \overline{\psi}_0 + \sum_{q=1}^{\min(n-1,m)} \sum_{J,|J|=q} \frac{d\overline{P}_{j_1}}{\overline{P}_{j_1}} \wedge \cdots \wedge \frac{d\overline{P}_{j_q}}{\overline{P}_{j_q}} \wedge \overline{\psi}_J$$

such that

$$\overline{\varphi} = \overline{\nabla}\,\overline{\psi}.$$

Put

$$\psi = \overline{\psi}_0 + \sum_{q=1}^{\min(n-1,m)} \sum_{J,|J|=q} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_q}}{P_{j_q}} \wedge \overline{\psi}_J.$$

Then

$$\varphi - \nabla\psi \in F_{\mu-1}\Omega^n.$$

By continuing this process we finally arrive at (i).

About (ii):

By assumption there exists $\psi \in F_\mu \Omega^{n-1}$ such that

$$\varphi - \varphi^* = \nabla\psi,$$

where ψ has by Lemma 4 the expression

$$\psi = \psi_0 + \sum_{q=1}^{\min(m,n-1)} \sum_{J,|J|=q} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_q}}{P_{j_q}} \wedge \psi_J \tag{37}$$

such that $\psi_0 \in F_\mu \Omega^{n-1}(\boldsymbol{C}^n)$, $\psi_J \in F_\mu \Omega^{n-q-1}(\boldsymbol{C}^n)$.

Suppose that $\mu > (l-1)n$. Then by taking the homogeneous part of highest degree we have

$$0 = \overline{\nabla}\,\overline{\psi}.$$

Due to Proposition 3 there exists $\chi \in F_\mu \Omega^{n-2}(\log \overline{D})$ such that

$$\overline{\psi} = \overline{\nabla}\,\overline{\chi}.$$

Hence $\psi - \nabla\overline{\chi} \in F_{\mu-1}\Omega^{n-1}$ and $\nabla\psi = \nabla(\psi - \nabla\overline{\chi})$. By continuing this process, we finally arrive at (ii). □

Remark. Theorem 18 may be generalized as the following conjecture under a weaker condition.

Let m polynomials P_k of degree l_k such that $l_1 \geq l_2 \geq \cdots \geq l_m \geq 1$ satisfy the two conditions $\mathcal{C}_1, \mathcal{C}_2$. We can similarly define the filtration F_μ for the logarithmic forms $\Omega^{\cdot}$ and ∇ as in Section 1. Then we have the isomorphism

$$H^n(\Omega^{\cdot}, \nabla) \cong H^n(F_\mu \Omega^{\cdot}, \nabla) \quad (\mu \geq (l_1 - 1)n).$$

It is evident that

$$\nabla F_{\mu,q}\Omega^{p-1} \subset F_{\mu,q+1}\Omega^{p}.$$

Moreover the following is true:

Proposition 19. *Suppose that* $\psi \in F_{(l-1)n,q}\Omega^{p-1}$ $(2 \leq p \leq n,\ 1 \leq q \leq p-1)$ *satisfies*

$$\nabla\psi \equiv 0 \quad \mathrm{mod}\, F_{(l-1)n,q}\Omega^{p}.$$

Then we have

$$\psi \equiv 0 \quad \mathrm{mod}\, \nabla F_{(l-1)n,q-1}\Omega^{p-2} + F_{(l-1)n,q-1}\Omega^{p-1},$$

i.e.,

$$\nabla^{-1}(F_{(l-1)n,q}\Omega^{p}) \cap F_{(l-1)n,q}\Omega^{p-1} = \nabla F_{(l-1)n,q-1}\Omega^{p-2} + F_{(l-1)n,q-1}\Omega^{p-1},$$

so that we have

$$\nabla F_{(l-1)n,q}\Omega^{p-1} \cap F_{(l-1)n,q}\Omega^{p} = \nabla F_{(l-1)n,q-1}\Omega^{p-1}. \tag{38}$$

We want to prove this Proposition by induction on m. Before proving it we give three Lemmas.

Lemma 20. *There exist* $\chi^{(0)} \in F_{(l-1)n,q-1}\Omega^{p-2}$, $\psi^{(1)} \in F_{(l-1)n,q}\Omega^{p-1}$:

$$\chi^{(0)} \equiv \sum_{J\subset[1,m-1],|J|=q-1} \frac{P_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{P_{j_{q-1}}}{P_{j_{q-1}}} \wedge \chi^{(0)}_{J} \quad \mathrm{mod}\, F_{(l-1)n,q-2}\Omega^{p-2}$$

$$\psi^{(1)} \equiv \sum_{J\subset[1,m-1],|J|=q} \frac{P_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{P_{j_q}}{P_{j_q}} \wedge \psi^{(1)}_{J} \quad \mathrm{mod}\, F_{(l-1)n,q-1}\Omega^{p-1}$$

such that

$$\psi \equiv \nabla\chi^{(0)} + \psi^{(1)} \quad \mathrm{mod}\, F_{(l-1)n,q-1}\Omega^{p-1}. \tag{39}$$

Proof. $\psi \in F_{(l-1)n,q}\Omega^{p-1}$ can be described as

$$\psi \equiv \sum_{J=\{j_1,\ldots,j_q\}\subset[1,m]} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_q}}{P_{j_q}} \wedge \psi_J \quad \mathrm{mod}\, F_{(l-1)n,q-1}\Omega^{p-1},$$

where $\psi_J \in F_{(l-1)n}\Omega^{p-1-q}(\boldsymbol{C}^n)$, so that

$$\nabla\psi \equiv \sum_{k=1}^{m} \lambda_k \frac{dP_k}{P_k} \wedge \sum_{J\subset[1,m],|J|=q} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_q}}{P_{j_q}} \wedge \psi_J \quad \mathrm{mod}\, F_{(l-1)n,q}\Omega^{p}.$$

Hence in the representation (2) for $\varphi = \nabla\psi$ and $K = \{j_1, \dots, j_{q+1}\} \subset [1, m]$,

$$\varphi_K = \sum_{\nu=1}^{q+1} (-1)^{\nu-1} \lambda_{j_\nu} \psi_{\partial_\nu K} \tag{40}$$

($\partial_\nu K$ means the deletion of the suffix j_ν from K). Suppose further that

$$\nabla\psi \equiv 0 \mod F_{(l-1)n,q}\Omega^p. \tag{41}$$

Lemma 6 implies for each K

$$\varphi_K = \sum_{\nu=1}^{q+1} \left(P_{j_\nu} \theta_{j_\nu;K} + dP_{j_\nu} \wedge \theta'_{j_\nu;K} \right) \in \mathcal{F}^{p-q-1}_{(l-1)n}(K), \tag{42}$$

where

$$\theta_{j_\nu;K} \in F_{(l-1)n-l}\Omega^{p-q-1}(\boldsymbol{C}^n), \quad \theta'_{j_\nu;K} \in F_{(l-1)n-l}\Omega^{p-q-2}(\boldsymbol{C}^n).$$

Case (i): $J \subset [1, m-1]$, $K = \{J, m\}$.
From (40), (42)

$$\begin{aligned} \varphi_K &= \sum_{\nu=1}^{q} (-1)^{\nu-1} \lambda_{j_\nu} \psi_{\partial_\nu J, m} + (-1)^q \lambda_m \psi_J \\ &\equiv P_m \theta_{m;K} + dP_m \wedge \theta'_{m;K} \mod \mathcal{F}^{p-q-1}_{(l-1)n}(J). \end{aligned}$$

Hence from (42)

$$(-1)^q \lambda_m \psi_J \equiv -\sum_{\nu=1}^{q} (-1)^{\nu-1} \lambda_{j_\nu} \psi_{\partial_\nu J, m} + \varphi_K + P_m \theta_{m;K} + dP_m \wedge \theta'_{m;K} \mod \mathcal{F}^{p-q-1}_{(l-1)n}(J). \tag{43}$$

Case (ii): $K \subset [1, m-1]$.
From (40), (42), (43)

$$\begin{aligned} \varphi_K &= \sum_{\nu=1}^{q+1} (-1)^{\nu-1} \lambda_{j_\nu} \psi_{\partial_\nu K} \\ &\equiv \frac{(-1)^q}{\lambda_m} \Bigg[\sum_{\nu=1}^{q+1} (-1)^{\nu-1} \lambda_{j_\nu} \Bigg\{ -\sum_{1 \le \kappa < \nu \le q} (-1)^{\kappa-1} \lambda_\kappa \psi_{\partial_\kappa \partial_\nu K, m} \\ &\qquad - (-1)^\kappa \lambda_\kappa \sum_{1 \le \nu \le \kappa \le q} \psi_{\partial_\nu \partial_\kappa K, m} + \varphi_{\partial_\nu K, m} \Bigg\} \Bigg] \mod \mathcal{F}^{p-q-1}_{(l-1)n}(K) \end{aligned}$$

$$\equiv 0 \quad \mathrm{mod}\, \mathcal{F}^{p-q-1}_{(l-1)n}(K). \tag{44}$$

On the other hand, from (38), (42), (43)

$$\begin{aligned}
\psi \equiv & \sum_{J\subset[1,m-1],|J|=q} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_q}}{P_{j_q}} \wedge \psi_J \\
& + \sum_{J\subset[1,m-1],|J|=q-1} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_{q-1}}}{P_{j_{q-1}}} \wedge \frac{dP_m}{P_m} \wedge \psi_{J,m} \quad \mathrm{mod}\, F_{(l-1)n,q-1}\Omega^{p-1} \\
& = \frac{(-1)^q}{\lambda_m} \sum_{J\subset[1,m-1],|J|=q} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_q}}{P_{j_q}} \wedge \left\{ \sum_{\nu=1}^{q} (-1)^\nu \lambda_{j_\nu} \psi_{\partial\nu J,m} + \varphi_{J,m} \right\} \\
& + \sum_{J\subset[1,m-1],|J|=q-1} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_{q-1}}}{P_{j_{q-1}}} \wedge \frac{dP_m}{P_m} \wedge \psi_{J,m} \\
& = \sum_{k=1}^{m} \lambda_k \frac{dP_k}{P_k} \wedge \chi^{(0)} + \psi^{(1)},
\end{aligned}$$

where

$$\chi^{(0)} = -\frac{(-1)^q}{\lambda_m} \sum_{J\subset[1,m-1],|J|=q-1} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_{q-1}}}{P_{j_{q-1}}} \wedge \psi_{J,m} \in F_{(l-1)n,q-1}\Omega^{p-2},$$

$$\psi^{(1)} = \frac{(-1)^q}{\lambda_m} \sum_{J\subset[1,m-1],|J|=q} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_q}}{P_{j_q}} \wedge \varphi_{J,m} \in F_{(l-1)n,q}\Omega^{p-1},$$

which shows Lemma 20. □

From the equality

$$\nabla\psi \equiv 0 \quad \mathrm{mod}\, F_{(l-1)n,q}\Omega^p,$$

the following Lemma is valid.

Lemma 21. *We have*

$$\nabla\psi^{(1)} \equiv 0 \quad \mathrm{mod}\, F_{(l-1)n,q}\Omega^p.$$

Namely if we write $\psi^{(1)}$ *as*

$$\psi^{(1)} \equiv \sum_{J\subset[1,m-1],|J|=q} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_q}}{P_{j_q}} \wedge \psi^{(1)}_J \quad \mathrm{mod}\, F_{(l-1)n,q-1}\Omega^{p-1},$$

then for $J = \{j_1, \ldots, j_q\} \subset [1, m-1]$ *we have*

$$\psi_J^{(1)} \equiv 0 \mod \mathcal{F}_{(l-1)n}^{p-1-q}(J, m) \tag{45}$$

and for $K = \{j_1, \ldots, j_{q+1}\} \subset [1, m-1]$ *we have*

$$\sum_{\nu=1}^{q+1} (-1)^{\nu-1} \lambda_{j_\nu} \psi_{\partial_\nu K}^{(1)} \equiv 0 \mod \mathcal{F}_{(l-1)n}^{p-q-1}(K). \tag{46}$$

Continuing this process we can conclude the following assertion:

Lemma 22. *There exist* $\chi^{(s)} \in F_{(l-1)n,q-1}\Omega^{p-2}$, $\psi^{(s)} \in F_{(l-1)n,q}\Omega^{p-1}$ $(s = 1, 2, 3, \ldots)$:

$$\chi^{(s)} = \sum_{J=\{j_1,\ldots,j_{q-1}\}\subset[1,m-s-1]} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_{q-1}}}{P_{j_{q-1}}} \wedge \chi_J^{(s)} \in F_{(l-1)n,q-1}\Omega^{p-2},$$

$$\psi^{(s)} = \sum_{J=\{j_1,\ldots,j_q\}\subset[1,m-s]} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_q}}{P_{j_q}} \wedge \psi_J^{(s)} \in F_{(l-1)n,q}\Omega^{p-1},$$

$$\psi_J^{(s)} \equiv 0 \mod \bigcap_{k=m-s+1}^{m} \mathcal{F}_{(l-1)n}^{p-q-1}(J, k),$$

such that

$$\nabla\psi^{(s)} \equiv 0 \mod F_{(l-1)n,q}\Omega^p, \tag{47}$$

$$\psi^{(s)} \equiv \nabla\chi^{(s)} + \psi^{(s+1)} \mod F_{(l-1)n,q-1}\Omega^{p-1}. \tag{48}$$

Proof of Proposition 19. From (39), (47), (48) we get

$$\psi \equiv \sum_{s=0}^{p-q+1} \nabla\chi^{(s)} \mod F_{(l-1)n,q-1}\Omega^{p-1}.$$

Since $\sum_{s=0}^{p-q+1} \chi^{(s)} \in F_{(l-1)n,q-1}\Omega^{p-2}$, Proposition 19 is proved. □

Proposition 23. *Suppose that* $\psi \in F_{(l-1)n,q}\Omega^{p-1}$ $(2 \le p \le n,\ 1 \le q \le p-1)$ *satisfies*

$$\nabla\psi = 0.$$

Then there exists $\chi \in F_{(l-1)n,q-1}\Omega^{p-2}$ *such that*

$$\psi = \nabla\chi,$$

i.e.,

$$\mathrm{Ker}\nabla \cap F_{(l-1)n,q}\Omega^{p-1} = \nabla F_{(l-1)n,q-1}\Omega^{p-2}. \tag{49}$$

Proof. Indeed from Proposition 19, ψ can be described as

$$\psi = \nabla\chi^{(0)} + \psi^{(1)} \quad (\chi^{(0)} \in F_{(l-1)n,q-1}\Omega^{p-2}, \psi^{(1)} \in F_{(l-1)n,q-1}\Omega^{p-1}).$$

By hypothesis $\nabla\psi^{(1)} = 0$. By the same Proposition we have

$$\psi^{(1)} = \nabla\chi^{(1)} + \psi^{(2)} \quad (\chi^{(1)} \in F_{(l-1)n,q-2}\Omega^{p-2}, \psi^{(2)} \in F_{(l-1)n,q-2}\Omega^{p-1}).$$

Repeating this process there exist $\chi^{(s)} \in F_{(l-1)n,q-s-1}\Omega^{p-2}, \psi^{(s)} \in F_{(l-1)n,q-s}\Omega^{p-1}$ such that

$$\begin{aligned} \nabla\psi^{(s)} &= 0, \\ \psi^{(s)} &= \nabla\chi^{(s)} + \psi^{(s+1)} \quad (s = 1, 2, 3, \ldots). \end{aligned}$$

Since $\psi^{(s)} = 0$ $(s \geq p - q)$, we have

$$\psi^{(p-q-1)} = \nabla\chi^{(p-q-1)}.$$

Thus setting $\chi = \sum_{s=0}^{p-q-1} \chi^{(s)}$, we have

$$\psi = \nabla\chi \equiv 0 \mod F_{(l-1)n,q-1}\Omega^{p-1}. \qquad \square$$

Corollary 24. *For $1 \leq q \leq p - 1$, we have*

$$\dim \mathrm{Ker}\nabla \cap F_{(l-1)n,q}\Omega^{p-1} = \sum_{k=0}^{q-1}(-1)^k \dim F_{(l-1)n,q-1-k}\Omega^{p-2-k}, \tag{50}$$

$$\dim \nabla F_{(l-1)n,q}\Omega^{p-1} = \sum_{k=0}^{q}(-1)^k \dim F_{(l-1)n,q-k}\Omega^{p-1-k}. \tag{51}$$

Remark. Theorem 18, Proposition 19, Proposition 23, Corollary 24 are still true for μ $(\mu \geq (l-1)n)$ instead of $\mu = (l-1)n$, seeing that the above proofs can proceed in the same way. In the sequel we shall only consider the case $\mu = (l-1)n$.

It is convenient to define $F_{(l-1)n,q}\Omega^p$ for $q = -1$ as follows:

Definition 25.

$$\begin{aligned} &F_{(l-1)n,-1}\Omega^p = \{\psi \in F_{(l-1)n,0}\Omega^p | \nabla\psi \in F_{(l-1)n,0}\Omega^{p+1}\} \quad (0 \leq p \leq n-1), \\ &F_{(l-1)n,-1}\Omega^n = \nabla F_{(l-1)n,0}\Omega^{n-1} \cap F_{(l-1)n,0}\Omega^n. \end{aligned}$$

By definition we have

$$\nabla F_{(l-1)n,-1}\Omega^p = \nabla F_{(l-1)n,0}\Omega^p \cap F_{(l-1)n,0}\Omega^{p+1} \quad (0 \leq p \leq n-1).$$

Hence (38) is also true for $q=0$.

LEMMA 26. *Suppose* $0 \le p \le n-1$.

(i) *Case* $(l-1)n - lm < 0$, *even more, case* $m \ge n$. *We always have*

$$F_{(l-1)n,-1}\Omega^p \cong \{0\}. \tag{52}$$

(ii) *Case* $(l-1)n - lm \ge 0$. *If* $p \le m$ *then* (52) *does not hold, while if* $p > m$ *then* (52) *holds true for* $p > (l-1)(n-m)$, *but it does not hold for* $p \le (l-1)(n-m)$.

PROOF. Suppose first that $(l-1)n - lm < 0$. $\psi \in F_{(l-1)n,-1}\Omega^p$ $(0 \le p \le n-1)$ can be described as

$$\psi = P_1 \cdots P_m \left(\psi_0 + \sum_{q=1}^{p} \sum_{J=\{j_1,\dots,j_q\} \subset [1,m], |J|=q} \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_q}}{P_{j_q}} \wedge \psi_J \right),$$

where $\psi_J \in F_{(l-1)n-lm}\Omega^{p-q}(\boldsymbol{C})$, i.e., $\deg \psi_J \le (l-1)n - lm - p + q$. Hence ψ_0 and ψ_J vanish for all J.

On the other hand, suppose $(l-1)n - lm \ge 0$. If $p \le m$, then for $|J| = p$, ψ_J are possibly nonzero. If $p > m$, then for $|J| = m$, (52) holds true or does not hold according as $(l-1)(n-m) - p < 0$ or $(l-1)(n-m) - p \ge 0$. □

DEFINITION 27. We can find a subspace V_q of $F_{(l-1)n,q}\Omega^n$ such that

$$F_{(l-1)n,q}\Omega^n = V_q \oplus \left(\nabla F_{(l-1)n,q-1}\Omega^{n-1} + F_{(l-1)n,q-1}\Omega^n \right) \quad (1 \le q \le \min(n,m)), \tag{53}$$

$$F_{(l-1)n,0}\Omega^n = V_0 \oplus \nabla F_{(l-1)n,0}\Omega^{n-1} \cap F_{(l-1)n,0}\Omega^n, \tag{54}$$

i.e.,

$$V_q \cong \frac{F_{(l-1)n,q}\Omega^n}{\nabla F_{(l-1)n,q-1}\Omega^{n-1} + F_{(l-1)n,q-1}\Omega^n} \quad (0 \le q \le \min(n,m)).$$

We note that, if $m \le n$,

$$V_m \cong \{0\}.$$

In fact, an arbitrary $\varphi \in F_{(l-1)n,m}\Omega^n$ can be expressed by

$$\frac{dP_1}{P_1} \wedge \cdots \wedge \frac{dP_m}{P_m} \wedge \varphi_{12\cdots m}$$

for $\varphi_{12\cdots m} \in F_{(l-1)n}\Omega^{n-m}(\boldsymbol{C}^n)$. We may assume $\lambda_1 \ne 0$. If we take

$$\psi = \frac{1}{\lambda_1} \frac{dP_2}{P_2} \wedge \cdots \wedge \frac{dP_m}{P_m} \wedge \varphi_{12\cdots m} \in F_{(l-1)n,m-1}\Omega^{n-1},$$

then $\varphi - \nabla\psi \equiv 0 \mod F_{(l-1)n,m-1}\Omega^n$.

Theorem 28. *We have the isomorphism*

$$\tilde{\rho}:\ H^n(F_{(l-1)n}\Omega^{\cdot},\nabla) \cong \bigoplus_{q=0}^{\min(n,m)} V_q,$$

so that the commutative diagram:

$$\begin{array}{ccc} F_{(l-1)n}\Omega^n & \xrightarrow{\ \rho\ } & W_0 \oplus \sum_{q=1}^{\min(n,m)} \sum_{J,|J|=q} W_J \\ \downarrow{\scriptstyle \mathcal{H}} & & \downarrow{\scriptstyle \mathcal{H}} \\ H^n(F_{(l-1)n}\Omega^{\cdot},\nabla) & \xrightarrow{\ \tilde{\rho}\ } & \bigoplus_{q=0}^{\min(n,m)} V_q \end{array}$$

where $\mathcal{H}$ are the projections and the equality

$$\dim V_q = (-1)^q\left(\sum_{\nu=1}^{q}(-1)^{\nu}\binom{m}{\nu}N_{\nu}^{(n-q+\nu)} + \tilde{N}_0^{(n-q)}\right) \quad (0 \le q \le \min(n,m)) \tag{55}$$

hold where

$$\tilde{N}_0^{(n-q)} = N_0^{(n-q)} - \dim F_{(l-1)n,-1}\Omega^{n-q}.$$

Proof. Indeed, for $0 \le q \le \min(n,m)$,

$$\begin{aligned} \dim V_q &= \dim F_{(l-1)n,q}\Omega^n - \dim \nabla F_{(l-1)n,q-1}\Omega^{n-1} - \dim F_{(l-1)n,q-1}\Omega^n \\ &\quad + \dim(\nabla F_{(l-1)n,q-1}\Omega^{n-1} \cap F_{(l-1)n,q-1}\Omega^n) \\ &= \dim F_{(l-1)n,q}\Omega^n - \dim \nabla F_{(l-1)n,q-1}\Omega^{n-1} - \dim F_{(l-1)n,q-1}\Omega^n \\ &\quad + \dim \nabla F_{(l-1)n,q-2}\Omega^{n-1} \\ &= \dim F_{(l-1)n,q}\Omega^n - \dim F_{(l-1)n,q-1}\Omega^n \\ &\quad - \sum_{k=0}^{q-1}(-1)^k\big(\dim F_{(l-1)n,q-1-k}\Omega^{n-1-k} - \dim F_{(l-1)n,q-2-k}\Omega^{n-1-k}\big) \\ &= \sum_{k=0}^{q}(-1)^k\big(\dim F_{(l-1)n,q-k}\Omega^{n-k} - \dim F_{(l-1)n,q-k-1}\Omega^{n-k}\big) \\ &= \sum_{k=0}^{q-1}(-1)^k N_{q-k}^{(n-k)}\binom{m}{q-k} + (-1)^q\tilde{N}_0^{(n-q)}. \end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{q=0}^{\min(n,m)} \dim V_q &= \sum_{q=0}^{\min(n,m)} \big(\dim F_{(l-1)n,q}\Omega^n - \dim \nabla F_{(l-1)n,q-1}\Omega^{n-1} \\ &\qquad - \dim F_{(l-1)n,q-1}\Omega^n + \dim(\nabla F_{(l-1)n,q-1}\Omega^{n-1} \cap F_{(l-1)n,q-1}\Omega^n)\big) \\ &= \sum_{q=0}^{\min(n,m)} \big(\dim F_{(l-1)n,q}\Omega^n - \dim \nabla F_{(l-1)n,q-1}\Omega^{n-1} \\ &\qquad - \dim F_{(l-1)n,q-1}\Omega^n + \dim \nabla F_{(l-1)n,q-2}\Omega^{n-1}\big) \\ &= \dim F_{(l-1)n}\Omega^n - \dim \nabla F_{(l-1)n}\Omega^{n-1} \\ &= \dim H^n(F_{(l-1)n}\Omega^{\cdot}, \nabla). \end{aligned}$$ □

Example 1. Case $n = 1$.
We have

$$\dim V_0 = l - 1, \quad \dim V_1 = l(m-1),$$
$$H^1(\Omega^{\cdot}, \nabla) \cong H^1(F_{l-1}\Omega^{\cdot}, \nabla) \cong V_0 \oplus V_1, \quad \dim H^1(\Omega^{\cdot}, \nabla) = lm - 1.$$

Example 2. Case $m = 1$.
We have

$$V_k \cong \{0\} \quad (1 \le k \le n),$$
$$H^n(\Omega^{\cdot}, \nabla) \cong V_0, \quad \dim V_0 = \tilde{N}_0^{(n)} = (l-1)^n.$$

In fact, it follows from Proposition 3 and Lemma 4 that

$$\begin{aligned}\tilde{N}_0^{(n)} &= N_0^{(n)} - \dim F_{(l-1)n,-1}\Omega^n = N_0^{(n)} - \sum_{\nu=1}^{n} (-1)^{\nu-1} \dim F_{(l-1)n-\nu l,0}\Omega^{n-\nu} \\ &= (l-1)^n,\end{aligned}$$

since

$$\dim F_{(l-1)n-\nu l,0}\Omega^{n-\nu} = \binom{(l-1)n + \nu(1-l)}{n}.$$

Example 3. Case $l = 1$.
We have

$$V_k \cong \{0\} \quad (0 \le k \le n-1),$$
$$H^n(\Omega^{\cdot}, \nabla) \cong H^n(F_0\Omega^{\cdot}, \nabla) \cong V_n, \quad \dim V_n = \binom{m-1}{n}.$$

Example 4. Case $l = 2$.
In view of Lemma 26, we have $N_0^{(p)} = \tilde{N}_0^{(p)}$ for $p \ge n - m + 1$ and

$$N_0^{(n)}=1,\quad N_0^{(n-1)}=n(n+1),\quad N_0^{(n-2)}=\frac{1}{4}(n+2)(n+1)n(n-1),$$
$$N_1^{(n)}=\frac{1}{2}n(n+3),\quad N_1^{(n-1)}=\frac{1}{24}n(n-1)(n+1)(3n+14),$$
$$N_2^{(n)}=\frac{1}{24}n(n-1)(n^2+11n+22).$$

Suppose first (i) $m\le n$.
Theorem 28 shows that

$$\dim V_q=\sum_{\nu=0}^{q}(-1)^\nu\binom{m}{q-\nu}N_{q-\nu}^{(n-\nu)}\quad(0\le q\le m-1),$$
$$\dim V_m=0.$$

For example,

$$\cdot m=1:\ \dim V_0=1,\quad \dim H^n(\Omega^\cdot,\nabla)=1.$$
$$\cdot m=2:\ \dim V_0=1,\quad \dim V_1=2n,\quad \dim H^n(\Omega^\cdot,\nabla)=2n+1.$$
$$\cdot m=3:\ \dim V_0=1,\quad \dim V_1=\frac{1}{2}n(n+7),\quad \dim V_2=\frac{3}{2}n(n-1),$$
$$\dim H^n(\Omega^\cdot,\nabla)=2n^2+2n+1.$$

Suppose next (ii) $m\ge n+1$.
Then it follows that

$$\dim V_q=\sum_{\nu=0}^{q}(-1)^\nu\binom{m}{q-\nu}N_{q-\nu}^{(n-\nu)}\quad(0\le q\le n),$$
$$\dim V_q=0\quad(q\ge n+1).$$

In particular, for $m=n+1$,

$$\dim V_0=1,$$
$$\dim V_1=\frac{1}{2}n(n+1)^2,$$
$$\dim V_2=\frac{1}{48}n(n+1)^2(n-1)(n^2+4n-4),$$

$\dim V_k$ is a polynomial in n of degree $3k$, or $3(n-k)+1$,

$$\dim V_{n-1}=\frac{1}{6}n(n+1)(n^2+3n-1),$$
$$\dim V_n=n+1,$$

and from Proposition 1 and Theorem 18,

$$\dim H^n(\Omega^\cdot, \nabla) = \dim H^n(F_n\Omega^\cdot, \nabla) = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{(n+1)_{n-\nu}}{(n-\nu)!}.$$

For example, we have the decomposition formula into V_q:

$\cdot\, n = 1: \;\dim H^1(F_1\Omega^\cdot, \nabla) = 1 + 2 = 3.$

$\cdot\, n = 2: \;\dim H^2(F_2\Omega^\cdot, \nabla) = 1 + 9 + 3 = 13.$

$\cdot\, n = 3: \;\dim H^3(F_3\Omega^\cdot, \nabla) = 1 + 24 + 34 + 4 = 63.$

$\cdot\, n = 4: \;\dim H^4(F_4\Omega^\cdot, \nabla) = 1 + 50 + 175 + 90 + 5 = 321.$

4. Gauss-Manin Connection.

We take the multiplicative function

$$\Phi(x) = \prod_{k=1}^{m} P_k^{\lambda_k}(x).$$

The integral of $\Phi\varphi$ attached to $\varphi \in F_{(l-1)n}\Omega^n$ over a twisted n dimensional cycle $\mathfrak{z}$ in $\mathcal{M}$ can be defined as a pairing between the cohomology class $[\varphi] \in H^n(F_{(l-1)n}\Omega^n, \nabla)$ and the homology class $[\mathfrak{z}]$ of $\mathfrak{z}$:

$$\langle \varphi, \mathfrak{z} \rangle = \int_{\mathfrak{z}} \Phi\varphi$$

which is abbreviated by $\langle \varphi \rangle$ in the sequel (see [**3**] for details).

Theorem 28 shows that there exists the unique element $\mathcal{H}_q(\varphi) \in V_q$ such that

$$\varphi \sim \mathcal{H}(\varphi) = \sum_{q=0}^{\min(n,m)} \mathcal{H}_q(\varphi) \in \bigoplus_{q=0}^{\min(n,m)} V_q. \tag{56}$$

We fix $\varphi \in V_q$ as

$$\varphi = \frac{dP_{j_1}}{P_{j_1}} \wedge \cdots \wedge \frac{dP_{j_q}}{P_{j_q}} \wedge \varphi_J \quad (J = \{j_1, \ldots, j_q\}, \varphi_J \in F_{(l-1)n}\Omega^{n-q}(\boldsymbol{C}^n)). \tag{57}$$

We want to derive the differentiation formulae for $\langle \varphi \rangle$ with respect to the coefficients of P_k. We may assume $k = 1$ without losing generality.

Suppose that $P_1(x)$ has the expression:

$$P_1(x) = \sum_{\boldsymbol{\nu}=(\nu_1,\ldots,\nu_n), |\boldsymbol{\nu}| \le l} a_{\boldsymbol{\nu}} x^{\boldsymbol{\nu}}.$$

We assume for simplicity that any φ_J does not depend on $a_{\boldsymbol{\nu}}$.

(i) Case $1 \notin J$.
Then we have

$$\frac{\partial}{\partial a_{\boldsymbol{\nu}}}\langle\varphi\rangle = \lambda_1\Big\langle\frac{x^{\boldsymbol{\nu}}}{P_1}\varphi\Big\rangle = \lambda_1\Big\langle\frac{x^{\boldsymbol{\nu}}}{P_1}\frac{dP_{j_1}}{P_{j_1}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge\varphi_J\Big\rangle.$$

Seeing that deg $\varphi_J \le (l-2)n+q$ ($i.e.$, $\mathrm{tdeg}\,\varphi_J \le (l-1)n$), we have

$$\frac{x^{\boldsymbol{\nu}}}{P_1}\varphi \in F_{(l-1)n}\Omega^n.$$

Hence we have

$$\frac{x^{\boldsymbol{\nu}}}{P_1}\varphi \sim \mathcal{H}\Big(\frac{x^{\boldsymbol{\nu}}}{P_1}\varphi\Big),$$

that is,

$$\frac{\partial}{\partial a_{\boldsymbol{\nu}}}\langle\varphi\rangle = \lambda_1\Big\langle\mathcal{H}\Big(\frac{x^{\boldsymbol{\nu}}}{P_1}\varphi\Big)\Big\rangle, \tag{58}$$

where $\mathcal{H}((x^{\boldsymbol{\nu}}/P_1)\varphi)$ belongs to $V_0 \oplus V_1 \oplus \cdots \oplus V_{q+1}$.

(ii) Case $1 \in J$.
We may assume that $j_1 = 1$, i.e., $J = \{1, j_2, \ldots, j_q\}$. Then we have

$$\begin{aligned}\frac{\partial}{\partial a_{\boldsymbol{\nu}}}\langle\varphi\rangle = (\lambda_1 - 1)&\Big\langle x^{\boldsymbol{\nu}}\frac{dP_1}{P_1^2}\wedge\frac{dP_{j_2}}{P_{j_2}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge\varphi_J\Big\rangle\\ &+\Big\langle\frac{d(x^{\boldsymbol{\nu}})}{P_1}\wedge\frac{dP_{j_2}}{P_{j_2}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge\varphi_J\Big\rangle.\end{aligned} \tag{59}$$

On the other hand, if we take

$$\psi = \frac{x^{\boldsymbol{\nu}}}{P_1}\frac{dP_{j_2}}{P_{j_2}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge\varphi_J,$$

then

$$\begin{aligned}0 = \langle\nabla\psi\rangle = (\lambda_1 - 1)&\Big\langle x^{\boldsymbol{\nu}}\frac{dP_1}{P_1^2}\wedge\frac{dP_{j_2}}{P_{j_2}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge\varphi_J\Big\rangle\\ &+\sum_{k\notin J}\lambda_k\Big\langle\frac{x^{\boldsymbol{\nu}}}{P_1}\frac{dP_k}{P_k}\wedge\frac{dP_{j_2}}{P_{j_2}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge\varphi_J\Big\rangle\\ &+(-1)^{q-1}\Big\langle\frac{1}{P_1}\frac{dP_{j_2}}{P_{j_2}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge d(x^{\boldsymbol{\nu}}\varphi_J)\Big\rangle.\end{aligned} \tag{60}$$

By subtracting (60) from (59) side by side, we get

$$\begin{aligned}\frac{\partial}{\partial a_{\boldsymbol{\nu}}}\langle\varphi\rangle &= -\sum_{k\notin J}\lambda_k\left\langle \frac{x^{\boldsymbol{\nu}}}{P_1}\frac{dP_k}{P_k}\wedge\frac{dP_{j_2}}{P_{j_2}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge\varphi_J\right\rangle \\ &\quad -(-1)^{q-1}\left\langle\frac{1}{P_1}\frac{dP_{j_2}}{P_{j_2}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge d(x^{\boldsymbol{\nu}}\varphi_J)\right\rangle \\ &\quad +\left\langle\frac{d(x^{\boldsymbol{\nu}})}{P_1}\wedge\frac{dP_{j_2}}{P_{j_2}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge\varphi_J\right\rangle \\ &= -\sum_{k\notin J}\lambda_k\left\langle \frac{x^{\boldsymbol{\nu}}}{P_1}\frac{dP_k}{P_k}\wedge\frac{dP_{j_2}}{P_{j_2}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge\varphi_J\right\rangle \\ &\quad +(-1)^q\left\langle\frac{x^{\boldsymbol{\nu}}}{P_1}\wedge\frac{dP_{j_2}}{P_{j_2}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge d\varphi_J\right\rangle.\end{aligned}$$

As $(x^{\boldsymbol{\nu}}/P_1)(dP_k/P_k)\wedge dP_{j_2}/P_{j_2}\wedge\cdots\wedge dP_{j_q}/P_{j_q}\wedge\varphi_J$ and $(x^{\boldsymbol{\nu}}/P_1)(dP_{j_2}/P_{j_2})\wedge\cdots\wedge dP_{j_q}/P_{j_q}\wedge d\varphi_J$ both belong to $F_{(l-1)n}\Omega^n$, we get the formula

$$\begin{aligned}\frac{\partial}{\partial a_{\boldsymbol{\nu}}}\langle\varphi\rangle &= -\sum_{k\notin J}\lambda_k\left\langle\mathcal{H}\left(\frac{x^{\boldsymbol{\nu}}}{P_1}\frac{dP_k}{P_k}\wedge\frac{dP_{j_2}}{P_{j_2}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge\varphi_J\right)\right\rangle \\ &\quad +(-1)^q\left\langle\mathcal{H}\left(\frac{x^{\boldsymbol{\nu}}}{P_1}\frac{dP_{j_2}}{P_{j_2}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge d\varphi_J\right)\right\rangle,\end{aligned}\tag{61}$$

where

$$\mathcal{H}\left(\frac{x^{\boldsymbol{\nu}}}{P_1}\frac{dP_k}{P_k}\wedge\frac{dP_{j_2}}{P_{j_2}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge\varphi_J\right),\quad \mathcal{H}\left(\frac{x^{\boldsymbol{\nu}}}{P_1}\frac{dP_{j_2}}{P_{j_2}}\wedge\cdots\wedge\frac{dP_{j_q}}{P_{j_q}}\wedge d\varphi_J\right)$$

both belong to $V_0\oplus\cdots\oplus V_{\min(q+1,m)}$.

The differentiation with respect to the other coefficients of P_k can be written similarly.

In this way we have proved

THEOREM 29. *The differentiations for φ with respect to the coefficients of each P_k preserves $F_{(l-1)n}\Omega^n$. Therefore we can express the Gauss–Manin connection for the integral $\langle\varphi,\mathfrak{z}\rangle$ in the form* (58), (61) *through the projection $\mathcal{H}$.*

If we take, as a basis of V_q, $e_1^{(q)},\dots,e_{\kappa_q}^{(q)}$ ($\kappa_q=\dim V_q$), then the above Theorem shows that the differential of $\langle e_\nu^{(q)}\rangle$ with respect to the coefficients $\mathfrak{a}$ of the polynomials $P_1,\dots,P_m$ satisfies Gauss–Manin connection

$$d_{\mathfrak{a}}\langle e_\nu^{(q)}\rangle=\sum_{r=0}^{\min(q+1,m)}\sum_{\iota=1}^{\kappa_r}\omega_{r,\nu}^{(q,\iota)}\langle e_\iota^{(r)}\rangle,$$

where $(\omega_{r,\nu}^{(q,\iota)})$ denotes a suitable matrix valued (with values in $\mathfrak{gl}_\kappa(\boldsymbol{C})$, $\kappa = \sum_{q=0}^{\min(n,m)} \kappa_q$) rational differential 1-form over the field of the coefficients $\mathfrak{a}$.

References

[1] K. Aomoto, Les équations aux différences linéaires et les intégrales des fonctions multiformes, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **22** (1975), 271–297.

[2] K. Aomoto, Une correction et un complément à l'article "Les équations aux différences linéaires et les intégrales des fonctions multiformes", J. Fac. Sci. Univ. Tokyo Sect. IA Math., **26** (1979), 519–523.

[3] K. Aomoto and M. Kita, Theory of Hypergeometric Functions, Springer Monogr. Math., Springer-Verlag, 2011.

[4] K. Aomoto, M. Kita, P. Orlik and H. Terao, Twisted de Rham cohomology groups of logarithmic forms, Adv. Math., **128** (1997), 119–152.

[5] D. Eisenbud, The Geometry of Syzygies, Grad. Texts in Math., **229**, Springer-Verlag, 2005.

[6] W. Gröbner, Moderne Algebraische Geometrie, Springer-Verlag, 1949.

[7] M. Kita and M. Noumi, On the structure of cohomology groups attached to the integral of certain many-valued analytic functions, Japan J. Math. (N.S.), **9** (1983), 113–157.

[8] K. Saito, On a generalization of de-Rham lemma, Ann. Inst. Fourier (Grenoble), **26** (1976), 165–170.

Kazuhiko Aomoto
5-1307 Hara, Tenpaku-ku
Nagoya-shi 468-0015, Japan
E-mail: kazuhiko@aba.ne.jp

Yoshinori Machida
Numazu College of Technology
3600 Ooka, Numazu-shi
Shizuoka 410-8501, Japan
E-mail: machida@numazu-ct.ac.jp