Symmetries of order four on K3 surfaces

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Abstract. In this paper we study automorphisms of order four on K3 surfaces. We give a classification of the non-symplectic ones when either the square of the automorphism is symplectic, or its fixed locus contains a curve of positive genus, or its fixed locus contains at least a curve and all the curves fixed by its square are rational. We provide partial results in the other cases.

Introduction.

Let X be a K3 surface over $\mathbb{C}$ with an order four automorphism. Such automorphism acts on the one-dimensional vector space $H^{2,0}(X)$ of holomorphic two-forms of X either as the identity, minus the identity or as the multiplication by $\pm i$. Accordingly, the automorphism is called symplectic, with symplectic square or purely non-symplectic. Symplectic automorphisms of finite order have been investigated by several authors, their fixed locus only contains isolated points and their action on the K3 lattice $H^2(X,\mathbb{Z})$ is known to be independent from the surface (cf. [**12**], [**13**], [**23**]). Non-symplectic automorphisms have been studied in [**20**], [**23**] (see also [**36**] for a survey on the topic) and the fixed locus has been identified if the order is prime in [**2**], [**3**], [**32**]. In [**33**] Taki classified order four non-symplectic automorphisms acting as the identity on the Picard lattice of the surface. Moreover, Schütt [**28**] studied the special case when the transcendental lattice of the surface has rank four.

This paper mainly deals with purely non-symplectic automorphisms σ of order four under the assumption that their square is the identity on the Picard lattice. By the Torelli type theorem, this holds for the generic element of the family of K3 surfaces carrying an order four non-symplectic automorphism with a given action on its second cohomology group. The fixed locus $\mathrm{Fix}(\sigma)$ of such an automorphism σ is the disjoint union of smooth curves and points. We give a complete classification of $\mathrm{Fix}(\sigma)$ when either it contains a curve of positive genus or it contains a curve and all the curves fixed by σ^2 are rational. More precisely, we denote by r and l the ranks of the eigenspaces of σ^* in $H^2(X,\mathbb{Z})$ with eigenvalues 1 and -1, by n and k the number of isolated points and of smooth rational curves in $\mathrm{Fix}(\sigma)$ and by $2a$ the number of smooth curves fixed by σ^2 and interchanged by σ. We prove the following result.

THEOREM 0.1.   *Let σ be a purely non-symplectic order four automorphism on a K3 surface X such that $(\sigma^*)^2$ acts identically on $\mathrm{Pic}(X)$. Then*:

$$n = 2\alpha + 4, \quad r - l = 4\alpha + 2,$$

where $\alpha = \sum_{C \subset \mathrm{Fix}(\sigma)} 1 - g(C)$. Moreover the following hold.

- *If $\mathrm{Fix}(\sigma)$ contains a curve of genus $g = 1$, then all the other fixed curves are rational and we have the following possibilities for (r, k, a) :*

$$(6, 0, 0), \ (7, 0, 0), \ (10, 1, 0), \ (8, 0, 1), \ (9, 0, 2), \ (10, 0, 3).$$

- *If $\mathrm{Fix}(\sigma)$ contains a curve of genus $g > 1$, then all the other fixed curves are rational and we have the following possibilities for (r, k, a, g) :*

$$(1, 0, 0, 3), \ (4, 0, 0, 2), \ (2, 0, 1, 3), \ (5, 0, 1, 2), \ (6, 0, 2, 2),$$

  where all the cases occur.
- *If $\mathrm{Fix}(\sigma)$ contains a curve of genus $g = 0$ and all the curves fixed by σ^2 are rational, then we have the following possibilities for (r, k, a) :*

$$(10, 1, 0), \ (13, 2, 0), \ (11, 1, 1), \ (16, 3, 0),$$
$$(14, 2, 1), \ (12, 1, 2), \ (19, 4, 0), \ (13, 1, 3),$$

  where all the cases occur.

We prove Theorem 0.1 in several steps, mainly in Theorems 3.1, 4.1 and 5.1. We give examples showing the existence in Examples 3.2, 4.2, 4.3, 5.3. In the cases not considered in the theorem, i.e. when either $\mathrm{Fix}(\sigma)$ only contains isolated points or σ^2 fixes a curve of positive genus which is not fixed by σ, we provide partial results and we give several examples. More precisely, we show the following theorem, where we use the same notation as before.

THEOREM 0.2.   *Let σ be a purely non-symplectic order four automorphism on a K3 surface X such that $(\sigma^*)^2$ acts identically on $\mathrm{Pic}(X)$. Then the following hold.*

- *If σ^* acts as the identity on $\mathrm{Pic}(X)$, then the fixed locus of σ only contains smooth rational curves and points, σ^2 fixes a curve of genus $g \geq 1$ and we have the following possibilities for (r, k, g) :*

$$(2, 0, 10), \ (2, 0, 9), \ (6, 1, 7), \ (6, 1, 6), \ (10, 2, g), \ 3 \leq g \leq 6,$$
$$(14, 3, 3), \ (14, 3, 2), \ (18, 4, 2), \ (18, 4, 1),$$

  where all the cases occur.
- *If $\mathrm{Fix}(\sigma)$ is zero-dimensional, then it contains exactly 4 points. If $l > 0$, then the possible invariants of $\mathrm{Fix}(\sigma^2)$ are given in Table 6 and we have $3 \leq r \leq 11$, $a \leq 4$,*

$g \leq 8$, where g is the highest genus of a curve fixed by σ^2.

• *If we are not in one of the previous cases or in one of the cases of Theorem* 0.1, *i.e.* Fix(σ) *contains only* $k > 0$ *rational curves and isolated fixed points, the highest genus* g *of a curve fixed by* σ^2 *is positive and* $l > 0$, *then*
  – *if* $g > 1$ *there are* 63 *possible cases with* $k \leq 3$, $g \leq 7$ *and* $a \leq 4$;
  – *if* $g = 1$ *we have the following possibilities for* (r, k, a) :

$$(9, 1, 0), \ (12, 2, 0), \ (15, 3, 0), \ (10, 1, 1), \ (10, 1, 0).$$

We prove this result in Theorems 6.1, 7.1, 8.1 and 8.4 and give Examples in 6.3, 6.4, 7.2, 7.3, 8.2, 8.3, 8.5.

Finally, in Proposition 2 we consider the case when the automorphism has symplectic square.

The study of non-symplectic automorphisms of order four is interesting in relation with the Borcea–Voisin construction of Calabi–Yau varieties and the investigation of Mirror–Symmetry (cf. [**5**], [**35**]). In fact Borcea and Voisin consider the product between a K3 surface with a non-symplectic automorphism of order 2,3,4 or 6, and an elliptic curve with an automorphism of the same order. A resolution of a quotient variety is then a Calabi–Yau threefold. In [**11**] Garbagnati used non-symplectic automorphisms of order four to give examples of Calabi–Yau threefolds by means of this construction.

We now give a short description of the paper's sections. In Section 1 we give a general description of the fixed locus of σ. By means of Lefschetz's formulas we provide relations between the invariants n, k, g and the ranks of the eigenspaces of σ^* on the vector space $H^2(X, \mathbb{C})$.

In Section 2 we study elliptic fibrations $\pi : X \to \mathbb{P}^1$ such that σ preserves each fiber of π. In Corollary 1 the configuration of the singular fibers, which are of Kodaira type III, I$_0^*$ or III*, is related to the structure of the fixed locus of σ.

In Section 3 we assume that σ fixes pointwisely an elliptic curve E. In Theorem 3.1 we describe the singular fibers of the elliptic fibration with fiber E and the corresponding structure of the fixed locus of σ.

In Sections 4 and 5 we classify the case when the fixed locus of σ either contains a curve of positive genus, or it contains a rational curve and σ^2 fixes only rational curves.

In Section 6 we assume that σ^* is the identity on the Picard lattice and we give an independent proof of [**33**, Proposition 4.3].

In Section 7 we consider the case when σ only fixes isolated points and we provide families of examples.

In Section 8, we study the case when σ only fixes isolated points and rational curves, and σ^2 fixes a curve of positive genus.

1. The fixed locus.

Let X be a K3 surface with a *non-symplectic* automorphism σ of order four, i.e. such that the action of σ^* on the vector space $H^{2,0}(X) = \mathbb{C}\omega_X$ of holomorphic two-forms is not trivial. We call the automorphism *purely non-symplectic* if $\sigma^*\omega_X = \pm i\omega_X$, i.e. if its square is a non-symplectic involution.

We will denote by r, l, m the rank of the eigenspace of σ^* in $H^2(X, \mathbb{C})$ relative to the eigenvalues $1, -1$ and i respectively. Moreover, let

$$S(\sigma) = \{x \in H^2(X, \mathbb{Z}) \mid \sigma^*(x) = x\},$$

$$S(\sigma^2) = \{x \in H^2(X, \mathbb{Z}) \mid (\sigma^2)^*(x) = x\}, \quad T(\sigma^2) = S(\sigma^2)^{\perp} \cap H^2(X, \mathbb{Z}).$$

Observe that $r = \mathrm{rk}\, S(\sigma)$, $r + l = \mathrm{rk}\, S(\sigma^2)$ and $2m = \mathrm{rk}\, T(\sigma^2)$.

We start recalling the following result about non-symplectic involutions (see [**25**, Theorem 4.2.2], [**17**, Theorem 6.1] or [**3**, Theorem 4.1]).

THEOREM 1.1.   *Let τ be a non-symplectic involution on a K3 surface X. The fixed locus of τ is either empty, the disjoint union of two elliptic curves or the disjoint union of a smooth curve of genus $\gamma \geq 0$ and j smooth rational curves.*

Moreover, its fixed lattice $S(\tau) \subset \mathrm{Pic}(X)$ is a 2-elementary lattice with determinant 2^d such that:

- $S(\tau) \cong U(2) \oplus E_8$ *if the fixed locus of τ is empty;*
- $S(\tau) \cong U \oplus E_8(2)$ *if τ fixes two elliptic curves;*
- $2\gamma = 22 - \mathrm{rk}\, S(\tau) - d$ *and* $2j = \mathrm{rk}\, S(\tau) - d$ *otherwise.*

PROPOSITION 1.   *Let σ be a purely non-symplectic automorphism of order four on a K3 surface X. Then $\mathrm{Fix}(\sigma)$ is the disjoint union of smooth curves and n isolated points. Moreover, the following relations hold:*

$$n = 2\alpha + 4, \quad 4\alpha = r - l - 2 = 2(10 - l - m),$$

where $\alpha = \sum_{C \subset \mathrm{Fix}(\sigma)}(1 - g(C))$.

PROOF.   Since σ is purely non-symplectic, then σ^2 is a non-symplectic involution. Thus the fixed locus of σ is the disjoint union of smooth curves and isolated points by Theorem 1.1. The action of σ at a point in $\mathrm{Fix}(\sigma)$ can be locally diagonalized as follows (see [**23**, Section 5]):

$$A_{4,0} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{4,1} = \begin{pmatrix} -i & 0 \\ 0 & -1 \end{pmatrix}.$$

In the first case the point belongs to a smooth fixed curve, while in the second case it is an isolated fixed point. We will apply holomorphic and topological Lefschetz's formulas to obtain the last two relations in the statement. The holomorphic Lefschetz number of σ is

$$L(\sigma) = \sum_{j=0}^{2}(-1)^j \operatorname{tr}(\sigma^*|H^j(X, \mathcal{O}_X)) = 1 - i,$$

since σ^* acts as multiplication by i on $H^{2,0}(X)$. By [**4**, p. 567] one obtains:

$$L(\sigma) = \frac{n}{\det(I - \sigma^*|T_x)} + \frac{i-1}{2}\sum_{C \subset \operatorname{Fix}(\sigma)}(1 - g(C)) = \frac{n}{\det(I - A_{4,1})} + \alpha\frac{i-1}{2},$$

where n is the number of isolated fixed points, x is an isolated fixed point and T_x denotes the tangent space at x. Comparing the two formulas for $L(\sigma)$ we obtain the relation $n = 2\alpha + 4$. In particular this implies that the fixed locus of σ (and thus that of σ^2) is not empty. We now consider the topological Lefschetz fixed point formula

$$\chi(\operatorname{Fix}(\sigma)) = \sum_{j=0}^{4}(-1)^j \operatorname{tr}(\sigma^*|H^j(X, \mathbb{R})) = 2 + \operatorname{tr}(\sigma^*|H^2(X, \mathbb{R})).$$

Since $\operatorname{tr}(\sigma^*|H^2(X, \mathbb{R})) = r - l$, then:

$$\chi(\operatorname{Fix}(\sigma)) = n + 2\alpha = 2 + r - l.$$

Using the relation $n = 2\alpha + 4$ we obtain the two expressions for α in the statement. $\square$

We now provide a similar result in case σ^2 is symplectic.

PROPOSITION 2. *Let σ be a non-symplectic automorphism of order four on a K3 surface X such that σ^2 is symplectic. Then $\operatorname{Fix}(\sigma)$ contains $n \leq 8$ isolated fixed points and the possible values of the triple (r, l, n) are*

$$(6, 8, 0), \ (7, 7, 2), \ (8, 6, 4), \ (9, 5, 6), \ (10, 4, 8).$$

PROOF. The fixed locus of the symplectic involution σ^2 is given by 8 points [**23**]. Thus σ fixes at most $n \leq 8$ points. Moreover, the topological Lefschetz formula gives that $n = 2 + r - l$. Since the invariant lattice of a symplectic involution has rank 14 by [**23**], then $l + r = 14$. This gives the statement. $\square$

EXAMPLE 1.2. Consider the following family of quartics in $\mathbb{P}^3$:

$$a_1 x_0^4 + x_0^2(a_2 x_1^2 + a_3 x_2 x_3) + x_0 x_1(a_4 x_2^2 + a_5 x_3^2)$$
$$+ x_1^2(a_6 x_1^2 + a_7 x_2 x_3) + x_2^2(a_8 x_2^2 + a_9 x_3^2) + a_{10} x_3^4 = 0.$$

The generic element X_a of the family is a smooth quartic surface, hence a K3 surface, and carries the order four automorphism:

$$\sigma(x_0, x_1, x_2, x_3) = (x_0, -x_1, ix_2, -ix_3),$$

which has no fixed points and whose square fixes the eight intersection points between X_a and the lines $x_0 = x_1 = 0$, $x_2 = x_3 = 0$. Since the space of matrices in $GL_4(\mathbb{C})$ commuting with σ has dimension 4, then the family has $10 - 4 = 6$ moduli.

On the other hand, the family of plane quartics of the form

$$f(x_0, x_1) + g(x_2, x_3) = 0,$$

where f, g are homogeneous of degree 4, carries the order four automorphism $\tau(x_0, x_1, x_2, x_3) = (x_0, x_1, ix_2, ix_3)$, whose square is again symplectic and whose fixed locus contains exactly 8 points. The matrices commuting with τ are a space of dimension 8 so we get $10 - 8 = 2$ moduli.

Finally consider the elliptic fibration

$$y^2 = x^3 + a(t)x + b(t)$$

with $a(t) = a_8 t^8 + a_4 t^4 + a_0$, $b(t) = b_{10}t^{10} + b_6 t^6 + b_2 t^2$, $a_i, b_j \in \mathbb{C}$ and automorphism $(x, y, t) \mapsto (-x, iy, it)$. The automorphism has symplectic square and leaves invariant the fibers over $t = 0$ and $t = \infty$, each of them contains two isolated fixed points which gives $n = 4$. Here the automorphisms of $\mathbb{P}^1$ commuting with the automorphism $t \mapsto it$ are a space of dimension 1, moreover we can apply to the equation the transformation $(x, y) \mapsto (\lambda^2 x, \lambda^3 y)$ for a suitable $\lambda \in \mathbb{C}$ (and divide by λ^6). So we find $6 - 2 = 4$ moduli.

REMARK 1.3.   The moduli space of K3 surfaces carrying a purely non-symplectic automorphism of order four with a given action on the K3 lattice is known to be a complex ball quotient of dimension $m - 1$, see [**9**, Section 11]. The complex ball is given by:

$$B = \{[w] \in \mathbb{P}(V) : (w, \bar{w}) > 0\},$$

where V is the i-eigenspace of σ^* in $T(\sigma^2) \otimes \mathbb{C}$. This implies that the Picard lattice of the K3 surface corresponding to the generic point of such space equals $S(\sigma^2)$ (see [**9**, Theorem 11.2]). On the other hand, if the automorphism has symplectic square, then the period point of X belongs to the -1-eigenspace in $H^2(X, \mathbb{Z})$, so that $\mathrm{Pic}(X)$ contains $S(\sigma) \oplus T(\sigma^2)$, $\mathrm{rk}\,\mathrm{Pic}(X) \geq 14$ and, given the action on the K3 lattice, the dimension of the moduli space is equal to $l - 2$.

The following results will be useful later.

LEMMA 1.   *Let σ be a non-symplectic automorphism on a* K3 *surface X. Then the invariant lattice $S(\sigma)$ is a hyperbolic sublattice of* $\mathrm{Pic}(X)$.

PROOF.   If $x \in S(\sigma)$, then $(x, \omega_X) = (\sigma^*(x), \sigma^*(\omega_X)) = (x, \epsilon\omega_X)$, with $\epsilon \neq 1$ since σ is non-symplectic. Thus $x \in \mathrm{Pic}(X) = \omega_X^\perp \cap H^2(X, \mathbb{Z})$. By [**23**, Theorem 3.1] the surface X is algebraic, so that $\mathrm{Pic}(X)$ is hyperbolic by Hodge index theorem. This implies that $S(\sigma)$ is a hyperbolic lattice since, given an effective class $x \in \mathrm{Pic}(X)$ with $x^2 > 0$, then $\sum_{i=1}^n (\sigma^*)^i(x)$ (n is the order of σ) is a σ^*-invariant class with positive

self-intersection. □

LEMMA 2.   *Let τ be a non-symplectic involution, $\pi : X \to Y := X/\langle\tau\rangle$ be the quotient map and assume that the fixed locus of τ is the union of a curve C and j smooth rational curves $E_1, \ldots, E_j$. Then the invariant lattice $S(\tau)$ is generated by $\pi^* \operatorname{Pic}(Y)$ and by the classes of $E_1, \ldots, E_j$.*

PROOF.   First observe that $\pi^* \operatorname{Pic}(Y) \otimes \mathbb{Q} = S(\tau) \otimes \mathbb{Q}$. In fact, if $y \in \operatorname{Pic}(Y)$ then $\pi^*(y) = z + \tau^*(z)$ is invariant by τ^*. On the other hand, if $x \in S(\tau)$ then $x = (\pi^*\pi_*(x))/2 \in \pi^* \operatorname{Pic}(Y) \otimes \mathbb{Q}$. In particular $\rho := \operatorname{rk} S(\tau) = \operatorname{rk} \operatorname{Pic}(Y)$. Moreover we have $\pi^* \operatorname{Pic}(Y) \cong \operatorname{Pic}(Y)(2)$ as a lattice. Since the quotient surface Y is smooth and rational, then $\operatorname{Pic}(Y) = H^2(Y, \mathbb{Z})$ is unimodular, so $[S(\tau) : \pi^* \operatorname{Pic}(Y)]^2 = 2^\rho/2^d$. Since $2j = \rho - d$ by Theorem 1.1 we get $[S(\tau) : \pi^* \operatorname{Pic}(Y)] = 2^j$.

Observe that the curves $\pi(E_i)$ are (-4)-curves in Y and the sum of their classes is not equal to $-2K_Y$ (in fact $[\pi(C) + \sum_i \pi(E_i)] = -2K_Y$). It follows by [**21**, Lemma 2.2] that any subset of $\{E_1, \ldots, E_j\}$ is not an even set. This implies that the classes of the E_i's are not contained in $\pi^* \operatorname{Pic}(Y)$ and give independent elements in $S(\tau)/\pi^* \operatorname{Pic}(Y) \cong \mathbb{Z}/2\mathbb{Z}^{\oplus j}$. Thus the index of the lattice generated by $\pi^* \operatorname{Pic}(Y)$ and the classes of $E_1, \ldots, E_j$ in the lattice $S(\tau)$ is one. □

LEMMA 3.   *If $x \in S(\sigma^2)$, then $x \cdot \sigma^*(x)$ is even.*

PROOF.   We use Lemma 2 and write $x = \pi^*(y) + \sum a_i e_i$, where $a_i \in \mathbb{Z}$ and e_i is the class of E_i. Then $\sigma^*(x) = \pi^*\sigma^*(y) + \sum a_i \sigma^*(e_i)$, where we use the fact that $\pi^*\sigma^* = \sigma^*\pi^*$ and we denote again with σ the induced automorphism on Y. Then we get:

$$x \cdot \sigma^*(x) = \pi^*(y) \cdot \pi^*\sigma^*(y) + \sum a_i e_i \cdot \sum a_i \sigma^*(e_i) + \pi^*(y) \cdot \sum a_i \sigma^*(e_i) + \pi^*\sigma^*(y) \cdot \sum a_i e_i$$

and we have

$$\pi^*(y) \cdot \pi^*\sigma^*(y) = 2y \cdot \sigma^*(y),$$

which is even, and clearly the product $\sum a_i e_i \cdot \sum a_i \sigma^*(e_i)$ is even too since the E_i's are disjoint. Finally

$$\pi^*(y) \cdot \sum a_i \sigma^*(e_i) = \pi^*(\sigma^*)^2(y) \cdot \sum a_i \sigma^*(e_i) = \pi^*\sigma^*(y) \cdot \sum a_i e_i,$$

where the last equality follows from the fact that σ^* is an isometry. This concludes the proof. □

We formulate the following useful lemma that generalizes [**7**, Lemma 8.1]. Its proof is in fact the same as there, since it does not depend on the order of the automorphism.

LEMMA 4.   *Let $T = \sum_i R_i$ be a tree of smooth rational curves on a K3 surface X such that each R_i is invariant under the action of a purely non-symplectic automorphism σ of order k. Then, the points of intersection of the rational curves R_i are fixed by σ*

and the action at one fixed point determines the action on the whole tree.

PROOF. Since σ is purely non-symplectic, then it acts on the vector space of holomorphic two forms of X as the multiplication by a primitive k-th root of unity η. Let p be the unique point of intersection of two rational curves R_1 and R_2. This is clearly fixed by σ, and σ is locally given in p by the matrix (see [**23**, Section 5]):

$$\begin{pmatrix} \eta^{k+1-j} & 0 \\ 0 & \eta^j \end{pmatrix},$$

where the two eigenvectors correspond to the curves R_1 and R_2. Assume that R_2 corresponds to the second eigenvector, so that the action of σ on the tangent space at p in R_2 is $\mathbb{C} \to \mathbb{C}$, $z \mapsto \eta^j z$. Let now q be the point of intersection of R_2 with a rational curve $R_3 \neq R_1$. Since $R_2 \cong \mathbb{P}^1$ we can fix coordinates $(z_0 : z_1)$ so that the action of σ in $\mathbb{P}^1 \backslash \{q\}$ is $z_1/z_0 \mapsto \eta^j z_1/z_0$ (in these coordinates $p = 0$ and $q = \infty$). Thus the action on $\mathbb{P}^1 \backslash \{p\}$ is given by $z_0/z_1 \mapsto \eta^{k-j} z_0/z_1$. Since σ multiplies holomorphic two forms by η, then its local action at q is given by the matrix:

$$\begin{pmatrix} \eta^{k-j} & 0 \\ 0 & \eta^{j+1} \end{pmatrix},$$

so it is uniquely determined. $\square$

REMARK 1.4. In the case of an automorphism of order 4, with the assumption of Lemma 4, the local actions at the intersection points of the curves R_i appear in the following order:

$$\ldots, \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \ldots.$$

2. Elliptic fibrations.

In this section we will study elliptic fibrations on K3 surfaces carrying a purely non-symplectic automorphism of order four. The following result is proved by means of an argument contained in the proof of [**8**, Proposition 2.9].

PROPOSITION 3. *Let σ be an automorphism of a K3 surface X. If the rank of the invariant lattice of σ^* in $H^2(X, \mathbb{Z})$ is bigger than 4, then there exists a σ-invariant elliptic fibration $\pi : X \to \mathbb{P}^1$.*

PROOF. As before, we denote by $S(\sigma)$ the invariant lattice of σ^* in $H^2(X, \mathbb{Z})$. By [**30**, Corollary 2, p. 43], since $\operatorname{rk} S(\sigma) \geq 5$, there exists a primitive isotropic vector $x \in S(\sigma)$. After applying a finite number of reflections with respect to (-2)-curves we obtain a nef class x' which is uniquely determined by x (see [**27**, Section 6, Theorem 1]). Observe that x' is primitive and $x'^2 = 0$. It is easy to see that σ^* acts on the orbit of x with respect to the reflection group. Since x' is the unique nef member in the orbit and

any automorphism preserves nefness, then $\sigma^*(x') = x'$. The morphism associated to x' is a σ-invariant elliptic fibration on X. □

Let $\pi : X \to \mathbb{P}^1$ be an elliptic fibration such that any of its fibers is invariant for σ and contains at least a fixed point. The last assumption is not necessary if the fibration is jacobian: indeed if σ is fixed points free on the generic fiber, then it acts as a translation on it and it can be easily proved, by writing explicitly a holomorphic 2-form, that the automorphism would be symplectic. If σ has order four, then the generic fiber of π contains two fixed points for σ and four fixed points of σ^2. Thus π has two bisections (not necessarily irreducible): a curve $E_\sigma \subset \mathrm{Fix}(\sigma)$ and a curve $E_{\sigma^2} \subset \mathrm{Fix}(\sigma^2)$. We now describe the singular fibers of the elliptic fibration and the action of σ on them.

PROPOSITION 4. *Let X be a K3 surface with a non-symplectic order four auto-morphism σ and $\pi : X \to \mathbb{P}^1$ be an elliptic fibration such that σ preserves each fiber of π and has a fixed point on it. Then the singular fibers of π are of the following Kodaira types*:

- III : $R_1 \cup R_2$, *where either*
    - a) *the R_i's are exchanged by σ, E_{σ^2} intersects each R_i at one point and E_σ intersects in $R_1 \cap R_2$ or*
    - b) *the R_i's are σ-invariant, E_σ intersects each R_i at one point and E_{σ^2} intersects in $R_1 \cap R_2$.*
- I_0^* : $2R_1 + R_2 + R_3 + R_4 + R_5$, *where either*
    - a) *R_2, R_3 are σ-invariant (intersected by E_σ) and R_4, R_5 are exchanged by σ (intersected by E_{σ^2}) or*
    - b) *$R_2, \ldots, R_5$ are permuted by σ, and E_σ and E_{σ^2} intersect R_1.*
- III^* : $R_1 + 2R_2 + 3R_3 + 4R_4 + 2R_5 + 3R_6 + 2R_7 + R_8$, *where either*
    - a) *σ preserves each irreducible component of the fiber, $R_4 \subset \mathrm{Fix}(\sigma)$, R_2, R_7 contain two isolated fixed points, E_σ intersects R_1, R_8, and E_{σ^2} intersects R_5 or*
    - b) *σ exchanges the two branches of the fiber, E_{σ^2} intersects R_1, R_8 and E_σ intersects R_5.*

PROOF. By the previous argument, the restriction of σ to the generic fiber of π has order four and two fixed points. Thus any smooth fiber of π has j-invariant equal to 1. By the Kodaira classification it follows that the singular fibers of π are either of type $\mathrm{I}_0^*, \mathrm{III}$, or III^*. We now analyze the possible actions of σ on these fibers.

If F is a reducible fiber of type I_0^*, then the component R_1 is clearly σ-invariant. Observe that R_1 is not fixed by σ, since otherwise each R_i, $i = 2, \ldots, 5$, should contain a fixed point for σ in the intersection with either E_{σ^2} or E_σ. This is absurd because σ exchanges the two (distinct) points in $F \cap E_{\sigma^2}$. Thus σ has either order two or four on R_1. If $\sigma^2 = \mathrm{id}$ on R_1, then each R_i, $i = 2, \ldots, 5$ contains a fixed point of σ^2 in the intersection with either E_{σ^2} or E_σ, thus we are in case $\mathrm{I}_0^*a)$. If σ has order 4 on R_1, then σ permutes the curves R_i, $i = 2, \ldots, 5$ and R_1 contains two fixed points for σ in the intersection with E_σ and E_{σ^2}, giving case $\mathrm{I}_0^*b)$.

If F is a fiber of type III^*, then R_4 and R_5 are clearly σ-invariant. If σ preserves each irreducible component of F, then $R_4 \subset \mathrm{Fix}(\sigma)$ (since it contains 3 fixed points) and,

since E_{σ^2} contains at most a fixed point, E_σ intersects R_1, R_8 and E_{σ^2} intersects R_5. By Lemma 4, the only curve fixed by σ is R_4, thus the curves R_2 and R_7 contain each two isolated fixed points. This gives case III*a). Otherwise, if σ exchanges the two branches of the fiber, then $\sigma^2 = $ id on R_4, E_{σ^2} intersects R_1 and R_8 in two points exchanged by σ and E_σ intersects R_5. The case of a fiber of type III can be discussed in a similar way. □

We will denote by g_σ and g_{σ^2} the genus of E_σ and E_{σ^2} respectively, by n the number of isolated points in $\mathrm{Fix}(\sigma)$, by k the number of smooth rational curves in $\mathrm{Fix}(\sigma)$ and by $2a$ the number of smooth rational curves in $\mathrm{Fix}(\sigma^2)$ exchanged by σ.

COROLLARY 1. *Under the hypotheses of Proposition 4, we have the following possibilities for the invariants defined above.*

- If E_σ is irreducible and E_{σ^2} is reducible (*hence the union of two smooth rational curves exchanged by* σ) :

g_σ	n	k	a	reducible fibers
3	0	0	1	$8\,\mathrm{III}\,a)$
2	2	0	1	$6\,\mathrm{III}\,a) + \mathrm{I}_0^*\,a)$
	2	0	2	$5\,\mathrm{III}\,a) + \mathrm{III}^*\,b)$
1	4	0	1	$4\,\mathrm{III}\,a) + 2\,\mathrm{I}_0^*\,a)$
	4	0	2	$3\,\mathrm{III}\,a) + \mathrm{I}_0^*\,a) + \mathrm{III}^*\,b)$
	4	0	3	$2\,\mathrm{III}\,a) + 2\,\mathrm{III}^*\,b)$
0	6	1	1	$2\,\mathrm{III}\,a) + 3\,\mathrm{I}_0^*\,a)$
	6	1	2	$\mathrm{III}\,a) + 2\,\mathrm{I}_0^*\,a) + \mathrm{III}^*\,b)$
	6	1	3	$\mathrm{I}_0^*\,a) + 2\,\mathrm{III}^*\,b)$

- If E_σ is reducible and E_{σ^2} is irreducible: $a = 0$ and

g_{σ^2}	n	k	reducible fibers
3	8	2	$8\,\mathrm{III}\,b)$
2	8	2	$6\,\mathrm{III}\,b) + \mathrm{I}_0^*\,a)$
	10	3	$5\,\mathrm{III}\,b) + \mathrm{III}^*\,a)$
1	8	2	$4\,\mathrm{III}\,b) + 2\,\mathrm{I}_0^*\,a)$
	10	3	$3\,\mathrm{III}\,b) + \mathrm{I}_0^*\,a) + \mathrm{III}^*\,a)$
	12	4	$2\,\mathrm{III}\,b) + 2\,\mathrm{III}^*\,a)$
0	8	2	$2\,\mathrm{III}\,b) + 3\,\mathrm{I}_0^*\,a)$
	10	3	$\mathrm{III}\,b) + 2\,\mathrm{I}_0^*\,a) + \mathrm{III}^*\,a)$
	12	4	$\mathrm{I}_0^*\,a) + 2\,\mathrm{III}^*\,a)$

- *If E_σ and E_{σ^2} are both irreducible*:

g_σ	g_{σ^2}	n	k	a	reducible fibers
2	0	2	0	0	$6\,\mathrm{III}\,a) + 2\,\mathrm{III}\,b)$
1	1	4	0	0	$4\,\mathrm{III}\,a) + 4\,\mathrm{III}\,b)$
		4	0	0	$4\,\mathrm{I}_0^*\,b)$
1	0	6	1	0	$3\,\mathrm{III}\,a) + \mathrm{I}_0^*\,b) + \mathrm{III}^*\,a)$
		6	1	0	$4\,\mathrm{III}\,a) + \mathrm{III}\,b) + \mathrm{III}^*\,a)$
		4	0	0	$4\,\mathrm{III}\,a) + 2\,\mathrm{III}\,b) + \mathrm{I}_0^*\,a)$
		4	0	1	$3\,\mathrm{III}\,a) + 2\,\mathrm{III}\,b) + \mathrm{III}^*\,b)$
0	2	6	1	0	$2\,\mathrm{III}\,a) + 6\,\mathrm{III}\,b)$
		6	1	0	$4\,\mathrm{III}\,b) + 2\,\mathrm{I}_0^*\,b)$
0	1	8	2	0	$2\,\mathrm{III}\,a) + 3\,\mathrm{III}\,b) + \mathrm{III}^*\,a)$
		6	1	0	$2\,\mathrm{III}\,a) + 4\,\mathrm{III}\,b) + \mathrm{I}_0^*\,a)$
		6	1	1	$\mathrm{III}\,a) + 4\,\mathrm{III}\,b) + \mathrm{III}^*\,b)$
0	0	10	3	0	$2\,\mathrm{III}\,a) + 2\,\mathrm{III}^*\,a)$
		8	2	0	$2\,\mathrm{III}\,a) + \mathrm{III}\,b) + \mathrm{I}_0^*\,a) + \mathrm{III}^*\,a)$
		8	2	1	$\mathrm{III}\,a) + \mathrm{III}\,b) + \mathrm{III}^*\,a) + \mathrm{III}^*\,b)$
		6	1	0	$2\,\mathrm{III}\,a) + 2\,\mathrm{III}\,b) + 2\,\mathrm{I}_0^*\,a)$
		6	1	1	$\mathrm{III}\,a) + 2\,\mathrm{III}\,b) + \mathrm{I}_0^*\,a) + \mathrm{III}^*\,b)$
		6	1	2	$2\,\mathrm{III}\,b) + 2\,\mathrm{III}^*\,b)$

- *If both E_σ and E_{σ^2} are reducible: $n = 8$, $k = 2$, $a = 1$ and the reducible fibers are $4\,\mathrm{I}_0^*$ of type a).*

PROOF. Observe that the restrictions of π to E_σ and to E_{σ^2} are double covers of $\mathbb{P}^1$. If E_σ (or E_{σ^2}) is irreducible, then it contains $2g_\sigma + 2$ (or $2g_{\sigma^2} + 2$) ramification points. Since a smooth fiber of π contains exactly 4 fixed points for σ^2, then such ramification points belong to singular fibers of π, which are classified in Proposition 4. The ramification points of E_σ belong either to a fiber of type $\mathrm{III}\,a)$, $\mathrm{I}_0^*\,b)$ or $\mathrm{III}^*\,b)$, while the ramification points of E_{σ^2} belong either to a fiber of type $\mathrm{III}\,b)$, $\mathrm{I}_0^*\,b)$, or $\mathrm{III}^*\,a)$. This implies that $g_\sigma \leq 3$ (or $g_{\sigma^2} \leq 3$) since otherwise the Euler–Poincaré characteristic of the singular fibers would give at least $e(\mathrm{III})(2g_\sigma + 2) = 3(2g_\sigma + 2) > 24 = e(X)$ (similarly for g_{σ^2}). If E_σ and E_{σ^2} are both irreducible, this implies that $g_\sigma, g_{\sigma^2} \leq 2$. We obtain the tables in the statement by enumerating all cases which are compatible with Proposition 4 and Proposition 1. $\qquad\square$

The following result allows, in some cases, to prove that a given elliptic fibration is σ-invariant.

LEMMA 5. *Let X be a K3 surface with an automorphism σ and $\pi : X \to \mathbb{P}^1$ be an elliptic fibration whose fibers have class f. If σ^* fixes a class $x \in \mathrm{Pic}(X)$ with $x^2 > 0$, then*

$$(f \cdot \sigma^*(f))x^2 \leq 2(x \cdot f)^2.$$

Moreover, if in addition π is jacobian and there is a section s of π such that $s \cdot x = 0$, then the following holds

$$x^2 \leq \frac{2(x \cdot f)^2}{f \cdot \sigma^*(f) + 1}.$$

Proof. Let M be the sublattice of $\mathrm{Pic}(X)$ generated by x and $f + \sigma^*(f)$. By Hodge index theorem we have that $\det(M) = 2(x^2(f \cdot \sigma^*(f)) - 2(x \cdot f)^2) \leq 0$. This gives the first inequality. The second inequality follows from a similar argument with the lattice generated by $x, f + \sigma^*(f)$ and the class of the section not intersecting x. □

Theorem 2.1. *Let σ be a purely non-symplectic automorphism of order 4 on a K3 surface X such that $\mathrm{Pic}(X) = S(\sigma^2) \cong U \oplus L$ and σ^2 fixes a curve C of genus $g > 1$. Then X carries a jacobian elliptic fibration $\pi : X \to \mathbb{P}^1$ whose fibers are σ^2-invariant. Moreover*:

- *If L is isomorphic to a direct sum of root lattices of types A_1, D_{4n}, E_7 or E_8, then π has reducible fibers described by L and a unique section $E \subset \mathrm{Fix}(\sigma^2)$. Moreover, π is σ-invariant if $g > 4$ and the genus of a curve in $\mathrm{Fix}(\sigma)$ is at most 2.*
- *If L is not isomorphic to a direct sum of root lattices, then π has two sections $E, E' \subset \mathrm{Fix}(\sigma^2)$ and C intersects each fiber in two points. Moreover, π is σ-invariant if $g > 2$ and the genus of a curve in $\mathrm{Fix}(\sigma)$ is at most 2 if $a = 0$.*

In both cases the automorphism σ^2 acts as an involution on the simple components of the reducible fibers of π and on the fibers of types $\mathrm{I}^*_{4n}, \mathrm{III}^*, \mathrm{II}^*$ as in Figure 1, where σ^2 acts identically on dotted components and as an involution on the other ones.

Proof. The first half of the statement follows from [**16**, Lemma 2.1]. If σ^2 fixes a curve C of genus $g > 1$, then this curve is transversal to the fibers of π. This implies that σ^2 preserves each fiber of π and has 4 fixed points on it. If L is a root lattice, then π has a unique section E since its Mordell–Weil group $MW(\pi) \cong \mathrm{Pic}(X)/U \oplus L$ is trivial [**29**, Theorem 6.3]. This implies that E is fixed by σ^2 and that C intersects each fiber in three points.

Let x be the class of C and f be the class of a fiber of π. If $f \neq \sigma^*(f)$, then clearly $f \cdot \sigma^*(f) \geq 2$. It follows from Lemma 5 that $2g - 2 = x^2 \leq 2(x \cdot f)^2/(f \cdot \sigma^*(f) + 1) \leq 6$, thus $g \leq 4$. Observe that, if σ fixes a curve C of genus $g > 1$, then $f \neq \sigma^*(f)$ since otherwise the generic fiber would contain at most 2 fixed points by σ. Moreover in this case $f \cdot \sigma^*(f) \geq 4$ since each fiber contains at least 3 fixed points and the intersection $f \cdot \sigma^*(f)$ is even by Lemma 3. This implies $g \leq 2$ by Lemma 5.

If L is not a root lattice, then the Mordell–Weil group of π is not trivial, so that π has at least two sections $E, E' \subset \mathrm{Fix}(\sigma^2)$. Thus C intersects each fiber in two points. In this case, if $f \neq \sigma^*(f)$, then $g \leq 2$ by Lemma 5. If σ fixes C, then either π is σ-invariant or $f \cdot \sigma^*(f) \geq 2$. In the first case, since σ has two fixed points in each fiber, $\sigma(E) = E'$, so that $a \geq 1$. In the second case Lemma 5 implies, as before, that $g \leq 2$. □

Remark 2.2. By [**24**, Corollary 1.13.5] an even indefinite lattice S is isomorphic to a direct orthogonal sum $U \oplus L$, if $\mathrm{rk}(S) \geq 3 + \ell(S)$, where $\ell(S)$ is the minimal number of
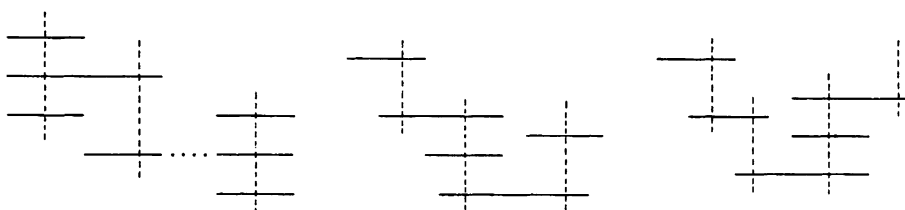
Figure 1. Action of σ^2 on reducible fibres of types $\mathrm{I}^*_{4n}, \mathrm{III}^*, \mathrm{II}^*$.

generators of the discriminant group of S. In case $S = S(\sigma^2)$ this condition is equivalent to ask that the number j of smooth rational curves fixed by σ^2 is at least 2 by Theorem 1.1.

3. Fix (σ) contains an elliptic curve.

We now assume that σ fixes an elliptic curve C. In this case the K3 surface X has an elliptic fibration $\pi_C : X \to \mathbb{P}^1$ having C as a smooth fiber. Observe that all curves fixed by σ^2, since they are disjoint from C, are contained in the fibers of π_C. In particular $\alpha \geq 0$. We will now classify the reducible fibers of π_C.

THEOREM 3.1. *Let σ be a purely non-symplectic order four automorphism on a K3 surface X with $\mathrm{Pic}(X) = S(\sigma^2)$ and $\pi_C : X \to \mathbb{P}^1$ be an elliptic fibration with a smooth fiber $C \subset \mathrm{Fix}(\sigma)$. Then σ preserves π_C and acts on its base as an order four automorphism with two fixed points corresponding to the fiber C and a fiber C' which is either smooth, of Kodaira type I_{4M} or IV^*. The corresponding invariants of σ are given in Table* 1.

Table 1. The case $g = 1$.

m	r	l	n	k	a	type of C'
6	6	4	4	0	0	I_0
5	7	5	4	0	0	I_4
4	10	4	6	1	0	IV^*
4	8	6	4	0	1	I_8 or IV^*
3	9	7	4	0	2	I_{12}
2	10	8	4	0	3	I_{16}

PROOF. We first observe that σ^2 is not the identity on the base of π_C, since otherwise it would act as the identity on the tangent space at a point of C, contradicting the fact that σ^2 is non-symplectic. Hence σ has order four on $\mathbb{P}^1$ and has two fixed points, corresponding to C and another fiber C'. If C' is irreducible, then $\alpha = 0$ and $n = 4$ by Proposition 1, which implies that C' is smooth elliptic and σ has order two on it.

We now assume that C' is reducible and we classify the possible Kodaira types for it (see also [**25**, Section 4.2]). Since $n \geq 4$ by Proposition 1, then C' contains at least two (disjoint) smooth rational curves fixed by σ^2. This excludes the Kodaira types $\mathrm{I}_2, \mathrm{I}_3, \mathrm{III}, \mathrm{IV}$ for C'. Since $S(\sigma^2) = \mathrm{Pic}(X)$, then any smooth rational curve is invariant

for σ^2. Moreover, observe that if a component of C' is "external", i.e. it only intersects one other component, then it is fixed by σ^2.

If C' is of type I_N^*, then the four external components of C' are fixed by σ^2 by the previous remark and the same holds for the multiplicity two components intersecting them, since they contain at least 3 fixed points. This gives a contradiction since the fixed curves of σ^2 do not intersect. The case when C' is either of type II^* or III^* can be excluded by a similar argument.

If C' is of type IV^*, then the central component of multiplicity 3 is clearly invariant for σ. If the central component is fixed by σ, then $k = 1$, $a = 0$ and $n = 6$ by Proposition 1. Otherwise two branches of the fiber are exchanged, and the same Proposition gives $k = 0$, $n = 4$, $a = 1$.

Finally we assume that C' is of type I_N, $N \geq 4$. By the previous remark C' contains at least two components fixed by σ^2, this implies that all components are preserved by σ^2 and a component which is not fixed intersects two fixed ones. Moreover, it follows from Proposition 1 that the number of components of C' in $\text{Fix}(\sigma^2)$ is even, since it equals $j = k + n/2 + 2a = 2 + 2\alpha + 2a$. Thus C' is of type I_{4M} for some positive integer M.

If $a = 0$, then $n = 4M - 2k$. Since $n = 2k + 4$ by Proposition 1, this gives $k = M - 1$ and $n = 2M + 2$. On the other hand by applying Lemma 4 we obtain for $M \geq 2$ that $k = 4M/4 = M$, which is clearly a contradiction. So if $a = 0$ then $M = 1$.

If $a = M - 1$, $M \geq 2$, then σ acts as an order two symmetry on the set of components of C', so that $k = 0$ and $n = 4$. Observe that σ can not act as a rotation of I_{4M} because otherwise $n = 0$, contradicting the fact that $n \geq 4$.

If $M = 1$ we have a fiber I_4. Since $n \geq 4$ and the fixed points are contained in the fiber C' the only possibility is that $k = 0$ and $n = 4$. Observe that the case with $a = 1$ is not possible, otherwise the fiber C' would contain two isolated fixed points for σ^2, which is not possible, so $a = 0$. As before one remarks that σ can not act as a rotation of I_4 since otherwise $n = 0$ contradicting the fact that $n \geq 4$. $\qquad\square$

EXAMPLE 3.2. We now assume that the fibration $\pi_C : X \to \mathbb{P}^1$ in Theorem 3.1 has a σ-invariant section. Then a Weierstrass equation for the fibration is of the following form:

$$y^2 = x^3 + \alpha(t)x + \beta(t),$$

where $\alpha(t) = ft^8 + at^4 + b$, $\beta(t) = gt^{12} + ct^8 + dt^4 + e$ and

$$\sigma(x, y, t) = (x, y, it).$$

The fibers preserved by σ are over $0, \infty$ and the action at infinity is

$$(x/t^4, y/t^6, 1/t) \mapsto (x/t^4, -y/t^6, -i/t).$$

The discriminant polynomial of π_C is:

$$\Delta(t) := 4\alpha(t)^3 + 27\beta(t)^2 = g_1 t^{24} + g_2 t^{20} + g_3 t^{16} + g_4 t^{12} + O(t^8),$$

where

$$g_1 = 4f^3 + 27g^2, \quad g_2 = 12f^2a + 54gc, \quad g_3 = 12f^2b + 54gd + 12fa^2 + 27c^2,$$

$$g_4 = 24fab + 4a^3 + 54ge + 54cd.$$

For a generic choice of the coefficients of $\alpha(t)$ and $\beta(t)$ the fibration has 24 fibers of type I_1 over the zeros of $\Delta(t)$, σ fixes pointwisely the fiber over 0 and it acts as an involution on the fiber over ∞ (both fibers are smooth). If $g_1 = 0$ the fibration acquires a fiber of type I_4 at ∞ by a generic choice of the parameters, if $g_1 = g_2 = g_3 = 0$ we generically get a fiber of type I_{12} and for $g_1 = g_2 = g_3 = g_4 = 0$ we get a fiber of type I_{16}. If $g_1 = g_2 = 0$ one gets two possible solutions: if $f = g = 0$ the fibration acquires a fiber of type IV^*, otherwise it gets a fiber of type I_8.

More examples for the case $g = 1$ will be given in Examples 4.2 and 4.3.

4. Fix (σ) contains a curve of genus > 1.

We now assume that $\mathrm{Fix}(\sigma)$ contains a curve C of genus $g > 1$. By Proposition 1 we have

$$\mathrm{Fix}(\sigma^2) = C \cup (E_1 \cup \cdots \cup E_k) \cup (F_1 \cup F_1' \cup \cdots \cup F_a \cup F_a') \cup (G_1 \cup \cdots \cup G_{n/2}),$$

$$\mathrm{Fix}(\sigma) = C \cup E_1 \cup \cdots \cup E_k \cup \{p_1, \ldots, p_n\},$$

where E_i, F_i, G_i are smooth rational curves such that $\sigma(F_i) = F_i'$, $\sigma(G_i) = G_i$ and each G_i contains exactly two isolated fixed points of σ.

LEMMA 6. $k \leq r + m - 8$, $l - m = 2a$.

PROOF.   The curves C, E_i, $F_i \cup F_i'$, G_i are σ-invariant and are orthogonal to each other, thus their classes in $\mathrm{Pic}(X)$ give independent elements in $S(\sigma)$. Then $r = \mathrm{rk}(S(\sigma)) \geq 1 + k + a + n/2$ and this gives the inequality. An easy computation shows that

$$\mathcal{X}(\mathrm{Fix}(\sigma^2)) - \mathcal{X}(\mathrm{Fix}(\sigma)) = 4a.$$

On the other hand, by Proposition 1 and topological Lefschetz fixed point formula applied to σ^2:

$$\mathcal{X}(\mathrm{Fix}(\sigma)) = 24 - 2m - 2l, \quad \mathcal{X}(\mathrm{Fix}(\sigma^2)) = 24 - 4m.$$

Comparing these equalities, we obtain the statement. □

THEOREM 4.1.   *Let X be a K3 surface and σ be a purely non-symplectic automorphism of order four on it such that $\mathrm{Pic}(X) = S(\sigma^2)$. If $\mathrm{Fix}(\sigma)$ contains a curve of genus $g > 1$ then the invariants associated to σ are as in Table 2. All cases in the Table do exist, see Example 4.2 and Example 4.3.*

Table 2.   The case $g > 1$.

m	r	l	n	k	a	g
7	1	7	0	0	0	3
6	4	6	2	0	0	2
	2	8	0	0	1	3
5	5	7	2	0	1	2
4	6	8	2	0	2	2

PROOF.   If $r = \mathrm{rk}\, S(\sigma) \geq 5$ then, by Proposition 3, X carries a σ-invariant elliptic fibration π. If $C \subset \mathrm{Fix}(\sigma)$ has genus $g > 1$, then C is transversal to the fibers of π, so that any fiber of π is preserved by σ and we are in the hypotheses of Corollary 1. Thus $g = n = 2$, $k = 0$ and $a \in \{1, 2\}$ by Corollary 1 (the case $g = 3, n = k = 0$, $a = 1$ has $r = 2$ and the case $g = n = 2, k = a = 0$ has $r = 4$).

We now assume that $r < 5$. Since $l + 2m \geq 18$, $l + m = 10 - 2\alpha \leq 14$ by Proposition 1 and $l - m = 2a$ by Lemma 6, then $4 \leq m \leq 7$ and $a \leq 3$.

By Proposition 1 and Lemma 6 the vector (r, k, g, a) either appears in the rows of Table 2 or is equal to one among $(4, 0, 3, 3), (3, 0, 3, 2), (4, 1, 3, 0), (4, 2, 4, 0)$. In any case we can compute $S(\sigma^2)$ (up to isomorphism) by means of the classification theorem of 2-elementary lattices [**25**, Theorem 4.3.1]:

(r, k, g, a)	$S(\sigma^2)$
$(4, 0, 3, 3)$	$U \oplus E_8 \oplus D_4$
$(3, 0, 3, 2)$	$U \oplus D_8 \oplus A_1^{\oplus 2}$
$(4, 1, 3, 0)$	$U \oplus D_4 \oplus A_1^{\oplus 4}$ or $U(2) \oplus D_4^{\oplus 2}$
$(4, 2, 4, 0)$	$U \oplus D_4^{\oplus 2}$ or $U \oplus D_6 \oplus A_1^{\oplus 2}$

If $S(\sigma^2) \cong U \oplus R$, where R is a direct sum of root lattices, then $g \leq 2$ by Theorem 2.1, giving a contradiction. In the case $(4, 1, 3, 0)$ with $S(\sigma^2) \cong U(2) \oplus D_4^{\oplus 2}$ we still have an isomorphism $S(\sigma^2) \cong U \oplus R$ (where R is not a sum of root lattices) by Remark 2.2 since $j = 2$. By Theorem 2.1, since $g > 2$, there is a jacobian elliptic fibration π whose fibers are invariant for σ and with two sections E, E' fixed by σ^2. Since $a = 0$ these sections are fixed by σ, so that each fiber of π contains four fixed points, giving a contradiction. $\qquad \square$

EXAMPLE 4.2 (plane quartics).   This construction is due to Kondō [**17**]. Let C be a smooth plane quartic, defined as the zero set of a homogeneous polynomial $f_4 \in \mathbb{C}[x_0, x_1, x_2]$ of degree four. The fourfold cover of $\mathbb{P}^2$ branched along C is a K3 surface with equation

$$t^4 = f_4(x_0, x_1, x_2).$$

The covering automorphism

$$\sigma(x_0, x_1, x_2, t) = (x_0, x_1, x_2, it)$$

is a non-symplectic automorphism of order four whose fixed locus is the plane section $t = 0$, which is isomorphic to the curve C. Thus we have $a = n = k = 0$ and $g = 3$. In case C has nodes or cusps, then the fourfold cover X of $\mathbb{P}^2$ branched along C has rational double points of type A_3 and E_6 at the inverse images of a node and of a cusp of C respectively. The minimal resolution $\tilde{X}$ of X is a K3 surface and the covering automorphism of X lifts to a non-symplectic automorphism of $\tilde{X}$. If C has a node, then the central component of the exceptional divisor of type A_3 is fixed by σ^2 and contains two fixed points for σ. If C has a cusp, then the exceptional curve is of type E_6, σ^2 fixes its two simple components and σ exchanges them. Thus, if C is irreducible with x nodes and y cusps, then the invariants of σ are $g = 3 - x - y$, $a = y$, $n = 2(x + y)$ and $k = 0$. Taking $(x, y) = (0, 0), (1, 0)$ and $(0, 1)$ we obtain examples for the first, second and fourth case in Table 2. If $(x, y) = (2, 0), (1, 1)$ or $(0, 2)$ we obtain examples with $g = 1$ corresponding to the cases in Table 1 with a fiber C' of type I_4, I_8 and I_{12} respectively. If C is the union of a cubic and a line we obtain the case in Table 1 with a fiber C' of type IV^* and $(k, n, a) = (1, 6, 0)$. In [**1**, Proposition 1.7] the lattice $S(\sigma^2)$ has been computed in case C is irreducible and generic with x nodes and y cusps.

EXAMPLE 4.3 (hyperelliptic genus three curves). Let $\mathbb{F}_4$ be a Hirzebruch surface and $e, f \in \text{Pic}(\mathbb{F}_4)$ be the class of the rational curve E with $E^2 = -4$ and the class of a fiber respectively. A smooth curve C with class $2e + 8f$ is a hyperelliptic genus three curve. Let Y be the double cover of $\mathbb{F}_4$ branched along C and X be the double cover of Y branched along $C \cup R_1 \cup R_2$, where $R_1 \cup R_2$ is the inverse image of the curve E. The surface X is a K3 surface (see [**18**] and [**1**, Section 3]) with a non-symplectic automorphism σ of order 4 whose fixed locus is the inverse image of the curve C and such that $\sigma(R_1) = R_2$. Observe that σ^2 fixes $C \cup R_1 \cup R_2$. In this case we have $g = 3$, $n = k = 0$ and $a = 1$.

As in the previous example, if C has at most nodes and cusps, then the minimal resolution $\tilde{X}$ of X is again a K3 surface with a non-symplectic automorphism of order 4. If C is irreducible with x nodes and y cusps, then the invariants of σ are $g = 3 - x - y$, $a = y + 1$, $n = 2(x + y)$, $k = 0$. Taking C with a node and a cusp we obtain examples for the fourth and the last case respectively in Table 2. If $(x, y) = (2, 0), (1, 1)$ or $(0, 2)$ we obtain examples with $g = 1$ corresponding to the cases in Table 1 with a fiber C' of type I_8, I_{12} and I_{16} respectively. In [**18**, Section 4.9] the lattice $S(\sigma^2)$ has been computed in case C is irreducible and generic with x nodes and y cusps.

5. Fix (σ^2) only contains rational curves.

In this section we assume that the curves fixed by σ^2 are rational and that at least one of them is fixed by σ.

THEOREM 5.1. *Let X be a K3 surface and σ be a purely non-symplectic automorphism of order four on it. If $\text{Fix}(\sigma)$ contains a smooth rational curve and all curves fixed by σ^2 are rational, then the invariants associated to σ are as in Table 3. All cases in the table do exist, see Example 5.3.*

Table 3.   The case $g = 0$.

m	r	l	n	k	a
4	10	4	6	1	0
3	13	3	8	2	0
	11	5	6	1	1
2	16	2	10	3	0
	14	4	8	2	1
	12	6	6	1	2
1	19	1	12	4	0
	13	7	6	1	3

Proof.   By Proposition 1 and Lemma 6 we have that $l + m \leq 8$ and $l - m = 2a$, thus the possible cases are those appearing in Table 3 and $(r, k, a) = (17, 3, 1), (15, 2, 2)$. In both cases $m = 1$ and the surface X is isomorphic to Vinberg's K3 surface. Moreover, by [20, Lemma 1.5, (2)] or Remark 2.2, X has a σ-invariant jacobian elliptic fibration, since $\ell(S(\sigma)) \leq \mathrm{rk}(S(\sigma)^\perp) \leq 7 \leq \mathrm{rk}\, S(\sigma) - 3$. By Example 5.2, if σ preserves an elliptic fibration on X, then $a \in \{0, 3, 4\}$. Thus the two cases do not appear.   □

EXAMPLE 5.2 (Vinberg's K3 surface).   If $m = 1$, then $S(\sigma^2) = \mathrm{Pic}(X)$ has maximal rank and X is isomorphic to the unique K3 surface with

$$T(X) = T(\sigma^2) \cong \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

since it can be easily proved that, up to isometry, this is the only rank two, even, positive definite and 2-elementary lattice. The automorphism group of this K3 surface is known to be infinite and has been computed by Vinberg in [34, Section 2.4]. In particular, it is known that X is birationally isomorphic to the following quartics in $\mathbb{P}^3$:

$$x_0^4 = -x_1 x_2 x_3 (x_1 + x_2 + x_3),$$
$$x_0^4 = x_2^2 x_3^2 + x_3^2 x_1^2 + x_1^2 x_2^2 - 2x_1 x_2 x_3 (x_1 + x_2 + x_3),$$

which are degree four covers of $\mathbb{P}^2$ branched along the union of four lines in general position and an irreducible quartic with three cusps respectively. The two covering automorphisms $x_0 \mapsto ix_0$ induce non-conjugate order four non-symplectic automorphisms on X: the first one has $a = 0$, $n = 12$ and fixes 4 smooth rational curves (the proper transforms of the lines), the second one has $a = 3$ (coming from the cusps), $n = 6$ and fixes one smooth rational curve (the proper transform of the quartic). These give the last two examples in Table 3.

All elliptic fibrations $\pi : X \to \mathbb{P}^1$ are known to be jacobian by [15, Theorem 2.3] and have been classified by Nishiyama in [26, Theorem 3.1]. We recall the classification in Table 4, where R is the lattice generated by the components of the reducible fibers not intersecting the zero section and MW denotes the Mordell–Weil group of π.

Table 4.   Elliptic fibrations of Vinberg's K3 surface.

No.	R	MW	a
1)	$E_8^{\oplus 2} \oplus A_1^{\oplus 2}$	0	$0, 4$
2)	$E_8 \oplus D_{10}$	0	0
3)	$D_{16} \oplus A_1^{\oplus 2}$	$\mathbb{Z}/2\mathbb{Z}$	$0, 3$
4)	$E_7^{\oplus 2} \oplus D_4$	$\mathbb{Z}/2\mathbb{Z}$	$0, 3, 4$
5)	$E_7 \oplus D_{10} \oplus A_1$	$\mathbb{Z}/2\mathbb{Z}$	$--$
6)	$A_{17} \oplus A_1$	$\mathbb{Z}/3\mathbb{Z}$	0
7)	D_{18}	0	0
8)	$D_{12} \oplus D_6$	$\mathbb{Z}/2\mathbb{Z}$	$0, 3$
9)	$D_8^{\oplus 2} \oplus A_1^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$0, 3, 4$
10)	$A_{15} \oplus A_3$	$\mathbb{Z}/4\mathbb{Z}$	$0, 3, 4$
11)	$E_6 \oplus A_{11}$	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$0, 3$
12)	$D_6^{\oplus 3}$	$(\mathbb{Z}/2\mathbb{Z})^2$	3
13)	$A_9^{\oplus 2}$	$\mathbb{Z}/5\mathbb{Z}$	0

In each case we will compute the possible values taken by a. We will apply the height formula [**31**, Theorem 8.6] and the notation therein. Moreover, we will apply Theorem 2.1 to determine the number of fixed curves of σ^2 contained in the reducible fibers.

1) All curves fixed by σ^2 are contained in the reducible fibers. The automorphism σ either preserves the II* fibers or it exchanges them, giving $a = 0$ or $a = 4$ respectively.

2) The reducible fibers contain exactly 8 curves fixed by σ^2 and the two remaining fixed curves of σ^2 are transversal to the fibers, one of them is the unique section, the other is a 3-section. This implies that both are σ-invariant and σ preserves all components of I$_6^*$, giving $a = 0$.

3) The reducible fibers contain exactly 7 curves fixed by σ^2. The remaining fixed curves of σ^2 are transversal to the fibers and give two sections s_0, s_1 (assume s_0 to be the zero section) and a 2-section. The translation by the order two element in the Mordell–Weil group gives rise to a symplectic automorphism of order two. Since such automorphism has 8 fixed points by [**23**], then it acts on the fiber I$_{14}^*$ as a reflection with respect to the central component, i.e. s_0 and s_1 intersect the fiber in simple components not meeting the same component. This implies that σ acts on the components of I$_{14}^*$ either as the identity or as a reflection with respect to the central component, giving $a = 0$ and $a = 3$ respectively.

4) As before, the reducible fibers only contain 7 curves fixed by σ^2 and the fibration has two sections s_0, s_1 and a 2-section fixed by σ^2. If σ preserves each fiber, then either s_0, s_1 are fixed by σ or they are exchanged giving $a = 0$ and $a = 3$ respectively by Corollary 1. Otherwise, if σ has order two on the basis of π, then it exchanges the two fibers of type III*, so that either $a = 3$ or $a = 4$.

5) Since the fibration has three reducible fibers of distinct types, then σ acts as the identity on $\mathbb{P}^1$. This is not possible by Proposition 4, thus this fibration can not be σ-invariant.

6) The fiber of type I$_{18}$ contains 9 curves fixed by σ^2 and is clearly σ-invariant. Since

$18 \not\equiv 0 \pmod 4$, then $a = 0$.

7) The fiber of type I_{14}^* contains 8 curves fixed by σ^2, the remaining two curves fixed by σ^2 give a section and a 3-section. Thus $a = 0$.

8) The reducible fibers contain 7 curves fixed by σ^2, the remaining fixed curves of σ^2 give two sections s_0, s_1. By the height formula, s_0 and s_1 meet simple components intersecting distinct components in the I_8^* fiber and intersecting the same component in the I_2^* fiber. Thus either $a = 0$ or $a = 3$.

9) The reducible fibers contain 6 curves fixed by σ^2 and the fibration has four sections s_0, s_1, s_2, s_3 fixed by σ^2. Observe that the elliptic fibration has two fibers of type I_4^*, two of type I_2 and no other singular fibers (since the Euler characteristic of X is 24). Let Θ_i^1, Θ_i^2, $i = 0, 1, 2, 3$ denote the "external" components of the I_4^* fibers and Θ_i^3, Θ_i^4, $i = 0, 1$, denote the components of the I_2 fibers, see also the notation of [**31**, Theorem 8.6, p. 229]. By the height formula, we can assume that s_1 intersects $\Theta_1^1, \Theta_2^2, \Theta_1^3, \Theta_1^4$, s_2 intersects $\Theta_2^1, \Theta_1^2, \Theta_1^3, \Theta_1^4$ and s_3 intersects $\Theta_3^1, \Theta_3^2, \Theta_0^3, \Theta_0^4$. Observe that σ either preserves the fibers of type I_4^* and exchanges the fibers of type I_2, or it exchanges both pairs of reducible fibers, or it exchanges the fibers of type I_4^* and it preserves those of type I_2. In the first case $a = 0, 3$ or 4, according to the action of σ on the sections s_i's. In the second case $a = 4$ (observe that the 4 fixed points of σ are contained in two smooth fibers). The last case does not appear since otherwise σ should preserve all components of the fibers I_2, which gives a contradiction since the components intersecting s_1, s_2 contain no fixed points.

10) In this case all fixed curves by σ^2 are contained in the two reducible fibers, thus σ has order 4 on $\mathbb{P}^1$. Observe that the fibration has four sections s_0, s_1, s_2, s_3 preserved by σ^2, so that each of them intersects the curves fixed by σ^2. This remark and the height formula imply that we can assume that s_1 intersects Θ_8^1, Θ_0^2, s_2 intersects Θ_4^1, Θ_2^2 and s_3 intersects $\Theta_{12}^1, \Theta_2^2$ (again we use the notation of [**31**, Theorem 8.6, p. 229]). This implies that σ either preserves all components of the reducible fibers ($a = 0$), or it acts on the sections as a permutation $(s_2 s_3)$ or $(s_0 s_1)$ ($a = 3$), or as a permutation of type $(s_0 s_1)(s_2 s_3)$ ($a = 4$).

11) In this case all fixed curves by σ^2 are contained in the two reducible fibers, thus σ has order 4 on $\mathbb{P}^1$. By the height formula we can assume that s_1 intersects Θ_1^1, Θ_4^2 and s_2 intersects Θ_2^1, Θ_8^2. This implies that σ either preserves all components of reducible fibers ($a = 0$), or it acts on the sections as a transposition ($a = 3$).

12) The reducible fibers contain 6 curves fixed by σ^2, the remaining fixed curves give four sections $s_i, i = 0, 1, 2, 3$. Observe that σ has order two on $\mathbb{P}^1$, it exchanges two fibers of type I_2^* and it preserves the third one. By the height formula we can assume that s_1 intersects $\Theta_1^1, \Theta_3^2, \Theta_3^3$, s_2 intersects $\Theta_3^1, \Theta_3^2, \Theta_1^3$ and s_3 intersects $\Theta_3^1, \Theta_1^2, \Theta_2^3$. This implies that σ acts on the sections as a transposition, so that $a = 3$.

13) All fixed curves by σ^2 are contained in the two reducible fibers and σ preserves the two fibers of type I_{10} (since otherwise $a = 5$, which is not possible), so that $a = 0$.

EXAMPLE 5.3.   Let $\tilde{X}$ and C be in Example 4.2. Taking C with 3 nodes or with 2 nodes and a cusp we obtain examples for the first and third case in Table 3 respectively. If C is the union of two conics (or the union of a line and a nodal cubic), the union of a conic and two lines or the union of a line and a cuspidal cubic, we obtain examples for

the second, fourth and fifth case in the table respectively. Finally, as mentioned in the previous example, if C is the union of four lines or has 3 cusps, we obtain the last two cases in the table. Similarly, if $\tilde{X}$ and C are as in Example 4.3 and C has 3 nodes, 2 nodes and a cusp, or 1 node and two cusps, we obtain examples for the third, sixth and the last case in Table 3 respectively.

6. The case $l = 0$.

In this section we will assume that $l = 0$, i.e. σ^* acts as the identity on $S(\sigma^2)$.

PROPOSITION 5. *Let σ be a non-symplectic automorphism on a* K3 *surface X such that $l = 0$. Then $\mathrm{Fix}(\sigma)$ is the disjoint union of smooth rational curves and points, $r = 2$ (mod 4) and $a = 0$.*

PROOF. By Theorem 3.1, if $\mathrm{Fix}(\sigma)$ contains an elliptic curve, then $l \geq 4$. On the other hand, if $\mathrm{Fix}(\sigma)$ contains a curve of genus $g > 1$, then $l = 2a + m > 0$ by Lemma 6. Thus the fixed locus of σ only contains smooth rational curves. Observe that $r = 2 \,(\mathrm{mod}\, 4)$ by Proposition 1. If a is not zero, then there are two rational curves F_1, F_1' fixed by σ^2 such that $\sigma(F_1) = F_1'$. If $f = [F_1] - [F_1']$ in $\mathrm{Pic}(X)$, then $\sigma^*(f) = -f$ and $f \neq 0$, contradicting $l = 0$. $\square$

LEMMA 7. *Let L be a lattice which is the direct sum of lattices isomorphic to $U \oplus U$, $U \oplus U(2)$, E_8 or D_{4k}, $k \geq 1$. Then L has an isometry τ with $\tau^2 = -\,\mathrm{id}$ acting trivially on $A_L = \mathrm{Hom}(L, \mathbb{Z})/L$.*

PROOF. It is known that the Weyl group of a lattice L isomorphic to either E_8 or D_{4k}, $k \geq 1$, contains an isometry τ with $\tau^2 = -\,\mathrm{id}$ acting trivially on A_L [**6**]. An isometry τ of $U \oplus U$ as in the statement can be defined as follows:

$$\tau : \; e_1 \mapsto e_2, \; e_2 \mapsto -e_1, \; e_3 \mapsto e_4, \; e_4 \mapsto -e_3,$$

where e_1, e_2 and e_3, e_4 are the natural generators of the first and second copy of U. Such an action can be defined similarly on $U \oplus U(2)$. $\square$

By Proposition 5 the fixed locus of σ and of its square are as follows:

$$\mathrm{Fix}(\sigma^2) = C \cup (E_1 \cup \cdots \cup E_k) \cup (G_1 \cup \cdots \cup G_{n_1/2}),$$
$$\mathrm{Fix}(\sigma) = E_1 \cup \cdots \cup E_k \cup \{p_1, \ldots, p_n\},$$

where C is a curve of genus g, E_i, G_i are smooth rational curves such that $\sigma(G_i) = G_i$ and each G_i contains exactly two isolated fixed points of σ. We will denote by $n_2 = n - n_1$ the number of isolated fixed points of σ contained in C.

THEOREM 6.1. *Let σ be a non-symplectic automorphism on a* K3 *surface X such that $S(\sigma^2) = S(\sigma) = \mathrm{Pic}(X)$. Then the invariants of the fixed locus of σ and the lattices $S(\sigma^2)$ and $T(\sigma^2)$ (up to isomorphism) appear in Table 5. Moreover, all cases in the table do exist.*

PROOF.   By Proposition 1 we have that

$$\mathcal{X}(\mathrm{Fix}(\sigma^2)) - \mathcal{X}(\mathrm{Fix}(\sigma)) = 2 - 2g - n_2 = -2m.$$

Moreover, since σ acts on C as an involution with n_2 fixed points, then $g \geq n_2/2 - 1$ by Riemann–Hurwitz formula. These remarks, together with Proposition 1 and the classification theorem of 2-elementary even lattices [**24**, Theorem 3.6.2] give the list of cases appearing in the table and some more with $S(\sigma^2)$ isomorphic to one of the following lattices:

$$U \oplus A_1^{\oplus 4}, \ U \oplus D_6 \oplus A_1^{\oplus 2}, \ U \oplus D_4 \oplus A_1^{\oplus 4}, \ U \oplus E_8 \oplus A_1^{\oplus 4}, \ U \oplus E_8 \oplus E_7 \oplus A_1,$$
$$U \oplus E_7 \oplus A_1, \ (2) \oplus A_1.$$

In the first five cases X has a σ-invariant jacobian elliptic fibration $\pi : X \to \mathbb{P}^1$ with more than two reducible fibers by Theorem 2.1. Since σ fixes > 2 points in the basis of π, then it preserves each fiber of π. This gives a contradiction since σ should have two fixed points in each fiber while the fibration has a unique section fixed by σ.

We now show that the case $S(\sigma^2) \cong (2) \oplus A_1$ does not appear. Let e, f be the generators of $S(\sigma^2)$ with $e^2 = 2$, $f^2 = -2$, $e \cdot f = 0$. The class e is nef and the associated morphism is a degree two map $\pi : X \to \mathbb{P}^2$ which is the minimal resolution of the double cover branched along an irreducible plane sextic S with a node at the image of the curve with class $\pm f$. Since $\sigma^*(e) = e$, then σ induces a projectivity $\bar{\sigma}$ of $\mathbb{P}^2$ with $\bar{\sigma}^2 = \mathrm{id}$ (since it fixes $\pi(C)$). This implies that, up to a choice of coordinates, a birational model of X and σ are given as follows:

$$X : \ w^2 = f_6(x_0, x_1, x_2), \quad \sigma(x_0, x_1, x_2, w) = (-x_0, x_1, x_2, iw),$$

where f_6 is a homogeneous degree six polynomial with $\sigma(f_6) = -f_6$ and singular at one point. Observe that such polynomial f_6 contains $x_0 = 0$ as a component, giving a contradiction. If $S(\sigma) \cong U \oplus E_7 \oplus A_1 \cong (2) \oplus A_1 \oplus E_8$, then X is the minimal resolution of the double cover of $\mathbb{P}^2$ branched along an irreducible sextic with a node and a triple point of type E_8. We can exclude this case by an argument similar to the previous one.

If $T(\sigma^2)$ is any lattice appearing in Table 5, then it carries an isometry τ with $\tau^2 = -\mathrm{id}$ and acting trivially on the discriminant group by Lemma 7. It follows that the isometries $\mathrm{id}_{S(\sigma^2)}$ and τ glue to give an order four isometry ρ of L_{K3}. By the Torelli-type Theorem [**22**, Theorem 3.10] there exists a K3 surface X with a non-symplectic automorphism σ of order four such that $\sigma^* = \rho$ up to conjugacy. Thus all cases in the table do exist. $\square$

REMARK 6.2.   In all cases of Table 5 the lattices $S(\sigma^2)$ and $T(\sigma^2)$ are 2-elementary even lattices with $x^2 \in \mathbb{Z}$ for all $x \in L^\vee = \mathrm{Hom}(L, \mathbb{Z})$. A lattice theoretical proof of this fact and an alternative proof of Theorem 6.1 are given by Taki in [**33**, Proposition 2.4].

Table 5. The case $l = 0$.

m	r	n_1	n_2	k	g	$S(\sigma^2)$	$T(\sigma^2)$
10	2	2	2	0	10	U	$U \oplus U \oplus E_8^{\oplus 2}$
	2	0	4	0	9	$U(2)$	$U \oplus U(2) \oplus E_8^{\oplus 2}$
8	6	2	4	1	7	$U \oplus D_4$	$U \oplus U \oplus E_8 \oplus D_4$
	6	0	6	1	6	$U(2) \oplus D_4$	$U \oplus U(2) \oplus E_8 \oplus D_4$
6	10	6	2	2	6	$U \oplus E_8$	$U \oplus U \oplus E_8$
	10	4	4	2	5	$U(2) \oplus E_8$	$U \oplus U(2) \oplus E_8$
	10	2	6	2	4	$U \oplus D_4^{\oplus 2}$	$U \oplus U \oplus D_4^{\oplus 2}$
	10	0	8	2	3	$U(2) \oplus D_4^{\oplus 2}$	$U \oplus U(2) \oplus D_4^{\oplus 2}$
4	14	6	4	3	3	$U \oplus E_8 \oplus D_4$	$U \oplus U \oplus D_4$
	14	4	6	3	2	$U(2) \oplus E_8 \oplus D_4$	$U \oplus U(2) \oplus D_4$
2	18	10	2	4	2	$U \oplus E_8^{\oplus 2}$	$U \oplus U$
	18	8	4	4	1	$U(2) \oplus E_8^{\oplus 2}$	$U \oplus U(2)$

EXAMPLE 6.3. Consider the elliptic fibration $\pi : X \to \mathbb{P}^1$ in Weierstrass form given by

$$y^2 = x^3 + a(t)x + b(t),$$

where $a(t)$ is an even polynomial and $b(t)$ is an odd polynomial in t. Observe that it carries the order four automorphism

$$(x, y, t) \mapsto (-x, iy, -t).$$

For generic coefficients X is a K3 surface, the fixed locus of σ^2 is the union of the curve of genus 10 defined by $y = 0$ and the section $x = z = 0$ and σ fixes four points on them (in the fibers over $t = 0, \infty$). It follows by Theorem 1.1, Proposition 1 and [**25**, Theorem 4.3.1] that $S(\sigma^2) \cong U$, $r = 2$, $l = 0$ and $m = 10$.

A geometric construction of this family of K3 surfaces can be given as follows. Let Y be the Hirzebruch surface $\mathbb{F}_4$ and $e, f \in \mathrm{Pic}(Y)$ be the classes of the (-4)-curve and of a fiber respectively. Observe that a section of $-2K_Y = 4e + 12f$ is the disjoint union of the (-4)-curve E and a curve C with class $3e + 12f$ (e is in the base locus of $4e + 12f$). The generic such C is smooth and the double cover $p : X \to Y$ branched along $C \cup E$ is a K3 surface. We denote by $\tilde{C}$ and $\tilde{E}$ the pull-backs of C and E by p, observe that $g(\tilde{C}) = 10$ and $g(\tilde{E}) = 0$. We will consider the affine coordinates $t = u_1/u_2$ and $x = v_1/v_2$, where u_1, u_2 give a basis of $H^0(Y, f)$ and $v_1 \in H^0(Y, e + 4f)$, $v_2 \in H^0(Y, e)$ are non-zero sections. Let $\iota \in \mathrm{Aut}(Y)$ be the involution on Y given by $(t, x) \mapsto (-t, -x)$ and let $\varphi = 0$ be the equation of $\tilde{C}$ in local coordinates. If $\varphi(-t, -x) = -\varphi(t, x)$, then $\varphi(t, x) = x^3 + a(t)x + b(t)$ where a is an even, degree 8 and b is an odd, degree 11 polynomial in t. In this case ι lifts to an order 4 automorphism σ on X, in local coordinates:

$$X : \ y^2 = \varphi(t, x), \qquad \sigma(t, x, y) = (-t, -x, iy).$$

The ruling of Y induces a jacobian elliptic fibration on X having $\tilde{C}$ as a trisection and $\tilde{E}$ as a section. The Weierstrass equation of such fibration and the action of σ on it are clearly the same as the ones given for π at the beginning of this example.

EXAMPLE 6.4. Consider the involution $\iota : ((x_0, y_0), (x_1, y_1)) \mapsto ((x_0, -y_0), (x_1, -y_1))$ of $\mathbb{P}^1 \times \mathbb{P}^1$ and let f be a bihomogeneous polynomial of degree $(4, 4)$ such that $\iota(f) = -f$. If $C = \{f = 0\}$ is a smooth curve, then the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along C

$$X : \ w^2 = f(x_0, x_1, y_0, y_1)$$

is a K3 surface and carries the order four automorphism:

$$\sigma : \ (((x_0, y_0), (x_1, y_1)), w) \mapsto (((x_0, -y_0), (x_1, -y_1)), iw).$$

The fixed locus of σ^2 is the genus 9 curve defined by $w = 0$ and σ fixes 4 points on it. It follows by Theorem 1.1, Proposition 1 and [**25**, Theorem 4.3.1] that $S(\sigma^2) \cong U(2)$ (it is the pull-back of the Picard group of the quadric), $r = 2$, $l = 0$ and $m = 10$.

7.   Fix (σ) only contains isolated points.

Let σ be a purely non-symplectic automorphism of order four on a K3 surface X having only isolated fixed points. It follows from Proposition 1 that $\text{Fix}(\sigma)$ contains exactly four points $p_1, \ldots, p_4$. Moreover, the fixed locus of σ^2 is as follows:

$$\text{Fix}(\sigma^2) = C \cup (F_1 \cup F_1') \cup \cdots \cup (F_a \cup F_a') \cup G_1 \cup \cdots \cup G_{n_1/2},$$

where each G_i is a smooth rational curve which contains 2 fixed points of σ and C is a smooth genus g curve which contains the remaining $n_2 := 4 - n_1$ fixed points.

THEOREM 7.1. *Let σ be a purely non-symplectic automorphism of order 4 having only isolated fixed points on a K3 surface X. Then σ fixes exactly 4 points. Moreover, if $\text{Pic}(X) = S(\sigma^2)$ and $l > 0$, then the invariants of $\text{Fix}(\sigma^2)$ and the lattice $S(\sigma^2)$ (up to isomorphism) appear in Table 6.*

PROOF. Since $\alpha = k = 0$, it follows from Proposition 1 that $n = 4$, $r = l + 2$ and $m = 10 - l$. The fixed locus of σ^2 contains $j = 2a + n_1/2$ smooth rational curves and a curve of genus g. Thus by Theorem 1.1 we obtain $r + l = 11 - g + 2a + n_1/2$.

Observe that the case $l = 1, n_1 = 4$ does not exist since in this case, by Theorem 1.1 and [**25**, Theorem 4.3.1] (see Figure 1 in [**3**]), a curve fixed by σ^2 has genus ≤ 8.

We now assume that $S(\sigma^2) \cong U \oplus R$, where R is a direct sum of root lattices. If $g > 4$, then by Theorem 2.1 the surface X carries a σ-invariant jacobian elliptic fibration π with reducible fibers of type R. The automorphism σ acts as an involution on $\mathbb{P}^1$ and preserves two fibers F, F' of π. The curve $C \subset \text{Fix}(\sigma^2)$ of genus g intersects each fiber at 3 points and $C \cap F$, $C \cap F'$ are σ-invariant. This implies that C contains at least two fixed points. Moreover, π has a unique section fixed by σ^2 which is σ-invariant. Thus

Table 6. The case $\alpha = k = 0$, $l > 0$.

m	r	n_1	g	a	$S(\sigma^2)$
9	3	2	8	0	$U \oplus A_1^{\oplus 2}$
		0	7	0	$U(2) \oplus A_1^{\oplus 2}$
8	4	2	6	0	$U \oplus A_1^{\oplus 4}$
		0	5	0	$U(2) \oplus A_1^{\oplus 4}$
7	5	2	4	0	$U \oplus A_1^{\oplus 6}$
		0	3	0	$U(2) \oplus A_1^{\oplus 6}$
6	6	4	3	0	$U(2) \oplus D_4^{\oplus 2}$
		2	2	0	$U \oplus A_1^{\oplus 8}$
		0	1	0	$U(2) \oplus A_1^{\oplus 8}, U \oplus E_8(2)$
		0	3	1	$U \oplus D_4 \oplus A_1^{\oplus 4}, U(2) \oplus D_4^{\oplus 2}$
		2	4	1	$U \oplus D_4^{\oplus 2}, U \oplus D_6 \oplus A_1^{\oplus 2}$
5	7	4	1	0	$U(2) \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 2}$
		2	0	0	$U \oplus A_1^{\oplus 10}$
		0	1	1	$U(2) \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 2}$
		2	2	1	$U \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 2}$
		0	3	2	$U \oplus E_7 \oplus A_1^{\oplus 3}$
		2	4	2	$U \oplus E_8 \oplus A_1^{\oplus 2}$
4	8	2	0	1	$U \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 4}$
		4	1	1	$U(2) \oplus D_6^{\oplus 2}, U \oplus D_4^{\oplus 3}$
		0	1	2	$U(2) \oplus D_6^{\oplus 2}, U \oplus D_4^{\oplus 3}$
		2	2	2	$U \oplus D_6^{\oplus 2}, U(2) \oplus D_4 \oplus E_8$
		0	3	3	$U \oplus E_8 \oplus D_4$
3	9	2	0	2	$U \oplus D_6^{\oplus 2} \oplus A_1^{\oplus 2}$
		4	1	2	$U(2) \oplus E_7^{\oplus 2}$
		0	1	3	$U(2) \oplus E_7^{\oplus 2}$
		2	2	3	$U \oplus E_7^{\oplus 2}$
2	10	2	0	3	$U \oplus E_7^{\oplus 2} \oplus A_1^{\oplus 2}, U \oplus D_8^{\oplus 2}$
		4	1	3	$U \oplus E_8 \oplus E_7 \oplus A_1, U(2) \oplus E_8^{\oplus 2}$
		0	1	4	$U \oplus E_8 \oplus E_7 \oplus A_1, U(2) \oplus E_8^{\oplus 2}$
1	11	2	0	4	$U \oplus E_8^{\oplus 2} \oplus A_1^{\oplus 2}$

$n_1 = n_2 = 2$. In particular $a = 0$ if the rational curves fixed by σ^2 are at most 2.

Observe that, since C intersects the generic fiber of π in three points, then it is trigonal. If $g \geq 3$, this implies that C is not hyperelliptic, hence the canonical morphism of C is an embedding in $\mathbb{P}^{g-1}$. The involution σ on C is thus induced by an automorphism of the projective space. If $g = 3$, this implies that C is isomorphic to a plane quartic and, since an involution of $\mathbb{P}^2$ fixes a line, then $n_2 > 0$.

If $(m, r, g, a) = (8, 4, 6, 0)$ and $S(\sigma^2) \cong U(2) \oplus D_4$, then X is the double cover of $\mathbb{P}^2$ branched along the union of a smooth quintic and a line [**19**, Section 4.2]. By [**10**, Corollary 2.4.3] an involution on a smooth plane quintic has quotient of genus two,

equivalently it has 6 fixed points. Thus this case does not appear.

If $(m, r, g, a) = (7, 5, 6, 1)$, then $S(\sigma^2) \cong U \oplus D_6$. Observe that σ^2 fixes the section of π and two irreducible components of the fiber of type I_2^*. Since $a = 1$ and the fibration is σ-invariant, then σ should act as a reflection on the fiber I_2^*, but this is not possible since the fibration has a unique section. Thus this case does not appear.

The cases in the table are then obtained by taking all possible values for l, using the equations and remarks at the beginning of the proof. The lattices can be computed by means of Theorem 1.1, Proposition 1, and the classification theorem of 2-elementary lattices [25, Theorem 4.3.1]. □

EXAMPLE 7.2. Let $C, E \subset \mathbb{F}_4$ as in Example 6.3. If C has rational double points, then the minimal resolution X of the double cover of $\mathbb{F}_4$ branched along $C \cup E$ is still a K3 surface. If C has two nodes exchanged by ι, then ι lifts to an order four automorphism σ of X such that σ^2 fixes a curve of genus 8 (the pull-back of the proper transform of C) and a smooth rational curve $\tilde{E}$ (the pull-back of the (-4)-curve E). Observe that here $l > 0$ since the exceptional divisors over the two nodes are exchanged by σ. The lattice $S(\sigma^2)$ in this case is isomorphic to $U \oplus A_1^{\oplus 2}$.

If C has two triple points exchanged by ι, then σ^2 fixes the pull-back of the proper transform of the curve C, $\tilde{E}$ and the central components of the resolution trees over the two singular points, which are of type $\tilde{D}_4$. In this case $a = 1$, since such components are exchanged by σ, and the lattice $S(\sigma^2)$ is isomorphic to $U \oplus D_4^{\oplus 2}$. Similarly, we get examples if the triple points of C are simple singularities of type D_n ($n \geq 6$), E_7, E_8.

Considering C with ordinary nodes (up to 10) and triple points exchanged by ι we obtain several examples for the cases in the table with $S(\sigma^2) \cong U \oplus R$, where R is a direct sum of root lattices.

Similarly, we can construct examples for the cases of type $U(2) \oplus R$ by generalizing Example 6.4 to the case when the curve C in $\mathbb{P}^1 \times \mathbb{P}^1$ has simple singularities.

EXAMPLE 7.3. Consider the jacobian elliptic fibration $\pi : X \to \mathbb{P}^1$ defined as follows:

$$y^2 = x(x^2 + a(t^4)x + b(t^4)),$$

where a, b are polynomials of degree 1 and 2 respectively. Observe that π has a 2-torsion section $t \mapsto (0, 0, t)$. The translation by this section gives a symplectic involution τ on X. Moreover, X has the order four non-symplectic automorphisms $\sigma : (x, y, z, t) \mapsto (x, y, z, it)$ and $\sigma' := \sigma \circ \tau$. For generic a, b, π has 8 fibers of type I_2 and 8 fibers of type I_1. The automorphisms σ and σ' act with order four on $\mathbb{P}^1$, preserve the two smooth fibers over $t = 0$ and $t = \infty$ and act as an involution over $t = \infty$. Moreover, σ fixes pointwisely the fiber over $t = 0$, while σ' acts as an order two translation on it.

For special choices of a and b we can obtain examples with reducible fibers of type I_{4M} over $t = 0$ or $t = \infty$. For example, if $a(t^4) = t^4$ and $b(t^4) = 1$, then π has a smooth fiber over $t = 0$ and a fiber of type I_{16} over $t = \infty$. The automorphism σ fixes pointwisely the fiber over $t = 0$ and acts on the fiber over $t = \infty$ as a reflection which leaves invariant the components Θ_0, Θ_8 (see the notation in Example 5.2), giving $k = 0, a = 3, n = 4$.

The symplectic involution τ acts over $t = \infty$ as a rotation sending Θ_0 to Θ_8. Finally, the automorphism σ' acts over $t = 0$ as a translation and over $t = \infty$ as a reflection which leaves invariant the components Θ_4, Θ_{12}, giving $k = 0$, $a = 3$, $n_2 = 0$, $n_1 = 4$ (see Table 6). For more details on this example see also [**14**, Proposition 4.7].

8. The other cases.

At this point of the classification the cases left out are those with

$$\text{Fix}(\sigma) = E_1 \cup \cdots \cup E_k \cup \{p_1, \ldots, p_n\},$$

$$\text{Fix}(\sigma^2) = C \cup (E_1 \cup \cdots \cup E_k) \cup (F_1 \cup F_1' \cup \cdots \cup F_a \cup F_a') \cup (G_1 \cup \cdots \cup G_{n_1/2}),$$

where C is a curve of genus $g > 0$, $k > 0$, $l > 0$ and $n_2 = n - n_1$ is the number of fixed points on C. Observe that in this case $\alpha = k$.

THEOREM 8.1. *Assume that $g = g(C) > 1$. Then $g \leq m$ and we are in one of the following cases*:

$m + l$	k	$g \leq$	$a \leq$
4	3	3	2
6	2	5	3
8	1	7	4

PROOF. By computing the difference $\chi(\text{Fix}(\sigma^2)) - \chi(\text{Fix}(\sigma))$ topologically and using the Lefschetz's formula, one gets the relation

$$g - 2a = m - l + 1 - \frac{n_2}{2}. \tag{1}$$

By Theorem 1.1 we also have:

$$g + j = g + 2a + k + \frac{n_1}{2} \leq 11.$$

By the previous two relations and Proposition 1 we get

$$g \leq 5 - k + \frac{m - l}{2} = m,$$

and

$$4a \leq 8 - 2k + n_2 + l - m \leq 12 + l - m, \tag{2}$$

where we obtain the last inequality using $n_2 \leq n = 2k + 4$. By Proposition 1 we have that $m + l$ is even and $m + l \leq 8$, so that $m + l = 4, 6, 8$.

We now show that the case $m + l = 4$, $k = 3$ and $a = 3$ is not possible. By (2) we get $l \geq m$ and by (1) with $n_2 \leq 2k + 4 = 10$ we get $g \leq 2 + m - l$. If $l > m$ then

$g < 2$, which is not possible. If $l = m$ then we have $g = 2$, hence again $n_2 = 10$ by (1), contradicting the Riemann–Hurwitz formula. The case $m + l = 6$, $a = 4$ can be excluded similarly.   □

EXAMPLE 8.2.   Consider the following elliptic K3 surface $\pi : X \to \mathbb{P}^1$ in Weierstrass form:

$$y^2 = x^3 - a(t)x, \quad \deg a(t) = 8,$$

with the order four automorphism

$$\sigma(x, y, t) = (-x, iy, t).$$

The automorphism σ fixes the two sections $x = y = 0$, $x = z = 0$ (hence $k \geq 2$), while σ^2 also fixes the curve $C : y = x^2 - a(t) = 0$. The discriminant of the fibration is $\Delta(t) = 4a^3(t)$, hence for a generic polynomial $a(t)$ the fibration has 8 fibers of type III (more precisely these are of type IIIb)). Moreover, the curve C has genus 3 and σ has 8 fixed points on it, so that: $k = 2$, $n_2 = 8$, $a = 0$. One can compute also that $l = 0$, so that this case appears in Table 5. We now study how $a(t)$ can split:

i)   If $a(t) = a_1(t)^2 a_6(t)$, then π has a fiber of type I$_0^*$ and 6 of type IIIb). In this case the ramification points on C are 6, so that $g(C) = 2$. Here $k = 2$, $n_1 = 2$, $n_2 = 6$, $a = 0$. Thus we have an example in Theorem 8.1.

ii) If $a(t) = a_1(t)^2 b_1(t)^2 a_4(t)$, then π has two fibers of type I$_0^*$ and four of type IIIb). Here $k = 2$, $n_1 = 4$, $n_2 = 4$, $a = 0$ and $g = 1$. This case appears in Theorem 8.4.

iii) If $a(t) = a_1(t)^2 b_1(t)^2 c_1(t)^2 a_2(t)$, then we have 3 fibers of type I$_0^*$ and two of type IIIb), so that $k = 2$, $n_1 = 6$, $n_2 = 2$, $a = 0$ and $g = 0$. This case appears in Table 3.

iv) If $a(t) = a_1(t)^2 b_1(t)^2 c_1(t)^2 d_1(t)^2$, then we have four fibers of type I$_0^*$. In this case C splits into the union of two rational curves. In this case we have $k = 2$, $n = 8$, $a = 1$. We are again in one case of Table 3.

For generic $a(t)$ the Mordell–Weil group of X is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The translation by a generator of the group is a symplectic involution ι on X and the composition $\sigma' = \iota \circ \sigma$ is again a non-symplectic automorphism of order 4 on X which fixes C and exchanges the two sections of π.

EXAMPLE 8.3.   An easy computation with MAPLE shows that there are 63 possible cases in Theorem 8.1. One can produce some examples with the fixed locus described there starting from Example 6.3 (resp. Example 6.4) and imposing simple singularities on the curve C of genus 10 (resp. genus 9) such that at least one singular point is invariant for ι. For example, one can assume that the curve C in Example 6.3 has an ordinary triple point at an invariant point of ι and two nodes exchanged by ι. The Dynkin diagram of the resolution over the triple point, which is of type $\tilde{D}_4$, is σ-invariant: its simple components are preserved and the double component is fixed by σ. Thus $k = 1$, $n_2 = 4$, $n_1 = 2$, $a = 0$ and $g = 5$. Observe that if C has just an ordinary triple point at an invariant point, then we are in the case $g = 7$, $k = 1$, $n_1 = 2$, $n_2 = 4$ of Table 5. Similar examples can be constructed from Example 6.4.

We now consider the case when σ^2 fixes an elliptic curve.

THEOREM 8.4. *With the previous notation, if $g(C) = 1$ then we are in one of the cases appearing in Table* 7.

Table 7.   The case $g = 1$, $k > 0$, $l > 0$.

m	r	l	n_1	n_2	k	a	type of C'
5	9	3	2	4	1	0	I_4
4	12	2	4	4	2	0	I_8
3	15	1	6	4	3	0	I_{12}
4	10	4	2	4	1	1	IV^*
4	10	4	6	0	1	0	IV^*

PROOF.   Using the relations in the proof of Theorem 8.1 one can find the values in the table, we now show that these are the only possibilities.

Since σ preserves C, then there is a σ-invariant elliptic fibration $\pi_C : X \to \mathbb{P}^1$ with fiber C. Observe that σ has order four on the basis of π_C, since otherwise σ^2 would act as the identity on the tangent space at a point of C. Thus σ has two fixed points on $\mathbb{P}^1$, corresponding to the fiber C and a fiber C' of π_C. This implies that all rational curves fixed by σ are contained in C', so that C' is reducible (since $k > 0$). Observe that σ acts on C either as an involution with four fixed points or as a translation. By Proposition 1 we have $n \geq 6$, so that C' contains at least two fixed points of σ. This excludes the Kodaira types I_2, I_3, III, IV for C'. By similar arguments as in the proof of Theorem 3.1 also the types I_N^*, II* and III* can be excluded. If C' has Kodaira type IV* then σ either preserves each component or it exchanges two branches. In the first case $a = 0$ and either $n = n_1 = 6$ with $k = 1$ or $n_1 = n_2 = 4$ with $k = 2$. By Lemma 4 the last case is not possible since if each rational curve is σ-invariant, then IV* contains only one fixed rational curve. In the second case $a = 1$, $k = 1$, $n_1 = 2$ and $n_2 = 4$.

We now consider the case when C' is of type I_N, $N \geq 4$. Since C' contains at least a fixed curve for σ, then all components of I_N are preserved by σ, hence $a = 0$. By the previous remarks, since $n_2 = 4$ or 0, then either $n_1 = 2k$ or $n_1 = 2k + 4$ respectively.

If $n_2 = 4$, then $N = 2k + n_1 = 4k$. For $N > 12$ we get $k > 3$, which gives $m < 3$ by Proposition 1. By the equality (1) we get $m - l = 2$ so $m = 2$ and $l = 0$, contradicting the assumption $l > 0$. Hence we only have the cases in the table.

On the other hand, if $n_2 = 0$ we get $N = 2k + (2k + 4) = 4k + 4$. By equation (1) we get $l = m$ and $k + m = 5$ by Proposition 1. If $N \geq 16$ then $k \geq 3$, which gives $m < 3$ as before. If $m = l = 1$, then $k = 4$, $n_1 = 12$ and C' is of type I_{20}. If $m = l = 2$, then $k = 3$, $n_1 = 10$ and C' is of type I_{16}. If $m = l = 3$ (or $m = l = 4$) then $k = 1$ (respectively $k = 2$) and we have a fiber I_8 (respectively I_{12}). All these cases can be excluded by Lemma 4 since if $a = 0$ then a fiber of type I_{4M}, $M \geq 2$ has $k = M$.   □

EXAMPLE 8.5.   Observe that the elliptic K3 surface $\pi_C : X \to \mathbb{P}^1$ in Example 3.2 also carries the non-symplectic order four automorphism

$$\sigma'(x, y, t) = (x, -y, it),$$

obtained by composing the automorphism σ defined there with the non-symplectic involution $y \mapsto -y$. The automorphism $\sigma^2 = (\sigma')^2$ is an involution fixing the smooth elliptic curve over $t = 0$ and some rational curves in the singular fiber over $t = \infty$. Generically, σ' acts on the elliptic curve over $t = 0$ as an involution with 4 isolated fixed points and as the identity on the fiber over $t = \infty$. If the fiber over $t = \infty$ is reducible, then σ fixes at most rational curves, in particular $k = \alpha$. Observe that σ' preserves the section at infinity $t \mapsto (0 : 1 : 0; t)$ and has two fixed points on it, one on the fiber over $t = 0$, the other over $t = \infty$. Using this condition and the equality $n = 2\alpha + 4$ given by Proposition 1, one sees that $n = 6$, $k = 1$ if π_C has a fiber of type I_4, $(k, n, a) = (2, 8, 0)$ for a fiber of type I_8, $(3, 10, 0)$ for a fiber of type I_{12} and $(k, n, a) = (4, 12, 0)$ for a fiber of type I_{16}. In case π_C has a fiber of type IV^*, σ' acts as a reflection fixing one curve, giving $(k, n, a) = (1, 6, 1)$.

The cases with the fibers of type I_4, I_8, I_{12} and IV^* give examples with $l \neq 0$ that appear in Table 7. In case there is a fiber of type I_{16} we have instead $l = 0$ (see the last line of Table 5).

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