Mean value theorems for the double zeta-function

By Kohji Matsumoto and Hirofumi Tsumura

Abstract. We prove asymptotic formulas for mean square values of the Euler double zeta-function $\zeta_2(s_0, s)$, with respect to $\Im s$. Those formulas enable us to propose a double analogue of the Lindelöf hypothesis.

1. Introduction and the statement of results.

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers and $i = \sqrt{-1}$.

The Euler double zeta-function is defined by

$$\zeta_2(s_1, s_2) = \sum_{m=1}^{\infty} \frac{1}{m^{s_1}} \sum_{n=1}^{\infty} \frac{1}{(m+n)^{s_2}} = \sum_{k=2}^{\infty} \left(\sum_{m=1}^{k-1} \frac{1}{m^{s_1}} \right) \frac{1}{k^{s_2}} \tag{1.1}$$

which is absolutely convergent for $s_1, s_2 \in \mathbb{C}$ with $\Re s_2 > 1$ and $\Re(s_1 + s_2) > 2$ (Theorem 3 in [**9**]), and can be continued meromorphically to $\mathbb{C}^2$. The singularities are $s_2 = 1$ and $s_1 + s_2 = 2, 1, 0, -2, -4, \ldots$ (Theorem 1 in [**1**]). Euler himself considered the behaviour of this function when s_1, s_2 are positive integers. It was Atkinson [**3**] who first studied (1.1) from the analytic viewpoint, and he proved the analytic continuation of it. Recently the active research of (1.1) revived, because it is the simplest example of multiple zeta-functions. As for the studies on the analytic side of (1.1), for example, upper-bound estimates were discussed in [**5**], [**6**], [**7**], and functional equations were discovered in [**8**], [**12**].

It is the purpose of the present paper to prove certain mean square formulas for (1.1). Let

$$\zeta_2^{[2]}(s_1, s_2) = \sum_{k=2}^{\infty} \left| \sum_{m=1}^{k-1} \frac{1}{m^{s_1}} \right|^2 \frac{1}{k^{s_2}}. \tag{1.2}$$

Since the inner sum is $O(1)$ (if $\Re s_1 > 1$), $O(\log k)$ (if $\Re s_1 = 1$), or $O(k^{1-\Re s_1})$ (if $\Re s_1 < 1$), the series (1.2) is convergent when $\Re s_1 \geq 1$ and $\Re s_2 > 1$, or when $\Re s_1 < 1$ and $2\Re s_1 + \Re s_2 > 3$. Note that $\zeta_2^{[2]}(1, q)$ ($q \in \mathbb{N}_{\geq 2}$) was already studied by Borwein et al. (see [**4**]).

Hereafter we write s_0 and s instead of s_1 and s_2, respectively, and consider the mean square with respect to s, while s_0 is to be fixed.

THEOREM 1.1.   *For $s_0 = \sigma_0 + it_0 \in \mathbb{C}$ with $\sigma_0 > 1$ and $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$, $t \geq 2$, we have*

$$\int_2^T |\zeta_2(s_0, s)|^2 dt = \zeta_2^{[2]}(s_0, 2\sigma)T + O(1) \qquad (T \to \infty). \tag{1.3}$$

THEOREM 1.2.   *For $s_0 = \sigma_0 + it_0 \in \mathbb{C}$ with $\sigma_0 > 1$ and $s = \sigma + it \in \mathbb{C}$ with $1/2 < \sigma \leq 1$, $t \geq 2$ and $\sigma_0 + \sigma > 2$, we have*

$$\int_2^T |\zeta_2(s_0, s)|^2 dt = \zeta_2^{[2]}(s_0, 2\sigma)T + O(T^{2-2\sigma} \log T) + O(T^{1/2}). \tag{1.4}$$

The most important result in the present paper is the following Theorem 1.3, which describes the situation under the condition $3/2 < \sigma_0 + \sigma \leq 2$.

THEOREM 1.3.   *Let $s_0 = \sigma_0 + it_0 \in \mathbb{C}$ with $1/2 < \sigma_0 < 3/2$ and $s = \sigma + it \in \mathbb{C}$ with $1/2 < \sigma \leq 1$, $t \geq 2$ and $3/2 < \sigma_0 + \sigma \leq 2$. Assume that when t moves from 2 to T, the point (s_0, s) does not encounter the hyperplane $s_0 + s = 2$ (which is a singular locus of ζ_2). Then*

$$\int_2^T |\zeta_2(s_0, s)|^2 dt = \zeta_2^{[2]}(s_0, 2\sigma)T$$

$$+ \begin{cases} O(T^{4-2\sigma_0-2\sigma} \log T) + O(T^{1/2}) & \left(\frac{1}{2} < \sigma_0 < 1, \frac{1}{2} < \sigma < 1\right) \\[2mm] O(T^{2-2\sigma_0}(\log T)^2) + O(T^{1/2}) & \left(\frac{1}{2} < \sigma_0 < 1, \sigma = 1\right) \\[2mm] O(T^{2-2\sigma}(\log T)^3) + O(T^{1/2}) & \left(\sigma_0 = 1, \frac{1}{2} < \sigma < 1\right) \\[2mm] O(T^{1/2}) & (\sigma_0 = 1, \sigma = 1) \\[2mm] O(T^{2-2\sigma} \log T) + O(T^{1/2}) & \left(1 < \sigma_0 < \frac{3}{2}, \frac{1}{2} < \sigma < 1\right). \end{cases} \tag{1.5}$$

REMARK 1.4.   In Theorems 1.2 and 1.3, the error terms $O(T^{1/2})$ are coming from the simple application of the Cauchy-Schwarz inequality. It is plausible to expect that we can reduce these error terms by more elaborate analysis.

It is interesting to compare our theorems with the classical results on the mean square of the Riemann zeta-function $\zeta(s)$. It is known that

$$\int_2^T |\zeta(\sigma + it)|^2 dt \sim \zeta(2\sigma)T \qquad \left(\sigma > \frac{1}{2}\right) \tag{1.6}$$

and

$$\int_2^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \sim T \log T \tag{1.7}$$

(see Titchmarsh [**15**, Theorems 7.2, 7.3]). These simple results suggest two important observations.

(a) First, it is trivial that $\zeta(\sigma + it)$ is bounded with respect to t in the region of absolute convergence $\sigma > 1$, but (1.6) and (1.7) suggest that $\zeta(\sigma + it)$ seems not so large in the strip $1/2 \leq \sigma \leq 1$, too. In fact, the well-known Lindelöf hypothesis predicts that

$$\zeta(\sigma + it) = O(t^\varepsilon) \qquad \left(\frac{1}{2} \leq \sigma < 1\right) \tag{1.8}$$

for any $\varepsilon > 0$. (For $\sigma = 1$, even a stronger estimate has already been known.) Formulas (1.6) and (1.7) support this hypothesis.

(b) The second observation is that the coefficient $\zeta(2\sigma)$ on the right-hand side of (1.6) tends to infinity as $\sigma \to 1/2$, hence the form of the formula should be changed at $\sigma = 1/2$, which is in fact embodied by (1.7). This is one of the special features of the "critical line" $\Re s = 1/2$ in the theory of the Riemann zeta-function.

Our theorems proved in the present paper may be regarded as double analogues of (1.6). Since the coefficient $\zeta_2^{[2]}(s_0, 2\sigma)$ tends to infinity as $\sigma_0 + \sigma \to 3/2$, it is natural to raise, analogously to the above (a) and (b), the following two conjectures:

(i) (a double analogue of the Lindelöf hypothesis) For any $\varepsilon > 0$,

$$\zeta_2(s_0, s) = O(t^\varepsilon) \tag{1.9}$$

when (s_0, s) (which is not in the domain of absolute convergence) satisfies $\sigma_0 > 1/2$, $\sigma > 1/2$, $t \geq 2$, $\sigma_0 + \sigma \geq 3/2$ and $s_0 + s \neq 2$;

(ii) (the criticality of $\sigma_0 + \sigma = 3/2$) When $\sigma_0 + \sigma = 3/2$, the form of the main term of the mean square formula would not be CT (with a constant C; most probably, some log-factor would appear).

REMARK 1.5.   It is not easy to find the "correct" double analogue of the Lindelöf hypothesis. Nakamura and Pańkowski [**14**] raised the conjecture

$$\zeta_2(1/2 + it, 1/2 + it) = O(t^\varepsilon) \tag{1.10}$$

(actually they stated their conjecture for more general multiple case), and gave a certain result (their Proposition 6.3) which supports the conjecture. However, the value $\zeta_2(1/2 + it_1, 1/2 + it_2)$ is, if $t_1 \neq t_2$, not always small. In fact, Corollary 1 of Kiuchi, Tanigawa and Zhai [**7**] describes the situation when $\zeta_2(s_1, s_2)$ is not small. For example, if $t_2 \ll t_1^{1/6-\varepsilon}$, then

$$\zeta_2(1/2 + it_1, 1/2 + it_2) = \Omega\big(t_1^{1/3+\varepsilon}\big).$$

Our theorems imply that our conjecture (1.9) is true in mean. That is, (1.9) is reasonable in view of our theorems.

REMARK 1.6.   The above conjecture (ii) suggests that $\sigma_0 + \sigma = 3/2$ might be the double analogue of the critical line of the Riemann zeta-function $\Re s = 1/2$. On the other hand, in view of the result of Nakamura and Pańkowski mentioned above, we see that another candidate of the double analogue of the critical line is $\sigma_0 + \sigma = 1$. At present it is not clear which is more plausible.

REMARK 1.7.   We cannot expect the analogue of the Riemann hypothesis on the location of zeros. In fact, Theorem 5.1 of Nakamura and Pańkowski [**14**] asserts (in the double zeta case) that for any $1/2 < \sigma_1 < \sigma_2 < 1$, $\zeta_2(s, s)$ has $\asymp T$ non-trivial zeros in the rectangle $\sigma_1 < \sigma < \sigma_2$, $0 < t < T$.

The plan of the present paper is as follows. We first prove the simplest Theorem 1.1 in Section 2. To prove the other theorems, we need certain approximation formulas for $\zeta_2(s_0, s)$. Using the Euler-Maclaurin formula, we show the first approximation formula (Theorem 3.1) in Section 3, and using it, we prove Theorem 1.2 in Section 4. In Section 5 we introduce and discuss the double analogue of the Euler constant. The most difficult part of the present paper is the proof of Theorem 1.3. In Section 6 we show the second approximation formula (Theorem 6.3), by employing the method of Mellin-Barnes integral formula. Based on this second approximation formula, we give the proof of Theorem 1.3 in the final Section 7.

A possible direction of future study is to search for a strong type of approximate functional equation (that is, similar to [**15**, Theorem 4.16]) for the double zeta-function, based on our previous results on functional equations for the double zeta-function obtained in [**8**], [**12**]. If we could succeed in finding such an equation, we would be able to give a more precise version of mean value theorems for the double zeta-function.

A part of the results in this paper has been announced in [**13**].

2.   Proof of Theorem 1.1.

In this section, we give the proof of Theorem 1.1. Throughout this paper, we frequently use the following elementary estimations:

$$\sum_{m=1}^{k-1} \frac{1}{m} \ll \int_1^k u^{-1} du = \log k,$$

$$\sum_{m=1}^{k-1} \frac{1}{m^\sigma} \ll \int_0^k u^{-\sigma} du = \frac{k^{1-\sigma}}{1-\sigma} \qquad (0 < \sigma < 1),$$

$$\sum_{m=k}^{\infty} \frac{1}{m^\sigma} \ll \int_k^\infty u^{-\sigma} du = \frac{k^{1-\sigma}}{\sigma - 1} \qquad (\sigma > 1).$$

PROOF OF THEOREM 1.1.   Let $s_0 = \sigma_0 + it_0 \in \mathbb{C}$ with $\sigma_0 > 1$ and $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$. We set

$$S := \zeta_2(s_0, s)\overline{\zeta_2(s_0, s)} = \sum_{\substack{m_1 \geq 1 \\ n_1 \geq 1}} \frac{1}{m_1^{s_0}(m_1 + n_1)^{\sigma+it}} \sum_{\substack{m_2 \geq 1 \\ n_2 \geq 1}} \frac{1}{m_2^{\overline{s_0}}(m_2 + n_2)^{\sigma-it}}.$$

Taking out the terms corresponding to $m_1 + n_1 = m_2 + n_2$ and setting $k = m_1 + n_1$, we have

$$S = \sum_{k=2}^{\infty} \left(\sum_{m_1=1}^{k-1} \sum_{m_2=1}^{k-1} \frac{1}{m_1^{s_0} m_2^{\overline{s_0}}} \right) \frac{1}{k^{2\sigma}} + \sum_{\substack{m_1,m_2,n_1,n_2 \geq 1 \\ m_1+n_1 \neq m_2+n_2}} \frac{1}{m_1^{s_0} m_2^{\overline{s_0}}(m_1 + n_1)^{\sigma+it}(m_2 + n_2)^{\sigma-it}}$$

$$= \zeta_2^{[2]}(s_0, 2\sigma) + \sum_{\substack{m_1,m_2,n_1,n_2 \geq 1 \\ m_1+n_1 \neq m_2+n_2}} \frac{1}{m_1^{s_0} m_2^{\overline{s_0}}(m_1 + n_1)^{\sigma}(m_2 + n_2)^{\sigma}} \left(\frac{m_2 + n_2}{m_1 + n_1} \right)^{it}.$$

Hence we have

$$\int_2^T |\zeta_2(s_0, s)|^2 dt$$

$$= \zeta_2^{[2]}(s_0, 2\sigma)(T - 2) + \sum_{\substack{m_1,m_2,n_1,n_2 \geq 1 \\ m_1+n_1 \neq m_2+n_2}} \frac{1}{m_1^{s_0} m_2^{\overline{s_0}}(m_1 + n_1)^{\sigma}(m_2 + n_2)^{\sigma}} \int_2^T \left(\frac{m_2 + n_2}{m_1 + n_1} \right)^{it} dt.$$

The second term on the right-hand side is

$$\sum_{\substack{m_1,m_2,n_1,n_2 \geq 1 \\ m_1+n_1 \neq m_2+n_2}} \frac{1}{m_1^{s_0} m_2^{\overline{s_0}}(m_1 + n_1)^{\sigma}(m_2 + n_2)^{\sigma}}$$

$$\times \frac{e^{iT \log((m_2+n_2)/(m_1+n_1))} - e^{2i \log((m_2+n_2)/(m_1+n_1))}}{i \log((m_2 + n_2)/(m_1 + n_1))}$$

$$\ll \sum_{\substack{m_1,m_2,n_1,n_2 \geq 1 \\ m_1+n_1 < m_2+n_2}} \frac{1}{(m_1 m_2)^{\sigma_0}(m_1 + n_1)^{\sigma}(m_2 + n_2)^{\sigma}} \frac{1}{\log((m_2 + n_2)/(m_1 + n_1))}$$

$$= \left(\sum_{\substack{m_1,m_2,n_1,n_2 \geq 1 \\ m_1+n_1 < m_2+n_2 \leq 2(m_1+n_1)}} + \sum_{\substack{m_1,m_2,n_1,n_2 \geq 1 \\ m_2+n_2 > 2(m_1+n_1)}} \right) \frac{1}{(m_1 m_2)^{\sigma_0}}$$

$$\times \frac{1}{(m_1 + n_1)^{\sigma}(m_2 + n_2)^{\sigma} \log((m_2 + n_2)/(m_1 + n_1))}.$$

We denote the right-hand side by $V_1 + V_2$. Then we have

$$V_2 \ll \sum_{\substack{m_1,m_2,n_1,n_2 \geq 1 \\ m_2+n_2 > 2(m_1+n_1)}} \frac{1}{(m_1 m_2)^{\sigma_0}(m_1 + n_1)^{\sigma}(m_2 + n_2)^{\sigma}}$$

$$\ll \sum_{m_1,m_2,n_1,n_2 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}(n_1 n_2)^{\sigma}} = O(1).$$

As for V_1, setting $r = (m_2 + n_2) - (m_1 + n_1)$, we have

$$V_1 = \sum_{m_1,m_2,n_1 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{r=1}^{m_1+n_1} \frac{1}{(m_1+n_1)^\sigma (m_1+n_1+r)^\sigma} \frac{1}{\log((m_1+n_1+r)/(m_1+n_1))}.$$

Since $m_1 + n_1 + r \asymp m_1 + n_1$, we obtain

$$V_1 \ll \sum_{m_1,m_2,n_1 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \frac{1}{(m_1+n_1)^{2\sigma}} \sum_{r=1}^{m_1+n_1} \frac{1}{\log(1+(r/(m_1+n_1)))}$$

$$\ll \sum_{m_1,m_2,n_1 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \frac{1}{(m_1+n_1)^{2\sigma}} \sum_{r=1}^{m_1+n_1} \frac{m_1+n_1}{r}$$

$$\ll \sum_{m_1,m_2,n_1 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \frac{1}{(m_1+n_1)^{2\sigma-1}} \log(m_1+n_1)$$

$$\ll \sum_{m_2 \geq 1} \frac{1}{m_2^{\sigma_0}} \sum_{m_1,n_1 \geq 1} \frac{\log(m_1+n_1)}{m_1^{\sigma_0}(m_1+n_1)^{2\sigma-1}} = O(1),$$

because $\sigma_0 > 1$ and $\sigma > 1$. This completes the proof of Theorem 1.1.   □

REMARK 2.1.   The fundamental idea of the above proof of Theorem 1.1 is similar to that of the proof of [15, Theorem 7.2]. The basic structure of the proofs of Theorems 1.2 and 1.3 given below is the same, though the technical details are more complicated.

3.   The first approximation theorem.

Hardy and Littlewood proved the following well-known result (see [15, Theorem 4.11]). Let $\sigma_1 > 0$, $x \geq 1$ and $C > 1$. Suppose $s = \sigma + it \in \mathbb{C}$ with $\sigma \geq \sigma_1$ and $|t| \leq 2\pi x/C$. Then

$$\zeta(s) = \sum_{1 \leq n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \quad (x \to \infty). \tag{3.1}$$

Here we prove the double series analogue of (3.1) as follows.

THEOREM 3.1.   Let $s_0 = \sigma_0 + it_0 \in \mathbb{C}$, $s = \sigma + it \in \mathbb{C} \setminus \{1\}$, $x \geq 1$ and $C > 1$. Suppose $\sigma > \max(0, 2 - \sigma_0)$ and $|t| \leq 2\pi x/C$. Then

$$\zeta_2(s_0, s) = \sum_{m=1}^{\infty} \sum_{1 \leq n \leq x} \frac{1}{m^{s_0}(m+n)^s} - \frac{1}{1-s} \sum_{m=1}^{\infty} \frac{1}{m^{s_0}(m+x)^{s-1}}$$

$$+ \begin{cases} O(x^{-\sigma}) & (\sigma_0 > 1) \\ O(x^{-\sigma} \log x) & (\sigma_0 = 1) \\ O(x^{1-\sigma-\sigma_0}) & (\sigma_0 < 1) \end{cases} \quad (x \to \infty). \tag{3.2}$$

In order to prove this theorem, we quote the following lemma.

LEMMA 3.2 ([**15**, Lemma 4.10]). *Let $f(x)$ be a real function with a continuous and steadily decreasing derivative $f'(x)$ in (a, b), and let $f'(b) = \alpha$, $f'(a) = \beta$. Let $g(x)$ be a real positive decreasing function with a continuous derivative $g'(x)$, satisfying that $|g'(x)|$ is steadily decreasing. Then*

$$\sum_{a < n \leq b} g(n) e^{2\pi i f(n)} = \sum_{\substack{\nu \in \mathbb{Z} \\ \alpha - \eta < \nu < \beta + \eta}} \int_a^b g(x) e^{2\pi i (f(x) - \nu x)} dx$$

$$+ O(g(a) \log(\beta - \alpha + 2)) + O(|g'(a)|) \tag{3.3}$$

for an arbitrary $\eta \in (0, 1)$.

PROOF OF THEOREM 3.1. By the Euler-Maclaurin formula (see [**15**, Equation (2.1.2)]), we have

$$\sum_{a < l \leq b} \frac{1}{l^s} = \frac{b^{1-s} - a^{1-s}}{1 - s} - s \int_a^b \frac{y - [y] - 1/2}{y^{s+1}} dy + \frac{1}{2}(b^{-s} - a^{-s}) \tag{3.4}$$

for $0 < a < b$. At first assume $\sigma_0 > 1$, $\sigma > 1$. Setting $a = m + N$ (where $m \in \mathbb{N}$, $N \in \mathbb{N}_0$) in (3.4) and $b \to \infty$, we have

$$\sum_{l = m + N + 1}^{\infty} \frac{1}{l^s} = -\frac{(m + N)^{1-s}}{1 - s} - s \int_{m+N}^{\infty} \frac{y - [y] - 1/2}{y^{s+1}} dy - \frac{1}{2}(m + N)^{-s}.$$

Therefore we have

$$\sum_{m=1}^{\infty} \frac{1}{m^{s_0}} \sum_{n=1}^{\infty} \frac{1}{(m + n)^s} = \sum_{m=1}^{\infty} \frac{1}{m^{s_0}} \sum_{n=1}^{N} \frac{1}{(m + n)^s} - \sum_{m=1}^{\infty} \frac{(m + N)^{1-s}}{m^{s_0}(1 - s)}$$

$$- s \sum_{m=1}^{\infty} \frac{1}{m^{s_0}} \int_{m+N}^{\infty} \frac{y - [y] - 1/2}{y^{s+1}} dy - \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m^{s_0}(m + N)^s}$$

$$= A_1 - A_2 - A_3 - A_4, \tag{3.5}$$

say. The terms A_1 and A_4 are absolutely convergent in the region $\sigma_0 + \sigma > 1$, and in this region

$$A_4 = O\left(\sum_{m=1}^{\infty} \frac{1}{m^{\sigma_0}(m + N)^{\sigma}} \right). \tag{3.6}$$

The integral in A_3 is absolutely convergent if $\sigma > 0$, and is $O(\sigma^{-1}(m+N)^{-\sigma})$. Therefore A_3 can be continued to the region $\sigma > 0$, $\sigma_0 + \sigma > 1$ and

$$A_3 = O\left(\sum_{m=1}^{\infty} \frac{|s|/\sigma}{m^{\sigma_0}(m+N)^{\sigma}} \right) \tag{3.7}$$

there. The term A_2 is absolutely convergent for $\sigma_0 + \sigma > 2$, $s \neq 1$. Therefore we see that the right-hand side of (3.5) gives the meromorphic continuation to the desired region.

Hereafter in this proof we assume $N > x$. The term A_1 can be rewritten as

$$\sum_{m=1}^{\infty} \sum_{n \leq x} \frac{1}{m^{s_0}(m+n)^s} + \sum_{m=1}^{\infty} \sum_{x < n \leq N} \frac{e^{-it\log(m+n)}}{m^{s_0}(m+n)^{\sigma}}. \tag{3.8}$$

Fix $m \in \mathbb{N}$ and set

$$f(x) = \frac{t}{2\pi}\log(m+x), \quad g(x) = (m+x)^{-\sigma},$$

$(a,b) = (x,N)$ in Lemma 3.2. Then we have

$$(\alpha, \beta) = \left(\frac{t}{2\pi(m+N)}, \frac{t}{2\pi(m+x)} \right).$$

We see that

$$|f'(x)| = \frac{|t|}{2\pi(m+x)} \leq \frac{|t|}{2\pi x} \leq \frac{1}{C} < 1.$$

When $\sigma > 0$, the function $g(x)$ is decreasing and so we can apply Lemma 3.2. By taking a small η, we obtain from (3.3) that

$$\sum_{x < n \leq N} \frac{e^{it\log(m+n)}}{(m+n)^{\sigma}} = \int_x^N \frac{1}{(m+u)^{\sigma-it}}du + O((m+x)^{-\sigma}).$$

Considering complex conjugates on the both sides, we have

$$\sum_{x < n \leq N} \frac{1}{(m+n)^s} = \sum_{x < n \leq N} \frac{e^{-it\log(m+n)}}{(m+n)^{\sigma}} = \int_x^N \frac{1}{(m+u)^s}du + O((m+x)^{-\sigma})$$

$$= \frac{(m+N)^{1-s} - (m+x)^{1-s}}{1-s} + O((m+x)^{-\sigma}). \tag{3.9}$$

In other words, denoting the above error term by $E(s; x, m, N)$, we find that this function is entire in s (the point $s = 1$ is a removable singularity) and satisfies

$$E(s; x, m, N) = O((m+x)^{-\sigma}) \tag{3.10}$$

uniformly in N in the region $\sigma > 0$. Using (3.9), we find that the second term of (3.8) is equal to

$$\frac{1}{1-s}\sum_{m=1}^{\infty}\frac{1}{m^{s_0}(m+N)^{s-1}} - \frac{1}{1-s}\sum_{m=1}^{\infty}\frac{1}{m^{s_0}(m+x)^{s-1}} + \sum_{m=1}^{\infty}\frac{E(s;x,m,N)}{m^{s_0}} \quad (3.11)$$

(where the first two sums are convergent in $\sigma_0 + \sigma > 2$, while the last sum is convergent in $\sigma_0 + \sigma > 1$ because of (3.10)), whose first term is cancelled with A_2. Therefore now we have

$$\zeta_2(s_0,s) = \sum_{m=1}^{\infty}\sum_{n\leq x}\frac{1}{m^{s_0}(m+n)^s} - \frac{1}{1-s}\sum_{m=1}^{\infty}\frac{1}{m^{s_0}(m+x)^{s-1}}$$

$$+ \sum_{m=1}^{\infty}\frac{E(s;x,m,N)}{m^{s_0}} - A_3 - A_4 \quad (3.12)$$

in the region $\sigma > \max(0, 2 - \sigma_0)$, $s \neq 1$. Letting $N \to \infty$, and noting (3.6), (3.7) and (3.10), we obtain the proof of Theorem 3.1. □

4. Proof of Theorem 1.2.

In this section, using Theorem 3.1, we give the proof of Theorem 1.2.

PROOF OF THEOREM 1.2. Let $s_0 = \sigma_0 + it_0 \in \mathbb{C}$ with $\sigma_0 > 1$ and $s = \sigma + it \in \mathbb{C} \setminus \{1\}$ with $1/2 < \sigma \leq 1$, $\sigma_0 + \sigma > 2$. Setting $C = 2\pi$ and $x = t$ in (3.2), we easily see that the second term on the right-hand side is $O(t^{-\sigma})$, so we have

$$\zeta_2(s_0,s) = \sum_{m=1}^{\infty}\sum_{1\leq n\leq t}\frac{1}{m^{s_0}(m+n)^s} + O(t^{-\sigma}) \quad (t \to \infty). \quad (4.1)$$

We denote the first term on the right-hand side by $\Sigma_1(s_0,s)$. Let $M(n_1,n_2) = \max\{n_1, n_2, 2\}$. Then

$$\int_2^T |\Sigma_1(s_0,s)|^2 dt$$

$$= \int_2^T \sum_{m_1\geq 1}\sum_{n_1\leq t}\frac{1}{m_1^{s_0}(m_1+n_1)^{\sigma+it}}\sum_{m_2\geq 1}\sum_{n_2\leq t}\frac{1}{m_2^{\overline{s_0}}(m_2+n_2)^{\sigma-it}}dt$$

$$= \sum_{m_1\geq 1}\sum_{m_2\geq 1}\frac{1}{m_1^{s_0}m_2^{\overline{s_0}}}\sum_{n_1\leq T}\sum_{n_2\leq T}\frac{1}{(m_1+n_1)^{\sigma}(m_2+n_2)^{\sigma}}\int_{M(n_1,n_2)}^T\left(\frac{m_2+n_2}{m_1+n_1}\right)^{it}dt$$

$$= \sum_{m_1\geq 1}\sum_{m_2\geq 1}\frac{1}{m_1^{s_0}m_2^{\overline{s_0}}}\sum_{n_1\leq T}\sum_{\substack{n_2\leq T\\m_1+n_1=m_2+n_2}}\frac{1}{(m_1+n_1)^{2\sigma}}(T - M(n_1,n_2))$$

$$+ \sum_{m_1\geq 1}\sum_{m_2\geq 1}\frac{1}{m_1^{s_0}m_2^{\overline{s_0}}}\sum_{n_1\leq T}\sum_{\substack{n_2\leq T\\m_1+n_1\neq m_2+n_2}}\frac{1}{(m_1+n_1)^{\sigma}(m_2+n_2)^{\sigma}}$$

$$\times \frac{e^{iT\log((m_2+n_2)/(m_1+n_1))} - e^{iM(n_1,n_2)\log((m_2+n_2)/(m_1+n_1))}}{i\log((m_2+n_2)/(m_1+n_1))}. \tag{4.2}$$

We denote the first and the second term on the right-hand side by $S_1 T - S_2$ and S_3, respectively. As for S_1, setting $k = m_1 + n_1 (= m_2 + n_2)$, we have

$$S_1 = \sum_{k=2}^{\infty} \left(\sum_{m_1=1}^{k-1} \sum_{m_2=1}^{k-1} \frac{1}{m_1^{s_0} m_2^{\overline{s_0}}} \right) \frac{1}{k^{2\sigma}}$$

$$- \sum_{\substack{m_1 \geq 1 \\ m_2 \geq 1}} \frac{1}{m_1^{s_0} m_2^{\overline{s_0}}} \left\{ \sum_{\substack{n_1 > T \\ n_2 \leq T \\ m_1+n_1=m_2+n_2}} + \sum_{\substack{n_1 \leq T \\ n_2 > T \\ m_1+n_1=m_2+n_2}} + \sum_{\substack{n_1 > T \\ n_2 > T \\ m_1+n_1=m_2+n_2}} \right\} \frac{1}{(m_1+n_1)^{2\sigma}}.$$

We further denote the second term on the right-hand side by $-(U_1 + U_2 + U_3)$, which is equal to $-(U_1 + U_3) - (\overline{U_1} + U_3) + U_3$ because $U_2 = \overline{U_1}$. Since $\sigma_0 > 1$, we have

$$U_1 + U_3 \ll \sum_{\substack{m_1 \geq 1 \\ m_2 \geq 1}} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{n_1 > T} \frac{1}{(m_1+n_1)^{2\sigma}}$$

$$\ll \sum_{\substack{m_1 \geq 1 \\ m_2 \geq 1}} \frac{1}{(m_1 m_2)^{\sigma_0}} \int_T^{\infty} \frac{du}{(m_1+u)^{2\sigma}}$$

$$\ll \sum_{\substack{m_1 \geq 1 \\ m_2 \geq 1}} \frac{1}{(m_1 m_2)^{\sigma_0}(m_1+T)^{2\sigma-1}} \ll T^{1-2\sigma}.$$

Similarly we obtain $\overline{U_1} + U_3, U_3 \ll T^{1-2\sigma}$. Therefore we have

$$S_1 T = \zeta_2^{[2]}(s_0, 2\sigma)T + O(T^{2-2\sigma}). \tag{4.3}$$

As for S_2, since

$$M(n_1, n_2) = \max\{n_1, n_2, 2\} \leq m_1 + n_1 (= m_2 + n_2),$$

we have

$$S_2 \ll \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{\substack{n_1 \leq T \\ n_2 \leq T \\ m_1+n_1=m_2+n_2}} \frac{1}{(m_1+n_1)^{2\sigma-1}}$$

$$\ll \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{n_1 \leq T} \frac{1}{(m_1+n_1)^{2\sigma-1}}$$

$$\ll \sum_{m_1 \geq 1} \frac{1}{m_1^{\sigma_0}} \sum_{m_2 \geq 1} \frac{1}{m_2^{\sigma_0}} \sum_{n_1 \leq T} \frac{1}{n_1^{2\sigma-1}}$$

$$\ll \begin{cases} T^{2-2\sigma} & \left(\dfrac{1}{2} < \sigma < 1\right) \\ \log T & (\sigma = 1), \end{cases}$$

because $\sigma_0 > 1$.

As for S_3, we have

$$S_3 \ll \sum_{m_1,m_2 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{\substack{n_1,n_2 \leq T \\ m_1+n_1 < m_2+n_2 \leq 2(m_1+n_1)}} \frac{1}{(m_1+n_1)^\sigma (m_2+n_2)^\sigma} \frac{1}{\log \frac{m_2+n_2}{m_1+n_1}}$$

$$+ \sum_{m_1,m_2 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{\substack{n_1,n_2 \leq T \\ m_2+n_2 > 2(m_1+n_1)}} \frac{1}{(m_1+n_1)^\sigma (m_2+n_2)^\sigma} \frac{1}{\log \frac{m_2+n_2}{m_1+n_1}}.$$

We denote the first and the second term by W_1 and W_2, respectively. As for W_2, we have

$$W_2 \ll \sum_{m_1,m_2 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{\substack{n_1,n_2 \leq T \\ m_2+n_2 > 2(m_1+n_1)}} \frac{1}{(m_1+n_1)^\sigma (m_2+n_2)^\sigma}$$

$$\ll \sum_{m_1,m_2 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{n_1 \leq T} \frac{1}{n_1^\sigma} \sum_{n_2 \leq T} \frac{1}{n_2^\sigma}$$

$$\ll \begin{cases} T^{2-2\sigma} & \left(\dfrac{1}{2} < \sigma < 1\right) \\ (\log T)^2 & (\sigma = 1). \end{cases}$$

As for W_1, setting $r = (m_2 + n_2) - (m_1 + n_1)$, we have

$$W_1 \ll \sum_{m_1,m_2 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{n_1 \leq T} \sum_{r=1}^{m_1+n_1} \frac{1}{(m_1+n_1)^\sigma (m_1+n_1+r)^\sigma} \frac{1}{\log(1 + (r/(m_1+n_1)))}$$

$$\ll \sum_{m_1,m_2 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{n_1 \leq T} \frac{1}{(m_1+n_1)^{2\sigma}} \sum_{r=1}^{m_1+n_1} \frac{m_1+n_1}{r}$$

$$\ll \sum_{m_1,m_2 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{n_1 \leq T} \frac{1}{(m_1+n_1)^{2\sigma-1}} \log(m_1+n_1)$$

$$\ll \begin{cases} T^{2-2\sigma} \log T & \left(\dfrac{1}{2} < \sigma < 1\right) \\ (\log T)^2 & (\sigma = 1). \end{cases}$$

Combining these results, we obtain

$$\int_2^T |\Sigma_1(s_0,s)|^2 dt = \zeta_2^{[2]}(s_0, 2\sigma)T + \begin{cases} O(T^{2-2\sigma}\log T) & \left(\frac{1}{2} < \sigma < 1\right) \\ O((\log T)^2) & (\sigma = 1). \end{cases}$$

Therefore we have

$$\int_2^T |\zeta_2(s_0,s)|^2 dt$$

$$= \int_2^T |\Sigma_1(s_0,s) + O(t^{-\sigma})|^2 dt$$

$$= \int_2^T |\Sigma_1(s_0,s)|^2 dt + O\left(\int_2^T |\Sigma_1(s_0,s)| \, t^{-\sigma} dt\right) + O\left(\int_2^T t^{-2\sigma} dt\right). \qquad (4.4)$$

We see that the third term on the right-hand side is equal to $O(1)$ because $1/2 < \sigma \leq 1$. As for the second term, by the Cauchy-Schwarz inequality, we see that

$$\int_2^T |\Sigma_1(s_0,s)| \, t^{-\sigma} dt \ll \left(\int_2^T |\Sigma_1(s_0,s)|^2 dt\right)^{1/2} \cdot \left(\int_2^T t^{-2\sigma} dt\right)^{1/2}$$

$$= \left(\begin{cases} O(T) + O(T^{2-2\sigma}\log T) & (1/2 < \sigma < 1) \\ O(T) + O((\log T)^2) & (\sigma = 1) \end{cases}\right)^{1/2} \cdot O(1)^{1/2}$$

$$\ll T^{1/2}.$$

This completes the proof of Theorem 1.2. $\qquad\qquad\qquad\qquad\qquad\square$

5. The double analogue of the Euler constant.

Let γ be the Euler constant defined by

$$\gamma = \lim_{N\to\infty} \left(\sum_{n=1}^N \frac{1}{n} - \log N\right),$$

which satisfies that

$$\lim_{s\to 1} \left\{\zeta(s) - \frac{1}{s-1}\right\} = \gamma. \qquad (5.1)$$

Here we define analogues of the Euler constant corresponding to the double zeta-function as follows. For $s_0 \in \mathbb{C}$ with $\Re s_0 > 1$, we let

$$\gamma_2(s_0) = \lim_{N\to\infty} \sum_{m\geq 1} \frac{1}{m^{s_0}} \left\{\sum_{1\leq n\leq N} \frac{1}{(m+n)} - \log(m+N)\right\}. \qquad (5.2)$$

Then we obtain the following.

PROPOSITION 5.1. *For $s_0 \in \mathbb{C}$ with $\Re s_0 > 1$,*

$$\lim_{s \to 1} \left\{ \zeta_2(s_0, s) - \frac{\zeta(s_0)}{s - 1} \right\} = \gamma_2(s_0). \tag{5.3}$$

In particular,

$$\gamma_2(s_0) = \zeta(s_0)\gamma - \zeta_2(1, s_0) - \zeta(s_0 + 1). \tag{5.4}$$

PROOF. Applying (3.5) with $N = 0$, we have

$$\lim_{s \to 1} \left\{ \zeta_2(s_0, s) - \frac{\zeta(s_0)}{s - 1} \right\}$$

$$= \lim_{s \to 1} \left\{ \frac{\sum_{m \geq 1} m^{-s_0 - s + 1} - \zeta(s_0)}{s - 1} - s \sum_{m \geq 1} \frac{1}{m^{s_0}} \int_m^\infty \frac{u - [u] - 1/2}{u^{s+1}} du - \frac{1}{2} \sum_{m \geq 1} \frac{1}{m^{s_0 + s}} \right\}$$

$$= \zeta'(s_0) - \sum_{m \geq 1} \frac{1}{m^{s_0}} \int_m^\infty \frac{u - [u]}{u^2} du + \frac{1}{2} \sum_{m \geq 1} \frac{1}{m^{s_0}} \int_m^\infty \frac{1}{u^2} du - \frac{1}{2} \zeta(s_0 + 1),$$

where the third and the fourth terms are cancelled. Hence, from

$$\zeta'(s_0) = -\sum_{m \geq 1} \frac{\log m}{m^{s_0}},$$

the right-hand side of the above equation can be rewritten as

$$\zeta'(s_0) - \sum_{m \geq 1} \frac{1}{m^{s_0}} \lim_{K \to \infty} \sum_{k=m}^{K+m-1} \int_k^{k+1} \frac{u - [u]}{u^2} du$$

$$= \zeta'(s_0) - \sum_{m \geq 1} \frac{1}{m^{s_0}} \lim_{K \to \infty} \sum_{k=m}^{K+m-1} \int_k^{k+1} \left(\frac{1}{u} - \frac{k}{u^2} \right) du$$

$$= \zeta'(s_0) - \sum_{m \geq 1} \frac{1}{m^{s_0}} \lim_{K \to \infty} \left(\log(m + K) - \log m - \sum_{k=m}^{m+K-1} \frac{1}{k + 1} \right)$$

$$= \lim_{K \to \infty} \left\{ \sum_{m \geq 1} \frac{1}{m^{s_0}} \left(\sum_{n=1}^{K} \frac{1}{m + n} - \log(m + K) \right) \right\} = \gamma_2(s_0),$$

which implies (5.3). Note that Arakawa and Kaneko [**2**, Proposition 4] already showed that $\zeta_2(s_0, s)$, as a function in s, has a simple pole at $s = 1$ with its residue $\zeta(s_0)$, where $s_0 \in \mathbb{C}$ with $\Re s_0 > 1$. Suppose $s_0 \in \mathbb{C}$ with $\Re s_0 > 1$ and $\Re s > 1$. Then it is well-known that

$$\zeta(s_0)\zeta(s) = \zeta_2(s_0, s) + \zeta_2(s, s_0) + \zeta(s_0 + s).$$

By (5.1) and (5.3), we have

$$\zeta(s_0)\left(\frac{1}{s-1} + \gamma + o(s-1)\right) = \left(\frac{\zeta(s_0)}{s-1} + \gamma_2(s_0) + o(s-1)\right) + \zeta_2(s, s_0) + \zeta(s_0 + s).$$

Letting $s \to 1$, we obtain (5.4). This completes the proof. $\qquad\square$

6.   The second approximation theorem.

In the previous section, we gave the proof of Theorem 1.2 by use of (4.1) which comes from Theorem 3.1. However Theorem 3.1 holds under the conditions $\sigma > 0$ and $\sigma_0 + \sigma > 2$. Hence we cannot use it for $3/2 < \sigma_0 + \sigma \leq 2$. In order to prove a mean value result in the latter case, we have to prepare another approximate formula for $\zeta_2(s_0, s)$.

We begin with (3.12). As was discussed in the proof of Theorem 3.1, all but the second term on the right-hand side of (3.12) are convergent in $\sigma > 0, \sigma_0 + \sigma > 1$, so the remaining task is to study the second term.

First we assume $\sigma_0 + \sigma > 2$, $s \neq 1$. Then by the Euler-Maclaurin formula we have

$$\frac{1}{1-s} \sum_{m=1}^{\infty} \frac{1}{m^{s_0}(m+x)^{s-1}}$$

$$= \frac{1}{1-s} \int_1^{\infty} \frac{dy}{y^{s_0} x^{s-1}(1 + y/x)^{s-1}}$$

$$+ \frac{1}{1-s} \int_1^{\infty} \left(y - [y] - \frac{1}{2}\right)\left(-\frac{s_0}{y^{s_0+1}(y+x)^{s-1}} + \frac{1-s}{y^{s_0}(y+x)^s}\right) dy$$

$$+ \frac{1}{2(1-s)}(1+x)^{1-s}$$

$$= g(s_0, s; x) + Y_2 + Y_3, \tag{6.1}$$

say. Obviously Y_3 is defined for any $s \in \mathbb{C} \setminus \{1\}$ and satisfies $Y_3 = O(t^{-1}x^{1-\sigma})$. Next consider Y_2. We have

$$\frac{1}{1-s} \int_1^{\infty} \left(y - [y] - \frac{1}{2}\right) \frac{s_0}{y^{s_0+1}(y+x)^{s-1}} dy \ll \frac{1}{t} \int_1^{\infty} \frac{dy}{y^{\sigma_0+1}(y+x)^{\sigma-1}}$$

$$\ll t^{-1}x^{1-\sigma} \int_1^{\infty} \frac{dy}{y^{\sigma_0+1}} \ll t^{-1}x^{1-\sigma}$$

for $\sigma_0 > 0$, and

$$\frac{1}{1-s} \int_1^{\infty} \left(y - [y] - \frac{1}{2}\right) \frac{1-s}{y^{s_0}(y+x)^s} dy \ll \int_1^{\infty} \frac{dy}{y^{\sigma_0}(y+x)^{\sigma}}$$

$$\ll \left(\int_1^x + \int_x^\infty \right) \frac{dy}{y^{\sigma_0}(y+x)^\sigma}$$

$$\ll \int_1^x \frac{dy}{y^{\sigma_0}x^\sigma} + \int_x^\infty \frac{dy}{y^{\sigma_0+\sigma}}$$

$$= \begin{cases} O(x^{1-\sigma_0-\sigma}) & (0 < \sigma_0 < 1;\ \sigma_0 + \sigma > 1) \\ O(x^{-\sigma}\log x) & (\sigma_0 = 1;\ \sigma_0 + \sigma > 1) \\ O(x^{-\sigma}) & (\sigma_0 > 1;\ \sigma_0 + \sigma > 1). \end{cases}$$

Therefore now we find that $Y_2 + Y_3$ can be continued to the region $\sigma_0 > 0$, $\sigma_0 + \sigma > 1$ and $s \neq 1$, and in this region satisfies

$$Y_2 + Y_3 = O(t^{-1}x^{1-\sigma}) + \begin{cases} O(x^{1-\sigma_0-\sigma}) & (0 < \sigma_0 < 1;\ \sigma_0 + \sigma > 1) \\ O(x^{-\sigma}\log x) & (\sigma_0 = 1;\ \sigma_0 + \sigma > 1) \\ O(x^{-\sigma}) & (\sigma_0 > 1;\ \sigma_0 + \sigma > 1). \end{cases} \tag{6.2}$$

Next we consider $g(s_0, s; x)$. Here we invoke the classical Mellin-Barnes integral formula, that is

$$(1 + \lambda)^{-s} = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)} \lambda^z dz, \tag{6.3}$$

where s, λ are complex numbers with $\sigma = \Re s > 0$, $|\arg \lambda| < \pi$, $\lambda \neq 0$, c is real with $-\sigma < c < 0$, and the path (c) of integration is the vertical line $\Re z = c$. (Formula (6.3) has already been successfully used in the theory of multiple zeta-functions; see [**9**], [**10**], [**11**]).

LEMMA 6.1. *The function $g(s_0, s; x)$ can be continued meromorphically to the region $\sigma_0 < 3/2$ and $\sigma > 1/2$, and satisfies*

$$g(s_0, s; x) = \begin{cases} O(t^{-1}x^{1-\sigma} + t^{\sigma_0-2}x^{2-\sigma-\sigma_0} + t^{-1/2}x^{1/2-\sigma}) & (s_0 \neq 1) \\ O(t^{-1}x^{1-\sigma}(\log t + \log x) + t^{-1/2}x^{1/2-\sigma}) & (s_0 = 1) \end{cases}$$

in this region, except for the points on the singularities

$$s = 1, \quad s_0 + s = 2, 1, 0, -1, -2, -3, -4, \ldots. \tag{6.4}$$

PROOF. First we assume that $\sigma_0 > 1$ and $\sigma > 1$. Then, applying (6.3) with $\lambda = y/x$ and replacing s by $s - 1$ (because $\sigma - 1 > 0$), we have

$$g(s_0, s; x) = \frac{1}{(2\pi i)(1-s)} \int_1^\infty \frac{1}{y^{s_0}x^{s-1}} \int_{(c)} \frac{\Gamma(s-1+z)\Gamma(-z)}{\Gamma(s-1)} \left(\frac{y}{x}\right)^z dz\, dy, \tag{6.5}$$

where $1 - \sigma < c < 0$. Here we see that it is possible to change the order of the integral as follows. Since $1 - \sigma < c < 0 < \sigma_0 - 1$, we have $-\sigma_0 + c < -1$. This implies that (6.5)

is absolutely convergent with respect to y. Moreover, by the Stirling formula, we can easily check that (6.5) is absolutely convergent with respect to z. Therefore, changing the order of the integral on the right-hand side of (6.5), we obtain

$$
g(s_0, s; x) = \frac{x^{1-s}}{(2\pi i)(1-s)\Gamma(s-1)} \int_{(c)} \Gamma(s-1+z)\Gamma(-z)x^{-z} \int_1^\infty y^{z-s_0} dy\, dz
$$

$$
= \frac{x^{1-s}}{(2\pi i)(1-s)\Gamma(s-1)} \int_{(c)} \frac{\Gamma(s-1+z)\Gamma(-z)}{x^z(s_0-1-z)} dz.
$$

Now we temporarily assume that $1 < \sigma_0 < 3/2$. Then the pole $z = s_0-1$ of the integrand is located in the strip $c < \Re z < 1/2$. We shift the path (c) to $\Re z = 1/2$. Relevant poles are at $z = 0$ and $z = s_0 - 1$. Counting the residues of those poles, we obtain

$$
g(s_0, s; x) = \frac{x^{1-s}}{(1-s)\Gamma(s-1)} \left\{ \frac{\Gamma(s-1)}{s_0-1} + \frac{\Gamma(s+s_0-2)\Gamma(1-s_0)}{x^{s_0-1}} \right.
$$

$$
\left. + \frac{1}{(2\pi i)} \int_{(1/2)} \frac{\Gamma(s-1+z)\Gamma(-z)}{x^z(s_0-1-z)} dz \right\}
$$

$$
= \frac{x^{1-s}}{(1-s)(s_0-1)} + \frac{x^{1-s}}{(1-s)\Gamma(s-1)} \frac{\Gamma(s+s_0-2)\Gamma(1-s_0)}{x^{s_0-1}}
$$

$$
+ \frac{x^{1-s}}{(2\pi i)(1-s)\Gamma(s-1)} \int_{(1/2)} \frac{\Gamma(s-1+z)\Gamma(-z)}{x^z(s_0-1-z)} dz
$$

$$
= R_1 + R_2 + R_3, \tag{6.6}
$$

say. The last integral can be holomorphically continued to the region $\sigma_0 < 3/2$ and $\sigma > 1/2$ (because in this region the path does not meet the poles of the integrand). Therefore (6.6) gives the meromorphic continuation of $g(s_0, s; x)$ to this region. The possible singularities of R_1 and R_2 are $s_0 = 1$ and those listed as (6.4). But $s_0 = 1$ is actually not a singularity. Putting $s_0 = 1 + \delta$ and calculating the limit $\delta \to 0$, we find that

$$
R_1 + R_2 \big|_{s_0=1} = \frac{x^{1-s}}{1-s} \left(\log x - \gamma - \frac{\Gamma'}{\Gamma}(s-1) \right). \tag{6.7}
$$

We can easily check that $R_1 = O(t^{-1}x^{1-\sigma})$ and $R_2 = O(t^{\sigma_0-2}x^{2-\sigma-\sigma_0})$ by the Stirling formula, if $s_0 \neq 1$ and (s_0, s) is not on the singularities (6.4). If $s_0 = 1$, then from (6.7) we see that

$$
R_1 + R_2 = O\big(t^{-1}x^{1-\sigma}(\log t + \log x)\big).
$$

As for R_3, setting $z = 1/2 + iy$, we have

$$
R_3 \ll \frac{x^{1-\sigma}e^{\pi t/2}}{t \cdot t^{\sigma-3/2}} \int_{-\infty}^\infty \left| \frac{\Gamma(\sigma-1+it+1/2+iy)\Gamma(-1/2-iy)}{x^{1/2+iy}(\sigma_0+it_0-1-1/2-iy)} \right| dy
$$

$$\ll (tx)^{1/2-\sigma} e^{\pi t/2} \int_{-\infty}^{\infty} (|t+y|+1)^{\sigma-1} e^{-\pi|t+y|/2} (|y|+1)^{-2} e^{-\pi|y|/2} dy.$$

By Lemma 4 of [**10**], we find that the above integral is $O(t^{\sigma-1} e^{-\pi t/2})$, and hence $R_3 = O(t^{-1/2} x^{1/2-\sigma})$. This completes the proof of Lemma 6.1. $\qquad\square$

REMARK 6.2. By shifting the path more to the right, it is possible to prove that $g(s_0, s; x)$ can be continued meromorphically to the whole space $\mathbb{C}^2$.

From (6.2) and Lemma 6.1 we find that the right-hand side of (6.1) can be continued to the region $\sigma_0 < 3/2$, $\sigma > 1/2$, $\sigma_0 + \sigma > 1$, and satisfies the estimates proved above. On the other hand, the last three terms on the right-hand side of (3.12) are estimated by (3.6), (3.7), and (3.10), respectively.

Now set $x = t$. Then, using (3.10) we have

$$\sum_{m=1}^{\infty} \frac{E(s; x, m, N)}{m^{s_0}} \ll \sum_{m=1}^{\infty} \frac{1}{m^{\sigma_0}(m+t)^{\sigma}} \ll \sum_{m \le t} \frac{1}{m^{\sigma_0} t^{\sigma}} + \sum_{m>t} \frac{1}{m^{\sigma_0+\sigma}}$$

$$\ll \begin{cases} t^{1-\sigma_0-\sigma} & (0 < \sigma_0 < 1) \\ t^{-\sigma} \log t & (\sigma_0 = 1) \\ t^{-\sigma} & (\sigma_0 > 1), \end{cases}$$

while (3.6) and (3.7) imply that the contributions of A_3 and A_4 vanish when $N \to \infty$. Collecting all the information, we obtain the following.

THEOREM 6.3. *Let $s_0 = \sigma_0 + it_0 \in \mathbb{C}$ with $0 < \sigma_0 < 3/2$ and $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1/2$, $\sigma_0 + \sigma > 1$, $s \neq 1$, and $s_0 + s \neq 2$. Then*

$$\zeta_2(s_0, s) = \sum_{m=1}^{\infty} \sum_{n \le t} \frac{1}{m^{s_0}(m+n)^s} + \begin{cases} O(t^{1-\sigma_0-\sigma}) & (\sigma_0 < 1) \\ O(t^{-\sigma} \log t) & (\sigma_0 = 1) \\ O(t^{-\sigma}) & (\sigma_0 > 1). \end{cases} \qquad (6.8)$$

7. Proof of Theorem 1.3.

Based on these results, we finally give the proof of Theorem 1.3.

PROOF OF THEOREM 1.3. We let $s_0 \in \mathbb{C}$ with $1/2 < \sigma_0 < 3/2$ and $s \in \mathbb{C}$ with $1/2 < \sigma \le 1$ and $3/2 < \sigma_0 + \sigma \le 2$. We further assume that $s_0 + s \neq 2$. Similarly to Section 4, let

$$\Sigma_1(s_0, s) = \sum_{m=1}^{\infty} \sum_{1 \le n \le t} \frac{1}{m^{s_0}(m+n)^s}.$$

Then we can again obtain (4.2) and denote it by $S_1 T - S_2 + S_3$. As for S_1, we similarly set $k = m_1 + n_1 (= m_2 + n_2)$. Then we can write

$$S_1 = \sum_{k=2}^{\infty} \left(\sum_{m_1=1}^{k-1} \sum_{m_2=1}^{k-1} \frac{1}{m_1^{s_0} m_2^{\overline{s_0}}} \right) \frac{1}{k^{2\sigma}} - (U_1 + U_2 + U_3),$$

where $\overline{U_1} = U_2$. We have

$$U_1 + U_3 \ll \sum_{\substack{m_1 \geq 1 \\ m_2 \geq 1}} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{n_1 > T} \frac{1}{(m_1 + n_1)^{2\sigma}}$$

$$\ll \sum_{k=2}^{\infty} \frac{1}{k^{2\sigma}} \sum_{\substack{m_1=1 \\ k-m_1 > T}}^{k-1} \sum_{m_2=1}^{k-1} \frac{1}{(m_1 m_2)^{\sigma_0}},$$

where we set $k = m_1 + n_1 = m_2 + n_2$. Note that from the condition $k - m_1 > T$, we have $k > T$. Hence we obtain

$$U_1 + U_3 \ll \sum_{k > T} \frac{1}{k^{2\sigma}} \int_1^k u^{-\sigma_0} du \int_1^k v^{-\sigma_0} dv$$

$$\ll \begin{cases} \displaystyle\sum_{k>T} k^{2-2\sigma-2\sigma_0} = O(T^{3-2\sigma-2\sigma_0}) & \left(\dfrac{1}{2} < \sigma_0 < 1 \right) \\[2ex] \displaystyle\sum_{k>T} k^{-2\sigma} (\log k)^2 = O(T^{1-2\sigma} (\log T)^2) & (\sigma_0 = 1) \\[2ex] \displaystyle\sum_{k>T} k^{-2\sigma} = O(T^{1-2\sigma}) & \left(1 < \sigma_0 < \dfrac{3}{2} \right), \end{cases}$$

because $2 - 2\sigma - 2\sigma_0 < -1$, and the same estimate holds for $\overline{U_1} + U_3$ and U_3. As for the second estimate we used the integration by parts for

$$\sum_{k>T} k^{-2\sigma} (\log k)^2 \ll \int_T^{\infty} u^{-2\sigma} (\log u)^2 du.$$

Therefore

$$S_1 = \zeta_2^{[2]}(s_0, 2\sigma) + \begin{cases} O(T^{3-2\sigma-2\sigma_0}) & \left(\dfrac{1}{2} < \sigma_0 < 1 \right) \\[2ex] O(T^{1-2\sigma} (\log T)^2) & (\sigma_0 = 1) \\[2ex] O(T^{1-2\sigma}) & \left(1 < \sigma_0 < \dfrac{3}{2} \right). \end{cases} \tag{7.1}$$

Next we consider S_2. Using $M(n_1, n_2) = \max\{n_1, n_2, 2\}$, we have

$$S_2 = \sum_{\substack{m_1 \geq 1 \\ m_2 \geq 1}} \frac{1}{m_1^{s_0} m_2^{\overline{s_0}}} \sum_{\substack{n_1 \leq T \\ n_2 \leq T \\ m_1+n_1=m_2+n_2}} \frac{M(n_1, n_2)}{(m_1+n_1)^{2\sigma}}$$

$$= \sum_{k=2}^{\infty} \frac{1}{k^{2\sigma}} \sum_{\substack{m_1 \geq 1 \\ m_2 \geq 1}} \sum_{\substack{n_1 \leq T \\ n_2 \leq T \\ m_1+n_1=k \\ m_2+n_2=k}} \frac{M(n_1, n_2)}{m_1^{s_0} m_2^{\overline{s_0}}}$$

$$\ll \sum_{k \leq T} \frac{k}{k^{2\sigma}} \sum_{m_1=1}^{k-1} \sum_{m_2=1}^{k-1} \frac{1}{(m_1 m_2)^{\sigma_0}} + \sum_{k > T} \frac{T}{k^{2\sigma}} \sum_{m_1=1}^{k-1} \sum_{m_2=1}^{k-1} \frac{1}{(m_1 m_2)^{\sigma_0}}$$

$$\ll \begin{cases} \displaystyle\sum_{k \leq T} k^{1-2\sigma}(k^{1-\sigma_0})^2 + T \sum_{k>T} k^{-2\sigma}(k^{1-\sigma_0})^2 & \left(\dfrac{1}{2} < \sigma_0 < 1\right) \\[3mm] \displaystyle\sum_{k \leq T} k^{1-2\sigma}(\log k)^2 + T \sum_{k>T} k^{-2\sigma}(\log k)^2 & (\sigma_0 = 1) \\[3mm] \displaystyle\sum_{k \leq T} k^{1-2\sigma} + T \sum_{k>T} k^{-2\sigma} & \left(1 < \sigma_0 < \dfrac{3}{2}\right). \end{cases}$$

Therefore we obtain

$$S_2 = \begin{cases} O(T^{4-2\sigma_0-2\sigma}) & \left(\dfrac{1}{2} < \sigma_0 < 1, \ \dfrac{1}{2} < \sigma \leq 1\right) \\[3mm] O(T^{2-2\sigma}(\log T)^2) & \left(\sigma_0 = 1, \ \dfrac{1}{2} < \sigma < 1\right) \\[3mm] O((\log T)^3) & (\sigma_0 = 1, \ \sigma = 1) \\[3mm] O(T^{2-2\sigma}) & \left(1 < \sigma_0 < \dfrac{3}{2}, \ \dfrac{1}{2} < \sigma < 1\right), \end{cases} \tag{7.2}$$

where we have to note that $3/2 < \sigma_0 + \sigma < 2$ in the first case, and $\sigma \neq 1$ (because if $\sigma = 1$ then $\sigma_0 + \sigma > 2$) in the fourth case.

Finally we consider S_3. Similarly to the argument in Section 4, we have

$$S_3 \ll \sum_{m_1,m_2 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{\substack{n_1,n_2 \leq T \\ m_1+n_1 < m_2+n_2 \leq 2(m_1+n_1)}} \frac{1}{(m_1+n_1)^\sigma (m_2+n_2)^\sigma} \frac{1}{\log \frac{m_2+n_2}{m_1+n_1}}$$

$$+ \sum_{m_1,m_2 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{\substack{n_1,n_2 \leq T \\ m_2+n_2 > 2(m_1+n_1)}} \frac{1}{(m_1+n_1)^\sigma (m_2+n_2)^\sigma} \frac{1}{\log \frac{m_2+n_2}{m_1+n_1}},$$

which we denote by $W_1 + W_2$.

First estimate W_2. We have

$$W_2 \ll \sum_{m_1, m_2 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{\substack{n_1, n_2 \leq T \\ m_2 + n_2 > 2(m_1 + n_1)}} \frac{1}{(m_1 + n_1)^\sigma (m_2 + n_2)^\sigma}$$

$$= \sum_{\substack{m_1 \geq 1 \\ n_1 \leq T}} \frac{1}{m_1^{\sigma_0}(m_1 + n_1)^\sigma} \sum_{\substack{m_2 \geq 1, n_2 \leq T \\ m_2 + n_2 > 2(m_1 + n_1)}} \frac{1}{m_2^{\sigma_0}(m_2 + n_2)^\sigma}$$

$$= \sum_{\substack{m_1 \geq 1 \\ n_1 \leq T}} \frac{1}{m_1^{\sigma_0}(m_1 + n_1)^\sigma} \sum_{k > 2(m_1 + n_1)} \frac{1}{k^\sigma} \sum_{\substack{m_2 \geq 1, n_2 \leq T \\ m_2 + n_2 = k}} \frac{1}{m_2^{\sigma_0}}$$

$$= \sum_{\substack{m_1 \leq T \\ n_1 \leq T}} + \sum_{\substack{m_1 > T \\ n_1 \leq T}} = W_{21} + W_{22},$$

say. Consider W_{22}. Since $m_1 > T$, we have $k > 2T$, so $m_2 = k - n_2 \geq k - T > k/2$. Therefore the innermost sum of W_{22} is

$$\sum_{k - T \leq m_2 \leq k-1} \frac{1}{m_2^{\sigma_0}} \ll T k^{-\sigma_0},$$

and hence

$$W_{22} \ll T \sum_{\substack{m_1 > T \\ n_1 \leq T}} \frac{1}{m_1^{\sigma_0}(m_1 + n_1)^\sigma} \sum_{k > 2(m_1 + n_1)} k^{-\sigma_0 - \sigma}$$

$$\ll T \sum_{m_1 > T} m_1^{-\sigma_0} \sum_{n_1 \leq T} (m_1 + n_1)^{1 - \sigma_0 - 2\sigma}$$

$$\leq T \sum_{m_1 > T} m_1^{-\sigma_0} \sum_{n_1 \leq T} m_1^{1 - \sigma_0 - 2\sigma}$$

$$\ll T^2 \sum_{m_1 > T} m_1^{1 - 2\sigma_0 - 2\sigma}.$$

Since $1 - 2\sigma_0 - 2\sigma < -1$, we have

$$W_{22} \ll T^2 T^{2 - 2\sigma_0 - 2\sigma} = T^{4 - 2\sigma_0 - 2\sigma}. \tag{7.3}$$

As for W_{21}, we further divide the inner double sum of W_{21} into two parts D_1 and D_2 according to $2(m_1 + n_1) < k \leq 2T$ and $k > 2T$, respectively. We handle the innermost sum of D_2 similarly to the case of W_{22}. We have

$$D_2 \ll T \sum_{k > 2T} \frac{1}{k^\sigma} \frac{1}{k^{\sigma_0}} \ll T^{2 - \sigma_0 - \sigma}.$$

The innermost sum of D_1 is

$$\ll \sum_{m_2 \leq k-1} \frac{1}{m_2^{\sigma_0}} \ll \begin{cases} k^{1-\sigma_0} & \left(\frac{1}{2} < \sigma_0 < 1\right) \\ \log k & (\sigma_0 = 1) \\ 1 & \left(1 < \sigma_0 < \frac{3}{2}\right), \end{cases}$$

which gives

$$D_1 \ll \begin{cases} T^{2-\sigma_0-\sigma} & \left(\frac{1}{2} < \sigma_0 < 1,\ \frac{1}{2} < \sigma \leq 1\right) \\ T^{1-\sigma} \log T & \left(\sigma_0 = 1,\ \frac{1}{2} < \sigma < 1\right) \\ (\log T)^2 & (\sigma_0 = 1,\ \sigma = 1) \\ T^{1-\sigma} & \left(1 < \sigma_0 < \frac{3}{2},\ \frac{1}{2} < \sigma < 1\right). \end{cases}$$

Substituting the estimates of D_1 and D_2 into W_{21}, and estimating the remaining sum

$$\sum_{\substack{m_1 \leq T \\ n_1 \leq T}} \frac{1}{m_1^{\sigma_0}(m_1 + n_1)^\sigma} \leq \sum_{m_1 \leq T} \frac{1}{m_1^{\sigma_0}} \sum_{n_1 \leq T} \frac{1}{n_1^\sigma}$$

in the obvious way, we obtain

$$W_{21} \ll \begin{cases} T^{4-2\sigma_0-2\sigma} & \left(\frac{1}{2} < \sigma_0 < 1,\ \frac{1}{2} < \sigma < 1\right) \\ T^{2-2\sigma_0} \log T & \left(\frac{1}{2} < \sigma_0 < 1,\ \sigma = 1\right) \\ T^{2-2\sigma}(\log T)^2 & \left(\sigma_0 = 1,\ \frac{1}{2} < \sigma < 1\right) \\ (\log T)^4 & (\sigma_0 = 1,\ \sigma = 1) \\ T^{2-2\sigma} & \left(1 < \sigma_0 < \frac{3}{2},\ \frac{1}{2} < \sigma < 1\right). \end{cases} \tag{7.4}$$

Next consider W_1. We have

$$W_1 = \sum_{m_1, m_2 \geq 1} \frac{1}{(m_1 m_2)^{\sigma_0}} \sum_{\substack{n_1, n_2 \leq T \\ m_1+n_1 < m_2+n_2 \\ \leq 2(m_1+n_1)}} \frac{1}{(m_1+n_1)^\sigma (m_2+n_2)^\sigma} \frac{1}{\log\left(1 + \frac{m_2+n_2-m_1-n_1}{m_1+n_1}\right)}$$

$$\ll \sum_{m_1 \geq 1} \sum_{n_1 \leq T} \frac{1}{m_1^{\sigma_0}(m_1+n_1)^{2\sigma}} \sum_{n_2 \leq T} \sum_{\substack{m_2 \geq 1 \\ m_1+n_1 < m_2+n_2 \\ \leq 2(m_1+n_1)}} \frac{m_1+n_1}{m_2^{\sigma_0}(m_2+n_2-m_1-n_1)}$$

$$= \sum_{m_1 \leq 2T} \sum_{n_1 \leq T} + \sum_{m_1 > 2T} \sum_{n_1 \leq T} = W_{11} + W_{12},$$

say. Consider W_{12}. Since $m_1 > 2T$, we have $n_2 \leq T < m_1/2$, so $m_2 > m_1 + n_1 - n_2 > m_1/2$. Therefore, setting $r = m_2 + n_2 - m_1 - n_1$, we have

$$W_{12} \ll \sum_{m_1 \geq 2T} \sum_{n_1 \leq T} \frac{1}{m_1^{2\sigma_0}(m_1 + n_1)^{2\sigma-1}} \sum_{n_2 \leq T} \sum_{\substack{m_2 \geq 1 \\ m_1 + n_1 < m_2 + n_2 \\ \leq 2(m_1+n_1)}} \frac{1}{(m_2 + n_2 - m_1 - n_1)}$$

$$\ll \sum_{m_1 \geq 2T} \sum_{n_1 \leq T} \frac{1}{m_1^{2\sigma_0}(m_1 + n_1)^{2\sigma-1}} \sum_{n_2 \leq T} \sum_{r=1}^{m_1+n_1} \frac{1}{r}$$

$$\ll T \sum_{m_1 \geq 2T} \sum_{n_1 \leq T} \frac{1}{m_1^{2\sigma_0}(m_1 + n_1)^{2\sigma-1}} \log(m_1 + n_1)$$

$$\ll T \sum_{m_1 > 2T} m_1^{-2\sigma_0} \times \begin{cases} (m_1 + T)^{2-2\sigma} \log(m_1 + T) & \left(\frac{1}{2} < \sigma < 1 \right) \\ (\log(m_1 + T))^2 & (\sigma = 1) \end{cases}$$

$$\ll \begin{cases} T^{4-2\sigma_0-2\sigma} \log T & \left(\frac{1}{2} < \sigma < 1 \right) \\ T^{2-2\sigma_0}(\log T)^2 & (\sigma = 1). \end{cases} \tag{7.5}$$

Next, since $m_2 < 2(m_1 + n_1)$, the innermost sum of W_{11} is

$$\ll \sum_{m_2 < 2(m_1+n_1)} m_2^{-\sigma_0} \log(m_1 + n_1)$$

$$\ll \begin{cases} (m_1 + n_1)^{1-\sigma_0} \log(m_1 + n_1) & \left(\frac{1}{2} < \sigma_0 < 1 \right) \\ (\log(m_1 + n_1))^2 & (\sigma_0 = 1) \\ \log(m_1 + n_1) & \left(1 < \sigma_0 < \frac{3}{2} \right). \end{cases}$$

Therefore, when $1/2 < \sigma_0 < 1$, we have

$$W_{11} \ll \sum_{\substack{m_1 \leq 2T \\ n_1 \leq T}} m_1^{-\sigma_0}(m_1 + n_1)^{2-\sigma_0-2\sigma} \log(m_1 + n_1)$$

$$\ll \sum_{m_1 \leq 2T} m_1^{-\sigma_0}(m_1 + T)^{3-\sigma_0-2\sigma} \log(m_1 + T)$$

$$\ll T^{4-2\sigma_0-2\sigma} \log T,$$

because $2 < \sigma_0 + 2\sigma < 3$. Similarly, we have

$$W_{11} \ll \sum_{\substack{m_1 \leq 2T \\ n_1 \leq T}} m_1^{-1}(m_1 + n_1)^{1-2\sigma}(\log(m_1 + n_1))^2$$

$$\ll \begin{cases} T^{2-2\sigma}(\log T)^3 & \left(\frac{1}{2} < \sigma < 1\right) \\ (\log T)^4 & (\sigma = 1) \end{cases}$$

when $\sigma_0 = 1$, and

$$W_{11} \ll \sum_{\substack{m_1 \leq 2T \\ n_1 \leq T}} m_1^{-\sigma_0}(m_1 + n_1)^{1-2\sigma}\log(m_1 + n_1) \ll T^{2-2\sigma}\log T$$

when $1 < \sigma_0 < 3/2$. By (7.3), (7.4), (7.5) and the above estimates, we now obtain

$$S_3 = W_1 + W_2$$

$$\ll \begin{cases} T^{4-2\sigma_0-2\sigma}\log T & \left(\frac{1}{2} < \sigma_0 < 1, \ \frac{1}{2} < \sigma < 1\right) \\ T^{2-2\sigma_0}(\log T)^2 & \left(\frac{1}{2} < \sigma_0 < 1, \ \sigma = 1\right) \\ T^{2-2\sigma}(\log T)^3 & \left(\sigma_0 = 1, \ \frac{1}{2} < \sigma < 1\right) \\ (\log T)^4 & (\sigma_0 = 1, \ \sigma = 1) \\ T^{2-2\sigma}\log T & \left(1 < \sigma_0 < \frac{3}{2}, \ \frac{1}{2} < \sigma < 1\right). \end{cases} \tag{7.6}$$

Denote the right-hand side of the above by $\mathcal{E}(T)$. Combining (7.1), (7.2) and (7.6), we obtain

$$\int_2^T |\Sigma_1(s_0, s)|^2 dt = S_1 T - S_2 + S_3 = \zeta_2^{[2]}(s_0, 2\sigma)T + O(\mathcal{E}(T)). \tag{7.7}$$

Now, using the Cauchy-Schwarz inequality, we estimate the second term on the right-hand side of (4.4) with replacing $t^{-\sigma}$ by the error term on the right-hand side of (6.8). Denoting by $E(t)$ the error term on the right-hand side of (6.8), we have

$$\int_2^T |\Sigma_1(s_0, s)| \, E(t) dt \ll \left(\int_2^T |\Sigma_1(s_0, s)|^2 dt\right)^{1/2} \cdot \left(\int_2^T E(t)^2 dt\right)^{1/2}$$

$$= \{O(T) + O(\mathcal{E}(T))\}^{1/2} \cdot O(1)^{1/2} \ll T^{1/2}.$$

Thus we obtain the proof of Theorem 1.3. $\qquad\qquad\square$

References

[1] S. Akiyama, S. Egami and Y. Tanigawa, Analytic continuation of multiple zeta-functions and their values at non-positive integers, Acta Arith., **98** (2001), 107–116.

[2] T. Arakawa and M. Kaneko, Multiple zeta values, poly-Bernoulli numbers, and related zeta functions, Nagoya Math. J., **153** (1999), 189–209.

[3] F. V. Atkinson, The mean-value of the Riemann zeta function, Acta Math., **81** (1949), 353–376.

[4] D. Borwein, J. M. Borwein and R. Girgensohn, Explicit evaluation of Euler sums, Proc. Edinburgh Math. Soc. (2), **38** (1995), 277–294.

[5] H. Ishikawa and K. Matsumoto, On the estimation of the order of Euler-Zagier multiple zeta-functions, Illinois J. Math., **47** (2003), 1151–1166.

[6] I. Kiuchi and Y. Tanigawa, Bounds for double zeta-functions, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **5** (2006), 445–464.

[7] I. Kiuchi, Y. Tanigawa and W. Zhai, Analytic properties of double zeta-functions, Indag. Math. (N.S.), **21** (2011), 16–29.

[8] Y. Komori, K. Matsumoto and H. Tsumura, Functional equations and functional relations for the Euler double zeta-function and its generalization of Eisenstein type, Publ. Math. Debrecen, **77** (2010), 15–31.

[9] K. Matsumoto, On the analytic continuation of various multiple zeta-functions, In: Number Theory for the Millennium. II, Urbana, IL, 2000, (eds. M. A. Bennett, B. C. Berndt, N. Boston, H. G. Diamond, A. J. Hildebrand and W. Philipp), A K Peters, Natick, MA, 2002, pp. 417–440.

[10] K. Matsumoto, The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions. I, J. Number Theory, **101** (2003), 223–243.

[11] K. Matsumoto, Asymptotic expansions of double zeta-functions of Barnes, of Shintani, and Eisenstein series, Nagoya Math. J., **172** (2003), 59–102.

[12] K. Matsumoto, Functional equations for double zeta-functions, Math. Proc. Cambridge Philos. Soc., **136** (2004), 1–7.

[13] K. Matsumoto and H. Tsumura, Mean value theorems for double zeta-functions, In: Analytic Number Theory—Number Theory through Approximation and Asymptotics, RIMS, 2012, (ed. K. Chinen), RIMS Kôkyûroku, **1874**, RIMS, 2014, pp. 45–54.

[14] T. Nakamura and Ł. Pańkowski, Any non-monomial polynomial of the Riemann zeta-function has complex zeros off the critical line, preprint, arXiv:math.NT/1212.5890.

[15] E. C. Titchmarsh, The Theory of the Riemann Zeta-function. 2nd ed., Edited and with a preface by D. R. Heath-Brown, The Clarendon Press, Oxford University Press, New York, 1986.

Kohji Matsumoto

Graduate School of Mathematics
Nagoya University
Chikusa-ku
Nagoya 464-8602, Japan
E-mail: kohjimat@math.nagoya-u.ac.jp

Hirofumi Tsumura

Department of Mathematics and Information Sciences
Tokyo Metropolitan University
1-1, Minami-Ohsawa, Hachioji
Tokyo 192-0397, Japan
E-mail: tsumura@tmu.ac.jp