J. Math. Soc. Japan
Vol. 67, No. 1 (2015) pp. 183–193
doi: 10.2969/jmsj/06710183

Weak Neumann implies H^∞ for Stokes

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(Received Feb. 1, 2013)

Abstract. Let $\Omega \subset \mathbb{R}^n$ be a domain with uniform C^3 boundary and assume that the Helmholtz decomposition exists in $\mathbb{L}^q(\Omega) := L^q(\Omega)^n$ for some $q \in (1,\infty)$. We show that a suitable translate of the Stokes operator admits a bounded $\mathcal{H}^\infty$-calculus in $\mathbb{L}^p_\sigma(\Omega)$ for $p \in (\min\{q,q'\},\max\{q,q'\})$. For the proof we use a recent maximal regularity result for the Stokes operator on such domains ([**GHHS12**]) and an abstract result for the $\mathcal{H}^\infty$-calculus in complemented subspaces ([**KKW06**], [**KW13**]).

1. Introduction and main result.

Given an open set $\Omega \subset \mathbb{R}^n$, it is well known that the Stokes operator A_2 is a selfadjoint and semibounded operator in $\mathbb{L}^2_\sigma(\Omega)$. In particular, A_2 admits a bounded $\mathcal{H}^\infty$-calculus on $\mathbb{L}^2_\sigma(\Omega)$, where $\mathbb{L}^2_\sigma(\Omega)$ denotes the space of solenoidal vector fields in $\mathbb{L}^2(\Omega) := L^2(\Omega)^n$.

In this paper, we address the question whether these results can be generalized to corresponding results in $\mathbb{L}^q_\sigma(\Omega)$ for $q \neq 2$. An affirmative answer for bounded and exterior domains with sufficiently smooth boundary was given in [**NS03**] and [**Abe05b**]. Whereas the proof of the results given in [**NS03**] is based on a localization technique and the corresponding result in the halfspace and the bent halfspace, pseudo differential operator techniques for the reduced stokes operator were used in the latter reference. The same aproach was also used to treat layers, layer-like domains, and aperture domains, see [**Abe05c**] and [**Abe05a**]. Furthermore, in [**FR07**] the authors considered the case of an unbounded domain with several cylindrical exits at infinity. However, corresponding results for *general* unbounded domains seem to be unknown in general, unless $q = 2$. A key problem in the study of the Stokes problem in such general unbounded domains is that the Helmholtz decomposition of $\mathbb{L}^q(\Omega)$ into $\mathbb{L}^q_\sigma(\Omega) \oplus G^q(\Omega)$ is not possible for $q \neq 2$, in general. Here, $G^q(\Omega) \subset \mathbb{L}^q(\Omega)$ denotes the space of gradient fields. Indeed, Bogovskiĭ gave in [**Bog86**] examples of unbounded domains Ω with smooth boundaries for which the Helmholtz decomposition of $\mathbb{L}^q(\Omega)$ exists only for a restricted range of q.

One way to overcome the difficulties for unbounded domains described above was shown in [**FKS05**], [**FKS09**] by Farwig, Kozono and Sohr who replaced the usual $\mathbb{L}^q(\Omega)$-space by

2010 *Mathematics Subject Classification.* Primary 35Q30, 47A60.
Key Words and Phrases. Stokes operator, H^∞-functional calculus, general unbounded domains, Helmholtz decomposition, fractional powers.

$$\tilde{\mathbb{L}}^q(\Omega) := \begin{cases} \mathbb{L}^2(\Omega) \cap \mathbb{L}^q(\Omega), & 2 \le q < \infty, \\ \mathbb{L}^2(\Omega) + \mathbb{L}^q(\Omega), & 1 < q < 2 \end{cases}$$

for domains $\Omega \subset \mathbb{R}^n$ with uniform $C^{1,1}$-boundaries. In particular, they showed that the Stokes operator in $\tilde{\mathbb{L}}^q_\sigma(\Omega)$ is $\mathcal{R}$-sectorial. Using this and an abstract result by Kalton, Kunstmann and Weis ([**KKW06**]), the second author proved in [**Kun08**] that the Stokes operator admits a bounded $\mathcal{H}^\infty$-calculus in $\tilde{\mathbb{L}}^q_\sigma(\Omega)$, see Remark 1.2 for details.

For the *usual* space $\mathbb{L}^q(\Omega)$, such a general result does not exist in the literature. However, there are results of the type "Existence of the Helmholtz decomposition implies $\mathcal{H}^\infty$-calculus of the Stokes operator". Indeed, under the additional assumption that the homogeneous Sobolev space $\widehat{W}^{1,q}(\Omega)$ admits a certain decomposition such a result was proved in [**AT09**] and [**Abe10**] for general domains with a sufficiently smooth boundary. The proof is again based on pseudo differential operator theory and the reduced Stokes operator. In this paper we prove a similar result without the latter additional assumption on $\widehat{W}^{1,q}(\Omega)$. However, we need to impose higher regularity on the boundary of Ω than [**Abe10**]. Similar to the results shown in [**Kun08**], our proof is based on an abstract result by Kalton, Kunstmann and Weis ([**KKW06**], [**KW13**]) and $\mathcal{R}$-sectoriality of the Stokes operator proved in [**GHHS12**]. A similar approach was also exploited in [**KKW06**], [**KW13**] for the Stokes operator on *bounded* domains.

In order to state our main result, let us recall the definition of the Helmholtz decomposition. Given a domain $\Omega \subset \mathbb{R}^n$ and $q \in (1,\infty)$, we set

$$\begin{aligned} G^q(\Omega) &:= \big\{u \in \mathbb{L}^q(\Omega) : u = \nabla\pi \text{ for some } \pi \in W^{1,q}_{\mathrm{loc}}(\Omega)\big\}, \\ \mathbb{L}^q_\sigma(\Omega) &:= \overline{\{u \in C_c^\infty(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega\}}^{\|\cdot\|_q}. \end{aligned}$$

We say that the *Helmholtz projection* exists for $\mathbb{L}^q(\Omega)$ whenever $\mathbb{L}^q(\Omega)$ can be decomposed into

$$\mathbb{L}^q(\Omega) = \mathbb{L}^q_\sigma(\Omega) \oplus G^q(\Omega),$$

where $\oplus$ denotes the topological direct sum. In this case, there exists a unique projection operator P_q from $\mathbb{L}^q(\Omega)$ onto $\mathbb{L}^q_\sigma(\Omega)$ having $G^q(\Omega)$ as its null space. Let q' denote the Hölder conjugate exponent, i.e. $q' = q/(q-1)$. It is well known (see, e.g., [**Gal94**, Lemma III.1.2]) that the Helmholtz projection exists for $\mathbb{L}^q(\Omega)$ if and only if, for every $f \in \mathbb{L}^q(\Omega)$, there exists a function $u \in \widehat{W}^{1,q}(\Omega) := \{v \in L^1_{\mathrm{loc}}(\Omega) : \nabla v \in \mathbb{L}^q(\Omega)\}$, unique up to constants, satisfying

$$\langle \nabla u, \nabla\varphi\rangle = \langle f, \nabla\varphi\rangle, \quad \varphi \in \widehat{W}^{1,q'}(\Omega). \tag{1}$$

Problem (1) is called the *weak Neumann problem*.

Now, let us assume that $\Omega \subset \mathbb{R}^n$, $n \ge 2$, is a domain with a uniform $C^{1,1}$-boundary (cf. Section 2 below for the precise definition) and that the Helmholtz projection P_q exists on $\mathbb{L}^q(\Omega)$ for some $q \in (1,\infty)$. We define the Stokes operator A_q in $\mathbb{L}^q_\sigma(\Omega)$ as

$$\begin{aligned} D(A_q) &:= W^{2,q}(\Omega)^n \cap W_0^{1,q}(\Omega)^n \cap \mathbb{L}^q_\sigma(\Omega), \\ A_q u &:= P_q \Delta u \quad \text{for } u \in D(A_q). \end{aligned} \tag{2}$$

In this situation, Proposition 2.1 and 2.2 below show that, for $p \in [\min\{q,q'\}, \max\{q,q'\}]$ where q' denotes the Hölder conjugate exponent, the Helmholtz projection exists in $\mathbb{L}^p(\Omega)$ and that $-A_p$ generates an analytic semigroup in $\mathbb{L}^p_\sigma(\Omega)$. These semigroups are consistent, hence also resolvent operators $R(\lambda, A_p)$ are consistent for large $\operatorname{Re}\lambda$ and $p \in [\min\{q,q'\}, \max\{q,q'\}]$. Finally, we are able to state the main result of this paper.

THEOREM 1.1. *Let $n \geq 2$, $q \in (1,\infty)$. Assume that $\Omega \subset \mathbb{R}^n$ is a domain with uniform C^3-boundary and that the Helmholtz projection P_q exists for $\mathbb{L}^q(\Omega)$. Then, there exists $\lambda_0 > 0$ such that Stokes operator $\lambda_0 - A_p$ admits a bounded $\mathcal{H}^\infty$-calculus in $\mathbb{L}^p_\sigma(\Omega)$, where $p \in (\min\{q,q'\}, \max\{q,q'\})$ and q' denotes the Hölder conjugate exponent. These functional calculi are consistent.*

We make a few comments on our method of proof and the regularity assumptions for Ω.

REMARK 1.2. (a) As pointed out above, we use the abstract result from [**KKW06**], [**KW13**], and thus need information on the Stokes operator and its domain. The assumption of a uniform C^3-boundary is the same as in [**GHHS12**] and is needed here to have the Stokes semigroup and $\mathcal{R}$-sectoriality of $\lambda_0 - A_p$ in $\mathbb{L}^p_\sigma(\Omega)$.

(b) In [**Kun08**, Theorem 1.1, Corollary 1.2], Ω had been assumed to have a uniform $C^{1,1}$-boundary. The proof of [**Kun08**, Theorem 1.1] given there contains a gap, since application of [**KKW06**, Theorem 8.2] has to be replaced by application of [**KW13**, Theorem 1.3], cited as Theorem 2.3 below. This means that, compared to [**KKW06**, Theorem 8.2], one has to check an additional property. Below we shall see that this is possible under the assumption that Ω has a uniform $C^{2+\mu}$-boundary for some $\mu \in (0,1)$. The additional property for the application of Theorem 2.3 is checked by the proof of (8) below. Actually, this will be the main work in the proof of Theorem 1.1.

(c) The same correction ($\partial\Omega$ has to be assumed uniformly $C^{2+\mu}$ for some $\mu \in (0,1)$ instead of uniformly $C^{1,1}$) has to be applied to the assumptions of [**Kun10**, Theorem 1.1, Theorem 1.5, Theorem 1.6], since the proofs of these results used the assertion of [**Kun08**, Corollary 1.2].

This paper is organized as follows. In the next section we discuss some basic facts of the Helmholtz decomposition and the Stokes operator. Moreover, we recall the abstract result from [**KKW06**] and [**KW13**]. Finally, we present in the last section the proof of our main result.

2. Preliminaries.

We start with some comments on existence and consistency of the Helmholtz projection for domains $\Omega \subset \mathbb{R}^n$ with a uniform C^1-boundary. We recall that Ω is said to have uniform $\mathcal{F}$-boundary, where $\mathcal{F} \in \{C^1, C^{1,1}, C^{2+\mu}, C^3\}$, if there are constants $\alpha, \beta, K > 0$ such that, for each $x_0 \in \partial\Omega$, there is a Cartesian coordinate system with origin at x_0

and coordinates $y = (y', y_n)$, $y' = (y_1, \dots, y_{n-1})$ and a function $h \in \mathcal{F}$, defined on $\{y' : |y'| \le \alpha\}$ and with $\|h\|_{\mathcal{F}} \le K$, such that, for the neighborhood

$$U_{\alpha,\beta,h}(x_0) = \{y = (y', y_n) \in \mathbb{R}^n : |y_n - h(y')| < \beta, |y'| < \alpha\}$$

of x_0 we have $U_{\alpha,\beta,h}(x_0) \cap \partial\Omega = \{(y', h(y')) : |y'| < \alpha\}$ and

$$U_{\alpha,\beta,h}(x_0) \cap \Omega = \{(y', y_n) : h(y') - \beta < y_n < h(y'), |y'| < \alpha\}.$$

In order to simplify notation, we assume that the Helmholtz decomposition in $\mathbb{L}^q(\Omega)$ exists for some $q > 2$ (the case $q < 2$ can be treated similarly or by dualization) and denote its Hölder conjugate exponent by q'. Then the Helmholtz decomposition exists for $p \in [q', q]$, and the family of Helmholtz projections $(P_p)_{p\in[q',q]}$ is consistent, i.e.

$$P_p f = P_2 f, \quad p \in [q', q],\ f \in C_c^\infty(\Omega)^n. \tag{3}$$

This easily follows from [**FKS07**]. Indeed, for $f \in C_c^\infty(\Omega)^n$ there exists $f_\sigma \in \mathbb{L}^2_\sigma(\Omega) \cap \mathbb{L}^p_\sigma(\Omega)$ and $g \in \widehat{W}^{1,2}(\Omega) \cap \widehat{W}^{1,p}(\Omega)$ such that

$$f = f_\sigma + \nabla g.$$

Since the decomposition is unique by assumption, we see that P_2 and P_q are consistent. Hence, the claim follows for $p \in [2, q]$ from interpolation theory. By [**GHHS12**, Lemma 5.1], the Helmholtz decomposition exists for the Hölder conjugate exponent q' as well. Since consistency of P_2 and $P_{q'}$ follows from a duality argument, (3) holds. Summing up, we proved the following proposition.

Proposition 2.1. *Let $n \ge 2$, $q \in (1, \infty)$. Assume that $\Omega \subset \mathbb{R}^n$ is a domain with uniform C^1-boundary and that the Helmholtz projection P_q exists for $\mathbb{L}^q(\Omega)$. Then, the Helmholtz projection P_p exists for $p \in [\min\{q, q'\}, \max\{q, q'\}]$, where q' denotes the Hölder conjugate exponent. Moreover, the family $(P_p)_{p\in[\min\{q,q'\},\max\{q,q'\}]}$ is consistent.*

Finally, combining [**GHHS12**, Proof of Theorem 2.1] and Proposition 2.1, we obtain the following result.

Proposition 2.2. *Let $n \ge 2$, $q \in (1, \infty)$. Assume that $\Omega \subset \mathbb{R}^n$ is a domain with uniform C^3-boundary and that the Helmholtz projection P_q exists for $\mathbb{L}^q(\Omega)$. Then, there exists $\lambda_0 > 0$ such that $\lambda_0 - A_p$ is $\mathcal{R}$-sectorial in $\mathbb{L}^p_\sigma(\Omega)$ for $p \in [\min\{q, q'\}, \max\{q, q'\}]$, where q' denotes the Hölder conjugate exponent of q. Moreover, the resolvents of A_p are consistent.*

The rest of this preliminary section is devoted to the abstract result from [**KKW06**], [**KW13**] where we change notation a little.

Let B be a sectorial operator in a Banach space X, in particular, B has dense domain and range and is injective. For $\alpha \in \mathbb{R}$ we define *homogeneous* fractional spaces by completion:

$$\dot{X}_{\alpha,B} := (D(B^\alpha), \|B^\alpha \cdot\|)^\sim.$$

Then $X \cap \dot{X}_{\alpha,B} = D(B^\alpha)$. If $0 \in \rho(A)$ then $\dot{X}_{\alpha,B}$ coincides with $X_{\alpha,B}$, given by

$$X_{\alpha,B} := \begin{cases} D(B^\alpha), & \alpha \geq 0, \\ (X, \|(1+B)^\alpha \cdot\|)^\sim, & \alpha < 0. \end{cases}$$

For the abstract result we assume that (X_0, X_1) is an interpolation couple, the spaces $X_\theta := [X_0, X_1]_\theta$, $\theta \in (0,1)$, are obtained by complex interpolation, and there is family $(B_\theta)_{\theta\in[0,1]}$ of sectorial operators B_θ in X_θ satisfying the consistency condition

$$(1+B_\theta)^{-1}x = (1+B_{\tilde{\theta}})^{-1}x, \quad x \in X_\theta \cap X_{\tilde{\theta}},\ \theta, \tilde{\theta} \in [0,1].$$

We assume that (Y_0, Y_1) is another interpolation couple with scale $Y_\theta = [Y_0, Y_1]_\theta$ of complex interpolation spaces and a family $(A_\theta)_{\theta\in[0,1]}$ of sectorial operators satisfying a similar consistency condition. We shall use the following corrected version ([**KW13**, Theorem 1.3]) of [**KKW06**, Theorem 8.2]. Recall that a R-sectorial operator $A : D(A) \to X$ is *almost* R-sectorial, where the latter means that $\{\lambda AR(\lambda, A)^2 : \lambda \in \Sigma\}$ is R-bounded for a sector Σ.

Proposition 2.3 (cf. [**KW13**, Theorem 1.3]). *Let, in the situation described above, (X_0, X_1) be an interpolation couple of reflexive and B-convex spaces and assume that, for $j = 0, 1$, $P_j : X_j \to Y_j$ are compatible surjections with compatible right inverses $J_j : Y_j \to X_j$. Assume, for $j = 0, 1$, that B_j has an H^∞-calculus on X_j and that A_j is almost R-sectorial on Y_j. Assume moreover that there are $\alpha < 0 < \beta$ such that*

$$P_0((X_0)^{\cdot}_{\alpha,B_0}) = (Y_0)^{\cdot}_{\alpha,A_0}, \quad P_1((X_1)^{\cdot}_{\beta,B_1}) = (Y_1)^{\cdot}_{\beta,A_1} \tag{4}$$

and

$$J_0 : (Y_0)^{\cdot}_{\alpha,A_0} \to (X_0)^{\cdot}_{\alpha,B_0}, \quad J_1 : (Y_1)^{\cdot}_{\beta,A_1} \to (X_1)^{\cdot}_{\beta,B_1}. \tag{5}$$

Then, for $\theta \in (0,1)$, the operator A_θ has an H^∞-calculus on the complex interpolation space $Y_\theta = [Y_0, Y_1]_\theta$.

Here the equality $P(\dot{X}_{\gamma,B}) = \dot{Y}_{\gamma,A}$ is meant in the following sense: the projection $P : X \to Y$, restricted to $X \cap \dot{X}_{\gamma,B} = D(B^\gamma)$, has a continuous extension $\tilde{P} : \dot{X}_{\gamma,B} \to \dot{Y}_{\gamma,A}$ which is surjective. This also implies that P is compatible with the interpolation couples $(X, \dot{X}_{\gamma,B})$ and $(Y, \dot{Y}_{\gamma,A})$.

Similarly, the embedding $J : \dot{Y}_{\gamma,A} \hookrightarrow \dot{X}_{\gamma,B}$ is meant to indicate that the right inverse $J : Y \to X$, restricted to $Y \cap \dot{Y}_{\gamma,A} = D(A^\gamma)$, has a continuous extension $\tilde{J} : \dot{Y}_{\gamma,A} \to \dot{X}_{\gamma,B}$. As a consequence, $\tilde{J}$ is a right inverse of $\tilde{P}$.

In the application below we shall use $X_1 = \mathbb{L}^2(\Omega)$, $X_0 = \mathbb{L}^q(\Omega)$ and $Y_1 = \mathbb{L}^2_\sigma(\Omega)$, $Y_0 = \mathbb{L}^q_\sigma(\Omega)$. The operator B will be a translate of the Laplace operator with Dirichlet boundary conditions and A will be the corresponding translate of the Stokes operator.

We shall not use θ as an explicit parameter, but $p \in [\min\{q, q'\}, \max\{q, q'\}]$, and we shall rather index operators, i.e. A_p, B_p, than spaces and simply write X and Y.

3. Proof of the main result.

Let $\Omega \subset \mathbb{R}^n$ be a domain with uniform C^3-boundary, and assume that $q \in (1, \infty)$ is such that the Helmholtz projection P_q exists in $\mathbb{L}^q(\Omega)$. By Proposition 2.1 these assumptions hold also for the conjugate exponent q'. Hence we may assume $q > 2$, and it suffices to show the assertion for $p \in (2, q)$. We fix $\lambda_0 > 0$ according to Proposition 2.2 and such that $0 \in \rho(\lambda_0 - A_p)$ for $p \in [2, q]$.

We denote by $B := \lambda_0 - \Delta$ the Laplace operator with Dirichlet boundary conditions in $\mathbb{L}^2(\Omega)$ which has domain

$$D(B) = W^{2,2}(\Omega)^n \cap W_0^{1,2}(\Omega)^n.$$

Denote by P the Helmholtz projection in $\mathbb{L}^2(\Omega)$ onto $\mathbb{L}^2_\sigma(\Omega)$. By abuse of notation we shall write $A := \lambda_0 - P\Delta$ with domain

$$D(A) = D(B) \cap \mathbb{L}^2_\sigma(\Omega) = W^{2,2}(\Omega)^n \cap W_0^{1,2}(\Omega)^n \cap \mathbb{L}^2_\sigma(\Omega).$$

Then B and A are self-adjoint operators in $\mathbb{L}^2(\Omega)$ and $\mathbb{L}^2_\sigma(\Omega)$ ([**FKS09**]), respectively, and generate exponentially stable semigroups.

For $X_0 := \mathbb{L}^q(\Omega)$ and $Y_0 := \mathbb{L}^q_\sigma(\Omega)$, the Helmholtz projection $P_q : X_1 \to Y_1$ has a continuous and surjective extension $\widetilde{P_q} : X_{-1,B_q} \to Y_{-1,A_q}$. This can be shown by the argument given in [**KKW06**], which we recall here for completeness. By [**GHHS12**], we have

$$D(A_q) = D(B_q) \cap \mathbb{L}^q_\sigma(\Omega) = W^{2,q}(\Omega)^n \cap W_0^{1,q}(\Omega)^n \cap \mathbb{L}^q_\sigma(\Omega), \tag{6}$$

and for the dual operators: $(A_q)^* = A_{q'}$, $(B_q)^* = B_{q'}$, and $(P_q)^* = \iota_{q'}$ where $\iota_{q'} : \mathbb{L}^{q'}_\sigma(\Omega) \to \mathbb{L}^{q'}(\Omega)$ denotes the inclusion. By [**KKW06**, Proposition 5.5] we thus have to show $D(A_{q'}) \subset D(B_{q'})$, which is clear from (6), and

$$C^{-1}\|A_{q'}g\|_{q'} \le \|B_{q'}g\|_{q'} \le C\|A_{q'}g\|_{q'}, \quad g \in D(A_{q'}), \tag{7}$$

for some constant C. Since $A_{q'} = P_{q'}B_{q'}$, the first estimate in (7) is clear. For the second estimate in (7), we use $0 \in \rho(A_{q'}) \cap \rho(B_{q'})$ and (6) again:

$$\|B_{q'}g\|_{p'} \le C'\|g\|_{W^{2,q'}} \le C''\|A_{q'}g\|_{q'}.$$

Hence the assumption (4) holds for P_q in place of P_0.

A right inverse of P_q is given by $J_q := B_q\iota_q A_q^{-1}$. Here, (6) is used again. Since $B_q : X \to X_{-1,B_q}$ and $A_q : Y \to Y_{-1,A_q}$ act as isomorphisms, $J_q : Y \to X$ has a continuous extension $Y_{-1,A_q} \to X_{-1,B_q}$. This means that the assumption (5) of Theorem 2.3 is satisfied in $X_0 = \mathbb{L}^q(\Omega)$, $Y_0 = \mathbb{L}^q_\sigma(\Omega)$ for $\alpha = -1$.

Now we turn to the other endpoint for the interpolation procedure, namely to the situation in $X_1 := \mathbb{L}^2(\Omega)$, $Y_1 := \mathbb{L}^2_\sigma(\Omega)$. For reasons explained in Remark 1.2 (b), we only assume that Ω has a uniform $C^{1,1}$-boundary for the moment. For $s \in (0, 1/4)$ the proof of [**Kun08**, Lemma 4.3] shows that

$$D(A^s) = D(B^s) \cap \mathbb{L}^2_\sigma(\Omega) = W^{2s,2}(\Omega)^n \cap \mathbb{L}^2_\sigma(\Omega) = PW^{2s,2}(\Omega)^n.$$

This means that the assumption (4) on $P_1 := P$ in Theorem 2.3 is satisfied for any $\beta \in (0, 1/4)$. It rests to verify the assumption (5) on J_1, i.e.

$$B\iota A^{-1} : D(A^\beta) \to D(B^\beta) \quad \text{for some } \beta \in (0, 1/4). \tag{8}$$

This means we have to show

$$D(A^{s+1}) \subset D(B^{s+1}) \quad \text{for some small } s > 0. \tag{9}$$

We shall proceed in several steps.

Claim 1. For $s \in (0, 1/4)$ one has $D(B^{s+1}) \cap \mathbb{L}^2_\sigma(\Omega) \subset D(A^{s+1})$.

Proof of Claim 1. For the proof we use that $D(A^{s+1}) = \{u \in D(A) : Au \in D(A^s)\}$. Let $u \in D(B^{s+1}) \cap \mathbb{L}^2_\sigma(\Omega)$. Then $Bu \in D(B^s)$ and $u \in D(B) \cap \mathbb{L}^2_\sigma(\Omega) = D(A)$. This implies $Au = PBu \in PD(B^s) = D(A^s)$, and we have shown $u \in D(A^{s+1})$. Claim 1 is proved. □

Now we note that the additional assumption "Ω has a uniform $C^{2+\mu}$-boundary" implies that

$$D(B^{s+1}) = W^{2(1+s),2}(\Omega)^n \cap W^{1,2}_0(\Omega)^n, \quad 0 < s < \min\{1/4, \mu/2\}.$$

Hence it suffices to show, for small $s > 0$:

$$\begin{aligned}&\text{For any } f \in D(A^s) = W^{2s,2}(\Omega)^n \cap \mathbb{L}^2_\sigma(\Omega)\\ &\text{there exists } u \in D(B^{s+1}) \cap \mathbb{L}^2_\sigma(\Omega) = W^{2(1+s),2}(\Omega)^n \cap W^{1,2}_0(\Omega)^n \cap \mathbb{L}^2_\sigma(\Omega)\\ &\text{such that } Au = f.\end{aligned} \tag{10}$$

Of course, it is clear that the solution of $Au = f$ is unique, namely $u = A^{-1}f$. If f runs through $D(A^s)$ then u runs through $D(A^{s+1})$, and the desired inclusion (9) would be proved.

We rewrite (10): $Au = f$ means $\lambda_0 u - P\Delta u = f$ or

$$\lambda_0 u - \Delta u + \nabla p = f$$

where $\nabla p = (I - P)\Delta u$. Hence we have to study the problem

$$\lambda_0 u - \Delta u + \nabla p = f, \qquad \operatorname{div} u = 0, \qquad u|_{\partial\Omega} = 0,$$

and should prove the following regularity property:

$$f \in W^{2s,2}(\Omega)^n \Longrightarrow \nabla p \in W^{2s,2}(\Omega),\ u \in W^{2(1+s),2}(\Omega)^n. \tag{11}$$

Observe that, under the assumption $f \in W^{2s,2}(\Omega)^n$, one has equivalence of the two properties on the right hand side of (11). Formulated in this way, we face an elliptic regularity problem similar to those studied in [**Soh01**, III.1.5].

So let $u \in W^{2,2}(\Omega)^n \cap W_0^{1,2}(\Omega)^n$ and $\nabla p \in \mathbb{L}^2(\Omega)$ be a solution to the equation

$$\begin{aligned} \lambda_0 u - \Delta u + \nabla p &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0. \end{aligned}$$

Assume now that $f \in W^{2s,2}(\Omega)^n$ and that Ω has a uniform $C^{2+\mu}$-boundary.

CLAIM 2. $u \in W^{2(1+s),2}(\Omega)^n$ and $\nabla p \in W^{2s,2}(\Omega)^n$, if $s \in (0, \min\{1/4, \mu/2\})$ is small.

PROOF OF CLAIM 2. Fix a partition of unity $(\varphi_j)_j$ subordinated to a family of balls (B_j) of fixed radius r and with the properties of [**FKS07**, pp. 242/243]. As in [**FKS07**, Section 2.2] we let $U_j = B_j \cap \Omega$. We have $\sum_j \varphi_j = 1$ on Ω.

Let $u_j := \varphi_j u$ and $f_j := \varphi_j f$. Then

$$\lambda_0 u_j - \varphi_j \Delta u + \varphi_j \nabla p = f_j, \qquad \operatorname{div} u_j = \nabla\varphi_j \cdot u, \qquad u_j|_{\partial U_j} = 0.$$

Moreover,

$$\begin{aligned} \Delta u_j = \Delta(\varphi_j u) &= (\Delta\varphi_j)u + 2\nabla\varphi_j \cdot \nabla u + \varphi_j \Delta u, \\ \nabla(\varphi_j(p - M_j)) &= (\nabla\varphi_j)(p - M_j) + \varphi_j \nabla p, \end{aligned}$$

where $M_j := \int_{U_j} p\,dx$. Hence

$$\lambda_0 u_j - \Delta u_j + \nabla(\varphi_j(p - M_j)) = f_j - (\Delta\varphi_j)u - 2\nabla\varphi_j \cdot \nabla u + (\nabla\varphi_j)(p - M_j).$$

Here, $f_j \in W^{2s,2}(U_j)^n$, $(\Delta\varphi_j)u \in W^{2,2}(U_j)^n$, $2\nabla\varphi_j \cdot \nabla u \in W^{1,2}(U_j)^n$, and $(\nabla\varphi_j)(p - M_j) \in W^{1,2}(U_j)^n$. For the boundary conditions, we have $u_j|_{\partial U_j} = 0$ and

$$\int_{U_j} \operatorname{div} u_j\,dx = \int_{\partial U_j} \nu_{U_j} \cdot (\varphi_j u) d\sigma = 0,$$

since $u|_{\partial\Omega \cap \partial U_j} = 0$ and $\varphi_j = 0$ on a neighborhood of $\partial U_j \setminus \partial\Omega$.

Claim 2 will follow from

CLAIM 3. For $g \in W^{2s,2}(U)^n$ and $h \in W^{1+2s,2}(U)$ with $\int_U h\,dx = 0$ there is a unique solution $(v,\nabla q)\in W^{2+2s,2}(U)^n\times W^{2s,2}(U)^n$ of the problem

$$\begin{aligned}\lambda_0 v-\Delta v+\nabla q&=g\quad \text{in } U,\\ \operatorname{div} v&=h\quad \text{in } U,\\ v|_{\partial U}&=0\end{aligned}$$

and, for some $C>0$, we have the estimate

$$\|v\|_{W^{2+2s,2}(U)^n}+\|\nabla q\|_{W^{2s,2}(U)^n}\le C\big(\|g\|_{W^{2s,2}(U)^n}+\|h\|_{W^{1+2s,2}(U)}\big).$$

Of course, we need this for $U=U_j$ with a constant C which is uniform in j.

Claim 3 will be proved below. We apply Claim 3 to $U=U_j$, $v=u_j$, $q=\varphi_j(p-M_j)$, and

$$g=f_j-(\Delta\varphi_j)u-2\nabla\varphi_j\cdot\nabla u-(\nabla\varphi_j)(p-M_j),\qquad h=\nabla\varphi_j\cdot u,$$

and obtain

$$\begin{aligned}\|u_j\|_{W^{2+2s,2}(U_j)^n}\le C\big(&\|f_j\|_{W^{2s,2}(U_j)^n}+\|(\Delta\varphi_j)u\|_{W^{2s,2}(U_j)^n}+\|\nabla\varphi_j\cdot\nabla u\|_{W^{2s,2}(U_j)^n}\\ &+\|(\nabla\varphi_j)(p-M_j)\|_{W^{2s,2}(U_j)^n}+\|\nabla\varphi_j\cdot u\|_{W^{1+2s,2}(U_j)}\big).\end{aligned}$$

We study the terms involving u and p on the right hand side:

$$\begin{aligned}\|(\Delta\varphi_j)u\|_{W^{2s,2}(U_j)^n}&\le C\|u\|_{W^{2s,2}(U_j)^n}\le C\|u\|_{W^{2,2}(U_j)^n},\\ \|\nabla\varphi_j\cdot\nabla u\|_{W^{2s,2}(U_j)^n}&\le C\|\nabla u\|_{W^{2s,2}(U_j)^{n\times n}}\le C\|\nabla u\|_{W^{1,2}(U_j)^{n\times n}}\le C\|u\|_{W^{2,2}(U_j)^n},\\ \|\nabla\varphi_j\cdot u\|_{W^{1+2s,2}(U_j)}&\le C\|u\|_{W^{1+2s,2}(U_j)^n}\le C\|u\|_{W^{2,2}(U_j)^n},\\ \|(\nabla\varphi_j)(p-M_j)\|_{W^{2s,2}(U_j)^n}&\le C\|p-M_j\|_{W^{1,2}(U_j)}\le C\|\nabla p\|_{\mathbb{L}^2(\Omega)},\end{aligned}$$

where we used Poincaré's inequality in U_j in the last step. Here, constants C do not depend on j.

By finite intersection of the U_j and the properties of the φ_j we thus obtain

$$\|u\|_{W^{2+2s,2}(\Omega)^n}\le\sum_j\|u_j\|_{W^{2+2s,2}(U_j)^n}\le C\big(\|f\|_{W^{2s,2}(\Omega)^n}+\|u\|_{W^{2,2}(\Omega)^n}+\|\nabla p\|_{\mathbb{L}^2(\Omega)}\big).$$

By [**FKS09**] we have

$$\|u\|_{W^{2,2}(\Omega)^n}+\|\nabla p\|_{\mathbb{L}^2(\Omega)}\le C'\|f\|_{\mathbb{L}^2(\Omega)},$$

and Claim 2 is proved. □

Proof of Claim 3. Claim 3 can certainly be proved directly. One can, however, also use the extrapolation argument below. Then one only needs

CLAIM 4. For $g \in \mathbb{L}^2(U)$ and $h \in W^{1,2}(U)$ with $\int_U h\,dx = 0$ there is a unique solution $(v, \nabla q) \in W^{2,2}(U)^n \times \mathbb{L}^2(U)$ of the problem

$$\begin{aligned} \lambda_0 v - \Delta v + \nabla q &= g \quad \text{in } U, \\ \operatorname{div} v &= h \quad \text{in } U, \\ v|_{\partial U} &= 0 \end{aligned}$$

and, for some $C > 0$, we have the estimate

$$\|v\|_{W^{2,2}(U)^n} + \|\nabla q\|_{\mathbb{L}^2(U)} \leq C\big(\|g\|_{\mathbb{L}^2(U)} + \|h\|_{W^{1,2}(U)}\big).$$

Again, we need this for $U = U_j$ with a constant C which is uniform in j.

The assertion of Claim 4 holds as a close inspection of the proof of [**FS94**, Theorem 3.1(i), p. 624] shows.

For the extrapolation argument (which uses the same idea as in [**KW13**]) we consider the map

$$((v_j), (q_j)) \mapsto (\lambda_0 v_j - \Delta v_j + \nabla q_j, \operatorname{div} v_j)$$

which is bounded as a map

$$\begin{aligned} l^2 - \bigoplus_j \big(W^{2+2s,2}(U_j)^n \cap W_0^{1,2}(U_j)^n\big) \times \widehat{W}^{1+2s,2}(U_j) \\ \to l^2 - \bigoplus_j W^{2s,2}(U_j)^n \times \widehat{W}^{1+2s,2}(U_j) \end{aligned} \tag{12}$$

for small $|s|$. Here, for a sequence of Banach spaces (X_j) the Banach space $l^2 - \bigoplus_j X_j$ is defined by

$$l^2 - \bigoplus_j X_j =: \left\{(x_j)_{j\in\mathbb{N}} : x_j \in X_j \text{ and } \left(\sum_{j\in\mathbb{N}} \|x_j\|_{X_j}^2\right)^{1/2} < \infty\right\}.$$

Moreover, we note that, under our assumptions,

$$\widehat{W}^{1+2s,2}(U_j) = \left\{h \in W^{1+2s,2}(U_j) : \int_{U_j} h\,dx = 0\right\}$$

for small $|s|$. The spaces on the left and on the right of (12) form complex interpolation scales, respectively. By Claim 4 the map is an isomorphism for $s = 0$. By [**KM98**, Theorem 2.7] this also holds for small $s > 0$, and Claim 3 is proved for small $s > 0$. □

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